Seria 33, EDDP, Curs 13, 11.01.2021

Ecuații cu derivate parțiale de ordinul al doilea.

Fie $n \in \mathbb{N}^*$, $n \geq 2$, $x = (x_1, \dots, x_n) \in D \subset \mathbb{R}^n$. Se cere determinarea unei funcții $u : D \to \mathbb{R}$ care verifică:

$$\boxed{F\left(x, u, \partial_1 u, \dots, \partial_n u, \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j=\overline{1,n}}\right) = 0} \quad (1)$$

unde $F : G \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\frac{n(n+1)}{2}} \longrightarrow \mathbb{R}$ este o funcție arbitrară.

Ec. (1) cu derivate parțiale de ordinul al doilea se numește cvasiliniară dacă are forma:

$$\boxed{\sum_{i,j=1}^{n} a_{ij}(x) \underbrace{\frac{\partial^2 u}{\partial x_i \partial x_j}}_{\partial_i \partial_j u} + f(x, u, \partial_1 u, \dots, \partial_n u) = 0} \quad (2)$$

unde $f : G_1 \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ este o funcție arbitrară.

Ec. (2) îi asociem forma pătratică:

$$g(t_1, \dots, t_n) = \sum_{\substack{i=1 \\ j=1}}^{n} a_{ij}(x)\, t_i t_j \quad , x \in D. \quad (3)$$

Fie $x_0 \in D$ fixat $\Rightarrow$ $g(t_1, \dots, t_n)$ are coef constanți: $a_{ij}(x_0)$, $i,j = \overline{1,n}$

Se știe de la algebră că: $\exists$ o transformare liniară a coordonatelor $(t_1, \dots, t_n)$ în coordonate $(s_1, \dots, s_n)$ dată prin:

$$t_i = \sum_{p=1}^{n} b_{ip} s_p \quad , i = \overline{1,n} \quad (4)$$

adică:

$$\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \\ b_{n1} & \dots & b_{nn} \end{pmatrix}}_{B \in \mathcal{M}_n(\mathbb{R})} \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix},$$

astfel încât g să se scrie în formă canonică;

adică: $\tilde{g}(s_1,\dots,s_n) = \sum_{i=1}^{m} s_i^2 - \sum_{j=m+1}^{r} s_j^2 \quad (5)$

unde $\underline{m \geq 0}$, $\underline{0 \leq r \leq n}$.
$m \leq n$

Pt. (5) avem cazurile:

(I) $\boxed{\begin{matrix} r=0 \\ m=n \end{matrix}}$ sau $\boxed{\begin{matrix} m=0 \\ r=n \end{matrix}}$ $\Rightarrow$ ec. (2) este de tip <u>eliptic</u>.

$\tilde{g}(s_1,\dots,s_n) = \sum_{i=1}^{n} s_i^2$ $\qquad$ $\tilde{g}(s_1,\dots,s_n) = -\sum_{j=1}^{n} s_j^2$

(II) $\boxed{\begin{matrix} r=n \\ 0<m<n \end{matrix}}$ $\Rightarrow \tilde{g}(s_1,\dots,s_n) = \sum_{i=1}^{m} s_i^2 - \sum_{j=m+1}^{n} s_j^2 \Rightarrow$

$\Rightarrow$ ec. (2) este de tip <u>hiperbolic</u>.

(III) $\boxed{\begin{matrix} r<n \\ 0\leq m<n \end{matrix}}$ $\Rightarrow \tilde{g}(s_1,\dots,s_n) = \sum_{i=1}^{m} s_i^2 - \sum_{j=m+1}^{r} s_j^2 \Rightarrow$

$\Rightarrow$ ec. (2) este de tip <u>parabolic</u>

<u>Cazuri particulare</u>

- pt (II): $m=n-1$: $\tilde{g}(s_1,\dots,s_n) = \sum_{i=1}^{n-1} s_i^2 - s_n^2 \Rightarrow$

$\Rightarrow$ ec (2) este de tip <u>hiperbolic normal</u>.

- pt (III): $r=n-1 \Rightarrow$ ec. (2) este tip <u>parabolic normal</u>.

