ALJABAR LINIER

Vector – Cross Product

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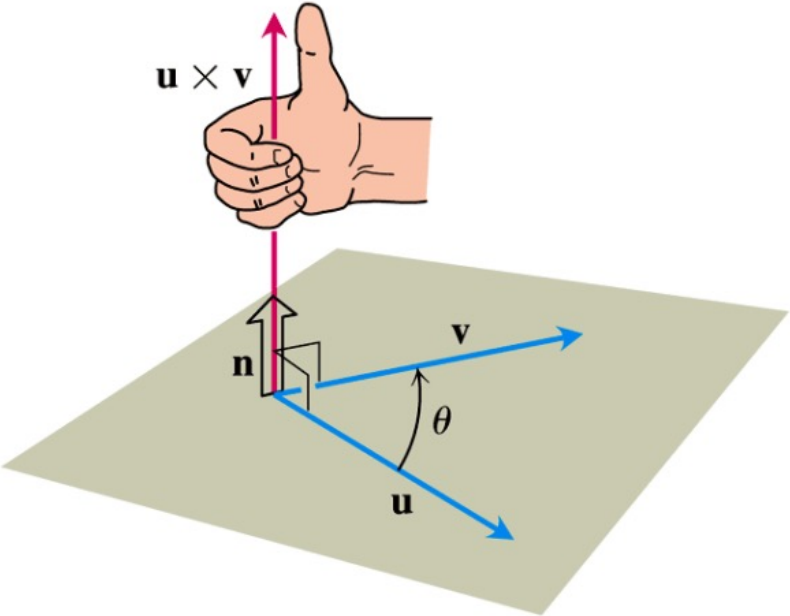
Cross Product

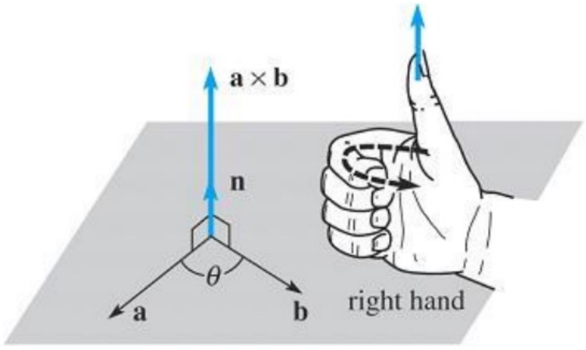
Deskripsi Geometri dari *Cross Product*

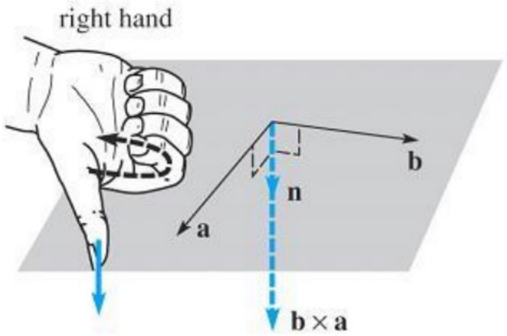
- $\boldsymbol{u} \times \boldsymbol{v}$ tegak lurus terhadap $\boldsymbol{u}$ dan $\boldsymbol{v}$
- Panjanga dari $\boldsymbol{u} \times \boldsymbol{v}$ adalah
  $|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}| sin\ \theta$
- Arah ditunjukkan dengan aturan tangan kanan



Aturan Tangan Kanan

- Letakkan 4 jari pada arah vektor pertama.
- Tekuk jari tersebut pada arah vektor kedua.
- Ibu jari menjukkan arah dari *cross product*





Definisi Aljabar dari Cross Product

Cross product dari $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ dab $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$ adalah

$$\boldsymbol{u} \times \boldsymbol{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$

Sehingga:

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{u} = \langle u_2 v_3 - u_3 v_2, u_3, v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \cdot \langle u_1, u_2, u_3 \rangle$$

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{u} = u_2 v_3 u_1 - u_3 v_2 u_1 + u_3 v_1 u_2 - u_1 v_3 u_2 + u_1 v_2 u_3 - u_2 v_1 u_3$$

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{u} = 0$$

Begitu juga:

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{v} = 0$$

Nilai Cross Product Berdasarkan Determinan Vektor #1

Diberikan vektor $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ dan $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$, nilai cross product di definisikan dengan determinan vektor,

$$\boldsymbol{u} \times \boldsymbol{v} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$

