Seria 33, EDDP, Curs 12, 04.01.2021

Problema Cauchy pentru ecuații neliniare, cu derivate parțiale de ordinul întâi.

Se cere determinarea unei funcții $u : D \subset \mathbb{R}^n \to \mathbb{R}$ care verifică:

$$(1) \quad \begin{cases} F(x, u, \partial_1 u, \dots, \partial_n u) = 0 \\ u(x) = u_0(x) \quad \text{pe } S = \{x \in \mathbb{R}^n \mid h(x) = 0\} \\ \quad \text{pentru } x \in S \cap D. \end{cases}$$

unde
$$\left.\begin{array}{l} F : G \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \\ u_0 : S \cap D \to \mathbb{R} \\ h : D_1 \subset \mathbb{R}^n \to \mathbb{R}. \end{array}\right\} \text{funcții cunoscute.}$$

Algoritmul de rezolvare a problemei (1):

- se scrie o parametrizare pentru S:

$$\begin{cases} x_1 = \alpha_1(s_1, \dots, s_{n-1}) \\ \vdots \\ x_n = \alpha_n(s_1, \dots, s_{n-1}) \end{cases}$$

$$s = (s_1, \dots, s_{n-1}) \in H \subset \mathbb{R}^{n-1}$$

Avem $(\alpha_1(s), \dots, \alpha_n(s)) \in S$, $\forall s \in H$.

- calculăm: $\varphi(s) = u_0(\alpha_1(s), \dots, \alpha_n(s))$.
- notăm (2) $\boxed{p_j = \partial_j u, \ j = \overline{1,n}}$ și calculăm $\gamma_1, \dots, \gamma_n$ valorile inițiale pentru p_j, rezolvând sistemul:

$$(3) \quad \begin{cases} F(\alpha_1(s), \dots, \alpha_n(s), \varphi(s), \gamma_1, \dots, \gamma_n) = 0 \\ (\gamma_1, \dots, \gamma_n) \begin{pmatrix} \frac{\partial \alpha_1}{\partial s_j} \\ \vdots \\ \frac{\partial \alpha_n}{\partial s_j} \end{pmatrix} = \frac{\partial \varphi}{\partial s_j}, \quad j = \overline{1, n-1} \end{cases}$$

din care se obține: $\boxed{\gamma_j = \gamma_j(s), \ j = \overline{1,n}}$, adică sunt valorile derivatelor pe S.

• scriem sistemul caracteristic pt (1):

$$(4)\quad \begin{cases} \left.\begin{aligned} \frac{dx_1}{dt} &= \frac{\partial F}{\partial p_1} \\ &\vdots \\ \frac{dx_n}{dt} &= \frac{\partial F}{\partial p_n} \end{aligned}\right] \\ \left.\begin{aligned} \frac{dp_1}{dt} &= -\frac{\partial F}{\partial x_1} - p_1 \cdot \frac{\partial F}{\partial u} \\ &\vdots \\ \frac{dp_n}{dt} &= -\frac{\partial F}{\partial x_n} - p_n \frac{\partial F}{\partial u} \end{aligned}\right] \\ \frac{du}{dt} = p_1 \frac{\partial F}{\partial p_1} + \dots + p_n \frac{\partial F}{\partial p_n} \\ \left.\begin{aligned} x_1(0) &= \alpha_1(s) \\ x_2(0) &= \alpha_2(s) \\ &\vdots \\ x_n(0) &= \alpha_n(s) \end{aligned}\right] \\ \left.\begin{aligned} p_1(0) &= \gamma_1(s) \\ &\vdots \\ p_n(0) &= \gamma_n(s) \end{aligned}\right] \\ u(0) = \varphi(s) \end{cases}$$

• rezolvăm sistemul caracteristic, din care se obținem:

$$(5)\quad \begin{cases} x_j = \tilde{x}_j(t,s) & , j=\overline{1,n} \\ p_j = \tilde{p}_j(t,s) & , j=\overline{1,n} \\ u = \tilde{u}(t,s) \end{cases} \Rightarrow$$

soluția parametrică a prob. Cauchy (1) este

$$\begin{cases} x_j = \tilde{x}_j(t,s), \ j=\overline{1,n} \\ u = \tilde{u}(t,s) \end{cases} \quad (6)$$

• pentru a scrie soluția u ca funcție de $(x_1, x_2, \dots, x_n)$ din $x_j = \tilde{x}_j(t,s)$, $j=\overline{1,n}$ exprimăm $t, s_1, \dots, s_{n-1}$ în funcție de $x_1, \dots, x_n$ și le înlocuim în $u = \tilde{u}(t,s)$.

