Grupa 331, Seminar 8, 26.11.2020, EDDP

I Se cere mulțimea soluțiilor următoarelor sisteme de ecuații diferențiale:

(1) $\begin{cases} x_1' = x_1 + x_2 \\ x_2' = 3x_2 - 2x_1 \end{cases}$, $t \in \mathbb{R}$, $x \in \mathbb{R}^2$, $x = (x_1, x_2)$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow x' = Ax$$

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$$

$\det(A - \lambda I_2) = 0$.

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda) + 2 = 0$$

$$\lambda^2 - 3\lambda - \lambda + 3 + 2 = 0$$

$$\lambda^2 - 4\lambda + 5 = 0$$

$$\Delta = (-4)^2 - 4 \cdot 1 \cdot 5 = 16 - 20 = -4$$

$$\lambda_{1,2} = \frac{4 \pm 2i}{2} = \frac{\cancel{2}(2 \pm i)}{\cancel{2}} = 2 \pm i$$

$n = 2$

$\lambda_1 = 2 + i$, $m_1 = 1$

$\lambda_2 = 2 - i$, $m_2 = 1$

$k = n = 2$; $m_1 + m_2 = 1 + 1 = 2$

$\boxed{\lambda_1 = 2+i}$, $\boxed{\lambda_2 = 2 - i = \overline{\lambda_1}}$, $m_1 = m_2 = 1 \Rightarrow$ se vor obține 2 soluții pt sist. fundamental.

Determinăm $u \in \mathbb{C}^2$, $u \neq 0_{\mathbb{C}^2}$ a.î. $Au = \lambda_1 u \Rightarrow$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (2+i) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} u_1 + u_2 = (2+i) u_1 \\ -2u_1 + 3u_2 = (2+i) u_2 \end{cases} \Rightarrow \begin{cases} u_1(1-2-i) + u_2 = 0 \\ -2u_1 = (-1+i) u_2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} u_2 = (1+i) u_1 \\ u_2 = \frac{-2}{-1+i} u_1 = \frac{-2(-1-i)}{(-1)^2 - i^2} u_1 = \frac{\cancel{2}(1+i)}{\cancel{2}} u_1 \end{cases}$$

$\boxed{i^2 = -1}$

Deci: $u_2 = (1+i) u_1 \Rightarrow$

$$\Rightarrow u = \begin{pmatrix} u_1 \\ (1+i)u_1 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 1+i \end{pmatrix}, u_1 \in \mathbb{R}$$

Cele 2 soluţii din sistemul fundam:

$$\varphi_1(t) = Re\left(e^{\lambda_1 t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}\right)$$
$$\varphi_2(t) = Im\left(e^{\lambda_1 t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}\right).$$

Avem

$$e^{\lambda_1 t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix} = e^{2t+it}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) =$$

$$= e^{2t}e^{it}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = e^{2t}(\cos t + i\sin t)\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \Rightarrow$$

$$\boxed{e^{it} = \cos t + i\sin t}$$

$$\Rightarrow \varphi_1(t) = e^{2t}\left(\cos t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \Rightarrow \boxed{\varphi_1(t) = e^{2t}\begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix}}$$

$$\varphi_2(t) = e^{2t}\left(\cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \Rightarrow \boxed{\varphi_2(t) = e^{2t}\begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix}}$$

Deci: $S_A = \{ c_1\varphi_1 + c_2\varphi_2 \mid c_1, c_2 \in \mathbb{R} \}$

sau $X(t) = \phi(t) = \text{coloane}(\varphi_1(t), \varphi_2(t)) =$

matricea fundamentală de soluţii

$$= e^{2t}\begin{pmatrix} \cos t & \sin t \\ \cos t - \sin t & \sin t + \cos t \end{pmatrix}$$

<u>Verif</u>: $\det X(t) \neq 0, \forall t \in \mathbb{R}$

$$\left| e^{2t} \cdot \begin{pmatrix} \cos t & \sin t \\ \cos t - \sin t & \sin t + \cos t \end{pmatrix} \right| = e^{2t} \cdot e^{2t}(\cancel{\cos t \sin t} + \cos^2 t - \cancel{\sin t \cos t} + \sin^2 t) =$$

$$= e^{4t} \neq 0 \quad \forall t \in \mathbb{R}.$$

$$\boxed{\begin{gathered}\det(\alpha A) = \alpha^n \det(A) \\ A \in M_n(\mathbb{R}) \\ \alpha \in \mathbb{R}\end{gathered}}$$

Avem: $S_A = \left\{ x(t) = \phi(t) \cdot C \mid C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^2 \right\}.$

(2) $\begin{cases} x_1' = 2x_1 - x_2 - x_3 \\ x_2' = 3x_1 - 2x_2 - 3x_3 \\ x_3' = -x_1 + x_2 + 2x_3 \end{cases}$, $n = 3$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} ; \quad x' = Ax$$

$p_A(\lambda) = \det(A - \lambda I_3) = 0$.