<u>Exemplu</u>: $\underline{n=3}$:

$$\partial_1^2 u + 2\partial_1\partial_2 u - 2\partial_1\partial_3 u + 2\partial_2^2 u + 6\partial_3^2 u = 0$$

ec. cvasiliniară de ordin 2 în $\mathbb{R}^3$,

cu coef constanți: $a_{11}=1$; $a_{12}=a_{21}=\frac{2}{2}=1$

$a_{13}=a_{31}=\frac{-2}{2}=-1$

$a_{22}=2$; $a_{23}=a_{32}=\frac{0}{2}=0$

$a_{33}=6$.

forma pătratică asociată este:

$g(t_1,t_2,t_3) = t_1^2 + 2t_1t_2 - 2t_1t_3 + 2t_2^2 + 6t_3^2$

Pt. forma canonică a lui g folosim metoda Gauss:

$$g(t_1,t_2,t_3)=\left(t_1^2+2t_1t_2-2t_1t_3\right)+2t_2^2+6t_3^2=$$

$$=\left(t_1^2+t_2^2+t_3^2+2t_1t_2-2t_1t_3-2t_2t_3\right)-t_2^2-t_3^2+$$

$$+2t_2t_3+2t_2^2+6t_3^2=$$

$$=\left(t_1+t_2-t_3\right)^2+\left(t_2^2+2t_2t_3\right)+5t_3^2=$$

$$=\left(t_1+t_2-t_3\right)^2+\left(t_2^2+2t_3t_2+t_3^2\right)-t_3^2+5t_3^2=$$

$$=\left(t_1+t_2-t_3\right)^2+\left(t_2+t_3\right)^2+\left(2t_3\right)^2\Rightarrow$$

$$\Rightarrow\begin{cases}s_1=t_1+t_2-t_3\\ s_2=t_2+t_3\\ s_3=2t_3\end{cases}\Leftrightarrow\begin{cases}t_1=s_1-s_2+\frac{1}{2}s_3+\frac{1}{2}s_3\\ t_2=s_2-\frac{1}{2}s_3\\ t_3=\frac{1}{2}s_3\end{cases}\Rightarrow$$

$$\Rightarrow\begin{pmatrix}t_1\\t_2\\t_3\end{pmatrix}=\underbrace{\begin{pmatrix}1&-1&1\\0&1&-\frac{1}{2}\\0&0&\frac{1}{2}\end{pmatrix}}_{B}\begin{pmatrix}s_1\\s_2\\s_3\end{pmatrix}\Rightarrow\tilde{g}(s_1,s_2,s_3)=s_1^2+s_2^2+s_3^2$$

$\to$ ec. este de tip eliptic.

$(r=0,\ m=n=3)$.

Folosind forma canonică, se poate face o schimbare de variab. în ec.:

$$x=(x_1,\dots,x_n)\longrightarrow y=(y_1,\dots,y_n)$$

$$u=u(x)\qquad\qquad y=y(x)$$

$$\qquad\qquad\qquad\tilde{u}=\tilde{u}(y)$$

$$(x,u)\qquad\qquad(y,\tilde{u})$$

Din forma canonică: $t=Bs\Longrightarrow$

$\Rightarrow\boxed{y=B^T x.}\Rightarrow$ Ec în y și $\tilde{u}$

este de forma: $\dfrac{\partial^2\tilde{u}}{\partial y_1^2}+\dots+\dfrac{\partial^2\tilde{u}}{\partial y_n^2}=0.$

în cazul ec. eliptice.

Pt. exemplul considerat avem:

$$y=\begin{pmatrix}1&0&0\\-1&1&0\\1&-\frac{1}{2}&\frac{1}{2}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\Rightarrow\begin{cases}y_1=x_1\\ y_2=-x_1+x_2\\ y_3=x_1-\frac{1}{2}x_2-\frac{1}{2}x_3\end{cases}\qquad(6)$$

Avem: $u(x_1,x_2,x_3) = \tilde{u}(y_1(x_1,x_2,x_3), y_2(x_1,x_2,x_3), y_3(x_1,x_2,x_3))$