Contoh, hitung $\boldsymbol{u} \times \boldsymbol{v}$ dan $\boldsymbol{v} \times \boldsymbol{u}$ untuk $\boldsymbol{u} = \langle 2, 3, 5 \rangle$ dan $\boldsymbol{v} = \langle 6, 7, 9 \rangle$.

$$\boldsymbol{u} \times \boldsymbol{v} = \left\langle \begin{vmatrix} 3 & 5 \\ 7 & 9 \end{vmatrix}, \begin{vmatrix} 5 & 2 \\ 9 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 6 & 7 \end{vmatrix} \right\rangle$$

$$\boldsymbol{u} \times \boldsymbol{v} = \langle 3 \cdot 9 - 7 \cdot 5, 5 \cdot 6 - 9 \cdot 2, 2 \cdot 7 - 6 \cdot 3 \rangle = \langle -8, 12, -4 \rangle$$

Nilai Cross Product Berdasarkan Determinan Vektor #2

Selanjutnya, jika, $\boldsymbol{v} \times \boldsymbol{u}$

$$\boldsymbol{v} \times \boldsymbol{u} = \left\langle \begin{vmatrix} 7 & 9 \\ 3 & 5 \end{vmatrix} , \begin{vmatrix} 9 & 6 \\ 5 & 2 \end{vmatrix} , \begin{vmatrix} 6 & 7 \\ 2 & 3 \end{vmatrix} \right\rangle = \langle 8, -12, 4 \rangle$$

Maka,

$$\boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})$$

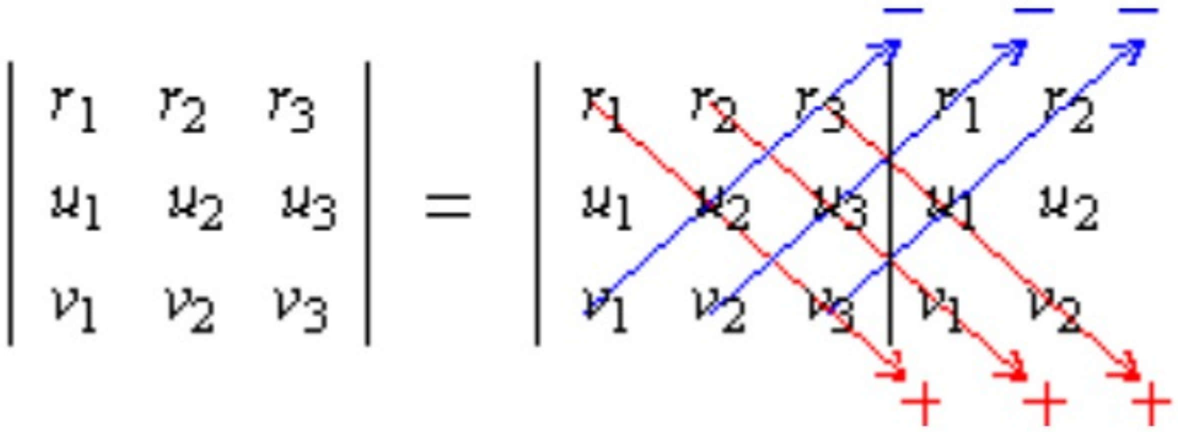
Sifat-sifat

Cross product di definisikan hanya dengan vektor 3 dimensi, vector $\boldsymbol{u}$ dan $\boldsymbol{v}$. Sehingga berlaku,

- $\boldsymbol{v} \times \boldsymbol{u} = -(\boldsymbol{u} \times \boldsymbol{v})$
- $\boldsymbol{u} \times \boldsymbol{u} = 0$
- $(k\boldsymbol{u}) \times \boldsymbol{v} = k(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u} \times k\boldsymbol{v}$
- $\boldsymbol{a} \times (\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{a} \times \boldsymbol{u} + \boldsymbol{a} \times \boldsymbol{v}$

Bagaimana jika menggunakan nilai determinan 3×3? #1

$$\begin{vmatrix} r_1 & r_2 & r_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = r_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + r_2 \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} + r_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$



Bagaimana jika menggunakan nilai determinan 3×3? #2

Jika kita letakkan, **i**, **j**, dan **k** pada baris pertama,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Bagaimana jika menggunakan nilai determinan 3×3? Contoh

