Grupa 331, Seminar 13, 14.01.2021, EDDP

1) Să se rezolve următoarele probleme Cauchy:

✓ a) $\begin{cases} x_2(\partial_2 u) - (\partial_1 u)(\partial_2 u) + x_1 x_2 - u = 0 \\ u(\underbrace{1}_{\alpha_1(s)}, \underbrace{s}_{\alpha_2(s)}) = \underbrace{s}_{\varphi(s)}, \ s \in \mathbb{R} \end{cases}$

b) $\begin{cases} x_1^2 + x_2^2 + \frac{1}{2}(\partial_1 u)^2 + \frac{1}{2}(\partial_2 u)^2 - 3u = 0. \\ u(s,2) = s^2 + 4, \ s > 0 \end{cases}$

c) $\begin{cases} (\partial_1 u)^2 + (\partial_2 u)^2 + (\partial_1 u)(\partial_2 u) - x_1(\partial_2 u) - x_1(\partial_1 u) + x_1 x_2 - 2u = 0 \\ u(x_1, x_2) = \frac{x_2^2}{2} \text{ pe } S = \{x \in \mathbb{R}^2 \mid x_1 = -1\}. \end{cases}$

a)

$F(x_1, x_2, u, \partial_1 u, \partial_2 u) = x_2 \partial_2 u - (\partial_1 u)(\partial_2 u) + x_1 x_2 - u$

Cond inițiale: $\begin{cases} \alpha_1(s) = 1 \\ \alpha_2(s) = s \end{cases}, s \in \mathbb{R}$

(parametrizarea suprafeței pe care este dată cond. inițială).

$\varphi(s) = s$

Avem: $\begin{cases} \partial_1 u = p_1 \\ \partial_2 u = p_2 \end{cases}$

determinăm γ_1, γ_2, valorile pentru p_1 și p_2 pe S, prin rezolvarea sistemului:

$$\begin{cases} F(\alpha_1(s), \alpha_2(s), \varphi(s), \gamma_1, \gamma_2) = 0 \\ \gamma_1 \cdot \alpha_1'(s) + \gamma_2 \cdot \alpha_2'(s) = \varphi'(s) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \alpha_2(s) \cdot \gamma_2 - \gamma_1 \gamma_2 + \alpha_1(s)\alpha_2(s) - \varphi(s) = 0 \\ \gamma_1 \cdot (1)' + \gamma_2 \cdot s' = s' \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} s\gamma_2 - \gamma_1\gamma_2 + \cancel{s} - \cancel{s} = 0. \\ \gamma_1 \cdot 0 + \gamma_2 \cdot 1 = 1 \Rightarrow \boxed{\gamma_2 = 1} \end{cases} \Rightarrow s - \gamma_1 = 0 \quad \boxed{\gamma_1 = s}$$

• scriem sistemul caracteristic; calculăm derivatele parțiale ale funcției $F(x_1, x_2, u, p_1, p_2) = x_2 p_2 - p_1 p_2 + x_1 x_2 - u$

$$\frac{\partial F}{\partial x_1}=x_2\,; \quad \frac{\partial F}{\partial x_2}=p_2+x_1\,; \quad \frac{\partial F}{\partial p_1}=-p_2\,; \quad \frac{\partial F}{\partial p_2}=x_2-p_1$$

$$\frac{\partial F}{\partial u}=-1.$$

Se obține sistemul caracteristic:

$$\begin{cases} \dfrac{dx_1}{dt}=-p_2 & \left(\dfrac{\partial F}{\partial p_1}\right) \\[2mm] \dfrac{dx_2}{dt}=x_2-p_1 & \left(\dfrac{\partial F}{\partial p_2}\right) \\[2mm] \dfrac{dp_1}{dt}=-x_2-p_1\cdot(-1) & \left(-\dfrac{\partial F}{\partial x_1}-p_1\dfrac{\partial F}{\partial u}\right) \\[2mm] \dfrac{dp_2}{dt}=-\cancel{p_2}-x_1-\cancel{p_2(-1)} & \left(-\dfrac{\partial F}{\partial x_2}-p_2\dfrac{\partial F}{\partial u}\right) \\[2mm] \dfrac{du}{dt}=p_1(-p_2)+p_2(x_2-p_1) & \left(p_1\dfrac{\partial F}{\partial p_1}+p_2\dfrac{\partial F}{\partial p_2}\right) \\ x_1(0)=\alpha_1(s) \\ x_2(0)=\alpha_2(s) \\ p_1(0)=\gamma_1(s) \\ p_2(0)=\gamma_2(s) \\ u(0)=\varphi(s) \end{cases}$$

$$\Rightarrow \begin{cases} x_1'=-p_2 & (1) \\ x_2'=x_2-p_1 & (2) \\ p_1'=-x_2+p_1 & (2) \\ p_2'=-x_1 & (1) \\ u'=p_2x_2-2p_1p_2 & (3) \\ x_1(0)=1 \\ x_2(0)=s \\ p_1(0)=s \\ p_2(0)=1 \\ u(0)=s. \end{cases}$$

t = variab. independentă

x_1, x_2, u, p_1, p_2 = variabile dependente.