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ 3 & -2-\lambda & -3 \\ -1 & 1 & 2-\lambda \end{vmatrix} = 0 \overset{C3 \leftarrow C3 - C2}{\Leftrightarrow} \begin{vmatrix} 2-\lambda & -1 & 0 \\ 3 & -2-\lambda & -3+2+\lambda \\ -1 & 1 & 2-\lambda-1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 0 \\ 3 & -2-\lambda & \lambda-1 \\ -1 & 1 & -(\lambda-1) \end{vmatrix} = 0 \Rightarrow (\lambda-1) \begin{vmatrix} 2-\lambda & -1 & 0 \\ 3 & -2-\lambda & 1 \\ -1 & 1 & -1 \end{vmatrix} = 0 \quad L2 \leftarrow L2 + L3$$

$$\Rightarrow (\lambda-1) \begin{vmatrix} 2-\lambda & -1 & 0 \\ 2 & -1-\lambda & 0 \\ -1 & 1 & -1 \end{vmatrix} = 0 \Rightarrow (\lambda-1)\big((\lambda+1)(2-\lambda) + 0 + 0 - 0 - 0 - 2\big) = 0$$

$$\Rightarrow (\lambda-1)(2\lambda - \lambda^2 + \cancel{2} - \lambda - \cancel{2}) = 0 \Rightarrow (\lambda-1)(-\lambda^2 + \lambda) = 0 \Rightarrow$$

$$\Rightarrow -\lambda(\lambda-1)(\lambda-1) = 0 \Rightarrow \underbrace{-\lambda(\lambda-1)^2 = 0}_{p_A(\lambda) = (-1)^3(\lambda-0)(\lambda-1)^2} \Rightarrow$$

$$\Rightarrow \begin{cases} \lambda_1 = 0, m_1 = 1 \\ \lambda_2 = 1, m_2 = 2 \end{cases}$$

- $\boxed{\lambda_1 = 0, m_1 = 1}$ ⇒ determinăm $u \in \mathbb{R}^3$, $u \neq 0_{\mathbb{R}^3}$ a.î.

$$A u = \underset{0}{\lambda_1} u \Rightarrow \begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} 2u_1 - u_2 - u_3 = 0 \\ 3u_1 - 2u_2 - 3u_3 = 0 \\ -u_1 + u_2 + 2u_3 = 0 \end{cases}$$

$$\begin{vmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{vmatrix} = -8 - 3 - 3 + 2 + 6 + 6 = -14 + 14 = 0 \Rightarrow$$

⇒ ele sunt dependente.

obs că lc. 1 - lc. 2 = lc. 3

$$\Delta_p = \begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} = -4+3 = -1 \neq 0 \Rightarrow u_3 \text{ secundară}$$

$$\Rightarrow \begin{cases} 2u_1 - u_2 = u_3 \\ 3u_1 - 2u_2 = 3u_3 \end{cases} \Rightarrow \left.\boxed{u_2 = 2u_1 - u_3}\right\} \Rightarrow$$

$$\Rightarrow 3u_1 - 4u_1 + 2u_3 = 3u_3 \Rightarrow -u_1 = u_3 \Rightarrow$$

$$\Rightarrow \boxed{u_1 = -u_3} \Rightarrow u_2 = -2u_3 - u_3$$

$$\boxed{u_2 = -3u_3} \Rightarrow$$

$$\Rightarrow u = \begin{pmatrix} -u_3 \\ -3u_3 \\ u_3 \end{pmatrix} = u_3 \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

Sol. în sist. fundam este $\varphi_1(t) = e^{\lambda_1 t} \cdot \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} \Rightarrow \boxed{\varphi_1(t) = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}}$