În cazul cvasi liniar, avem:

$$F(x, u, \partial_1 u, \dots, \partial_n u) = \sum_{k=1}^{n} a_k(x,u)\cdot \partial_k u - g(x,u) = 0.$$

şi deci:

$$F(x, u, p_1, \dots, p_n) = \sum_{k=1}^{n} a_k(x,u)\, p_k - g(x,u) \Rightarrow$$

$\Rightarrow \dfrac{\partial F}{\partial p_j} = a_j(x,u)$, $j=\overline{1,n}$ $\Rightarrow$ primele n ec. în (4)

sunt $\dfrac{dx_j}{dt} = a_j(x,u)$, $j=\overline{1,n}$

(adică, primele n ec. din sist. caract. pt ec. cvasiliniare).

Calc. $p_1 \dfrac{\partial F}{\partial p_1} + \dots + p_n \dfrac{\partial F}{\partial p_n} =$

$= p_1 \cdot a_1(x,u) + \dots + p_n \cdot a_n(x,u) = g(x,u) \Rightarrow$

$\Rightarrow \dfrac{du}{dt} = g(x,u) \Rightarrow$ a $(n+1)$-a ec. din sist. caracteristic pt ec. cvasiliniară.

Deci, în cazul ec. cvasiliniare nu este nevoie de $\dfrac{dp_j}{dt}$, $j=\overline{1,n}$. Adică, (4) se reduce la sistemul cunoscut pt. prob. Cauchy pt. ec. cvasiliniare.

Cazul particular $\boxed{n=2}$

$$(1) \Rightarrow \begin{cases} F(x_1, x_2, u, \partial_1 u, \partial_2 u) = 0 \\ u(x) = u_0(x) \quad \text{pe } S\cap D, \ S = \{x \in D_1 \mid h(x) = 0\} \end{cases}$$

$D_1 \subset \mathbb{R}^2$

unde $F: D \subset \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$

- o parametrizare pt S: $\begin{cases} x_1 = \alpha_1(s) \\ x_2 = \alpha_2(s) \end{cases}$; $s \in H \subset \mathbb{R}$, $s = (s_1)$, $s_1 = s$.
- $\varphi(s) = u_0(\alpha_1(s), \alpha_2(s))$
- Acum $p_1 = \partial_1 u$; $p_2 = \partial_2 u$

Se determină γ_1, γ_2 (valori pe S pt p_1, p_2) din sistemul (3):

$$\begin{cases} F(\alpha_1(s), \alpha_2(s), \varphi(s), \gamma_1, \gamma_2) = 0 \\ (\gamma_1, \gamma_2)\begin{pmatrix} \alpha_1'(s) \\ \alpha_2'(s) \end{pmatrix} = \varphi'(s) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \boxed{\begin{cases} F(\alpha_1(s), \alpha_2(s), \varphi(s), \gamma_1, \gamma_2) = 0 \\ \gamma_1 \cdot \alpha_1'(s) + \gamma_2 \alpha_2'(s) = \varphi'(s) \end{cases}} \quad (6)$$

Din (6) calculăm $\gamma_1(s)$, $\gamma_2(s)$.

• scriem sistemul caracteristic:

$$(7)\begin{cases} \dfrac{dx_1}{dt} = \dfrac{\partial F}{\partial p_1} \\ \dfrac{dx_2}{dt} = \dfrac{\partial F}{\partial p_2} \\ \dfrac{dp_1}{dt} = -\dfrac{\partial F}{\partial x_1} - p_1 \dfrac{\partial F}{\partial u} \\ \dfrac{dp_2}{dt} = -\dfrac{\partial F}{\partial x_2} - p_2 \dfrac{\partial F}{\partial u} \\ \dfrac{du}{dt} = p_1 \dfrac{\partial F}{\partial p_1} + p_2 \dfrac{\partial F}{\partial p_2} \\ x_1(0) = \alpha_1(s) \\ x_2(0) = \alpha_2(s) \\ p_1(0) = \gamma_1(s) \\ p_2(0) = \gamma_2(s) \\ u(0) = \varphi(s) \end{cases}$$

