Grupa 331, Seminar (3), EDDP, 22.10.2020

(11)-(12) (din tema seminar 2)

Se cere mulț. soluțiilor ec. dif:

1) $\frac{dx}{dt} = \frac{2t-x+1}{4t-2x+3}$, $(t,x)\in\Delta_1\subset\{(t,x) \mid 4t-2x+3>0\}$

2) $\frac{dx}{dt} = \frac{3t+x-5}{2t-x}$, $(t,x)\in\Delta_1\subset\{(t,x) \mid 2t-x<0\}$.

Ec de forma: $\frac{dx}{dt} = g\left(\frac{a_1t+b_1x+c_1}{a_2t+b_2x+c_2}\right)$

$a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$

$|a_1|+|a_2|>0$; $|b_1|+|b_2|>0$; $|c_1|+|c_2|>0$.

Calc: $\underline{d = a_1b_2 - b_1a_2}$. Avem 2 cazuri: $\begin{cases} I) & d=0 \\ II) & d\neq 0.\end{cases}$

(1) $\frac{dx}{dt} = \frac{2t-x+1}{4t-2x+3}$

$a_1=2$; $b_1=-1$; $c_1=1$

$a_2=4$; $b_2=-2$; $c_2=3$

$d = 2\cdot(-2)-4(-1) = 0$. $\Rightarrow$ cum avem $b_1\neq 0$, se face schimbarea de variabilă: $2t-x=y$

$$(t,x) \xrightarrow[x=2t-y]{} (t,y)$$

Ec. devine:

$$(2t-y)' = \frac{\cancel{2t}-(\cancel{2t}-y)+1}{\cancel{4t}-2(\cancel{2t}-y)+3}$$

$$2-y' = \frac{y+1}{2y+3} \Rightarrow y' = 2 - \frac{y+1}{2y+3}$$

$$y' = \frac{4y+6-y-1}{2y+3}$$

$$\left| \frac{dy}{dt} = \frac{3y+5}{2y+3}\cdot 1 \right|$$ ec. cu var. separabile $a(t)=1$; $b(y)=\frac{3y+5}{}$

• $b(y)=0 \Rightarrow y=-\frac{5}{3}$ sol. staționară pt ec în $y \Rightarrow$

$\Rightarrow \boxed{x_1(t)=2t+\frac{5}{3}}$, sol. particulară pt ec. în (t,x)

• $b(y)\neq 0 \Rightarrow$ sep. variabilele:

$$\frac{2y+3}{3y+5}\,dy = dt.$$

$$\int dt = t+C \Rightarrow A(t)=t.$$

$$\int \frac{2y+3}{3y+5}\,dy = \frac{1}{3}\int \frac{6y+9}{3y+5}\,dy = \frac{1}{3}\int \frac{2(3y+5)-1}{3y+5}\,dy$$

$$= \frac{1}{3}\left(\int 2\,dy - \int \frac{1}{3y+5}\,dy\right) =$$

$$= \frac{1}{3}\left(2y - \frac{1}{3}\ln|3y+5|\right) + C \Rightarrow B(y)=\frac{2}{3}y-\frac{1}{9}\ln|3y+5|$$