$\boxed{\lambda_2 = 1, \; m_2 = 2}$

Det. $p_0, p_1 \in \mathbb{R}^3$, vectori nu amândoi nuli

a.î. $\varphi(t) = (p_0 + p_1 t) e^{\lambda_2 t} = (p_0 + p_1 t) e^t$

să fie soluție a sistemului $x' = Ax$. $\Rightarrow$

$$\Rightarrow \left((p_0 + p_1 t) e^t \right)' = A \cdot (p_0 + p_1 t) e^t \Rightarrow$$

$$\Rightarrow (p_0 + p_1 t)' e^t + (p_0 + p_1 t) \underbrace{(e^t)'}_{= e^t} = (Ap_0 + Ap_1 \cdot t) e^t \;\Big|\; : e^t$$

$$\Rightarrow p_1 + p_0 + p_1 t = (Ap_0) + t(Ap_1)$$

Identif coef lui puterilor lui t $\Big| \Rightarrow \begin{cases} p_1 + p_0 = Ap_0 \\ p_1 = Ap_1 \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} p_1 = Ap_0 - p_0 \\ 0_{\mathbb{R}^3} = Ap_1 - p_1 \end{cases} \Rightarrow \begin{cases} \boxed{p_1 = (A - I_3) p_0} \\ 0_{\mathbb{R}^3} = (A - I_3) \cdot p_1 \end{cases} \quad \Big| \cdot (A - I_3) \text{ la stg}$$

$$\Rightarrow \underbrace{(A - I_3) p_1}_{0_{\mathbb{R}^3}} = (A - I_3)(A - I_3) p_0 \Rightarrow (A - I_3)^2 p_0 = 0_{\mathbb{R}^3} \Rightarrow$$

$$\Rightarrow p_0 \in \underbrace{\ker\left((A - I_3)^2 \right)}_{\text{subspațiu}} \subset \mathbb{R}^3$$

Luăm pt p_0 elem. unei baze din $\ker((A-I_3)^2)$

Prin definiție, $\ker((A-I_3)^2) = \{v \in \mathbb{R}^3 \mid (A-I_3)^2 v = 0_{\mathbb{R}^3}\}$

$$(A-I_3)^2 = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ -3 & 3 & 3 \\ 1 & -1 & -1 \end{pmatrix}$$

Din ec

$$(A-I_3)^2 v = 0_{\mathbb{R}^3} \Rightarrow \begin{pmatrix} -1 & 1 & 1 \\ -3 & 3 & 3 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \left.\begin{cases} -v_1 + v_2 + v_3 = 0 \\ -3v_1 + 3v_2 + 3v_3 = 0 \ /:3 \Rightarrow \text{ec.1} \\ v_1 - v_2 - v_3 = 0 \ /\cdot(-1) \Rightarrow \text{ec.1} \end{cases}\right| \Rightarrow v_1 = v_2 + v_3 \Rightarrow$$

$$\Rightarrow \forall v \in \ker((A-I_3)^2): \ v = \begin{pmatrix} v_2+v_3 \\ v_2 \\ v_3 \end{pmatrix} = v_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + v_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow$$

$v_2, v_3 \in \mathbb{R}$

$$\Rightarrow \ker((A-I_3)^2) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \Rightarrow \left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\} \text{ bază în } \ker((A-I_3)^2)$$

Alegem pt p_0 elem. bazei găsite.

$$\bullet\ p_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rightarrow p_1 = (A-I_3)p_0 = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \boxed{\varphi_2(t) = (p_0 + p_1 t)e^t = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^t}.$$

$$\bullet\ p_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow p_1 = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \boxed{\varphi_3(t) = (p_0 + p_1 t)e^t = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^t}$$

Matricea fundamentală de soluții $\boxed{X(t) = \Phi(t) = \begin{pmatrix} -1 & e^t & e^t \\ -3 & e^t & 0 \\ 1 & 0 & e^t \end{pmatrix}}$

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Deci: $S_A = \left\{ x(t) = \phi(t)\, c \;\middle|\; c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \in \mathbb{R}^3 \right\}$.

Verificăm că $\det X(t) \neq 0$

Avem $\det X(t) = \begin{vmatrix} -1 & e^t & e^t \\ -3 & e^t & 0 \\ 1 & 0 & e^t \end{vmatrix} =$

$= -e^{2t} + 0 + 0 - e^{2t} - 0 + 3e^{2t} = e^{2t} \neq 0$. $\forall t \in \mathbb{R}$.

