Grupa 331, Seminar 6, 12.11.2020

① (d1) $t^2x'' - 2x\cdot(tx') + tx' = 0$

$F(x, tx', t^2x'') = 0$

ec. Euler de ordin 2

$\left(F(x, tx^{(1)}, t^2x^{(2)}, \dots, t^n x^{(n)}) = 0 \right.$

$\left. \hookrightarrow \text{ec. Euler de ordin } n \right)$.

$(t, x) \xrightarrow{|t| = e^s} (s, y) \xrightarrow{y'(s) = z(y(s))} (y, z)$

sau

$(s, y) \xrightarrow{\frac{y'}{y} = z} (s, z)$

$x(t) = y(s(t))$

$s = \ln|t|$

$F_1(y, y', y'') = 0$

$x(t) = y(s(t))$

$\left.\begin{array}{l} tx' = y' \\ t^2x'' = y'' - y' \end{array}\right| \Rightarrow \underbrace{y'' - \cancel{y'} - 2\cdot y\cdot y' + \cancel{y'} = 0}_{F_1(y, y', y'') = 0.}$

$\Rightarrow \underline{y'' - 2yy' = 0}$.

$y'(s) = z(y(s))$

$y''(s) = z'(y(s))\cdot y'(s) = z'(y(s))\cdot z \Rightarrow$

$\Rightarrow y'' = z'z \Rightarrow z'z - 2yz = 0 \Rightarrow$

$\Rightarrow \underline{z\cdot(z' - 2y) = 0}$

Ⓘ $\underline{z = 0} \Rightarrow \underline{y' = 0} \Rightarrow \underset{y(s)}{y} = c_1 \Rightarrow \underset{x(t)}{x} = c_1,\ c_1 \in \mathbb{R}$

Ⓘⓘ $z' = 2y \Rightarrow \underset{\text{ec. de tip primitivă}}{\frac{dz}{dy} = 2y} \Rightarrow z = y^2 + c_1 \Rightarrow$

$\Rightarrow \underset{\text{ec. cu var. sep}}{\underline{y' = y^2 + c_1}} \Rightarrow \frac{dy}{ds} = y^2 + c_1$ ec. cu var. sep.

$\frac{dy}{ds} = \underbrace{(y^2 + c_1)}_{b(y)} \cdot \frac{1}{a(s)}$

$c_1 < 0$, $c_1 = 0$, $c_1 > 0$ (temă!)

② (f1) $\left(\frac{x}{t}\right)^2 + (x')^2 - 3tx'' - 2\frac{x}{t}\cdot x' = 0$

$F\left(\frac{x}{t}, x', tx''\right) = 0$

ec. omogenă de ordin 2

$\left(F\left(\frac{x}{t}, x^{(1)}, t x^{(2)}, t^2 x^{(3)}, \dots, t^{n-1} x^{(n)}\right) = 0 \right)$

(t,x) ec. omogenă $\xrightarrow{\frac{x}{t}=y}$ (t,y) ec. Euler.

$\boxed{\frac{x}{t}=y} \Rightarrow x = ty$

$\boxed{x' = y + ty'}$

$x'' = y' + y' + ty'' \;|\, t \Rightarrow \boxed{t x'' = 2ty' + t^2 y''}$

$$y^2 + (y+ty')^2 - 3(2ty' + t^2y'') - 2y\cdot(y+ty') = 0.$$

$$\cancel{y^2} + \cancel{y^2} + \cancel{2tyy'} + t^2(y')^2 - 6ty' - 3t^2y'' - \cancel{2y^2} - \cancel{2tyy'} = 0.$$

$$(ty')^2 - 6(ty') - 3(t^2y'') = 0$$

$$F(y, ty', t^2y'') = 0.$$

ec. Euler (temă, analog ex. 1).