$$\boxed{\int \frac{1}{my+n}\,dy = \frac{1}{m}\ln|my+n| + C}$$

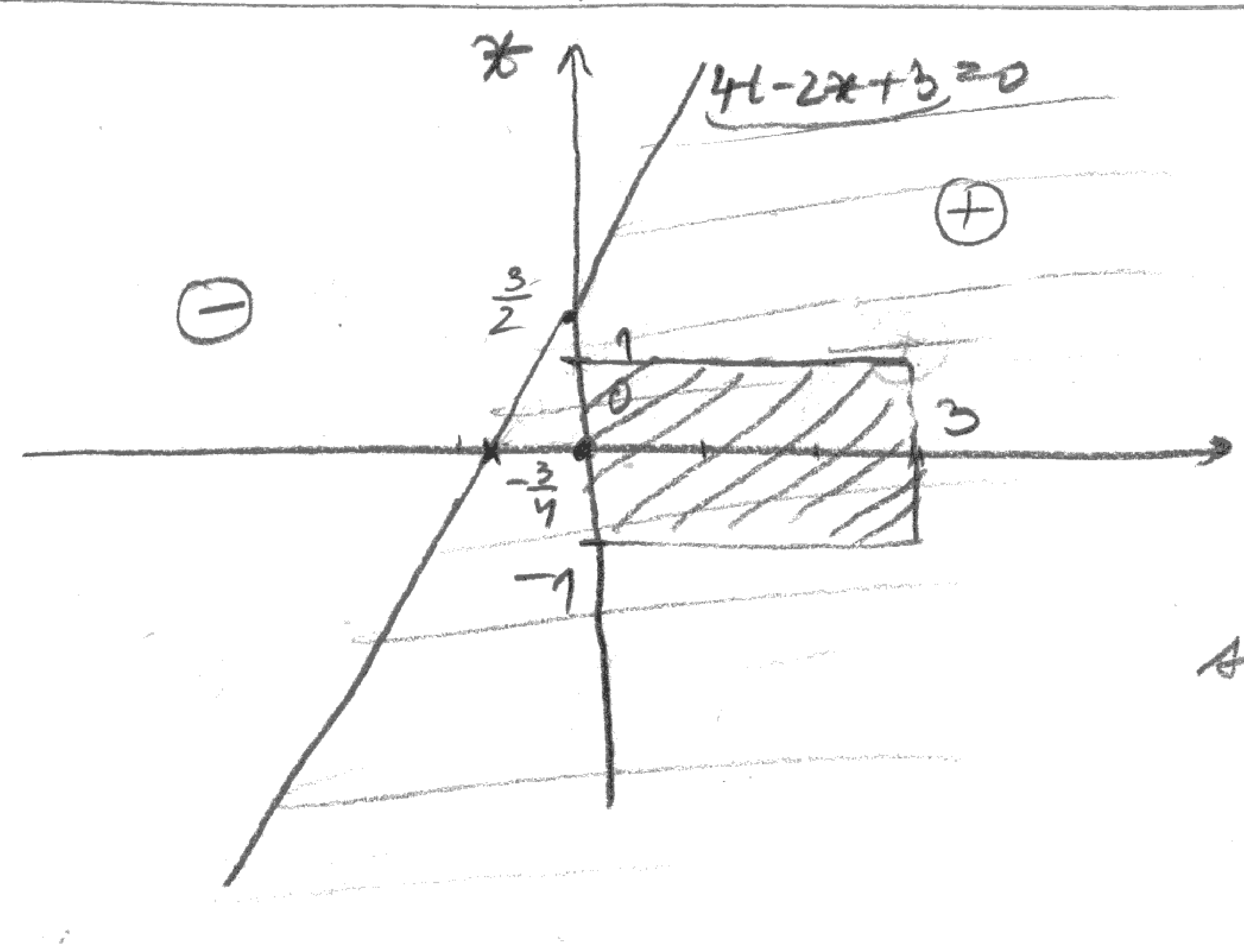
Mult sol. implicite pt ec în (t,y):

$$B(y)=A(t)+C$$

$$\boxed{\frac{2}{3}y-\frac{1}{9}\ln|3y+5| = t+C, \quad C\in\mathbb{R}.}$$

Mult sol. implicite pt ec. în (t,x):

$$\boxed{\frac{2}{3}(2t-x)-\frac{1}{9}\ln|6t-3x+5| = t+C, \quad C\in\mathbb{R}}$$



Exemplu de domeniu D_1:

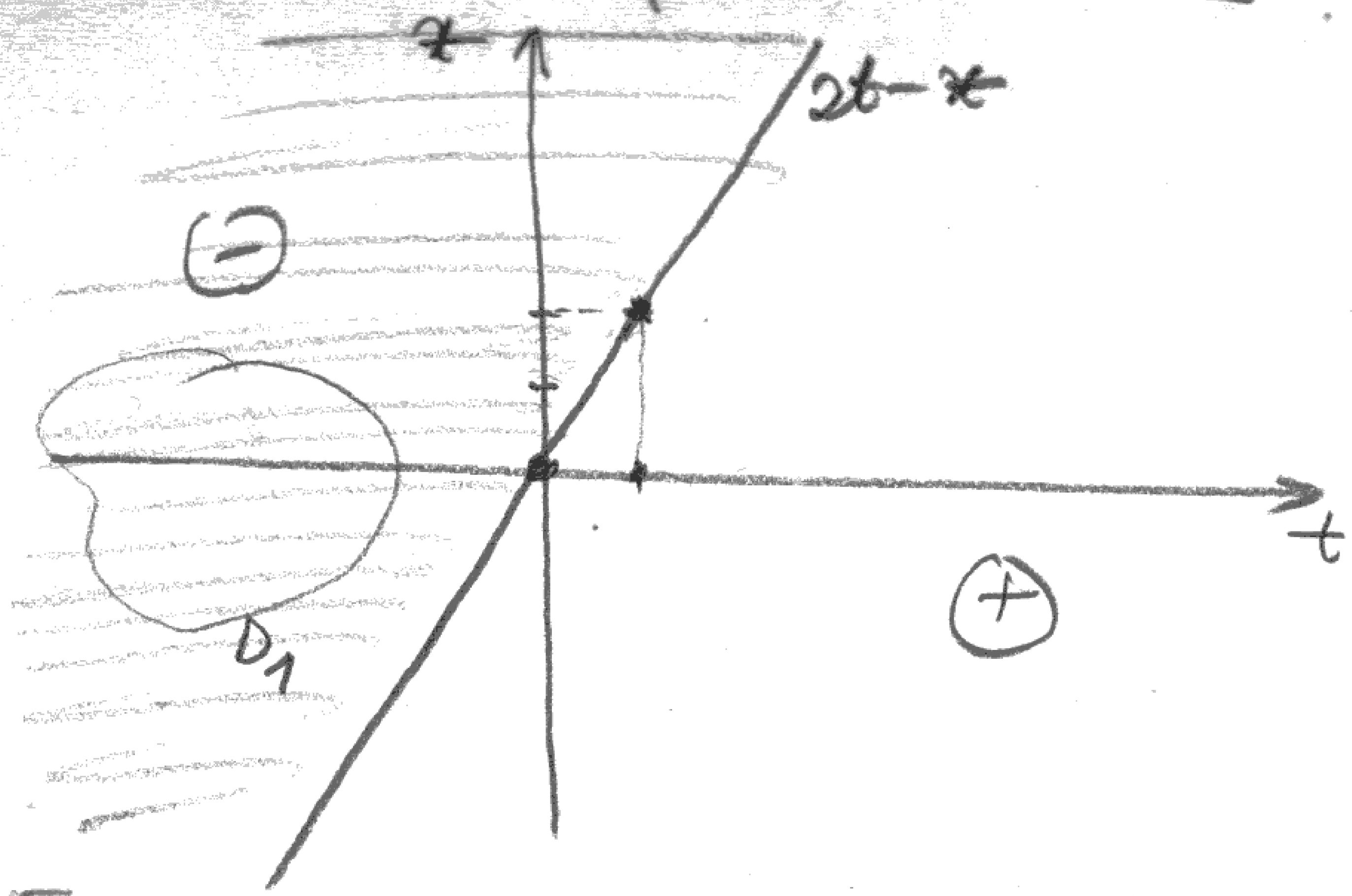
$$D_1 = \underbrace{[0,3]}_{t} \times \underbrace{[-1,1]}_{x}$$

$\Downarrow$

sol pant $x_1 : [0,3] \to \mathbb{R}$

$x_1 = 2t + \frac{5}{3}.$

2) $\frac{dx}{dt} = \frac{3t+x-5}{2t-x}$, $D_1 \subset \{(t,x) \mid 2t-x<0\}$



$a_1=3$, $b_1=1$, $c_1=-5$
$a_2=2$, $b_2=-1$, $c_2=0$

$d = 3\cdot(-1) - 2\cdot 1 = -5 \neq 0$.

