Grupa 331, Seminar 5, EDDP, 05.11.2020

1. Ec. dif implicită:

$$x = -t + \left(\frac{x'+1}{x'-1}\right)^2$$

Se cere mulțimea soluțiilor.

Avem $p = x'$ $\Rightarrow$ $x = -t + \left(\frac{p+1}{p-1}\right)^2$ $\Rightarrow$

Derivăm în raport t

$$\Rightarrow x' = -1 + 2\frac{p+1}{p-1}\cdot\left(\frac{p+1}{p-1}\right)'$$

$$p = -1 + 2\frac{p+1}{p-1}\left(p'\cdot\frac{1}{p-1} + (p+1)\frac{-1}{(p-1)^2}\cdot p'\right)$$

$$p+1 = p'\cdot\frac{2(p+1)}{p-1}\left(\frac{1}{p-1} - \frac{p+1}{(p-1)^2}\right)$$

• dacă $p+1=0 \Rightarrow p=-1 \Rightarrow x'=-1 \Rightarrow x=-t+c$, $c\in\mathbb{R}$

înloc în ec $\Rightarrow \cancel{-t}+c = \cancel{-t} + \left(\frac{\cancel{-1+1}}{-1-1}\right)^2 \Rightarrow$ $c=0$

$\Rightarrow \boxed{x=-t}$ (1)

• dacă $p+1\neq 0 \Rightarrow$ împărțim cu $(p+1) \Rightarrow$

$$1 = \frac{2p'}{p-1}\cdot\frac{-2}{(p-1)^2} \Rightarrow p' = \frac{(p-1)^3}{-4} \Rightarrow$$

$\Rightarrow \frac{dp}{dt} = \frac{(p-1)^3}{-4}$ răsturnăm ec. $\Rightarrow \frac{dt}{dp} = \frac{-4}{(p-1)^3} \Rightarrow$

ec. de tip primitivă în variabile (p, t)

var. independentă.

$$\Rightarrow t(p) = \int\frac{-4}{(p-1)^3}dp = -4\int(p-1)^{-3}dp = -\cancel{4}^{2}\,\frac{(p-1)^{-2}}{\cancel{-2}} + C \Rightarrow$$

$$\Rightarrow t = \frac{2}{(p-1)^2} + C \Rightarrow$$

$\Rightarrow$ mult. de soluții parametrice este:

$$\boxed{\begin{cases} x = -t + \left(\frac{p+1}{p-1}\right)^2 \\ t = \frac{2}{(p-1)^2} + C \end{cases}, C\in\mathbb{R}}\ (2)$$

Mulț. sol. ec. este (1) ∪ (2).

(2) Să se integreze ec. diferențiale de ordin ≥ 2, aplicând metode de reducere a ordinului:

a) $[1+(x')^2]\, x''' = 3x'(x'')^2$

b) $x'' \cos x + (x')^2 \sin x - x' = 0$.

c) $x^2 + (x')^2 - 2x x'' = 0$

d) $t^2 x'' - 2t x x' + t x' = 0$.

e) $t^2 x x'' - (t x')^2 - t x x' = 0$.

f) $\left(\frac{x}{t}\right)^2 - (x')^2 = 3t x'' + \frac{2 x x'}{t}$.

În general, ec dif de ordin k, în $\mathbb{R}$, sunt de forma $F(t, x, x^{(1)}, \dots, x^{(k)}) = 0$.

(b) $x \in (0, \frac{\pi}{2})$

$$x'' \cos x + (x')^2 \sin x - x' = 0.$$

$$k=2: \quad F(\cancel{t}, x, x', x'') = 0.$$

$$(t, x) \xrightarrow[x'(t) = y(x(t))]{x' = y} (x, y)$$

$$x''(t) = \left(y(x(t))\right)' = y'(x(t)) \cdot \underbrace{x'(t)}_{y} \Rightarrow \boxed{x'' = y'y}$$

Ec. devine:

$$y'y \cdot \cos x + y^2 \sin x - y = 0.$$

- dacă $y = 0 \Rightarrow x' = 0 \Rightarrow \boxed{x = C,\ C \in \mathbb{R}}$
- dacă $y \neq 0 \Rightarrow$ împărțim ec prin y:

$$y' \cos x + y \sin x - 1 = 0. \Rightarrow$$

$$y' = \frac{1 - y \sin x}{\cos x}$$

$$\frac{dy}{dx} = (-\operatorname{tg} x) y + \frac{1}{\cos x}$$

ec. liniară neomogenă (ec. afină): $\frac{dy}{dx} = a(x) \cdot y + b(x)$

• rez. ec. liniară omogenă atașată:

$$\frac{d\bar{y}}{dx} = (-tg\,x)\,\bar{y} \Rightarrow \bar{y}(x) = C \cdot e^{A(x)}$$

$$\int a(x)\,dx = -\int tg\,x\,dx = +\int \frac{(\cos x)'}{\cos x}\,dx = \ln|\cos x| + C_1$$

$$x \in \left(0, \frac{\pi}{2}\right) \Rightarrow A(x) = \ln(\cos x)$$

$$\bar{y}(x) = C\,e^{\ln(\cos x)} = C\cos x$$

• aplicăm met. variației constantelor:

determinăm $C : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$ aî

$y(x) = C(x) \cdot \cos x$ să fie soluția ec.