Probleme Cauchy pt ec. dif. de ordin 1

1. Fie problema Cauchy: $\begin{cases} \frac{dx}{dt} = t \sin x, \\ x(0) = \frac{\pi}{4} \end{cases}$ $(t,x) \in [-1,1]\times[0,\frac{\pi}{2}]$ (1)

   a) Verificare ipoteze TEU

   b) Calculați $\varphi_0, \varphi_1, \varphi_2$ din șirul aprox. succesive.

   c) Determinarea soluției prob. Cauchy (1).

   d) Pentru $t \in [0, \frac{\pi}{2}]$, construiți o schemă numerică de ordin 2 cu $N+1$ puncte echidistante.

2. Fie problema Cauchy: $\begin{cases} \frac{dx}{dt} = 2x + t, & (t,x) \in \mathbb{R}^2 \\ x(0) = 1 \end{cases}$

   a) Se cere șirul de aproximații succesive

   b) Soluția problemei.

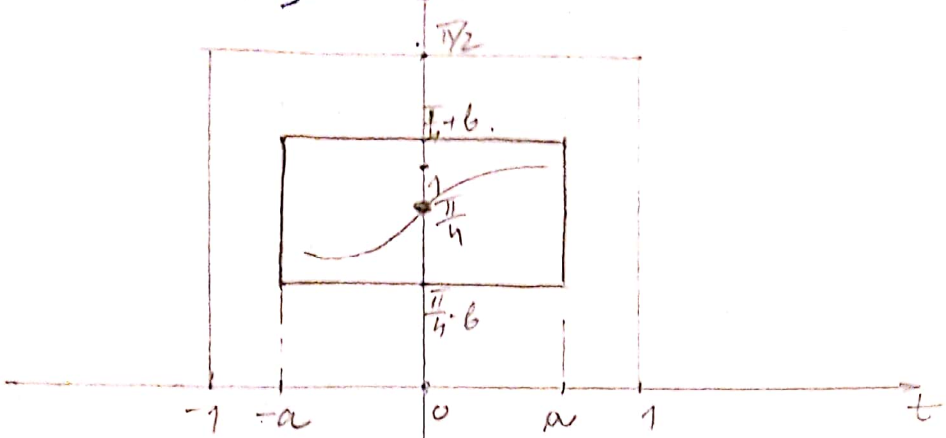
   c) Pt $N=2$, $t \in [0,1]$, calculați aproximarea soluției în $t=1$ folosind metoda Euler cu puncte echidistante.

1. $D = [-1,1]\times[0,\frac{\pi}{2}]$

$f: D \to \mathbb{R}$

$f(t,x) = t \sin x$

$t_0 = 0, \quad x_0 = \frac{\pi}{4}.$



1) $\exists a,b>0$ a.î. $D_{a,b} = [-a,a]\times[\frac{\pi}{4}-b, \frac{\pi}{4}+b] \subset D$.

Luăm: $\underline{a\in(0,1) \ , \ b\in(0,\frac{\pi}{4})}$.

De exemplu: $a=\frac{1}{2}$, $b=\frac{\pi}{8}$.

2) f continuă în ambele variabile pt că este produs de 2 funcții continue.

$$M = \sup_{(t,x)\in D_{a,b}} |f(t,x)| = \sup_{\substack{t\in[-a,a] \\ x\in[\frac{\pi}{4}-b, \frac{\pi}{4}+b]\subset[0,\frac{\pi}{2}]}} |t||\sin x| = \underline{a\cdot \sin(\tfrac{\pi}{4}+b)}$$

pe $[0,\frac{\pi}{2}]$ sin este crescătoare

3) $\frac{\partial f}{\partial x}(t,x) = t\cos x$ continuă

$$L = \sup_{(t,x)\in D_{a,b}} \left|\frac{\partial f}{\partial x}(t,x)\right| = \sup_{\substack{t\in[-a,a] \\ x\in[\frac{\pi}{4}-b, \frac{\pi}{4}+b]\subset[0,\frac{\pi}{2}]}} |t||\cos x| = \underline{a\cos(\tfrac{\pi}{4}-b)}$$

pe $[0,\frac{\pi}{2}]$ cos descrescătoare

Se verif. ip TEU $\Rightarrow$ $\forall\ \alpha\in\left(0, \min(a, \frac{b}{M})\right)$, $\exists!$