(3) → ec. pt u, care se integrează după ce aflăm p_1, p_2, x_1, x_2.

$$(1)\ \begin{cases} x_1'=-p_2 \\ p_2'=-x_1 \end{cases} \Rightarrow \begin{pmatrix} x_1' \\ p_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ p_2 \end{pmatrix}$$

sistem liniar cu coef constanți, care se poate rezolva cu met. valorilor proprii sau, rezolvăm sistemul asociind ec de ordinul al doilea, liniară, cu coef constanți:

$$x_1'' = (\mathrm{tr}\,A)\,x_1' - (\det A)\cdot x_1$$

$\mathrm{tr}\,A = 0+0=0$

$\det A = \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = 0-1=-1$

$\Rightarrow x_1'' = x_1$ ec. de ordinul al 2-lea, liniară cu coeficienți constanți.

Scrie ec. caracteristică pt $x_1''=x_1$. Se obține:

$r^2 = r^0$

$r^2 = 1 \Rightarrow \begin{cases} r_1 = 1, m_1 = 1 \\ r_2 = -1, m_2 = -1 \end{cases}$

$\Rightarrow$ un sistem fundamental de soluții este:

$\varphi_1(t) = e^{r_1 t} = e^{t}$

$\varphi_2(t) = e^{r_2 t} = e^{-t}$ $\Rightarrow$

$\Rightarrow x_1(t) = C_1 e^{t} + C_2 e^{-t}, \; C_1, C_2 \in \mathbb{R}$

din ec. 1 din sistemul (1) $\Rightarrow x_1' = -p_2 \Rightarrow$

$\Rightarrow p_2 = -x_1' \Rightarrow p_2(t) = -(C_1 e^{t} + C_2 e^{-t})' \Rightarrow$

$\Rightarrow p_2(t) = -C_1 e^{t} + C_2 e^{-t}$

Determinăm C_1, C_2 ai $\begin{cases} x_1(0) = 1 \\ p_2(0) = 1 \end{cases} \Rightarrow \begin{cases} C_1 + C_2 = 1 \\ -C_1 + C_2 = 1 \end{cases}$

$2C_2 = 2 \Rightarrow \boxed{C_2 = 1} \Downarrow \boxed{C_1 = 0}$

Deci, se obține: $\boxed{\begin{aligned} \tilde{x}_1(t,1) &= e^{-t} \\ \tilde{p}_2(t,1) &= e^{-t} \end{aligned}}$

La fel, rezolvăm sistemul al doilea:

$(2): \begin{cases} x_2' = x_2 - p_1 \\ p_1' = -x_2 + p_1 \end{cases} \Rightarrow \begin{pmatrix} x_2' \\ p_1' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{B} \begin{pmatrix} x_2' \\ p_1' \end{pmatrix} \Rightarrow$ ec. de

ordinul al doilea pt x_2 este $x_2'' = (\mathrm{tr}\,B)\,x_2' - (\det B)\,x_2 \Rightarrow$

$$tr\, B = 1+1 = 2$$

$$\det B = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 1-1 = 0 \Rightarrow x_2'' = 2x_2' \Rightarrow$$

$\Rightarrow$ ec. caract: $r^2 = 2r^1 \Rightarrow r(r-2) \Rightarrow r_1 = 0, m_1 = 1$; $r_2 = 2, m_2 = 1$

$\Rightarrow$ un sistem fundamental de soluții este:

$$\begin{cases} \varphi_1(t) = e^{0\cdot t} = 1 \\ \varphi_2(t) = e^{2t} \end{cases} \Rightarrow x_2(t) = C_1 \cdot 1 + C_2 e^{2t}, \quad C_1, C_2 \in \mathbb{R}$$

Acum prima ec din (2) $\Rightarrow p_1 = x_2 - x_2' \Rightarrow$

$$\Rightarrow p_1(t) = C_1 + C_2 e^{2t} - (C_1 + C_2 e^{2t})' =$$

$$= C_1 + C_2 e^{2t} - 0 - C_2 e^{2t} \cdot 2 \Rightarrow \boxed{p_1(t) = C_1 - C_2 e^{2t}}$$

