Grupa 331, Seminar 10, 10.12.2020, EDDP

Pt. examen: – fără 22, 28, 29, 31 Ianuarie 2021
– dacă se poate fără sâmbete și duminici.

① Fie sistemul
$$\begin{cases} x_1' = \frac{1}{t}x_1 + \frac{2}{t}x_2 \\ x_2' = -\frac{2}{t}x_1 - \frac{3}{t}x_2 + \ln t \end{cases}, \quad t>0. \qquad (1)$$

a) forma matricială

b) Arătați că prin schimbarea de variabilă
$$\boxed{t = e^s}$$
se obține un sistem (2) cu coef. constanți pentru partea liniară.

c) Soluția generală pt (2), apoi pt (1) soluția generală și soluția care verifică: $\begin{cases} x_1(1)=2 \\ x_2(1)=1 \end{cases}$

a) $x' = A(t)x + b(t)$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{t} & \frac{2}{t} \\ -\frac{2}{t} & -\frac{3}{t} \end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \ln t \end{pmatrix}$$

$$A(t) = \frac{1}{t}\begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} = \frac{1}{t}B \quad ; \quad b(t) = \begin{pmatrix} 0 \\ \ln t \end{pmatrix}$$

$$A : (0,\infty) \to \mathcal{M}_2(\mathbb{R}) \qquad , \; B = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$$
$$b : (0,\infty) \to \mathbb{R}^2$$

b) $x' = \frac{1}{t}Bx + b(t)$ (3)

$$\underset{\substack{\| \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}}{(t, x)} \xrightarrow[s = \ln t]{t = e^s} \underset{\substack{\| \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}}{(s, y)}$$

$$\boxed{x(t) = y(s(t))}$$

$$s(t) = \ln t \Rightarrow s'(t) = \frac{1}{t}$$

$$\boxed{x'(t) = (y(s(t)))' = y'(s(t))\cdot s'(t) = y'(s)\cdot\frac{1}{t} = \boxed{\frac{1}{e^s}y'(s)}}$$

Sist. (3) devine: $\frac{1}{e^s}y' = \frac{1}{e^s}By + b(e^s) \;\Big|\cdot e^s$

$$y' = By + \underbrace{e^s b(e^s)}_{\tilde{b}(s)} \Rightarrow \underbrace{\boxed{y' = By + \tilde{b}(s)}}_{(2)}$$

cu $\tilde{b}(s) = e^s b(e^s) = \begin{pmatrix} e^s \cdot 0 \\ e^s \cdot \ln e^s \end{pmatrix} = \begin{pmatrix} 0 \\ s e^s \end{pmatrix}$

c) Rezolvăm sistemul (2):

$$y' = By + \tilde{b}(s) \quad , \quad B = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$$

- rezolvăm sistemul liniar omogen atașat:

$$\bar{y}' = B\bar{y}$$

- valorile proprii pt B:

$$\det(B - \lambda I_2) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -2 & -3-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (1-\lambda)(-3-\lambda) + 4 = 0 \Rightarrow -3 - \lambda + 3\lambda + \lambda^2 + 4 = 0 \Rightarrow$

$\Rightarrow \lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda+1)^2 = 0 \Rightarrow$

$\Rightarrow \underline{|\lambda_1 = -1, \ m_1 = 2|} \Rightarrow$ determinăm $p_0, p_1 \in \mathbb{R}^2$, nu amândoi nuli a.î. $\varphi(s) = (p_0 + p_1 s) e^{-s}$ să verifice sistemul liniar omogen: $\bar{y}' = B\bar{y} \Rightarrow$

$\Rightarrow \left((p_0 + p_1 s) e^{-s}\right)' = B(p_0 + p_1 s) e^{-s} \Rightarrow$

$\Rightarrow p_1 e^{-s} + (p_0 + p_1 s) e^{-s} \cdot (-1) = (Bp_0 + Bp_1 s) e^{-s} \quad \Big| : e^{-s} \Rightarrow$