Pt. a arata că se obține forma canonică a ec. din exemplu, calculăm derivatele lui u în funcție de derivatele lui $\tilde{u}$:

$$\partial_1 u = \frac{\partial u}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\tilde{u}(y_1(x), y_2(x), y_3(x))\right) =$$

$$= \frac{\partial \tilde{u}}{\partial y_1}\cdot \underbrace{\frac{\partial y_1}{\partial x_1}}_{=1} + \frac{\partial \tilde{u}}{\partial y_2}\cdot \underbrace{\frac{\partial y_2}{\partial x_1}}_{=-1} + \frac{\partial \tilde{u}}{\partial y_3}\cdot \underbrace{\frac{\partial y_3}{\partial x_1}}_{1} \Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x_1} = \frac{\partial \tilde{u}}{\partial y_1} - \frac{\partial \tilde{u}}{\partial y_2} + \frac{\partial \tilde{u}}{\partial y_3}}$$

Calculând $\frac{\partial u}{\partial x_2}$, $\frac{\partial u}{\partial x_3}$, $\frac{\partial^2 u}{\partial x_1^2}$, $\frac{\partial^2 u}{\partial x_2^2}$, $\frac{\partial^2 u}{\partial x_3^2}$, $\frac{\partial^2 u}{\partial x_1 \partial x_2}$, $\frac{\partial^2 u}{\partial x_2 \partial x_3}$, $\frac{\partial^2 u}{\partial x_1 \partial x_3}$ și înlocuindu-le în ec din exemplu $\Rightarrow \frac{\partial^2 \tilde{u}}{\partial y_1^2} + \frac{\partial^2 \tilde{u}}{\partial y_2^2} + \frac{\partial^2 \tilde{u}}{\partial y_3^2} = 0$

<u>Cazul particular</u> $\boxed{n=2}$ $\Rightarrow$ Ec. cvasiliniară cu derivate parțiale de ordinul doi în 2 variabile se scrie:

$$(*)\quad a(x_1,x_2)\,\partial_1^2 u + 2b(x_1,x_2)\,\partial_1\partial_2 u + c(x_1,x_2)\,\partial_2^2 u + f(x_1,x_2,u,\partial_1 u,\partial_2 u) = 0 .$$

<u>Obs</u>: Din forma generală: $\begin{cases} a_{11}(x) = a(x) \\ a_{12}(x) = a_{21}(x) = b(x) \\ a_{22}(x) = c(x) \end{cases}$

<u>Clasificarea ca ec. eliptică, hiperbolică, parabolică în cazul</u> $\boxed{n=2}$ se face după următorul <u>algoritm</u>:

- calculăm $d(x_1,x_2) = b^2(x_1,x_2) - a(x_1,x_2)\cdot c(x_1,x_2)$
- Avem cazurile:
  - I) $d(x_1,x_2) > 0 \Rightarrow$ ec. de tip hiperbolic
  - II) $d(x_1,x_2) = 0 \Rightarrow$ ec. de tip parabolic
  - III) $d(x_1,x_2) < 0 \Rightarrow$ ec. de tip eliptic.

• I) $\boxed{d(x_1, x_2) > 0}$

• calculăm
$$\lambda_1(x_1, x_2) = \frac{b(x_1, x_2) - \sqrt{d(x_1, x_2)}}{a(x_1, x_2)} \in \mathbb{R}$$
$$\lambda_2(x_1, x_2) = \frac{b(x_1, x_2) + \sqrt{d(x_1, x_2)}}{a(x_1, x_2)} \in \mathbb{R}.$$

• se determină integrale prime pt. ec.:
$$\frac{dx_2}{dx_1} = \lambda_1(x_1, x_2)$$
şi
$$\frac{dx_2}{dx_1} = \lambda_2(x_1, x_2)$$

şi fie, integrale prime $\varphi_1(x_1, x_2) = C_1$
$\varphi_2(x_1, x_2) = C_2$

• pt. a aduce ec. (7) la forma canonică se face s.v.
$$\begin{cases} y_1 = \varphi_1(x_1, x_2) \\ y_2 = \varphi_2(x_1, x_2) \end{cases}$$
şi $u(x) = \tilde{u}(y)$.

