Grupa 331, EDDP, Seminar 1, 08.10.2020

Rezolvarea ecuației de tip primitivă, adică determinarea primitivelor unei funcții $f : I \subset \mathbb{R} \to \mathbb{R}$:

$$\int f(x)\,dx$$

Dacă F primitivă pt f, $F : I \to \mathbb{R}$, atunci:

1) F derivabilă pe I

2) $F'(x) = f(x)$, $\forall x \in I$

(P) Două primitive diferă printr-o constantă.

Dacă F primitivă: $\int f(x)\,dx = F(x) + \mathcal{C}$

mulțimea funcțiilor constante.

$(\mathcal{C} + \mathcal{C} = \mathcal{C}$

$\alpha\mathcal{C} = \mathcal{C}$, $\alpha \in \mathbb{R})$

Operații cu mulțimi de primitive

1) $\int (f_1(x) \pm f_2(x))\,dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx$

2) $\int \alpha f(x)\,dx = \alpha \int f(x)\,dx$, $\forall \alpha \in \mathbb{R}$.

Tabel de primitive

1) $\int 1\,dx = x + \mathcal{C}$

2) $\int x^r\,dx = \dfrac{x^{r+1}}{r+1} + \mathcal{C}$, $r \in \mathbb{R} \setminus \{-1\}$

3) $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + \mathcal{C}$

4) $\begin{cases} \int a^x\,dx = \dfrac{a^x}{\ln a} + \mathcal{C}, & a \in (0,\infty) \setminus \{1\} \\ \int e^x\,dx = e^x + \mathcal{C}, & \ln a = \log_e a,\ e \simeq 2{,}71\ldots \end{cases}$

5) $\begin{cases} \int \sin x\,dx = -\cos x + \mathcal{C} \\ \int \cos x\,dx = \sin x + \mathcal{C} \\ \int \operatorname{tg} x\,dx = -\ln|\cos x| + \mathcal{C} \end{cases}$

$$\int \operatorname{ctg} x\, dx = \ln|\sin x| + C$$

$$\int \frac{1}{\cos^2 x}\, dx = \int (1+\operatorname{tg}^2 x)\, dx = \operatorname{tg} x + C$$

$$\int \frac{1}{\sin^2 x}\, dx = \int (1+\operatorname{ctg}^2 x)\, dx = -\operatorname{ctg} x + C$$

6)
$$\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C$$

$$\int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$$

7)
$$\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \ln(x+\sqrt{x^2+a^2}) + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \ln|x-\sqrt{x^2+a^2}| + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \arcsin \frac{x}{a} + C$$

8)
$$\int \frac{x}{x^2+a^2}\, dx = \frac{1}{2} \ln(x^2+a^2) + C$$

$$\int \frac{x}{x^2-a^2}\, dx = \frac{1}{2} \ln|x^2-a^2| + C$$

9)
$$\int \frac{x}{\sqrt{x^2+a^2}}\, dx = \sqrt{x^2+a^2} + C$$

$$\int \frac{x}{\sqrt{x^2-a^2}}\, dx = \sqrt{x^2-a^2} + C$$

$$\int \frac{x}{\sqrt{a^2-x^2}}\, dx = -\sqrt{a^2-x^2} + C$$

Metode de integrare

1) Reducerea la formule din tabelul de primitive

2) Metoda de integrare prin părți

$$\boxed{\int u(x)\, v'(x)\, dx = u(x)v(x) - \int u'(x)\, v(x)\, dx}$$

provine din $\big(\underbrace{u(x)v(x)}_{f(x)}\big)' = u'(x)v(x) + u(x)v'(x)$

$$\int f'(x)dx = f(x) + C$$

3) <u>Prima metodă de schimbare de variabilă</u>

$$\boxed{\int \underbrace{g(u(x))\, u'(x)}_{f(x)} dx = G(u(x)) + C}$$, unde G este primitivă pt g

$u(x) = t$

$u'(x)dx = dt$

$$\int g(t)dt = G(t) + C$$

4) <u>A doua metodă de schimbare de variabilă</u>

$$\boxed{\int \underbrace{g(u(x))}_{f(x)} dx = H(u(x)) + C}$$, unde H este o primitivă pt $g\cdot(u^{-1})'$

