Grupa 331, Seminar (2), EDDP, 15.10.2020

(I) Să se determine mulțimea soluțiilor ecuațiilor diferențiale:

✓ 1) $\dfrac{dx}{dt} = \dfrac{2t\,(x^2+5x+6)}{t^2+4}$, $x \in \mathbb{R}$, $t \in \mathbb{R}$

2) $\dfrac{dx}{dt} = \dfrac{2tx(\ln x)}{(t^2+1)\ln(\ln x)}$; $x \in (3,+\infty)$, $t \in \mathbb{R}$

✓ 3) $\dfrac{dx}{dt} = \dfrac{x+x^3}{t\,(x^2-1)}$; $x \in (-1,1)$, $t \in (0,+\infty)$

4) $\dfrac{dx}{dt} = \dfrac{(x^3-1)(t+2)}{\sqrt{t^2+4}}$, $t \in \mathbb{R}$, $x \in \mathbb{R}$.

5) $\dfrac{dx}{dt} = \dfrac{e^{2e^t+t}\cdot(x+2)}{x^2+2x+3}$, $x \in \mathbb{R}$, $t \in \mathbb{R}$.

✓ 6) $\dfrac{dx}{dt} = \dfrac{2tx - x^2}{t^2}$, $t \in (0,+\infty)$; $x \in \mathbb{R}$

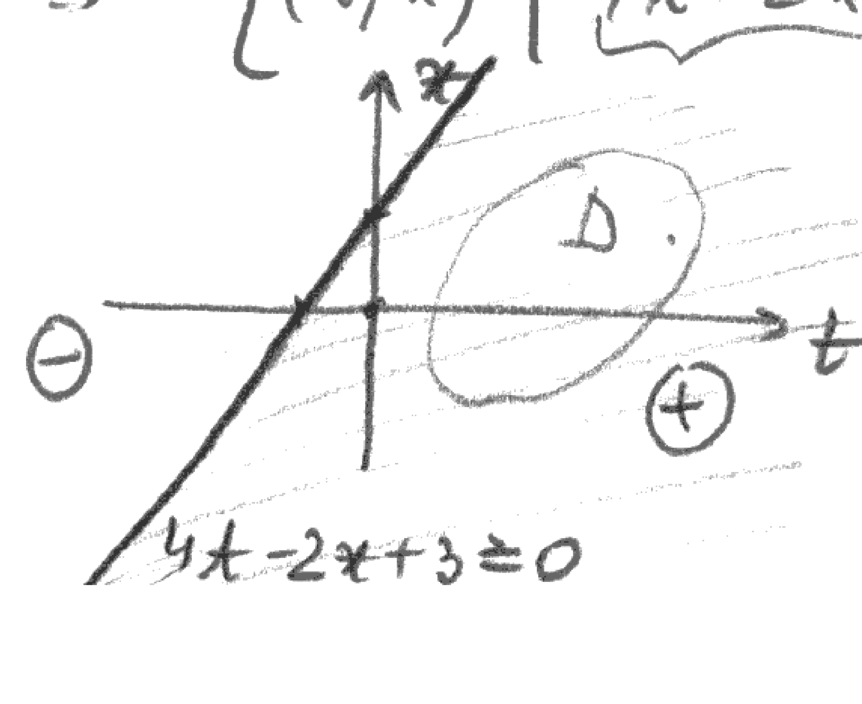
✓ 7) $\dfrac{dx}{dt} = x\cdot(\operatorname{tg} t) + \cos t$, $t \in \left(0, \frac{\pi}{2}\right)$; $x \in \mathbb{R}$

8) $\dfrac{dx}{dt} = \dfrac{3x-t^2}{t}$, $t \in (0,+\infty)$, $x \in \mathbb{R}$

✓ 9) $\dfrac{dx}{dt} = \dfrac{2t + xe^{-t}}{e^{-t}}$, $t \in \mathbb{R}$, $x \in \mathbb{R}$

10) $\dfrac{dx}{dt} = \dfrac{2xt}{x^2-t^2}$, $t \in (0,1)$, $x \in (1,+\infty)$

11) $\dfrac{dx}{dt} = \dfrac{2t-x+1}{4t-2x+3}$; $(t,x) \in D = \{(t,x) \mid 4t-2x+3>0\}$



12) $\dfrac{dx}{dt} = \dfrac{3t+x-5}{2t-x}$

[illegible]

(1) $\frac{dx}{dt} = \frac{2t(x^2+5x+6)}{t^2+4}$, $(t,x) \in \mathbb{R}^2$

ec. diferențială cu var sep. $\frac{dx}{dt} = a(t)\, b(x)$

$a: \mathbb{R} \to \mathbb{R}$
$a(t) = \frac{2t}{t^2+4}$
$b: \mathbb{R} \to \mathbb{R}$
$b(x) = x^2+5x+6$.