• se obține forma canonică: $\frac{\partial^2 \tilde{u}}{\partial y_1^2} - \frac{\partial^2 \tilde{u}}{\partial y_2^2} + \tilde{f}(y, \tilde{u}, \partial_1 \tilde{u}, \partial_2 \tilde{u}) = 0$

II) $\boxed{d(x_1, x_2) = 0}$ $\Rightarrow \lambda_1(x_1, x_2) = \frac{b(x_1, x_2)}{a(x_1, x_2)} = \lambda_2(x_1, x_2)$

• se determină o integrală primă a ec.:
$$\frac{dx_2}{dx_1} = \lambda_1(x_1, x_2), \text{ fie aceasta } \varphi_1(x) = C_1.$$

• se consideră schimbarea de variabilă:
$$\begin{cases} y_1 = \varphi_1(x_1, x_2) \\ y_2 = \varphi_2(x_1, x_2) \end{cases}$$
cu φ_2 convenabil ales $\left(\text{a.i. } \begin{vmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} \end{vmatrix} \neq 0. \right)$

şi $u(x) = \tilde{u}(y)$.

• se obține forma canonică:
$$\frac{\partial^2 \tilde{u}}{\partial y_1^2} + \tilde{f}(y, u, \partial_1 \tilde{u}, \partial_2 \tilde{u}) = 0$$
sau
$$\frac{\partial^2 \tilde{u}}{\partial y_2^2} + \tilde{f}(y, u, \partial_1 \tilde{u}, \partial_2 \tilde{u}) = 0.$$

III) $\boxed{d(x_1,x_2) < 0}$

• se calculăm $\lambda_1(x_1,x_2) = \dfrac{b(x_1,x_2) + i\sqrt{-d(x_1,x_2)}}{a(x_1,x_2)}$

$\lambda_2(x_1,x_2) = \overline{\lambda_1}(x_1,x_2)$

• se determină o integrală primă în complex pt ec.: $\dfrac{dx_2}{dx_1} = \lambda_1(x_1,x_2)$ și fie aceasta $\varphi_1(x) = C_1$

• se consideră transformarea:

$$\begin{cases} y_1 = \mathrm{Re}\left(\varphi_1(x_1,x_2)\right) \\ y_2 = \mathrm{Im}\left(\varphi_1(x_1,x_2)\right) \end{cases}$$

și $u(x) = \tilde{u}(y)$

• forma canonică a ec. este:

$$\frac{\partial^2 \tilde{u}}{\partial y_1^2} + \frac{\partial^2 \tilde{u}}{\partial y_2^2} + \tilde{f}(y, u, \partial_1 \tilde{u}, \partial_2 \tilde{u}) = 0.$$

<u>Exemple</u>: Să se aducă la forma canonică ec:

✓ 1) $\underbrace{\partial_1^2 u - 6\partial_1\partial_2 u + 10\partial_2^2 u} + \partial_1 u - 3\partial_2 u = 0$

2) $\underbrace{4\partial_1^2 u + 4\partial_1\partial_2 u + \partial_2^2 u} - 2\partial_2 u = 0$

3) $\underbrace{\partial_1^2 u + 2\partial_1\partial_2 u - 3\partial_2^2 u} + \partial_2 u = 0$

1) $\left.\begin{array}{l} a(x) = 1 \\ b(x) = \dfrac{-6}{2} = -3 \\ c(x) = 10. \end{array}\right|$ constante ; $x = (x_1, x_2)$

$d(x) = b^2(x) - a(x)\cdot c(x) = (-3)^2 - 1\cdot 10 =$
$= 9 - 10 = -1 < 0 \Rightarrow$

$\Rightarrow$ ec. este de tip eliptic.

$\lambda_1(x_1,x_2) = \dfrac{-3 + i\sqrt{1}}{1} = -3 + i$

$\lambda_2(x_1,x_2) = \overline{\lambda_1}$.

$$\frac{dx_2}{dx_1} = \lambda_1 \Rightarrow \frac{dx_2}{dx_1} = -3+i \Rightarrow$$

$$\Rightarrow dx_2 = (-3+i)\,dx_1 \Rightarrow$$

(am separat variabilele)

$$\Rightarrow \int dx_2 = (-3+i)\int dx_1 \Rightarrow$$

$$\Rightarrow x_2 = (-3+i)x_1 + C_1 \Rightarrow$$

$$\Rightarrow \underbrace{(3-i)x_1 + x_2}_{\varphi_1(x)} = C_1$$

Se obține transformarea: $\begin{cases} y_1 = Re(\varphi_1(x)) = 3x_1 + x_2 \\ y_2 = Im(\varphi_1(x)) = -x_1 \end{cases}$