• calculăm
$$\begin{cases} x_1 = \tilde{x}_1(t,s) \\ x_2 = \tilde{x}_2(t,s) \end{cases} (8)$$
$$\begin{cases} p_1 = \tilde{p}_1(t,s) \\ p_2 = \tilde{p}_2(t,s) \end{cases}$$
$$u = \tilde{u}(t,s) \quad (9)$$

soluția param. a prob. Cauchy

din (8) $\Rightarrow \begin{cases} t = \tilde{t}(x_1, x_2) \\ s = \tilde{s}(x_1, x_2) \end{cases} \xRightarrow{(9)} u(x_1, x_2) = \tilde{u}\left(\tilde{t}(x_1,x_2), \tilde{s}(x_1,x_2)\right)$

OBS: În cazul $n=2$, de obicei se folosesc notațiile: $x_1 = x$, $x_2 = y$; $p_1 = p$; $p_2 = q$.

<u>Temă</u>: Analog, particularizări pt $n=2$, particularizați alg pt prob. Cauchy (1) în cazul <u>$n=3$</u>.

<u>Exemplu</u>: Fie problema Cauchy:

$$\begin{cases} (\partial_1 u)^2 - 2(\partial_1 u)(\partial_2 u) + 2(\partial_2 u)^2 - 4u = 0 \\ u(x_1, x_2) = \frac{1}{2} x_2^2 \quad \text{pe } S = \{x \in \mathbb{R}^2 \mid x_1 = 0\}. \end{cases}$$

Se cere soluția problemei.

Avem:

$$F(x_1, x_2, u, \partial_1 u, \partial_2 u) = (\partial_1 u)^2 - 2(\partial_1 u)(\partial_2 u) + 2(\partial_2 u)^2 - 4u$$

$$u_0(x_1, x_2) = \frac{1}{2} x_2^2$$

- $S: x_1 = 0$

  $h(x_1, x_2) = x_1 \Rightarrow$ o parametrizare: $\begin{cases} x_1 = 0 = \alpha_1(s) \\ x_2 = s = \alpha_2(s) \end{cases}$ $s \in \mathbb{R}$.

- $\varphi(s) = u_0(0, s) = \frac{1}{2} s^2$

- $F(x_1, x_2, u, p_1, p_2) = p_1^2 - 2p_1 p_2 + 2p_2^2 - 4u$

  Rezolvăm sistemul:

$$\begin{cases} F(0, s, \frac{1}{2}s^2, \gamma_1, \gamma_2) = 0 \\ \gamma_1 \cdot \alpha_1'(s) + \gamma_2 \alpha_2'(s) = \varphi'(s) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \gamma_1^2 - 2\gamma_1\gamma_2 + 2\gamma_2^2 - \overset{2}{\cancel{4}} \cdot \frac{1}{\cancel{2}} s^2 = 0 \\ \gamma_1 \cdot 0 + \gamma_2 \cdot 1 = \frac{1}{\cancel{2}} \cancel{2} s. \Rightarrow \boxed{\gamma_2 = s} \Rightarrow p_2(0) = s \end{cases} \Rightarrow$$

$$\Rightarrow \gamma_1^2 - 2 \cdot \gamma_1 \cdot s + \cancel{2 \cdot s^2} - \cancel{2s^2} = 0.$$

$$\gamma_1^2 - 2 \cdot \gamma_1 s = 0 \Rightarrow \gamma_1(\gamma_1 - 2s) = 0$$ $\boxed{\gamma_1 = 0}$ sau $\boxed{\gamma_1 = 2s}$.

Avem 2 cazuri: I) $\begin{cases} \gamma_1 = 0 \\ \gamma_2 = s \end{cases}$ II) $\begin{cases} \gamma_1 = 2s \\ \gamma_2 = s \end{cases}$

Ecuațiile din sistemul caracteristic sunt aceleași, doar condițiile inițiale diferă.

<u>Considerăm</u> <u>II</u>: $\gamma_1(s) = 2s$

$\gamma_2(s) = s$.