$\Rightarrow \underbrace{p_1 - p_0}_{} - \underline{p_1} s = \underbrace{Bp_0}_{} + \underline{Bp_1} s \overset{\text{identificăm coef puterilor lui } s}{\Longrightarrow} \begin{cases} -p_1 = Bp_1 \\ p_1 - p_0 = Bp_0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} 0_{\mathbb{R}^2} = Bp_1 + p_1 \\ p_1 = Bp_0 + p_0 \end{cases} \Rightarrow \begin{cases} 0_{\mathbb{R}^2} = (B + I_2) p_1 \\ p_1 = (B + I_2) p_0 \ \Big/ (B+I_2) \text{ la stg} \end{cases}$

$\Rightarrow \underbrace{(B+I_2)p_1}_{= 0_{\mathbb{R}^2}} = (B+I_2)^2 p_0 \Rightarrow \underline{(B+I_2)^2 p_0 = 0_{\mathbb{R}^2}} \Rightarrow$

$\Rightarrow p_0 \in \ker\left((B+I_2)^2\right) = \{ v \in \mathbb{R}^2 \mid (B+I_2)^2 v = 0_{\mathbb{R}^2} \}$.

Calculăm $(B+I_2)^2 = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O_2 . \Rightarrow$

$\Rightarrow \ker\left((B+I_2)^2\right) = \mathbb{R}^2 = \text{Sp gen}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}. \Rightarrow$

$\Rightarrow$ pt. p_0 este suficient să considerăm elementele bazei canonice:

• $p_0 = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow p_1 = (B+I_2)p_0 = \begin{pmatrix}2 & 2\\-2 & -2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix} \Rightarrow$

$$\Rightarrow \varphi_1(s) = \left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}2\\-2\end{pmatrix}s\right)e^{-s} \Rightarrow \boxed{\varphi_1(s) = \begin{pmatrix}(1+2s)e^{-s}\\ -2se^{-s}\end{pmatrix}}$$

• $p_0 = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow p_1 = (B+I_2)p_0 = \begin{pmatrix}2 & 2\\-2 & -2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix} \Rightarrow$

$$\Rightarrow \varphi_2(s) = \left(\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}2\\-2\end{pmatrix}s\right)e^{-s} \Rightarrow \boxed{\varphi_2(s) = \begin{pmatrix}2s\cdot e^{-s}\\ (1-2s)e^{-s}\end{pmatrix}}$$

O matrice fundamentală de soluții este:

$$\Phi(s) = \begin{pmatrix}(1+2s)e^{-s} & 2se^{-s}\\ -2se^{-s} & (1-2s)e^{-s}\end{pmatrix}$$

Rezultă: $\bar{y} = \Phi(s)C$, $C \in \mathbb{R}^2$, $C = \begin{pmatrix}c_1\\c_2\end{pmatrix}$

• aplicăm met. variației constantelor:

determinăm $C = \begin{pmatrix}c_1\\c_2\end{pmatrix} : \underbrace{(-\infty, +\infty)}_{\mathbb{R}} \to \mathbb{R}^2$ $(s = \ln t,\ t \in (0,\infty))$

a.î. $\boxed{y(s) = \Phi(s)C(s)}$ soluție a sist. afin

$$y' = By + \tilde{b}(s) \Rightarrow$$

$$\Rightarrow \cancel{\underbrace{\Phi'(s)}_{B\Phi(s)}C(s)} + \Phi(s)C'(s) = \cancel{B\Phi(s)C(s)} + \tilde{b}(s) \Rightarrow$$