și $u(x) = \tilde{u}(y)$

Calculăm: $\frac{\partial u}{\partial x_1} = \frac{\partial \tilde{u}}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1} + \frac{\partial \tilde{u}}{\partial y_2}\cdot\frac{\partial y_2}{x_1}$

$$\frac{\partial y_1}{\partial x_1} = 3;\quad \frac{\partial y_1}{\partial x_2} = 1;\quad \frac{\partial y_2}{\partial x_1} = -1;\quad \frac{\partial y_2}{\partial x_2} = 0.$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x_1} = 3\frac{\partial \tilde{u}}{\partial y_1} - \frac{\partial \tilde{u}}{\partial y_2}}$$

$$\boxed{\frac{\partial u}{\partial x_2}} = \frac{\partial \tilde{u}}{\partial y_1}\cdot 1 + \frac{\partial \tilde{u}}{\partial y_2}\cdot 0 = \boxed{\frac{\partial \tilde{u}}{\partial y_1}}$$

$$\frac{\partial^2 u}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(3\frac{\partial \tilde{u}}{\partial y_1} - \frac{\partial \tilde{u}}{\partial y_2}\right) =$$

$$= \frac{\partial}{\partial y_1}\left(3\frac{\partial \tilde{u}}{\partial y_1} - \frac{\partial \tilde{u}}{\partial y_2}\right)\cdot 3 + \frac{\partial}{\partial y_2}\left(3\frac{\partial \tilde{u}}{\partial y_1} - \frac{\partial \tilde{u}}{\partial y_2}\right)\cdot(-1) \Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x_1^2} = 9\frac{\partial^2 \tilde{u}}{\partial y_1^2} - 6\frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2} + \frac{\partial^2 \tilde{u}}{\partial y_2^2}}$$

$$\frac{\partial^2 u}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_2}\right) = \frac{\partial}{\partial x_2}\left(\frac{\partial \tilde{u}}{\partial y_1}\right) = \frac{\partial^2 \tilde{u}}{\partial y_1^2}\cdot 1 + \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2}\cdot 0 \Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x_2^2} = \frac{\partial^2 \tilde{u}}{\partial y_1^2}}$$

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_2}\right) = \frac{\partial}{\partial x_1}\cdot\left(\frac{\partial \tilde{u}}{\partial y_1}\right) = \frac{\partial^2 \tilde{u}}{\partial y_1^2}\cdot 3 + \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2}(-1) \Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x_1 \partial x_2} = 3\frac{\partial^2 \tilde{u}}{\partial y_1^2} - \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2}}$$

Ec. în $(y, \tilde{u})$ este:

$$\underline{9\frac{\partial^2 \tilde{u}}{\partial y_1^2}} - \cancel{6\frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2}} + \frac{\partial^2 \tilde{u}}{\partial y_2^2} \underline{- 18\frac{\partial^2 \tilde{u}}{\partial y_1^2}} + \cancel{6\frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2}} \underline{+ 10\frac{\partial^2 \tilde{u}}{\partial y_1^2}} + \cancel{3\frac{\partial \tilde{u}}{\partial y_1}} - \frac{\partial \tilde{u}}{\partial y_2} - \cancel{3\frac{\partial \tilde{u}}{\partial y_1}} = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial^2 \tilde{u}}{\partial y_1^2} + \frac{\partial^2 \tilde{u}}{\partial y_2^2} \underbrace{- \frac{\partial \tilde{u}}{\partial y_2}}_{\tilde{f}(y, \tilde{u}, \partial_1 \tilde{u}, \partial_2 \tilde{u})} = 0$$

Temă: 2, 3.

Pt. examen:

- vedeți modalitatea afișată pe MOODLE de la începutul semestrului
- consultații pe 02.02.2021; ora o voi anunța pe MOODLE; va avea loc pe Teams.
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