Dar $\begin{cases} x_2(0) = s \\ p_1(0) = s \end{cases} \Rightarrow \begin{cases} C_1 + C_2 = s \\ C_1 - C_2 = s \end{cases}$

$$2C_1 = 2s \Rightarrow \boxed{C_1 = s} \Rightarrow \boxed{C_2 = 0}$$

$$\Rightarrow \boxed{\begin{cases} \tilde{x}_2(t,s) = s \\ \tilde{p}_1(t,s) = s \end{cases}}$$

Ec (3) pentru u:

$$\frac{du}{dt} = p_2 x_2 - 2p_1 p_2$$

$$\frac{du}{dt} = e^{-t} \cdot s - 2s \cdot e^{-t}$$

$$\frac{du}{dt} = -s e^{-t}, \text{ ec. de tip primitivă}$$

$$\Rightarrow u = -s\int e^{-t} dt = -s \frac{e^{-t}}{-1} + C_5 = s e^{-t} + C_5 \Rightarrow$$

dar $u(0) = s$

$$\Rightarrow s \cdot 1 + C_5 = s \Rightarrow C_5 = 0. \Rightarrow$$

$$\Rightarrow \boxed{\tilde{u}(t,s) = s e^{-t}}$$

Soluția parametrică a prob Cauchy este:

$$\begin{cases} x_1 = e^{-t} \\ x_2 = s \\ u = s \cdot e^{-t} \end{cases} \Rightarrow \boxed{u(x_1, x_2) = x_1 \cdot x_2}$$

② Să se aducă la forma canonică ec. cvasiliniare cu derivate parțiale de ordinul al doilea în $\mathbb{R}^2$:

a) $3\partial_1^2 u - 5\partial_1\partial_2 u - 2\partial_2^2 u + 3\partial_1 u + \partial_2 u = 0$.

b) $x_2\, \partial_1^2 u + (x_1 - x_2)\partial_1\partial_2 u - x_1 \partial_2^2 u = 0$.

c) $\partial_1\partial_2 u + 2\partial_1 u + 3\partial_2 u + 6u = 0$.

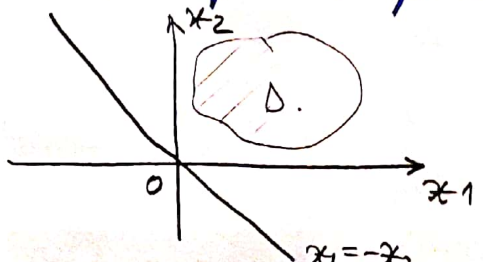
b) $a(x_1, x_2) = x_2$;

$b(x_1, x_2) = \dfrac{x_1 - x_2}{2}$; $c(x_1, x_2) = -x_1$

• calculăm

$$d(x_1, x_2) = b^2(x_1, x_2) - a(x_1, x_2) \cdot c(x_1, x_2)$$
$$= \left(\frac{x_1 - x_2}{2}\right)^2 - x_2(-x_1) =$$
$$= \frac{x_1^2 - 2x_1x_2 + x_2^2 + 4x_1x_2}{4} =$$
$$= \frac{x_1^2 + 2x_1x_2 + x_2^2}{4} = \frac{(x_1 + x_2)^2}{4} > 0 \Rightarrow$$

$\Rightarrow$ ec. este de tip hiperbolic pe $D \subset \mathbb{R}^2 \setminus \{(x_1, -x_1) \mid x_1 \in \mathbb{R}\}$



$$\lambda_1(x_1, x_2) = \frac{b(x_1, x_2) - \sqrt{d(x_1, x_2)}}{a(x_1, x_2)} = \frac{\frac{\cancel{x_1} - x_2}{2} - \frac{\cancel{x_1} + x_2}{2}}{x_2} =$$
$$= \frac{-2x_2}{2} \cdot \frac{1}{x_2} = -1$$

$$\lambda_2(x_1, x_2) = \frac{b(x_1, x_2) + \sqrt{d(x_1, x_2)}}{a(x_1, x_2)} = \frac{\frac{x_1 - \cancel{x_2}}{2} + \frac{x_1 + \cancel{x_2}}{2}}{x_2} =$$

$$= \frac{\cancel{2}x_1}{\cancel{2}} \cdot \frac{1}{x_2} = \frac{x_1}{x_2}$$

• determinare integrale prime pt ec:

$$\frac{dx_2}{dx_1} = -1 \quad \text{și} \quad \frac{dx_2}{dx_1} = \frac{x_1}{x_2}.$$

$$\Downarrow$$

$$dx_2 = -dx_1$$

$$\int dx_2 = -\int dx_1$$

$$x_2 = -x_1 + C_1$$

$$x_1 + x_2 = C_1$$

$$\boxed{\varphi_1(x_1, x_2) = x_1 + x_2}$$

separăm variabilele.