(3) $\begin{cases} x_1' = 2x_1 - x_2 + x_3 + e^t \\ x_2' = 3x_1 - 2x_2 - 3x_3 + 1 \\ x_3' = -x_1 + x_2 + 2x_3 - e^{2t} \end{cases}$

$x' = Ax + b(t)$;

$A = \begin{pmatrix} 2 & -1 & -1 \\ +3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}$; $b(t) = \begin{pmatrix} e^t \\ 1 \\ -e^{2t} \end{pmatrix}$

- Se determină soluția sist. liniar omogen atașat:

$\bar{x}' = A\bar{x} \Rightarrow \bar{x}(t) = \phi(t) \cdot C$, $C \in \mathbb{R}^3$

din ex (2)

$\phi(t) = \begin{pmatrix} -1 & e^t & e^t \\ -3 & e^t & 0 \\ 1 & 0 & e^t \end{pmatrix}$, $t \in \mathbb{R}$.

- Aplicăm met. variației const:

determinăm o funcție $C : \mathbb{R} \to \mathbb{R}^3$ ai $x(t) = \phi(t) \cdot C(t)$ să fie sol. sistemului afin $x' = Ax + b(t) \Rightarrow$

$\Rightarrow (\phi(t) \cdot C(t))' = A\phi(t)\, C(t) + b(t) \Rightarrow$

$\Rightarrow \cancel{\underbrace{\phi'(t)}_{A\phi(t)} C(t)} + \phi(t) C'(t) = \cancel{A\phi(t) C(t)} + b(t) \Rightarrow$

$\Rightarrow \phi(t)\, C'(t) = b(t) \Rightarrow$

$\Rightarrow \begin{pmatrix} -1 & e^t & e^t \\ -3 & e^t & 0 \\ 1 & 0 & e^t \end{pmatrix} \begin{pmatrix} C_1' \\ C_2' \\ C_3' \end{pmatrix} = \begin{pmatrix} e^t \\ 1 \\ -e^{2t} \end{pmatrix}$ rez. sistemul în nec. $C_1', C_2', C_3' \Rightarrow$

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$$\Rightarrow \begin{cases} C_1' + e^t C_2' + e^t C_3' = e^t \\ -3C_1' + e^t C_2' = 1 \\ C_1' + e^t C_3' = -e^{2t} \end{cases} \Rightarrow \boxed{C_2' = \frac{3C_1'}{e^t} = 3e^{-t}C_1'}$$

$$\Rightarrow C_3' = \frac{-e^{2t} - C_1'}{e^t} \Rightarrow \boxed{C_3' = -e^t - C_1' e^{-t}}$$

$$\Rightarrow \cancel{C_1'} + 3C_1' + (-e^{2t}) - \cancel{C_1'} = e^t$$

$$\boxed{C_1' = \frac{1}{3}(e^t + e^{2t})}$$

$$\boxed{C_2' = 1 + e^t}$$

$$C_3' = -e^t - \frac{1}{3}(1 + e^t) \Rightarrow \boxed{C_3' = \frac{1}{3}(1 + 4e^t)}$$

$\Rightarrow$ integrăm

$$\begin{cases} C_1 = \frac{1}{3}\int (e^t + e^{2t})\,dt = \frac{1}{3}\left(e^t + \frac{e^{2t}}{2}\right) + K_1 \\ C_2 = \int (1 + e^t)\,dt = t + e^t + K_2 \\ C_3 = \frac{1}{3}\int (1 + 4e^t)\,dt = \frac{1}{3}(t + 4e^t) + K_3 \end{cases} \quad K_1, K_2, K_3 \in \mathbb{R} \Rightarrow$$

$$\Rightarrow S_{A,b} = \left\{ x(t) = \phi(t)C(t) = \begin{pmatrix} -1 & e^t & e^t \\ -3 & e^t & 0 \\ 1 & 0 & e^t \end{pmatrix} \begin{pmatrix} \frac{1}{3}\left(e^t + \frac{e^{2t}}{2}\right) + K_1 \\ t + e^t + K_2 \\ \frac{1}{3}(t + 4e^t) + K_3 \end{pmatrix} \;\middle|\; K_1, K_2, K_3 \in \mathbb{R} \right\}.$$

<u>Temă</u>: Det. mulț. sol. sistemelor:

1) $\begin{cases} x_1' = 4x_1 - x_2 + 1 \\ x_2' = 3x_1 + x_2 - x_3 + t \\ x_3' = x_1 + x_3 \end{cases}$; $b(t) = \begin{pmatrix} 1 \\ t \\ 0 \end{pmatrix}$

2) $\begin{cases} x_1' = x_2 + 2e^t \\ x_2' = x_1 \end{cases}$; 3) $\begin{cases} x_1' = -x_1 + x_2 - 2x_3 \\ x_2' = 4x_1 + x_2 \\ x_3' = 2x_1 + x_2 - x_3 \end{cases} + e^{-t}$