• rezolvăm sistemul: $\begin{cases} 3t+x-5=0 \\ 2t-x=0 \end{cases}$

$5t \;/ = 5 \Rightarrow \boxed{t_0=1}$ $\boxed{x_0=2}$

$(t,x) \xrightarrow[\substack{s=t-1 \\ y=x-2}]{} (s,y)$ $\Leftrightarrow$ $\begin{matrix} t=s+1 \\ x=y+2 \end{matrix}$

$x(t) = y(s(t))$

$s(t)=t-1 \Rightarrow s'(t)=1$

$$\boxed{\frac{dx}{dt}(t) = \frac{dy}{ds}(s(t))\cdot s'(t) = \frac{dy}{ds}(s(t))}$$

Ec în (s,y):

$$\frac{dy}{ds} = \frac{3(s+1)+y+2-5}{2(s+1)-(y+2)}$$

$$\frac{dy}{ds} = \frac{3s+\cancel{3}+y+\cancel{2}-\cancel{5}}{2s+\cancel{2}-y-\cancel{2}} \Rightarrow \frac{dy}{ds} = \underbrace{\frac{3s+y}{2s-y}}_{f(s,y)} \Rightarrow$$

$$\Rightarrow \frac{dy}{ds} = \frac{3+\frac{y}{s}}{2-\frac{y}{s}} \quad \text{(ec. omogenă)}$$

Temă 1 $\frac{dx}{dt} = \frac{x}{t} + e^{\frac{x}{t}}$
(ec. omogenă)

$(s,y) \xrightarrow[y=sz \Leftrightarrow \frac{y}{s}=z]{} (s,z)$
$(y(s) = s\,z(s))$

$$(sz)' = \frac{3+z}{2-z} \Rightarrow z + sz' = \frac{3+z}{2-z} \Rightarrow$$

$$\Rightarrow z' = \frac{1}{s}\left(\frac{3+z}{2-z} \cdot \frac{2-z}{\cancel{2-z}} - z\right) \Rightarrow z' = \underbrace{\frac{1}{s}}_{a(s)} \underbrace{\frac{z^2-z+3}{2-z}}_{b(z)}$$

• $b(z) = 0 \Rightarrow z^2 - z + 3 = 0$

$\Delta = 1 - 12 < 0 \Rightarrow$ nu sunt soluții staționare pt. ec. în z.

• $b(z) \neq 0 \Rightarrow$ separăm variabilele:

$$\frac{2-z}{z^2-z+3}\,dz = \frac{1}{s}\,ds.$$

$$\int \frac{1}{s}\,ds = \ln|s| + c \Rightarrow \boxed{A(s) = \ln|s|}$$

$$\int \frac{2-z}{z^2-z+3}\,dz = \int \frac{2-z}{\left(z-\frac{1}{2}\right)^2 - \frac{1}{4} + 3}\,dz = \int \frac{2-z}{\left(z-\frac{1}{2}\right)^2 + \frac{11}{4}}\,dz$$