• $b(x) = 0 \Rightarrow x^2+5x+6=0$
$\Delta = 25-24 = 1$
$x_{1,2} = \frac{-5 \pm 1}{2}$ ⟨ $x_1 = -2$, $x_2 = -3$ $\Rightarrow$

$\Rightarrow$ sol. staționare: $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$
$\varphi_1(t) = -2$, $\forall t \in \mathbb{R}$ (1)
$\varphi_2(t) = -3$

• $b(x) \neq 0 \Rightarrow x \in \mathbb{R} \setminus \{-2,-3\} \Rightarrow$ separăm variabilele:

$$\frac{dx}{x^2+5x+6} = \frac{2t}{t^2+4}\, dt$$

$$\int \frac{dx}{x^2+5x+6} = \int \frac{(x+3)-(x+2)}{(x+2)(x+3)}\, dx = \int \frac{\cancel{x+3}}{(x+2)\cancel{(x+3)}}\, dx - \int \frac{\cancel{x+2}}{\cancel{(x+2)}(x+3)}\, dx =$$

$$= \int \frac{1}{x+2}\, dx - \int \frac{1}{x+3}\, dx = \ln|x+2| - \ln|x+3| + C = \ln\left|\frac{x+2}{x+3}\right| + C \Rightarrow$$

$$\Rightarrow B(x) = \ln\left|\frac{x+2}{x+3}\right|$$

$$\int \frac{2t}{t^2+4}\, dt = \ln(t^2+4) + C \Rightarrow A(t) = \ln(t^2+4)$$ $\Rightarrow$

$\Rightarrow$ mulț. de soluții implicite: $B(x) = A(t) + C \Rightarrow$

$$\Rightarrow \ln\left|\frac{x+2}{x+3}\right| = \ln(t^2+4) + \ln C, \quad C>0 \quad (2)$$

Mulț. sol. ec. este (1) $\cup$ (2)

Încercăm să explicităm (2):

$$\ln\left|\frac{x+2}{x+3}\right| = \ln\left(C(t^2+4)\right) \Rightarrow \left|\frac{x+2}{x+3}\right| = C(t^2+4) \Rightarrow$$
$C>0$

$$\Rightarrow \frac{x+2}{x+3} = \underbrace{\pm C}_{C_1}(t^2+4) \quad C_1, C_1 \in \mathbb{R}^* \Rightarrow x+2 = (x+3)C_1(t^2+4) \Rightarrow$$

$$\Rightarrow x\left(1 - C_1(t^2+4)\right) = 3C_1(t^2+4) - 2 \Rightarrow$$

$$\Rightarrow \boxed{x(t) = \frac{3C_1(t^2+4)-2}{1-C_1(t^2+4)}, \; C_1 \in \mathbb{R}^*} \quad (3)$$

Mulț. sol. ec. este $(1) \cup (3)$.

↑ mulț. de soluții explicite obținute din (2).

(3) $\frac{dx}{dt} = \frac{x+x^3}{t(x^2-1)}$, $t \in (0,+\infty)$, $\underline{x \in (-1,1)}$

$a : (0,+\infty) \to \mathbb{R}$
$a(t) = \frac{1}{t}$

$b : (-1,1) \to \mathbb{R}$
$b(x) = \frac{x+x^3}{x^2-1}$

• $b(x) = 0 \Rightarrow x + x^3 = 0 \Rightarrow x(x^2+1) = 0 \Rightarrow x = 0 \in (-1,1)$

$x^2+1=0$ nu are sol reale

$\Rightarrow$ o sol. staționară: $\varphi_1 : (0,\infty) \to \mathbb{R}$, $\varphi_1(t) = 0$, $\forall t \in (0,+\infty)$

• pt $b(x) \neq 0$, $x \in (-1,1) \setminus \{0\}$ separăm variabilele:

$$\frac{(x^2-1)\,dx}{x+x^3} = \frac{1}{t}\,dt$$

$$\int \frac{1}{t}\,dt = \ln|t| + C \underset{t>0}{=} \ln t + C \Rightarrow \boxed{A(t) = \ln t}$$

$$\int \frac{(x^2-1)}{x(x^2+1)}\,dx = J.$$

$$\frac{x^2-1}{x(x^2+1)} = \frac{\overset{x^2+1}{A}}{x} + \frac{\overset{x}{Bx+C}}{x^2+1}, \quad A,B,C \in \mathbb{R}$$ pe care le determinăm