$\varphi: [-\alpha,\alpha] \longrightarrow [\frac{\pi}{4}-b, \frac{\pi}{4}+b]$ sol. a prob. Cauchy(1).

b) Șirul de aprox. succesive:

$$\boxed{\varphi_0(t) = x_0 = \frac{\pi}{4},}$$

$\forall n\in\mathbb{N}^*$: $\varphi_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \varphi_n(s))\,ds \Rightarrow$

$$\Rightarrow \boxed{\varphi_{n+1}(t) = \frac{\pi}{4} + \int_0^t s\sin(\varphi_n(s))\,ds.}$$

$$\boxed{\varphi_0(t) = \frac{\pi}{4}}$$

$$\varphi_1(t) = \frac{\pi}{4} + \int_0^t s\cdot\sin(\varphi_0(s))\,ds = \frac{\pi}{4} + \int_0^t s\cdot\sin\frac{\pi}{4}\,ds =$$

$$= \frac{\pi}{4} + \frac{\sqrt{2}}{2}\int_0^t s\,ds = \frac{\pi}{4} + \frac{\sqrt{2}}{2}\,\frac{s^2}{2}\Big|_0^t = \frac{\pi}{4} + \frac{\sqrt{2}}{4}t^2 \Rightarrow$$

$$\Rightarrow \boxed{\varphi_1(t) = \frac{\pi}{4} + \frac{\sqrt{2}}{4}t^2}$$

$$\varphi_2(t) = \frac{\pi}{4} + \int_0^t s \sin(\varphi_1(s))\,ds = \frac{\pi}{4} + \int_0^t s \sin\left(\frac{\pi}{4} + \frac{\sqrt{2}}{4}s^2\right)ds$$

$$\frac{\pi}{4} + \frac{\sqrt{2}}{4}s^2 = z \Rightarrow \frac{\sqrt{2}}{4}\cdot 2s\,ds = dz \Rightarrow$$

$$\Rightarrow \frac{\sqrt{2}\cdot s}{2}ds = dz$$

s	0	t
z	$\frac{\pi}{4}$	$\frac{\pi}{4} + \frac{\sqrt{2}}{4}t^2$

$$\varphi_2(t) = \frac{\pi}{4} + \frac{\cancel{2}}{\cancel{\sqrt{2}}}\int_{\frac{\pi}{4}}^{\frac{\pi}{4}+\frac{\sqrt{2}}{4}t^2} \sin z\,dz = \frac{\pi}{4} + \sqrt{2}\,(-\cos z)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{4}+\frac{\sqrt{2}}{4}t^2} \Rightarrow$$

$$\Rightarrow \varphi_2(t) = \frac{\pi}{4} - \sqrt{2}\cos\left(\frac{\pi}{4} + \frac{\sqrt{2}}{4}t^2\right) + \sqrt{2}\cos\frac{\pi}{4} \Rightarrow \boxed{\varphi_2(t) = \frac{\pi}{4} + 1 - \sqrt{2}\cos\left(\frac{\pi}{4} + \frac{\sqrt{2}}{4}t^2\right)}$$

$$\varphi_3(t) = \frac{\pi}{4} + \int_0^t s \sin(\varphi_2(s))\,ds = \frac{\pi}{4} + \int_0^t s \sin\left(\frac{\pi}{4} + 1 - \sqrt{2}\cos\left(\frac{\pi}{4} + \frac{\sqrt{2}}{4}s^2\right)\right)ds.$$

c) temă! (ec. cu var. separabile)

d) Schema de ordin 2 construită prin metoda Taylor:

$$\begin{cases} x_0 \\ x_{j+1} = x_j + h\,\phi_2(h, t_j, x_j), \quad j = \overline{0, N-1} \end{cases}$$

unde
$$\phi_2(h, t_j, x_j) = f(t_j, x_j) + \frac{h}{2}\left[\frac{\partial f}{\partial t}(t_j, x_j) + \frac{\partial f}{\partial x}(t_j, x_j)\cdot f(t_j, x_j)\right].$$