$\Phi(s)C'(s) = \tilde{b}(s) \Rightarrow$ sistem alg. liniar în $c_1', c_2' \Rightarrow$

$$\Rightarrow \begin{pmatrix}(1+2s)e^{-s} & 2se^{-s}\\ -2se^{-s} & (1-2s)e^{-s}\end{pmatrix}\begin{pmatrix}c_1'\\c_2'\end{pmatrix} = \begin{pmatrix}0\\se^{s}\end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases}(1+2s)e^{-s}\cdot c_1' + 2se^{-s}c_2' = 0 \quad |: e^{-s} \Rightarrow (1+2s)c_1' + 2sc_2' = 0\\ -2se^{-s}c_1' + (1-2s)e^{-s}\cdot c_2' = se^{s}\end{cases}$$

$$1e^{-s}c_1' + 1\cdot e^{-s}c_2' = se^{s} \ (+) \quad |\cdot e^{s} \Rightarrow \underline{c_1' + c_2' = se^{2s}}$$

$$\Rightarrow \boxed{c_2' = se^{2s} - c_1'}$$

$$(1+2s)c_1' + 2s(se^{2s} - c_1') = 0 \Rightarrow c_1'(1+\cancel{2s}-\cancel{2s}) + 2s^2e^{2s} = 0$$

$$\Rightarrow \boxed{c_1' = -2s^2e^{2s}}\ ;\ c_2' = se^{2s} + 2s^2e^{2s} \Rightarrow \boxed{c_2' = (s+2s^2)e^{2s}}$$

Integrăm 2 ecuații de tip primitivă:

$$c_1' = -2s^2e^{2s} \Rightarrow c_1(s) = \int(-2s^2)e^{2s}\,ds = \int(-2s^2)\left(\frac{e^{2s}}{2}\right)'ds$$

$$= (-\cancel{2}s^2)\frac{e^{2s}}{\cancel{2}} - \int(-\cancel{4}s)\frac{e^{2s}}{\cancel{2}}\,ds =$$

$$= -s^2e^{2s} + 2\left(\int s\left(\frac{e^{2s}}{2}\right)'ds\right) =$$

$$= -s^2e^{2s} + \cancel{2}\left(s\cdot\frac{e^{2s}}{\cancel{2}} - \int 1\cdot\frac{e^{2s}}{\cancel{2}}\,ds\right) =$$

$$= -s^2e^{2s} + se^{2s} - \frac{e^{2s}}{2} + K_1 \Rightarrow$$

$$\Rightarrow \boxed{c_1(s) = \frac{e^{2s}}{2}\left(-2s^2+2s-1\right)+K_1}$$

$$c_2' = (s+2s^2)e^{2s} \Rightarrow c_2(s) = \int(s+2s^2)\left(\frac{e^{2s}}{2}\right)'ds =$$

$$= (s+2s^2)\frac{e^{2s}}{2} - \int(1+4s)\frac{e^{2s}}{2}\,ds =$$

$$= (s+2s^2)\frac{e^{2s}}{2} - \frac{1}{2}\int(1+4s)\left(\frac{e^{2s}}{2}\right)'ds =$$

$$= (s+2s^2)\frac{e^{2s}}{2} - \frac{1}{2}\left[(1+4s)\frac{e^{2s}}{2} - \int \cancel{4}^{2}\cdot\frac{e^{2s}}{\cancel{2}}\,ds\right] =$$

$$= (s+2s^2)\frac{e^{2s}}{2} - \frac{(1+4s)e^{2s}}{4} + \int e^{2s}\,ds =$$

$$= (s+2s^2)\frac{e^{2s}}{2} - \frac{(1+4s)e^{2s}}{4} + \frac{e^{2s}}{2} + K_2 \Rightarrow$$

$$\Rightarrow \boxed{c_2(s) = \frac{e^{2s}}{4}\left(4s^2-2s+1\right)+K_2}$$

Deci, soluția sistemului afin (2):

$$y(s) = \begin{pmatrix} (1+2s)e^{-s} & 2se^{-s} \\ -2se^{-s} & (1-2s)e^{-s} \end{pmatrix}\begin{pmatrix} \frac{e^{2s}}{2}(-2s^2+2s-1)+K_1 \\ \frac{e^{2s}}{4}(4s^2-2s+1)+K_2 \end{pmatrix} \Rightarrow$$