$z - \frac{1}{2} = v \Rightarrow dz = dv$

$z = v + \frac{1}{2}$

$$\int \frac{2 - v - \frac{1}{2}}{v^2 + \frac{11}{4}}\,dv = \frac{3}{2}\int \frac{1}{v^2 + \left(\frac{\sqrt{11}}{2}\right)^2}\,dv - \int \frac{v}{v^2+\frac{11}{4}}\,dv$$

$$= \underbrace{\frac{3}{\cancel{2}} \frac{1}{\frac{\sqrt{11}}{\cancel{2}}} \operatorname{arctg}\left(\frac{2v}{\sqrt{11}}\right) - \frac{1}{2}\ln\left(v^2 + \frac{11}{4}\right)} + C \Rightarrow$$

$$\Rightarrow \boxed{B(z) = \frac{3}{\sqrt{11}} \operatorname{arctg}\left(\frac{2z-1}{\sqrt{11}}\right) - \frac{1}{2}\ln\left(z^2 - z + 3\right)}$$

Mult. sol. implicite:

- pt. ec. în (s, z):

$$\frac{3\sqrt{11}}{11} \operatorname{arctg}\left(\frac{(2z-1)\sqrt{11}}{11}\right) - \frac{1}{2}\ln\left(z^2 - z + 3\right) = \ln|s| + C,$$

$C \in \mathbb{R}$

- pt ec. în (s, y)

$$\frac{3\sqrt{11}}{11} \operatorname{arctg}\left(\frac{\left(2\frac{y}{s} - 1\right)\sqrt{11}}{11}\right) - \frac{1}{2}\ln\left(\frac{y^2}{s^2} - \frac{y}{s} + 3\right) = \ln|s| + C$$

$C \in \mathbb{R}$.

- pt. ec. în (t, x)

$$\frac{3\sqrt{11}}{11} \operatorname{arctg}\left(\frac{\left(2\frac{(x-2)}{t-1} - 1\right)\sqrt{11}}{11}\right) - \frac{1}{2}\ln\left(\frac{(x-2)^2}{(t-1)^2} - \frac{x-2}{t-1} + 3\right) =$$

$$= \ln|t-1| + C.$$

Tema 2: Mulț. sol. ec. dif.:

3) $\dfrac{dx}{dt} = \dfrac{-t+2x-2}{2t-4x+1}$; $(t,x) \in D_1 \subset \{(t,x) \mid 2t-4x+1>0\}$.

4) $\dfrac{dx}{dt} = \dfrac{t+2x-7}{2t-x-5}$; $(t,x) \in D_1 \subset \{(t,x) \mid 2t-x-5<0\}$

Ex I. Să se determine mulț. soluțiilor ec. dif. următoare

ec. Bernoulli

5) $x' - \dfrac{x}{t} = \dfrac{1}{x^2 t^2}$; $x>0,\ t>0$

6) $x' = -xt + x^2 \sin t$, $t \in (0, \frac{\pi}{2})$, $x>0$.

✓ 7) $x' = \dfrac{4x}{t} + t\sqrt{x}$, $x>0,\ t>0$.

ec. Riccati

8) $tx' = -x^2 + 4x - 3$, $\underbrace{\text{considerând 2 variante}}^{t>0,\ x\in\mathbb{R}}$:

a) ca ec. cu variabile separabile

b) ca ec. Riccati pentru care se determină o soluție particulară de forma $\varphi_0(t) = k$, k = constantă.

9) $x' = \dfrac{3t^2}{t^5-1} + \dfrac{t^4}{t^5-1} x - \dfrac{2t}{t^5-1} x^2$; $t \in (1,+\infty)$, $x \in \mathbb{R}$,

știind că are soluție particulară de forma: $\varphi_0(t) = n t^m$, $n, m \in \mathbb{R}$.

Ex. II: Fie ec. diferențială:

$$x' + p(t)\cdot x = q(t), \qquad (1)$$

unde $p, q : (0, 2\pi) \to \mathbb{R}$

a) Determinați funcțiile p și q dacă ec. (1) are soluțiile: $\varphi_1(t) = t$ și $\varphi_2(t) = t \sin t$

b) Să se determine mulț. soluțiilor ec. (1)

c) Să se determine soluția ec. (1) care verifică: $x(\pi) = 2\pi$.