Dengan determinan 3 dimensi, hitung $\boldsymbol{u} \times \boldsymbol{v}$ dan $\boldsymbol{v} \times \boldsymbol{u}$ untuk $\boldsymbol{u} = \langle 2, 1, 2 \rangle$ and $\boldsymbol{v} = \langle 3, 4, 5 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 3 & 4 & 5 \end{vmatrix} = \left| \begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 2 & 1 & 2 & 2 & 1 \\ 3 & 4 & 5 & 3 & 4 \end{array} \right. = 5\mathbf{i} + 6\mathbf{j} + 8\mathbf{k} - 3\mathbf{k} - 8\mathbf{i} - 10\mathbf{j}$$

$$\boldsymbol{u} \times \boldsymbol{v} = -3\boldsymbol{i}, -4\boldsymbol{j} + 5\boldsymbol{k}$$

The Triple Scalar Product

If $\mathbf{r} = \langle r_1, r_2, r_3 \rangle$, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

$$\mathbf{r} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} r_1 & r_2 & r_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\mathbf{r} \cdot (\mathbf{u} \times \mathbf{v}) = \left| \begin{array}{ccc|cc} r_1 & r_2 & r_3 & r_1 & r_2 \\ u_1 & u_2 & u_3 & u_1 & u_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 \end{array} \right. = \left| \begin{array}{ccc|cc} v_1 & v_2 & v_3 & v_1 & v_2 \\ r_1 & r_2 & r_3 & r_1 & r_2 \\ u_1 & u_2 & u_3 & u_1 & u_2 \end{array} \right. = (\mathbf{r} \times \mathbf{u}) \cdot \mathbf{v}$$

We call this identity the *triple scalar product:*

$$\mathbf{r} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{r} \times \mathbf{u}) \cdot \mathbf{v}$$

Latihan

Compute the cross product of $\mathbf{u} \times \mathbf{v}$ *and then compute the cross product of* $\mathbf{v} \times \mathbf{u}$. *Also, show that* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal to* $\mathbf{u} \times \mathbf{v}$.

1. $\mathbf{u} = \langle 2, 1, 0 \rangle$, $\mathbf{v} = \langle 3, 1, 0 \rangle$
2. $\mathbf{u} = \langle 2, 1, 0 \rangle$, $\mathbf{v} = \langle -1, 3, 0 \rangle$
3. $\mathbf{u} = \langle 3, 3, 0 \rangle$, $\mathbf{v} = \langle 2, 0, 0 \rangle$
4. $\mathbf{u} = \langle 0, 1, 0 \rangle$, $\mathbf{v} = \langle 0, 0, 1 \rangle$
5. $\mathbf{u} = \langle 1, 0, 0 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$
6. $\mathbf{u} = \langle 1, 0, 0 \rangle$, $\mathbf{v} = \langle 1, 0, 0 \rangle$
7. $\mathbf{u} = \langle 2, 3, 7 \rangle$, $\mathbf{v} = \langle 7, 3, 5 \rangle$
8. $\mathbf{u} = \langle 6, 2, 9 \rangle$, $\mathbf{v} = \langle 1, 0, 3 \rangle$
9. $\mathbf{u} = \langle 3, 4, 2 \rangle$, $\mathbf{v} = \langle 9, 12, 6 \rangle$
10. $\mathbf{u} = \langle 1, 1, 1 \rangle$, $\mathbf{v} = \langle -1, -1, -1 \rangle$


thank you
danke
謝謝
gracias
merci
obrigado
teşekkür ederim
ngiyabonga
شكراً جزيلاً
рахмат
Баярлалаа
спасибо
bedankt
dziękuję
sukriya
kop khun krap
terima kasih
go raibh maith agat
mochchakkeram
arigatô
dakujem
grazie
감사합니다
ありがとう
мерси
tapadh leat
dank je
hvala
mauruuru
sagolun
takk
taiku
faafetai lava
kiitos
dankie
dhanyavad
nanni
nandri
enkosi
bayarlalaa
gràcie
köszönöm
sobodi
dėkuji
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didi madloba
kam sah hamnida
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rahmat
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Referensi

- https://www2.math.upenn.edu/~wziller/math114f13/ch12-4+5-1.pdf
- https://math.etsu.edu/multicalc/prealpha/Chap1/Chap1-3/printversion.pdf