$$\Rightarrow y(s) = e^{-s}\begin{pmatrix} 1+2s & 2s \\ -2s & 1-2s \end{pmatrix}\begin{pmatrix} 2(-2s^2+2s-1) \\ 4s^2-2s+1 \end{pmatrix}\frac{e^{2s}}{4} + \underbrace{e^{-s}\begin{pmatrix} 1+2s & 2s \\ -2s & 1-2s \end{pmatrix}}_{\phi(s)}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

$$\Rightarrow y(s) = \frac{e^{s}}{4}\begin{pmatrix} -\cancel{4s^2}+4s-2-\cancel{8s^3}+\cancel{8s^2}-\cancel{2s}+\cancel{8s^3}-\cancel{4s^2}+\cancel{2s} \\ \cancel{8s^3}-\cancel{8s^2}+\cancel{4s}+\cancel{4s^2}-\cancel{2s}+1-\cancel{8s^3}+\cancel{4s^2}-\cancel{2s} \end{pmatrix} + \phi(s)\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

$$\Rightarrow \boxed{y(s) = \frac{e^{s}}{4}\begin{pmatrix} 4s-2 \\ 1 \end{pmatrix} + \phi(s)\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}},\ K_1, K_2 \in \mathbb{R}.$$

Pt. sistemul (1) avem:

$$x(t)=y(\ln t)=\frac{e^{\ln t}}{4}\begin{pmatrix}4\ln t-2\\1\end{pmatrix}+\phi(\ln t)\begin{pmatrix}K_1\\K_2\end{pmatrix},\ K_1,K_2\in\mathbb{R}$$

$$\Rightarrow \boxed{x(t)=\frac{t}{4}\begin{pmatrix}4\ln t-2\\1\end{pmatrix}+\begin{pmatrix}\frac{1+2\ln t}{t} & \frac{2\ln t}{t}\\ \frac{-\ln t}{t} & \frac{1-2\ln t}{t}\end{pmatrix}\begin{pmatrix}K_1\\K_2\end{pmatrix},\ K_1,K_2\in\mathbb{R}}$$

Soluția care verifică: $\begin{cases}x_1(1)=2\\x_2(1)=1\end{cases}\Rightarrow \frac{1}{4}\begin{pmatrix}-2\\1\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}K_1\\K_2\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$

$$\Rightarrow\begin{cases}-\frac{1}{2}+K_1=2\Rightarrow K_1=\frac{5}{2}\\ \frac{1}{4}+K_2=1\Rightarrow K_2=\frac{3}{4}.\end{cases}$$

<u>OBS</u>: Dacă am fi ștrut $\varphi_0(t)=\frac{t}{4}\begin{pmatrix}4\ln t-2\\1\end{pmatrix}$ soluție particulară, atunci nu mai facem variația constantelor, rezolvăm doar sistemul liniar omogen.

<u>Temă</u>: ② Aceleași cerințe pt: $\begin{cases}x_1'=-\frac{x_1+2x_2}{t}+t\cos t\\ x_2'=\frac{3x_1+4x_2}{t}\end{cases},\ t\in(0,\frac{\pi}{2})$

③ Fie sistemul: $\begin{cases}x_1'=\frac{1}{2}\left(1+\frac{1}{t}-\frac{1}{t^2}\right)x_1+\frac{1}{2}\left(1+\frac{1}{t}+\frac{1}{t^2}\right)x_2\\ x_2'=-\frac{1}{2}\left(1-\frac{1}{t}+\frac{1}{t^2}\right)x_1-\frac{1}{2}\left(1-\frac{1}{t}-\frac{1}{t^2}\right)x_2\end{cases}$ (4)

a) Arătați că $\varphi_1(t)=\begin{pmatrix}t+1\\1-t\end{pmatrix}$ este soluție a sistemului (4)

b) Determinați φ_2, folosind metoda reducerii dimensiunii, astfel încât $\{\varphi_1,\varphi_2\}$ să fie sistem fundamental de soluții pt. (4).