Calc.: $$\phi_2(h, t, x) = f(t, x) + \frac{h}{2}\left[\frac{\partial f}{\partial t}(t, x) + \frac{\partial f}{\partial x}(t, x)\cdot f(t, x)\right].$$

$$f(t, x) = t\sin x$$
$$\frac{\partial f}{\partial t}(t, x) = \sin x$$
$$\frac{\partial f}{\partial x}(t, x) = t\cos x.$$

$$\Rightarrow \phi_2(h, t, x) = t\sin x + \frac{h}{2}\left[\sin x + t\cos x \cdot t\sin x\right] \Rightarrow$$

$$\boxed{\begin{cases} x_0 \\ x_{j+1} = x_j + h\left[t_j \sin x_j + \frac{h}{2}\cdot \sin x_j \left(1 + t_j^2 \cos x_j\right)\right], \ j=\overline{0,N-1} \end{cases}}$$

$$\left.\begin{array}{l} t_j = t_0 + jh, \quad h = \frac{T}{N} \\ t_0 = 0, \quad t \in \left[0, \frac{\pi}{2}\right] \Rightarrow T = \frac{\pi}{2} \end{array}\right\} \Rightarrow$$

$$\Rightarrow t_j = 0 + j\frac{\pi}{2N} \Rightarrow \boxed{t_j = \frac{\pi}{2N}\cdot j, \ j = \overline{0,N}.}$$

(2) $\begin{cases} \frac{dx}{dt} = 2x + t \\ x(0) = 1 \end{cases}, \quad (t,x) \in \mathbb{R}^2$

a) verificare TEU (temă!)

$$\left.\begin{array}{l} t_0 = 0, \quad x_0 = 1 \\ f(t,x) = 2x + t \end{array}\right| \rightarrow \boxed{\begin{array}{l} \boxed{\varphi_0(t) = 1} \\ \varphi_{n+1}(t) = 1 + \int_0^t \left(s + 2\cdot\varphi_n(s)\right)ds, \ \forall n \in \mathbb{N}. \end{array}}$$

$$\varphi_1(t) = 1 + \int_0^t \left(s + 2\cdot\varphi_0(s)\right)ds = 1 + \int_0^t (s+2)\,ds = 1 + \frac{s^2}{2}\Big|_0^t + 2s\Big|_0^t$$

$$\Rightarrow \boxed{\varphi_1(t) = 1 + \frac{t^2}{2} + 2t} = \left(1 + \frac{2t}{1!}\right) + \left(\frac{t^2}{2!}\right)$$

$$\varphi_2(t) = 1 + \int_0^t \left(s + 2\varphi_1(s)\right)ds = 1 + \int_0^t \left(s + 2\left(1 + \frac{s^2}{2} + 2s\right)\right)ds =$$

$$= 1 + \frac{s^2}{2}\Big|_0^t + 2s\Big|_0^t + 2\cdot\frac{s^3}{2\cdot 3}\Big|_0^t + 2\cdot 2\frac{s^2}{2}\Big|_0^t \Rightarrow$$

$$\boxed{\varphi_2(t) = 1 + \frac{t^2}{2} + 2t + 2\frac{t^3}{3!} + 2^2\frac{t^2}{2!}}$$

$$\varphi_3(t) = 1 + \int_0^t \left(s + 2\varphi_2(s)\right)ds = 1 + \int_0^t \left[s + 2\left(1 + \frac{s^2}{2} + 2s + 2\frac{s^3}{3!} + 2^2\frac{s^2}{2!}\right)\right]ds$$