(7) $x' = \frac{4x}{t} + t\, x^{\frac{1}{2}}$, $x>0$, $t>0$.

ec. Bernoulli : $\frac{dx}{dt} = a(t)\,x + b(t)\,x^{\alpha}$

$a(t) = \frac{4}{t}$; $b(t) = t$; $\alpha = \frac{1}{2}$.

- rez. ec. liniară omogenă atașată:

$$\frac{d\bar{x}}{dt} = \frac{4}{t}\bar{x} \Rightarrow \bar{x} = C\cdot e^{A(t)}$$

$$\int \frac{4}{t}dt = 4\ln|t| + C = 4\ln t + C = $$
$$= \ln t^4 + C.$$

$$\Rightarrow A(t) = \ln t^4 \Rightarrow \bar{x}(t) = C\,e^{\ln t^4} = Ct^4$$

- aplicăm met. var. constantelor:

det. $C:(0,+\infty) \to \mathbb{R}$ a.î. $\underbrace{x(t) = C(t)\,t^4}$ să fie soluția ec. Bernoulli:

$$(C(t)\,t^4)' = \frac{4}{\cancel{t}}\cdot \underbrace{C(t)\,t^{4^3}} + t\cdot (C(t)\,t^4)^{\frac{1}{2}}$$

$$C'(t)\,t^4 + \cancel{C(t)\cdot 4t^3} = \cancel{4t^3 C(t)} + \cancel{t}\cdot C^{\frac{1}{2}}(t)\cdot t^2$$

$$C'(t) = C^{\frac{1}{2}}(t)\cdot\frac{1}{t} \Rightarrow \frac{dC}{dt} = C^{\frac{1}{2}}\cdot\frac{1}{t}.$$

ec. cu var. separabile $\Rightarrow$

$a_1(t) = \frac{1}{t}$

$b_1(C) = C^{\frac{1}{2}}$

- $b_1(C) = 0 \Rightarrow C = 0$ (sol. staționară pt ec. în (t,C)) $\Rightarrow$ $\boxed{x(t) = 0}$ sol. particulară pt ec Bernoulli.

- pt $C \neq 0 \Rightarrow$ sep. variab. $\frac{dC}{C^{\frac{1}{2}}} = \frac{dt}{t} \Rightarrow$

$$\Rightarrow \int\frac{dC}{C^{\frac{1}{2}}} = \int\frac{dt}{t} \Rightarrow \frac{C^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \ln|t| + C_1$$

$$\Rightarrow C^{\frac{1}{2}} = \frac{1}{2}(\ln t + C_1) \Rightarrow C = \left(\frac{1}{2}(\ln t + C_1)\right)^2 \Rightarrow$$

Ec. Bernoulli are sol:

$$\boxed{x(t) = \frac{t^4}{4}(\ln t + c_1)^2, \quad c_1 \in \mathbb{R}}$$

<u>Obs</u> Ec. (4) se poate rezolva cu schimbare de variabilă:

$$x = y^{\frac{1}{1-\frac{1}{2}}} \Rightarrow x = y^2$$

$$(t,x) \xrightarrow{\;x = y^2\;} (t,y)$$

$$x = y^2 \Leftrightarrow y = x^{\frac{1}{2}}$$

Ec. (4) devine:

$$(y^2)' = \frac{4}{t}\cdot y^2 + t\cdot y^{\frac{1}{2}\cdot 2}$$

$$2y\cdot y' = \frac{4}{t}y^2 + ty \quad |:2y$$

$$y' = \underbrace{\frac{2}{t}}_{a_2(t)}y + \underbrace{\frac{t}{2}}_{b_2(t)}$$

ec. afină care se rezolvă cu met. variației constantelor. (temă!)

<u>Temă</u> 3: $\begin{cases} I\ (5,6)\ \&\ (8,9) \\ II \end{cases}$

<u>Obs</u>: $a: I \to \mathbb{R}$, $t_0 \in I$

a continuă

Atunci o primitivă a lui a este

$$A(t) = \int_{t_0}^{t} a(s)\,ds.$$