<u>Ecuații diferențiale liniare de ordin n</u>

Să se determine soluția generală pentru fiecare dintre următoarele ecuații:

4) $x^{(3)}-x=0$; 5) $x^{(3)}-x=e^t$; 6) $x''+8x'+16x=t^2$;

7) $x''-4x'+8x=e^{2t}\sin t$; 8) $x''+x=\frac{2}{t^3}+\ln t$, $t>0$.

5) $\underbrace{x^{(3)} - x}_{\text{}} = e^t$

• rezolvăm ec. liniară omogenă atașată.

$$\bar{x}^{(3)} - \bar{x} = 0.$$

• se scrie ec. caracteristică:

$$r^3 - r^0 = 0$$

$$r^3 - 1 = 0$$

$$(r-1)(r^2+r+1) = 0.$$

$$r_1 = 1, \quad m_1 = 1$$

$$r^2 + r + 1 = 0 \Rightarrow \quad r_2 = \frac{-1+i\sqrt{3}}{2}, \quad m_2 = 1$$

$$\Delta = 1 - 4 = -3 \qquad r_3 = \frac{-1-i\sqrt{3}}{2}, \quad m_3 = 1 \qquad r_3 = \bar{r}_2$$

$\underline{r_1 = 1, m_1 = 1} \Rightarrow \boxed{\varphi_1(t) = e^t}$

$$\boxed{\begin{matrix} r_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad m_2 = 1 \\ r_3 = \bar{r}_2 \end{matrix}} \Rightarrow \left.\begin{matrix} \varphi_2(t) = Re(e^{r_2 t}) \\ \varphi_3(t) = Im(e^{r_2 t}) \end{matrix}\right\} \Rightarrow \boxed{\begin{matrix} \varphi_2(t) = e^{-\frac{t}{2}} \cos\frac{t\sqrt{3}}{2} \\ \varphi_3(t) = e^{-\frac{t}{2}} \sin\frac{t\sqrt{3}}{2} \end{matrix}}$$

$$e^{r_2 t} = e^{-\frac{1}{2}t + i\frac{\sqrt{3}}{2}t} = e^{-\frac{1}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}t\right) + i \sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

Avem $\{\varphi_1, \varphi_2, \varphi_3\}$ sistem fundam. de soluții pentru ec. liniară omogenă: $\bar{x}^{(3)} - \bar{x} = 0 \Rightarrow$

$$\Rightarrow \quad \bar{x}(t) = C_1\varphi_1(t) + C_2\varphi_2(t) + C_3\varphi_3(t), \quad C_1, C_2, C_3 \in \mathbb{R}.$$

• pt. a afla sol ec. afine se face variația constantelor: det $C_1, C_2, C_3 : \mathbb{R} \to \mathbb{R}$ a.î.

$$x(t) = C_1(t)\varphi_1(t) + C_2(t)\varphi_2(t) + C_3(t)\varphi_3(t) \text{ sol}$$

ec. afine.

Folosind sistemul afin asociat ec. afine se obține pt C_1', C_2', C_3' un sistem algebric liniar:

$$\underbrace{\begin{cases} C_1'\varphi_1(t) + C_2'\varphi_2(t) + C_3'\varphi_3(t) = 0 \\ C_1'\varphi_1'(t) + C_2'\varphi_2'(t) + C_3'\varphi_3'(t) = 0 \\ C_1'\varphi_1''(t) + C_2'\varphi_2''(t) + C_3'\varphi_3''(t) = \underbrace{e^t}_{g(t)} \end{cases}}$$

$$\underset{\begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_1' & \varphi_2' & \varphi_3' \\ \varphi_1'' & \varphi_2'' & \varphi_3'' \end{pmatrix}}{\underset{\|}{\Phi(t)}} \begin{pmatrix} C_1' \\ C_2' \\ C_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ g(t) \end{pmatrix}$$

Se obține ec. de tip frecventă pt C_1', C_2', C_3'

Temă: Integrați ec. liniare atașate pt exemplele 6, 7, 8.