$u(x) = t \iff x = u^{-1}(t) = \varphi(t)$

$x = \varphi(t)$

$dx = \varphi'(t)dt$

$$\int \underbrace{g(t)\,\varphi'(t)}_{h(t)} dt = H(t) + C$$

<u>Aplicații</u>: Să se determine mulțimea primitivelor următoarelor funcții:

1) $f(x) = x^6 - 2x^3 + x + 2$

2) $f(x) = \sqrt[4]{x} - \sqrt[3]{x} + 2$

3) $f(x) = x^3\sqrt{x} + x^2\sqrt[3]{x} - 1$

4) $f(x) = \dfrac{(x-2)^3}{\sqrt{x}}$

5) $f(x) = 2^x + 3^x e^x$

6) $f(x) = \dfrac{1}{\sin^2 x \cos^2 x}$

7) $f(x) = \dfrac{\cos 2x}{\sin^2 x \cos^2 x}$

√ 8) $f(x) = \operatorname{ctg}^2 x$

√ 9) $f(x) = \dfrac{1}{8 - 2x^2}$

10) $f(x) = \dfrac{1}{3x^2 + 27}$

√ 11) $f(x) = \dfrac{1}{(x^2+1)(x^2-4)}$

12) $f(x) = \dfrac{\sqrt{x^2+4} + 2\sqrt{x^2-4}}{\sqrt{x^4-16}}$

13) $f(x) = \dfrac{2x+1}{x^2+3}$

14) $f(x) = \dfrac{x-1}{\sqrt{x^2+1}}$

√ 15) $f(x) = \dfrac{2-x}{\ldots}$

16) $f(x) = (x-1)e^x$

17) $f(x) = x^2 \ln x$

18) $f(x) = (x+2)\sin x$

19) $f(x) = e^x \sin x$

20) $f(x) = e^{3x} \cos 2x$

21) $f(x) = \sqrt{x^2+1}$

22) $f(x) = x\sqrt{x^2-4}$

23) $f(x) = \dfrac{2x+3}{x^2+3x+7}$

24) $f(x) = 2x\, e^{x^2-3}$

25) $f(x) = \dfrac{\ln(-\ln x)}{x \ln x}$

26) $f(x) = (x-1)(x+2)^{2020}$

27) $f(x) = (x+1)(x^2+2x+5)^{1000}$

28) $f(x) = \dfrac{e^{\operatorname{arctg} x} + x}{1+x^2}$

29) $f(x) = \dfrac{\sin 2x}{\cos^2 x + 9}$

30) $f(x) = \dfrac{\sin x}{\cos^2 x + 9}$

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15) $$\int \frac{2-x}{\sqrt{4-x^2}}\,dx = \int \frac{2}{\sqrt{4-x^2}}\,dx - \int \frac{x}{\sqrt{4-x^2}}\,dx =$$
$$= 2\int \frac{1}{\sqrt{2^2-x^2}}\,dx - \left(-\sqrt{4-x^2}\right) = 2\arcsin\frac{x}{2} + \sqrt{4-x^2} + C$$

11) $$\int \frac{1}{(x^2+1)(x^2-4)}\,dx = \frac{1}{5}\int \frac{(x^2+1)-(x^2-4)}{(x^2+1)(x^2-4)}\,dx =$$
$$= \frac{1}{5}\left(\int \frac{\cancel{(x^2+1)}}{\cancel{(x^2+1)}(x^2-4)}\,dx - \int \frac{\cancel{x^2-4}}{(x^2+1)\cancel{(x^2-4)}}\,dx\right) =$$
$$= \frac{1}{5}\int \frac{1}{x^2-2^2}\,dx - \frac{1}{5}\int \frac{1}{x^2+1^2}\,dx =$$
$$= \frac{1}{5}\cdot\frac{1}{2\cdot 2}\ln\left|\frac{x-2}{x+2}\right| - \frac{1}{5}\cdot\frac{1}{1}\operatorname{arctg}\left(\frac{x}{1}\right) + C =$$
$$= \frac{1}{20}\ln\left|\frac{x-2}{x+2}\right| - \frac{1}{5}\operatorname{arctg} x + C$$

8) $$\int \operatorname{ctg}^2 x\,dx = \int\left[(1+\operatorname{ctg}^2 x) - 1\right]dx =$$
$$= \int (1+\operatorname{ctg}^2 x)\,dx - \int 1\,dx = -\operatorname{ctg} x - x + C.$$