afine $\frac{dy}{dx} = (-tg\,x)\,y + \frac{1}{\cos x} \Rightarrow$

$$\Rightarrow \left(C(x)\cdot\cos x\right)' = (-tg\,x)\,C(x)\cos x + \frac{1}{\cos x}$$

$$C'(x)\cos x \cancel{- (\sin x)\,C(x)} = \cancel{-\frac{\sin x}{\cos x} \cdot C(x) \cdot \cos x} + \frac{1}{\cos x}$$

$\Rightarrow C'(x) = \frac{1}{\cos^2 x}$; $\frac{dC}{dx} = \frac{1}{\cos^2 x}$ $\Big|$ $\Rightarrow C(x) = \int \frac{1}{\cos^2 x}\,dx \Rightarrow$

ec. de tip primitivă

$C(x) = tg\,x + C_1 \Rightarrow$

$\Rightarrow y(x) = (tg\,x + C_1)\cos x$ $\Big/$ $\Rightarrow x'(t) = \sin x + C_1 \cos x \Rightarrow$

dar $x'(t) = y(x(t))$

$$\Rightarrow \frac{dx}{dt} = \underbrace{(\sin x + C_1\cos x)}_{b_1(x)} \cdot \underset{a_1(t)}{1}$$

ec. cu var. separabile

• $b_1(x) = 0 \Rightarrow \sin x + C_1 \cos x = 0$

$\sin x = -C_1 \cos x$ $\quad \big| : \cos x \neq 0$ $\left(x \in \left(0, \frac{\pi}{2}\right)\right)$

$tg\,x = -C_1$

$x = arctg(-C_1) \in \left(0, \frac{\pi}{2}\right)$ pt $-C_1 > 0$

$C_1 < 0$

$\Rightarrow$ sol. staționară $\boxed{x = arctg(-C_1),\ \text{pt } C_1 < 0}$ (3)

• $b_1(x) \neq 0 \Rightarrow$ separăm variabilele:

$$\frac{dx}{\sin x + C_1 \cos x} = dt$$

$$\int dt = t + C \Rightarrow A_1(t) = t$$

$$\int \frac{dx}{\sin x + C_1 \cos x} \underset{\uparrow}{=} \int \frac{dx}{\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} + C_1\frac{(1-tg^2\frac{x}{2})}{1+tg^2\frac{x}{2}}} =$$

$$\boxed{\begin{aligned} \sin x &= \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} \\ \cos x &= \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}} \end{aligned}}$$

$$= 2\int \frac{\frac{1}{2}\left(1+tg^2\frac{x}{2}\right)dx}{-C_1 tg^2\frac{x}{2} + 2tg\frac{x}{2} + C_1} = \mathcal{I} :$$

notăm $tg\frac{x}{2} = t \Rightarrow \left(1+tg^2\frac{x}{2}\right)\cdot\frac{1}{2}dx = dt \Rightarrow$

$$(tg\,x)' = \frac{1}{\cos^2 x} = 1 + tg^2 x$$

$$\mathcal{I} = 2\int \frac{dt}{-C_1t^2+2t+C_1} = 2\int \frac{dt}{-C_1\left(\underbrace{t^2-\frac{2}{C_1}t}-1\right)} =$$

$$= -\frac{2}{C_1}\int \frac{dt}{\left(t^2-\frac{2}{C_1}t+\frac{1}{C_1^2}\right)-\frac{1}{C_1^2}-1} =$$

$$= -\frac{2}{C_1}\int \frac{dt}{\left(t-\frac{1}{C_1}\right)^2 - \frac{1+C_1^2}{C_1^2}} = -\frac{2}{C_1}\int \frac{dt}{\left(t-\frac{1}{C_1}\right)^2 - \left(\frac{\sqrt{1+C_1^2}}{C_1}\right)^2}$$

$$u = t - \frac{1}{C_1} \Rightarrow du = dt$$

$$-\frac{2}{C_1}\int \frac{du}{u^2-\left(\frac{\sqrt{1+C_1^2}}{C_1}\right)^2} = -\frac{\cancel{2}}{\cancel{C_1}}\cdot\frac{1}{\cancel{2}\cdot\frac{\sqrt{1+C_1^2}}{\cancel{C_1}}}\cdot\ln\left|\frac{{}^{C_1/}u - \frac{\sqrt{1+C_1^2}}{C_1}}{{}^{C_1/}u + \frac{\sqrt{1+C_1^2}}{C_1}}\right| + C$$

$$= -\frac{1}{\sqrt{1+C_1^2}}\ln\left|\frac{C_1u-\sqrt{1+C_1^2}}{C_1u+\sqrt{1+C_1^2}}\right| + C \quad \Rightarrow 1$$