$$x_2 dx_2 = x_1 dx_1$$

$$\int x_2 dx_2 = \int x_1 dx_1.$$

$$\frac{x_2^2}{2} = \frac{x_1^2}{2} + C_2$$

$$\frac{x_2^2}{2} - \frac{x_1^2}{2} = C_2 \,/\cdot 2$$

$$\varphi_2(x_1, x_2) = x_2^2 - x_1^2$$

• se consideră transformarea de coordonate:

$$\begin{cases} y_1 = x_1 + x_2 \\ y_2 = -x_1^2 + x_2^2 \end{cases}$$

• ec. în $u(x)$ se transformă într-o ec în $\tilde{u}(y)$.

• calc. derivatele pt $\underline{u(x_1, x_2) = \tilde{u}(y_1(x_1, x_2), y_2(x_1, x_2))}$

Avem, întâi:

$$\frac{\partial y_1}{\partial x_1} = 1 \; ; \quad \frac{\partial y_1}{\partial x_2} = 1 \; ; \quad \frac{\partial y_2}{\partial x_1} = -2x_1 \; ; \quad \frac{\partial y_2}{\partial x_2} = 2x_2.$$

și apoi:

$$\partial_1 u = \frac{\partial u}{\partial x_1} = \frac{\partial \tilde{u}}{\partial y_1} \cdot 1 + \frac{\partial \tilde{u}}{\partial y_2}(-2x_1) = \frac{\partial \tilde{u}}{\partial y_1} - 2x_1 \frac{\partial \tilde{u}}{\partial y_2}$$

$$\partial_2 u = \frac{\partial u}{\partial x_2} = \frac{\partial \tilde{u}}{\partial y_1} \cdot 1 + \frac{\partial \tilde{u}}{\partial y_2} \cdot 2x_2 = \frac{\partial \tilde{u}}{\partial y_1} + 2x_2 \frac{\partial \tilde{u}}{\partial y_2}$$

$$\partial_1^2 u = \frac{\partial}{\partial x_1}(\partial_1 u) = \frac{\partial}{\partial x_1}\left(\frac{\partial \tilde{u}}{\partial y_1} - 2x_1 \frac{\partial \tilde{u}}{\partial y_2}\right) =$$

$$= \frac{\partial}{\partial y_1}\left(\frac{\partial \tilde{u}}{\partial y_1}\right) \cdot 1 + \frac{\partial}{\partial y_2}\left(\frac{\partial \tilde{u}}{\partial y_1}\right) \cdot (-2x_1) -$$

$$-2\left[\frac{\partial \tilde{u}}{\partial y_2} + x_1\left(\frac{\partial}{\partial y_1}\left(\frac{\partial \tilde{u}}{\partial y_2}\right) \cdot 1 + \frac{\partial}{\partial y_2}\left(\frac{\partial \tilde{u}}{\partial y_2}\right) \cdot (-2x_1)\right)\right]$$

$$\Rightarrow \partial_1^2 u = \frac{\partial^2 \tilde{u}}{\partial y_1^2} \underbrace{- 2x_1 \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2}} - 2\frac{\partial \tilde{u}}{\partial y_2} \underbrace{- 2x_1 \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2}} + 4x_1^2 \frac{\partial^2 \tilde{u}}{\partial y_2^2}$$

$$\Rightarrow \boxed{\partial_1^2 u = \frac{\partial^2 \tilde{u}}{\partial y_1^2} - 4x_1 \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2} + 4x_1^2 \frac{\partial^2 \tilde{u}}{\partial y_2^2} - 2\frac{\partial \tilde{u}}{\partial y_2}.}$$

$$\partial_1 \partial_2 u = \frac{\partial}{\partial x_1}(\partial_2 u) = \frac{\partial}{\partial x_1}\left(\underbrace{\frac{\partial \tilde{u}}{\partial y_1}} + 2x_2 \frac{\partial \tilde{u}}{\partial y_2} \right) =$$

$$= \frac{\partial^2 \tilde{u}}{\partial y_1^2} \cdot 1 + \underline{\frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2}} \cdot (-2x_1) + 2x_2 \left(\underline{\frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2}} \cdot 1 + \frac{\partial^2 \tilde{u}}{\partial y_2^2} (-2x_1) \right)$$