9) $$\int \frac{1}{8-2x^2}\,dx = \int \frac{1}{-2(x^2-4)}\,dx = -\frac{1}{2}\int \frac{1}{x^2-2^2}\,dx = -\frac{1}{2}\cdot\frac{1}{2\cdot 2}\ln\left|\frac{x-2}{x+2}\right| + C$$
$$= -\frac{1}{8}\ln\left|\frac{x-2}{x+2}\right| + C$$

17) $\int x^2 \ln x \, dx = \int \left(\frac{x^3}{3}\right)' \ln x \, dx = \frac{x^3}{3}\ln x - \int \frac{\cancel{x^3}^{\,2}}{3} \cdot \frac{1}{\cancel{x}} dx = J$

$u(x) = \ln x \qquad u'(x) = \frac{1}{x}$

$v'(x) = x^2 \qquad v(x) = \frac{x^3}{3}$

$\Rightarrow J = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3}\ln x - \frac{1}{3}\cdot\frac{x^3}{3} + C =$

$= \frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C.$

21) $\int \sqrt{x^2+1}\, dx = \int \frac{\left(\sqrt{x^2+1}\right)^2}{\sqrt{x^2+1}} dx = \int \frac{x^2+1}{\sqrt{x^2+1}} dx =$

$= \underbrace{\int \frac{x^2}{\sqrt{x^2+1}} dx}_{\uparrow} + \underbrace{\int \frac{1}{\sqrt{x^2+1}} dx}_{\text{formulă}} = J$

$u(x) = x \qquad u'(x) = 1$

$v'(x) = \frac{x}{\sqrt{x^2+1}} \qquad v(x) = \sqrt{x^2+1}$

$\Rightarrow$

$\Rightarrow J = x\sqrt{x^2+1} - \underbrace{\int \sqrt{x^2+1}\, dx}_{= J} + \ln\left(x + \sqrt{x^2+1}\right) \Rightarrow$

$\Rightarrow 2J = x\sqrt{x^2+1} + \ln\left(x + \sqrt{x^2+1}\right) \Rightarrow$

$\Rightarrow J = \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left(x+\sqrt{x^2+1}\right)\right) + C.$

23) $\int \frac{2x+3}{x^2+3x+7} dx = J$

$\underbrace{x^2+3x+7}_{u(x)} = t$

$(x^2+3x+7)' dx = dt$

$(2x+3)dx = dt$

$\int \frac{1}{t} dt = \ln|t| + C \Rightarrow J = \ln|x^2+3x+7| + C.$

26) $\int (x-1)(x+2)^{2020} dx = \int \left((x+2)-3\right)(x+2)^{2020} dx =$

$= \int (x+2)^{2021} dx - 3\int (x+2)^{2020} dx =$

$\ldots^{2021} \ldots \int (x+2)^{2020} (x+2)' dx = J$

$x+2=t$

$(x+2)'dx=dt \Rightarrow dx=dt$

$$I=\int t^{2021}dt-3\int t^{2020}dt=$$

$$=\frac{t^{2022}}{2022}-3\frac{t^{2021}}{2021}+C \Rightarrow$$

$$\Rightarrow I=\frac{(x+2)^{2022}}{2022}-3\frac{(x+2)^{2022}}{2021}+C.$$

27) $\int (x+1)(x^2+2x+5)^{1000}dx=\frac{1}{2}\int 2(x+1)(x^2+2x+5)^{1000}dx=I$

$x^2+2x+5=t$

$(2x+2)dx=dt$

$2(x+1)dx=dt$

$$\frac{1}{2}\int t^{1000}dt=\frac{1}{2}\frac{t^{1001}}{1001}+C \Rightarrow I=\frac{(x^2+2x+5)^{1001}}{2002}+C$$

29) $\int \frac{-\sin 2x}{\cos^2 x+9}dx=I$

$\sin 2x=2\sin x\cos x$

$\cos^2 x+9=t$

$(\cos^2 x+9)'dx=dt$

$2\cos x(\cos x)'dx=dt$

$-2\cos x\sin x\,dx=dt \Rightarrow -\sin 2x\,dx=dt$

$$-\int\frac{dt}{t}=-\ln|t|+C \Rightarrow$$

$$\Rightarrow I=-\ln|\cos^2 x+9|+C=$$

$$=-\ln(\cos^2 x+9)+C.$$