Calculăm derivatele funcției F:

$$\frac{\partial F}{\partial p_1} = 2p_1 - 2p_2 \quad ; \quad \frac{\partial F}{\partial p_2} = -2p_1 + 4p_2$$

$$\frac{\partial F}{\partial x_1} = 0 \ ; \quad \frac{\partial F}{\partial x_2} = 0 \ ; \quad \frac{\partial F}{\partial u} = -4.$$

Sistemul caracteristic:

$$\begin{cases} \dfrac{dx_1}{dt} = 2p_1 - 2p_2 \\ \dfrac{dx_2}{dt} = -2p_1 + 4p_2 \\ \dfrac{dp_1}{dt} = -0 - p_1(-4) \\ \dfrac{dp_2}{dt} = -0 - p_2(-4) \\ \dfrac{du}{dt} = p_1(2p_1 - 2p_2) + p_2 \cdot (-2p_1 + 4p_2) \\ x_1(0) = 0 \\ x_2(0) = s. \\ p_1(0) = 2s \\ p_2(0) = s. \\ u(0) = \frac{1}{2}s^2. \end{cases}$$

$\dfrac{dp_1}{dt} = 4p_1 \Rightarrow p_1(t) = C_1 \cdot e^{4t}$ \| $\Rightarrow 2s = C_1 \cdot e^0 \Rightarrow C_1 = 2s$

ec. liniară în p_1 — dar $p_1(0) = 2s$

$$\Rightarrow \boxed{\tilde{p}_1(t,s) = 2se^{4t}}$$

$\dfrac{dp_2}{dt} = 4p_2 \Rightarrow p_2(t) = C_2 \cdot e^{4t}$ \| $\Rightarrow s = C_2 e^0 \Rightarrow C_2 = s \Rightarrow$

ec. liniară în p_2 — $p_2(0) = s$

$$\Rightarrow \boxed{\tilde{p}_2(t,s) = s e^{4t}}$$

$\dfrac{dx_1}{dt} = 4se^{4t} - 2se^{4t} \Rightarrow \dfrac{dx_1}{dt} = 2se^{4t}$ \| $\Rightarrow$

ec. de tip primitivă

$$\Rightarrow x_1(t) = 2s\int e^{4t}dt = \cancel{2}s \cdot \frac{e^{4t}}{\cancel{4}_2} + C_3$$

dar $x_1(0) = 0 \Rightarrow 0 = s \cdot \dfrac{e^0}{2} + C_3 \Rightarrow C_3 = -\dfrac{s}{2} \Rightarrow$

$$\Rightarrow \boxed{\tilde{x}_1(t,s) = \frac{s}{2}\left(e^{4t}-1\right)}$$

$$\frac{dx_2}{dt} = -4se^{4t} + 4se^{4t} \Rightarrow \frac{dx_2}{dt} = 0 \Rightarrow \left.\begin{array}{l} x_2 = C_4 \\ \text{dar } x_2(0) = s \end{array}\right\} \Rightarrow$$

$$\Rightarrow \boxed{\tilde{x}_2(t,s) = s}$$

$$\frac{du}{dt} = 2p_1^2 - 2p_1p_2 - 2p_1p_2 + 4p_2^2 \Rightarrow \frac{du}{dt} = 2p_1^2 - 4p_1p_2 + 4p_2^2 \Rightarrow$$

$$\frac{du}{dt} = \cancel{2\cdot 4s^2 e^{8t}} \cancel{- 4\cdot 2s^2 e^{8t}} + 4\cdot s^2 e^{8t}$$

$$\frac{du}{dt} = 4s^2 e^{8t}$$

ec. de tip primitivă pt u

$$\Rightarrow u = 4s^2 \int e^{8t}\,dt = 4s^2 \frac{e^{8t}}{8} + C_5 \Rightarrow$$

$$\Rightarrow u(t) = \frac{s^2 e^{8t}}{2} + C_5$$

dar $u(0) = \frac{1}{2}s^2$

$$\Rightarrow \frac{s^2}{2} = \frac{s^2}{2} + C_5 \Rightarrow C_5 = 0$$

$$\Rightarrow \boxed{\tilde{u}(t,s) = \frac{s^2 e^{8t}}{2}}$$

Avem: soluția în formă parametrică:

$$\begin{cases} \tilde{x}_1(t,s) = \frac{s}{2}\left(e^{4t}-1\right) \\ \tilde{x}_2(t,s) = s \\ \tilde{u}(t,s) = \frac{s^2 e^{8t}}{2} \end{cases}$$

$$\begin{cases} \frac{s}{2}\left(e^{4t}-1\right) = x_1 \\ \boxed{s = x_2} \end{cases} \Rightarrow e^{4t} - 1 = \frac{2x_1}{x_2} \Rightarrow e^{4t} = \frac{2x_1}{x_2} + 1$$