$$= 1 + \frac{s^2}{2}\Big|_0^t + 2s\Big|_0^t + 2\cdot\frac{s^3}{2\cdot 3}\Big|_0^t + 2^2\cdot\frac{s^2}{2}\Big|_0^t + 2^2\cdot\frac{s^4}{3!\cdot 4}\Big|_0^t + 2^3\frac{s^3}{2!\cdot 3}\Big|_0^t \Rightarrow$$

$$\boxed{\varphi_3(t) = 1 + \frac{t^2}{2} + 2t + 2\frac{t^3}{3!} + 2^2\frac{t^2}{2!} + 2^2\frac{t^4}{4!} + 2^3\frac{t^3}{3!}}$$

Dem prin inducție că:

$$(*)\left[\varphi_n(t) = \left(1 + \frac{2t}{1!} + \frac{2^2t^2}{2!} + \frac{2^3t^3}{3!} + \dots + \frac{2^n t^n}{n!}\right) + \left(\frac{t^2}{2!} + 2\frac{t^3}{3!} + 2^2\frac{t^4}{4!} + \dots + 2^{n-1}\frac{t^{n+1}}{(n+1)!}\right)\right.$$

Presup adev pt n și dem pt $n+1$ că:

$$\varphi_{n+1}(t) = \left(1+\frac{2t}{1!}+\frac{2^2t^2}{2!}+\dots+\frac{2^{n+1}t^{n+1}}{(n+1)!}\right)+$$
$$+\left(\frac{t^2}{2!}+2\frac{t^3}{3!}+\dots+2^n\frac{t^{n+2}}{(n+2)!}\right).$$

Calculăm φ_{n+1} din rel. de recurență:

$$\varphi_{n+1}(t) = 1+\int_0^t\left[s+2\cdot\left(1+\frac{2s}{1!}+\dots+\frac{2^n s^n}{n!}\right)+\right.$$
$$\left.+2\left(\frac{s^2}{2!}+2\frac{s^3}{3!}+\dots+2^{n-1}\frac{s^{n+1}}{(n+1)!}\right)\right]ds =$$
$$=1+\frac{s^2}{2}\Big|_0^t+\left(2s+\frac{2^2\cdot s^2}{1!\cdot 2}+\dots+\frac{2^{n+1}s^{n+1}}{n!(n+1)}\right)\Big|_0^t+$$
$$+\left(\frac{2s^3}{2!\cdot 3}+2^2\frac{s^4}{3!\cdot 4}+\dots+2^n\frac{s^{n+2}}{(n+1)!(n+2)}\right)\Big|_0^t=$$
$$=\left(1+\frac{2t}{1!}+\frac{2^2t^2}{2!}+\dots+\frac{2^{n+1}t^{n+1}}{(n+1)!}\right)+\left(\frac{t^2}{2!}+\frac{2t^3}{3!}+2^2\frac{t^4}{4!}+\dots+\right.$$
$$\left.+\dots+2^n\frac{t^{n+2}}{(n+2)!}\right)\Rightarrow$$

$\Rightarrow$ adev că $(\varphi_n)_{n\geq 0}$ are forma generală $(*)$

c) temă! (vezi exemplu în curs)

<u>Temă</u>: Fie problema Cauchy $\begin{cases}\frac{dx}{dt}=3\sqrt[3]{x^2}, & (t,x)\in\mathbb{R}^2\\ x(t_0)=x_0, & (t_0,x_0)\in\mathbb{R}^2\end{cases}$

a) Arătați că pt $x_0\neq 0$, sunt verificate ipotezele TEU.
Determinați soluția problemei.

b) Arătați că pt $\boxed{x_0=0}$ nu se poate aplica TEU.
Câte soluții are problema Cauchy în acest caz?.