$x^2-1 = A(x^2+1) + x(Bx+C)$

$x^2-1 = Ax^2 + A + Bx^2 + Cx$

$x^2-1 = x^2(A+B) + Cx + A$

identificăm coeficienții $\Rightarrow \begin{cases} A+B=1 \\ C=0 \\ A=-1 \end{cases} \Rightarrow \boxed{\begin{matrix} A=-1 \\ C=0 \\ B=2 \end{matrix}}$

$$J = \int \left(\frac{-1}{x} + \frac{2x}{x^2+1} \right) dx = -\ln|x| + \ln(x^2+1) + C \Rightarrow$$

$$\Rightarrow \boxed{B(x) = \ln\left(\frac{x^2+1}{|x|}\right)}$$

Mulț. sol. implicite: $\ln\left(\frac{x^2+1}{|x|}\right) = \ln t + \ln C,\ C>0$

$$\Rightarrow \boxed{\frac{x^2+1}{|x|} = Ct,\ C>0}$$

⑥ $\frac{dx}{dt} = \underbrace{\frac{2tx - x^2}{t^2}}_{f(t,x)},\ t\in(0,\infty),\ x\in\mathbb{R}.$

$$f(\alpha t, \alpha x) = \frac{2\alpha t\alpha x - \alpha^2 x^2}{\alpha^2 t^2} = \frac{\cancel{\alpha^2}(2tx - x^2)}{\cancel{\alpha^2} t^2} = f(t,x)$$

$\Rightarrow$ f este funcție omogenă $\Rightarrow$ ec. dif. omogenă

Se face schimbarea de variabilă $\frac{x}{t} = y$

$$(t,x) \xrightarrow[x=ty]{} (t,y)$$

Ec. devine:

$$(ty)' = \frac{2t\cdot ty - t^2y^2}{t^2}$$

$$\underset{1}{t'}y + ty' = \frac{\cancel{t^2}(2y-y^2)}{\cancel{t^2}}$$

$$ty' = 2y - y^2 - y$$

$$y' = \frac{1}{t}\cdot(y-y^2)$$

ec. cu var. separabile $\Rightarrow$ temă

$$\Rightarrow y(t)$$

$$x(t) = t\,y(t)$$

⑦ $\frac{dx}{dt} = x(\operatorname{tg} t) + \cos t,\quad t\in\left(0,\frac{\pi}{2}\right),\ x\in\mathbb{R}.$

ec. afină: $\frac{dx}{dt} = a(t)x + b(t)$

$a, b: \left(0,\frac{\pi}{2}\right)\to\mathbb{R},\quad a(t) = \operatorname{tg} t$

$b(t) = \cos t.$

Folosim metoda variației constantelor.
Rezolvăm ec. liniară omogenă atașată:

$$\frac{d\bar{x}}{dt} = \bar{x}(tg\,t)$$

pt care sol este: $\bar{x}(t) = C \cdot e^{A(t)}$, $C \in \mathbb{R}$

A este primitiva pt a:

$$\int a(t)dt = \int tg\,t\,dt = -\ln|\cos t| + C \Rightarrow$$

$t \in (0, \frac{\pi}{2}] \Rightarrow \cos t > 0$

$$\Rightarrow A(t) = -\ln(\cos t) = \ln(\cos t)^{-1} = \ln\left(\frac{1}{\cos t}\right) \Rightarrow$$

$$\Rightarrow \bar{x}(t) = C \cdot e^{\ln\left(\frac{1}{\cos t}\right)} = C\frac{1}{\cos t} \Rightarrow \boxed{\bar{x}(t) = \frac{C}{\cos t}}$$

$$e^{\ln y} = y$$

Aplicăm met. var. constantelor:

determinăm $C : (0, \frac{\pi}{2}) \to \mathbb{R}$ a.î.

