Exercise 1

a) We have $u'(0) \approx \frac{u^1 - u^{-1}}{2\Delta t}$ with initial condition $u'(0) = V$, we get following expression for u^{-1}:

$$u^{-1} = u^1 - 2\Delta t V$$

Then we set this expression into $[D_t D_t u + \omega^2 u = f]^0$. This gives us:

$$\frac{u^1 - 2u^0 + u^{-1}}{\Delta t^2} + \omega^2 u^0 = f(0) \iff u^1 = \frac{1}{2}(u^0(2-(\Delta t\omega)^2) + \Delta t^2 f(0) + 2V\Delta t)$$

b) Let $u(t) = ct + d$.

$$u(0) = I, u'(0) = V \Rightarrow d = I, c = V$$

Then I calculate descrete doubble derivative of u by term:

$$c[D_t D_t t]^n = c\frac{n + \Delta t - 2n + n - \Delta t}{\Delta t^2} = 0$$

For the constant term:

$$[D_t D_t d]^n = \frac{d - 2d + d}{\Delta t^2} = 0$$

We are also asked to calculate sorce term. We know $u''(t) = 0$. When we put u into our equation we therefore get

$$f(t) = u''(t) + \omega^2 u(t) = \omega^2 u(t) = \omega^2(Vt + I)$$

If we use this f, we see that $u(t) = ct + d$ solves $[D_t D_t u + \omega^2 u = f]^n$
c) Everything in code.
d) Everything in code, notice that I make an additional symbolic variable b that is the coefficient of x^2, sine this is not determined by the ODE.

e) I check make a function cubic in code, and use the symbolic functions to check if cubic polynomial solves discrete equation. The rusult is that the cubic polynomial gives zero residual on $[D_t D_t u + \omega^2 u = f]^n$, but on the initial condition we get a residual $r = 2a$, where a is the coefficient to x^3. This means that cubic polynomial does not solve discrete equation.
f and g explained in code.

Exercise 21

a)Everything in code. b)Everything in code. c)We want an equation that is vertical only. This means $x(t) = 0$. This gives us the equation :

$$y'' = -\frac{\beta}{1-\beta}(1 - \frac{\beta}{|y-1|})(y-1) - \beta$$

If we assume $y < 1$ we get:

$$y'' = -\frac{\beta}{1-\beta}(y - 1 + \beta) - \beta = -\frac{\beta}{1-\beta}y$$

This gives us equation exercise was asking for. For initial conditions, we see:

$$x(0) = 0 \Rightarrow \Theta = 0 \Rightarrow y(0) = 1 - (\epsilon + 1)cos(\Theta) = -\epsilon$$

This means that the exact solution to the vertical motion is

$$y(t) = -\epsilon \cos(\sqrt{\frac{\beta}{1-\beta}}t)$$

With requirement $|\epsilon| < 1$. Rest of exercise is in code. d)Everything in code.