Avem $u = t - \frac{1}{C_1} = tg\frac{x}{2} - \frac{1}{C_1}$

$$\Rightarrow B_1(x) = -\frac{1}{\sqrt{1+C_1^2}} \ln \left| \frac{C_1 \operatorname{tg}\frac{x}{2} - 1 - \sqrt{1+C_1^2}}{C_1 \operatorname{tg}\frac{x}{2} - 1 + \sqrt{1+C_1^2}} \right| \Rightarrow$$

$\Rightarrow$ mulț sol implicite : $B_1(x) = A_1(t) + C_2 \Rightarrow$

$$\Rightarrow \boxed{-\frac{1}{\sqrt{1+C_1^2}} \ln \left| \frac{C_1 \operatorname{tg}\frac{x}{2} - 1 - \sqrt{1+C_1^2}}{C_1 \operatorname{tg}\frac{x}{2} - 1 + \sqrt{1+C_1^2}} \right| = t + C_2 \quad C_1, C_2 \in \mathbb{R}.} \quad (4)$$

Mulț sol. ec este (3) $\cup$ (4).

e) $t^2 x x'' - (t x')^2 + t x' = 0$.

ec. Euler de ordin 2 : $F(x, tx', t^2x'') = 0$.

$$\boxed{x \cdot (t^2 x'') - (t x')^2 + t x' = 0.}$$

$$(t,x) \xrightarrow[x(t) = y(s(t))]{|t| = e^s} (s,y)$$

$$x'(t) = (y(s(t)))' = y'(s(t)) \cdot s'(t) = \frac{1}{t} y'(s(t)) \Rightarrow$$

$s(t) = \ln|t| \Rightarrow s'(t) = \frac{1}{t}$

$$\Rightarrow \boxed{\underbrace{t x'}_{\frac{dx}{dt}} = \underbrace{y'}_{\frac{dy}{ds}}}$$

$$x'' = \left(\frac{1}{t} y'\right)' = \left(\frac{1}{t}\right)' y' + \frac{1}{t} \cdot (y'(s(t)))' =$$

$$= -\frac{1}{t^2} y' + \frac{1}{t} y''(s(t)) \underbrace{s'(t)}_{\frac{1}{t}} = -\frac{1}{t^2} y' + \frac{1}{t^2} y'' \quad \Big| \cdot t^2$$

$$\Rightarrow \boxed{t^2 x'' = -y' + y''}$$

Ec în (s,y) este: $y(y'' - y') - (y')^2 + y' = 0$.

$$y y'' - y y' - (y')^2 + y' = 0.$$

Ec în y : $F_1(y, y', y'') = 0$.

$$(s, y) \xrightarrow[y' = z]{} (y, z)$$

$$y'(s) = z(y(s))$$

$$y''(s) = z'(y(s)) \cdot \underline{y'(s)} \Rightarrow \boxed{y'' = z' z}$$

Ec în (y, z) : $y \cdot z' z - y z - z^2 + z = 0$,

$$y z \cdot z' = z(z + y - 1)$$

1) $z = 0 \Rightarrow y' = 0 \Rightarrow y(s) = C_1 \Rightarrow x(t) = y(\ln|t|) = C_1$

$$\boxed{x(t) = C_1, \ C_1 \in \mathbb{R}}$$

2) $z \neq 0 \Rightarrow$ împărțim prin $z \Rightarrow y z' = z + y - 1 \Rightarrow$

$$\Rightarrow \frac{dz}{dy} = \underbrace{\frac{1}{y}}_{a_2(y)} z + \underbrace{\frac{y-1}{y}}_{b_2(y)} \quad \text{ec afină}$$

$$\frac{d\bar{z}}{dy} = \frac{1}{y}\bar{z} \Rightarrow \bar{z} = C \cdot e^{\ln|y|} = C \cdot |y|, \ C \in \mathbb{R} \Rightarrow$$

$$\Rightarrow \bar{z} = Cy$$

variem $C \Rightarrow$ $\underline{z(y) = C(y) \cdot y}$ în ec. afină $\Rightarrow$

$$\Rightarrow \underline{C'(y) \cdot y} + \cancel{C(y) \cdot \underset{1}{y'}} = \cancel{\frac{1}{y} \cdot C(y) \cdot y} + \underline{\frac{y-1}{y}} \quad | : y$$

$$\Rightarrow C'(y) = \frac{y-1}{y^2} \Rightarrow C(y) = \int \left(\frac{1}{y} - \frac{1}{y^2}\right) dy \Rightarrow$$

ec. de tip primitivă

$$\Rightarrow C(y) = \ln|y| + \frac{1}{y} + C_1 \Rightarrow$$

$$\Rightarrow z(y) = y \ln|y| + 1 + C_1 y \Bigg| \Rightarrow y' = y\ln|y| + C_1 y + 1$$

Dar $y'(s) = z(y(s))$

ec. cu var separabile

$$\frac{dy}{ds} = \underbrace{\left(y\left(\ln|y| + C_1\right) + 1\right)}_{b_3(y)} \underbrace{1}_{a_3(s)} \quad (5)$$

<u>Concluzie</u>: din ec (t, x) $\Big|\frac{\text{Euler}}{\text{de ordin } 2}$, se reduce la integrarea ec. (5) cu variabile separabile;

<u>Temă</u>: 2(a, c, d, f)