$$\boxed{\partial_1 \partial_1 u = \frac{\partial^2 \tilde{u}}{\partial y_1^2} + 2(x_2 - x_1) \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2} - 4x_1 x_2 \frac{\partial^2 \tilde{u}}{\partial y_2^2}}$$

$$\partial_2^2 u = \frac{\partial}{\partial x_2}(\partial_2 u) = \frac{\partial}{\partial x_2}\left(\underbrace{\frac{\partial \tilde{u}}{\partial y_1}} + 2x_2 \frac{\partial \tilde{u}}{\partial y_2} \right) =$$

$$= \frac{\partial^2 \tilde{u}}{\partial y_1^2} \cdot 1 + \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2} \cdot 2x_2 + 2\frac{\partial \tilde{u}}{\partial y_2} + 2x_2 \left[\frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2} \cdot 1 + \frac{\partial^2 \tilde{u}}{\partial y_2^2} \cdot (2x_2) \right] \Rightarrow$$

$$\Rightarrow \boxed{\partial_2^2 u = \frac{\partial^2 \tilde{u}}{\partial y_1^2} + 4x_2 \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2} + 4x_2^2 \frac{\partial^2 \tilde{u}}{\partial y_2^2} + 2\frac{\partial \tilde{u}}{\partial y_2}}$$

Ec. în $u(x)$ devine:

$$x_2 \cdot \cancel{\frac{\partial^2 \tilde{u}}{\partial y_1^2}} - 4x_1 x_2 \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2} + \cancel{4x_2 x_1^2 \frac{\partial^2 \tilde{u}}{\partial y_2^2}} - 2x_2 \frac{\partial \tilde{u}}{\partial y_2} +$$

$$+ \cancel{(x_1 - x_2)} \frac{\partial^2 \tilde{u}}{\partial y_1^2} - 2(x_2 - x_1)^2 \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2} - 4x_1 x_2 \cancel{(x_1 - x_2)} \frac{\partial^2 \tilde{u}}{\partial y_2^2} -$$

$$- \cancel{x_1 \frac{\partial^2 \tilde{u}}{\partial y_1^2}} - 4x_1 x_2 \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2} - \cancel{4x_2^2 x_1 \frac{\partial^2 \tilde{u}}{\partial y_2^2}} - 2x_1 \frac{\partial \tilde{u}}{\partial y_2} = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2}\left(-\cancel{4x_1x_2} - 2x_2^2 + \cancel{4x_1x_2} - 2x_1^2 - 4x_1x_2\right) -$$

$$-2x_2 \frac{\partial \tilde{u}}{\partial y_2} - 2x_1 \frac{\partial \tilde{u}}{\partial y_2} = 0.$$

$$\Rightarrow -2(x_1+x_2)^2 \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2} - 2(x_1+x_2) \frac{\partial \tilde{u}}{\partial y_2} = 0. \Big| : -2(x_1+x_2)$$

$$\Rightarrow \underbrace{(x_1+x_2)}_{y_1} \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2} + \frac{\partial \tilde{u}}{\partial y_2} = 0 \Rightarrow \boxed{y_1 \frac{\partial^2 \tilde{u}}{\partial y_1 \partial y_2} + \frac{\partial \tilde{u}}{\partial y_2} = 0}$$

în cazul $n=2$

Obs. La forma canonică a ec. hiperbolice poate să rămână și doar derivata mixtă, în loc diferența derivatelor de ordin 2 în raport cu y_1 resp y_2.

Rezultă:

$$\frac{\partial}{\partial y_2}\left(y_1 \frac{\partial \tilde{u}}{\partial y_1} + \tilde{u}\right) = 0 \Rightarrow$$ (integrăm în rap cu y_2)

$$\Rightarrow y_1 \frac{\partial \tilde{u}}{\partial y_1} + \tilde{u} = C_1(y_1)$$

$$\frac{\partial}{\partial y_1}(y_1 \tilde{u}) = C_1(y_1) \Rightarrow y_1 \tilde{u} = \int C_1(y_1)\,dy_1 + C_2(y_2)$$

integrăm în rap. cu y_1

$$\Rightarrow y_1 \tilde{u} = f(y_1) + g(y_2) \Rightarrow$$

$$\Rightarrow \tilde{u}(y_1, y_2) = \frac{1}{y_1}\left(f(y_1) + g(y_2)\right) \Rightarrow$$

$$\Rightarrow u(x_1, x_2) = \tilde{u}(x_1+x_2, x_2^2 - x_1^2) =$$

$$= \frac{1}{x_1+x_2}\left(f(x_1+x_2) + g(x_2^2 - x_1^2)\right).$$