$x(t) = \frac{C(t)}{\cos t}$, să fie soluție a ec. inițiale (afine).

$$\Rightarrow \left(\frac{C(t)}{\cos t}\right)' = \frac{C(t)}{\cos t} \cdot tg\,t + \cos t \Rightarrow$$

$$\Rightarrow \left(C(t) \cdot \frac{1}{\cos t}\right)' = \frac{C(t)}{\cos t} \cdot \frac{\sin t}{\cos t} + \cos t \Rightarrow$$

$$\Rightarrow C'(t) \cdot \frac{1}{\cos t} + \cancel{C(t) \cdot \frac{-1}{\cos^2 t}(-\sin t)} = \cancel{\frac{C(t)\sin t}{\cos^2 t}} + \cos t \Rightarrow$$

$$\Rightarrow \boxed{C'(t) = \cos^2 t} \Rightarrow C(t) = \int \cos^2 t\,dt = \int \cos t \cdot \cos t\,dt =$$

$$= \int (\sin t)' \cos t\,dt =$$

$$= \sin t \cdot \cos t - \int \sin t \cdot (\cos t)'dt =$$

$$= \sin t \cos t + \int \sin^2 t\,dt = \sin t \cos t + \int (1 - \cos^2 t)dt \Rightarrow$$

$$\Rightarrow C(t) = \sin t \cos t + \int 1 dt - \underbrace{\int \cos^2 t\,dt}_{C(t)} \Rightarrow$$

$$\Rightarrow C(t) = \sin t \cos t + t \Rightarrow$$

$\Rightarrow C(t) = \dfrac{t + \sin t \cdot \cos t}{2} + C_1, \quad C_1 \in \mathbb{R}. \Rightarrow$

$\Rightarrow x(t) = \dfrac{\dfrac{t + \sin t \cdot \cos t}{2} + C_1}{\cos t}, \quad C_1 \in \mathbb{R}.$

$\Rightarrow x(t) = \dfrac{C_1}{\cos t} + \underbrace{\dfrac{t + \sin t \cdot \cos t}{2\cos t}}_{\text{soluția particulară}}, \quad C_1 \in \mathbb{R}.$

(5) $\dfrac{dx}{dt} = \dfrac{2t + xe^{-t}}{e^{-t}}, \quad t, x \in \mathbb{R}.$

$\dfrac{dx}{dt} = \dfrac{2t}{e^{-t}} + \dfrac{x\cancel{e^{-t}}}{\cancel{e^{-t}}} \Rightarrow \dfrac{dx}{dt} = x + 2te^{t}.$

$a, b: \mathbb{R} \to \mathbb{R}$
$a(t) = 1$
$b(t) = 2te^{t}.$

• Căutăm soluții (particulare) de forma $\varphi(t) = (mt+n)e^{t} \Rightarrow$ $m, n \in \mathbb{R}$

$\Rightarrow ((mt+n)\cdot e^{t})' = (mt+n)e^{t} + 2te^{t}, \quad \forall t \in \mathbb{R}$

$(mt+n)'e^{t} + (mt+n)(e^{t})' = ((mt+n) + 2t)e^{t}$

$me^{t} + (mt+n)e^{t} = ((m+2)t + n)e^{t} \quad | : e^{t} \ (e^{t} > 0)$

$m + mt + \cancel{n} = (m+2)t + \cancel{n}$

identif. coef $\Rightarrow \begin{cases} m = m+2 \\ m = 0 \end{cases} \Rightarrow 0 = 2$ fals $\Rightarrow$

$\Rightarrow$ nu există soluții de forma $(mt+n)e^{t}$.

• Căutăm sol de forma $\varphi(t) = mt^2e^{t}, \quad m \in \mathbb{R}^* \Rightarrow$

$\Rightarrow (mt^2e^{t})' = mt^2e^{t} + 2te^{t}, \quad \forall t \in \mathbb{R}$

$m(2te^{t} + t^2e^{t}) = (mt^2 + 2t)e^{t} \quad | : e^{t}$

$2tm + \cancel{mt^2} = \cancel{mt^2} + 2t \Rightarrow 2m = 2 \Rightarrow \boxed{m=1} \Rightarrow$

$\Rightarrow$ o soluție particulară este $\boxed{\varphi_0(t) = t^2e^{t}} \Rightarrow$

$\Rightarrow$ mult. sol. ec. este $x(t) = \varphi_0(t) + \bar{x}(t)$,

unde $\bar{x}$ este sol. ec. liniară omogenă atașată.

$$\frac{d\bar{x}}{dt} = \bar{x} \quad , \; a(t) = 1$$

$$\int 1\,dt = t + C \Rightarrow A(t) = t \Bigg| \Rightarrow$$

$$\Rightarrow \quad \bar{x}(t) = C \cdot e^{t} \Rightarrow$$

$$\Rightarrow \quad x(t) = Ce^{t} + t^{2}e^{t} = (C + t^{2})e^{t} \; , \; C \in \mathbb{R}.$$

Temă: 2, 4, 5, 8, 10, 11, 12