$$\boxed{t = \frac{1}{4}\ln\left(\frac{2x_1}{x_2} + 1\right)}$$

Rezultă că:

$$u(x_1,x_2) = \frac{1}{2}x_2^2 \cdot \left(\frac{2x_1}{x_2} + 1\right)^2 = \frac{1}{2}\cancel{x_2^2} \cdot \frac{(2x_1+x_2)^2}{\cancel{x_2^2}} \Rightarrow$$

$$\Rightarrow \boxed{u(x_1,x_2) = \frac{1}{2}\left(2x_1 + x_2\right)^2}$$

Dacă considerăm și cazul I avem: $\begin{cases} \gamma_1(s)=0 \\ \gamma_2(s)=s. \end{cases}$

În acest caz sistemul caract este:

$$\begin{cases} \frac{dx_1}{dt}=2p_1-2p_2 \\ \frac{dx_2}{dt}=-2p_1+4p_2 \\ \frac{dp_1}{dt}=4p_1 \\ \frac{dp_2}{dt}=4p_2 \\ \frac{du}{dt}=2p_1^2-4p_1p_2+4p_2^2 \\ x_1(0)=0 \\ x_2(0)=s \\ p_1(0)=0 \\ p_2(0)=s \\ u(0)=\frac{1}{2}s^2. \end{cases}$$

Se obține: $\tilde{p}_1(t,s)=0$

$\tilde{p}_2(t,s)=se^{4t}$

$$\frac{dx_1}{dt}=-2\cdot se^{4t} \Rightarrow x_1(t)=-\cancel{2}s\frac{e^{4t}}{\cancel{4}_2}+C_3 \Rightarrow \boxed{\tilde{x}_1(t,s)=-\frac{s}{2}(e^{4t}-1)}$$

dar $x_1(0)=0 \Rightarrow 0=-\frac{s}{2}+C_3 \Rightarrow C_3=\frac{s}{2}$

$$\frac{dx_2}{dt}=4se^{4t} \Rightarrow x_2(t)=\cancel{4}s\frac{e^{4t}}{\cancel{4}}+C_4$$

dar $x_2(0)=s \Rightarrow s=s+C_4 \Rightarrow C_4=0 \Rightarrow$

$$\Rightarrow \boxed{\tilde{x}_2(t,s)=se^{4t}}$$

$$\frac{du}{dt}=2\cdot 0^2-4\cdot 0\cdot se^{4t}+4\cdot s^2e^{8t}$$

$u=\cancel{4}s^2\frac{e^{8t}}{\cancel{8}_2}+C_5$

dar $u(0)=\frac{s^2}{2}$

$\Rightarrow \frac{s^2}{2}=\frac{s^2}{2}+C_5$

$\Rightarrow \begin{cases} C_5=0 \\ \boxed{\tilde{u}(t,s)=\frac{s^2}{2}e^{8t}} \end{cases}$

Sol parametrică:
$$\begin{cases} x_1=-\frac{s}{2}(e^{4t}-1) \\ x_2=se^{4t} \\ u=\frac{s^2}{2}e^{8t} \end{cases}$$

$$\begin{cases} 2x_1 = -se^{4t} + s \\ x_2 = se^{4t} \end{cases}$$

$$\boxed{2x_1 + x_2 = s.} \Rightarrow x_2 = (2x_1 + x_2)\, e^{4t} \Rightarrow$$

$$\Rightarrow \boxed{e^{4t} = \frac{x_2}{2x_1 + x_2}}$$

Deci:

$$u(x_1, x_2) = \frac{\cancel{(2x_1 + x_2)^2}}{2} \cdot \frac{x_2^2}{\cancel{(2x_1 + x_2)^2}} \Rightarrow$$

$$\Rightarrow \boxed{u(x_1, x_2) = \frac{x_2^2}{2}}$$

Verificare: $u(0, x_2) = \frac{x_2^2}{2}$

$\partial_1 u = 0$; $\partial_2 u = x_2$

în ec: $0^2 - 2 \cdot 0 \cdot x_2 + 2x_2^2 - \overset{2}{\cancel{4}} \cdot \frac{x_2^2}{\cancel{2}} = 0 \Leftrightarrow 0 = 0$. Adev.

Data următoare: test (quiz) - de 10 min (într-un interval de 15 min)

10 minute între $15^{30} - 15^{45}$