MAT4500 - Mandatory assignment

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Exercise 1

a) Let D^n,S^n,S^n_U,S^n_L,$f_L : D^n \to S^n_L$ and $f_U : D^n \to S^n_U$ be as defined in exercise. Want to prove that f_L and f_U are homomorphisms. D^n is a closed and bounded subset of a metric space and is therefore compact. By the same argument we see that S^n_U and S^n_L are Hausdorff spaces. Theorem 26.6 tells us that f_L and f_U are homomorphisms if they are bijective and continuous.

(1) f_L and f_U injective:
Let $x, y \in D^n$ with $x \neq y \Rightarrow \ \exists \ i \in \{1, ..., n\}$ such that $x_i \neq y_i$. This means that:

$$f_L(x)_i = x_i \neq y_i = f_L(y)_i \ \Rightarrow f_L(x) \neq f_L(y)$$

and

$$f_U(x)_i = x_i \neq y_i = f_U(y)_i \ \Rightarrow f_U(x) \neq f_U(y)$$

(2) f_L and f_U surjective:
Let $x \in S^n_U$ and $y \in S_L$ with $x = (x_1, .., x_{n+1})$ and
$y = (y_1, .., y_{n+1})$. Set $\bar{x} = (x_1, .., x_n)$ and $\bar{y} = (y_1, .., y_n)$. We clearly see that $\bar{x}, \bar{y} \in \mathbb{R}^n$, but need to show that
$||\bar{x}||, ||\bar{y}|| \leq 1$. However this is obvious, since
$||\bar{x}|| \leq ||x|| = 1$ and $||\bar{y}|| \leq ||y|| = 1$.

Now we need to show that $f_U(\bar{x}) = x$ and $f_L(\bar{y}) = y$. It suffices to prove $x_{n+1} = \sqrt{1 - ||\bar{x}||^2}$ and $y_{n+1} = -\sqrt{1 - ||\bar{y}||^2}$:

$$||x|| = 1 \Rightarrow \sum_{i=1}^{n+1} x_i^2 = 1 \Rightarrow x_{n+1}^2 = 1 - \sum_{i=1}^{n} x_i^2 \Rightarrow |x_{n+1}| = \sqrt{1 - ||\bar{x}||^2}$$

and

$$||y|| = 1 \Rightarrow \sum_{i=1}^{n+1} y_i^2 = 1 \Rightarrow y_{n+1}^2 = 1 - \sum_{i=1}^{n} y_i^2 \Rightarrow |y_{n+1}| = \sqrt{1 - ||\bar{y}||^2}$$

Since $x \in S_U^n$ we know that $x_{n+1} \geq 0$, $x_{n+1} = \sqrt{1 - ||\bar{x}||^2}$, and since $y \in S_L^n$ $y_{n+1} \leq 0$, $y_{n+1} = -\sqrt{1 - ||\bar{y}||^2}$.

We then know that $\forall\ x \in S_U^n\ \exists\ \bar{x} \in D^n\ s.t.\ f_U(\bar{x}) = x$, and that $\forall\ y \in S_L^n\ \exists\ \bar{y} \in D^n\ s.t.\ f_L(\bar{y}) = y$. This means that f_L and f_U surjective.

(3) f_L and f_U continuous:
Both f_L and f_U are vector functions, i.e. they are on the form $[F_1(x), .., F_{n+1}(x)]$. We know from calculus, that these types of functions are continuous if each component is continuous. We also know that $x \mapsto x_i$ and $x \mapsto \sqrt{1 - ||x||^2}$ are continuous. This means that f_L and f_U are continuous.

$(1) \wedge (2) \wedge (3) \Rightarrow f_L$ and f_U are homomorphisms.

b) Define $X = D^n \sqcup D^n = (D^n \times \{1\}) \cup (D^n \times \{2\})$, and the function $f : X \to S^n$ by:

$$f((x, i)) = \begin{cases} f_U(x) & i = 1 \\ f_L(x) & i = 2 \end{cases}$$

Where f_U and f_L is as in a). Note that when you take the Cartesian product between a set A and a singleton set $\{a\}$, $A \times \{a\}$ is homomorphic to A. This means that $D^n \times \{1\}$

and $D^n \times \{2\}$ are homomorphic to D^n. This again means that f_U and f_L are continuous functions when restricted to the two new sets. This again means that f restricted to $D^n \times \{1\}$ and $D^n \times \{2\}$ is continuous. Since these two sets obviously are closed in X, and have empty intersection, the pasting lemma 18.3 implies that f is continuous on the union of $D^n \times \{1\}$ and $D^n \times \{2\}$. But this is X.

Lets prove that f is surjective. Given $y \in S^n$, with $y_{n+1} \leq 0$, surjectivety of f_L implies that $\exists\ x \in D^n$ such that

$$f_L(x) = y \ \Rightarrow f((x,2)) = y$$

Repeat argument for lower part:

$$y_{n+1} > 0 \ \Rightarrow \ \exists\ x \in D^n\ s.t\ f_U(x) = y \Rightarrow f((x,1)) = y$$

Now assume $x \in D^n$ with $||x|| = 1$.

$$f((x,1)) = f_U(x) = (x_1, ..., x_n, \sqrt{1 - ||x||^2}) = (x_1, ..., x_n, 0)$$

We also have:

$$f((x,2)) = f_L(x) = (x_1, ..., x_n, -\sqrt{1 - ||x||^2}) = (x_1, ..., x_n, 0)$$

This means that $x^{(1)} \sim x^{(2)}$ in the meaning defined in the exercise.

c) Since $x^{(1)} \sim x^{(2)}$, it is obvious that $D^n \sqcup D^n / \sim$ is equal to $X^* = \{f^{-1}(\{y\}) : y \in S^n\}$ equipped with the quotient topology. Then by Corollary 22.3 in the book, $\exists$ an induced homomorphism $\bar{f} : X^* \to S^n$ if and only if f is quotient map. We already know that f is continuous and surjective, therefore we only need to prove that $f^{-1}(U)$ open in X $\iff$ U open in S^n.

Instead of proving the openness statement I look at the equivalent statement with U being closed. Assume U closed in S^n. f continuous implies that $f^{-1}(U)$ is closed.

Conversely assume $f^{-1}(U)$ closed in $X = D^n \sqcup D^n$. I now claim without proof that X is compact. This means that $f^{-1}(U)$ is compact. Since f is continuous $f(f^{-1}(U))$ is compact in S^n, and since S^n is Hausdorff $f(f^{-1}(U))$ is closed. Want to show that $f(f^{-1}(U)) = U$.

$$y \in U \;\Rightarrow\; \exists\, x \in X : f(x) = y \;\Rightarrow\; y = f(x) \in f(f^{-1}(U))$$

This means that $U \subset f(f^{-1}(U))$. Conversely:

$$y \in f(f^{-1}(U)) \;\Rightarrow\; \exists\, x \in f^{-1}(U) : y = f(x) \;\Rightarrow\; y \in U$$

This means $f(f^{-1}(U)) \subset U \;\Rightarrow f(f^{-1}(U)) = U$. Since $f(f^{-1}(U))$ is closed so is U.

I have now proved that f is quotient map, and as I explained above the statement in the exercise follows from this.

Exercise 2

a) Let (X, d) be a metric space, and let $K \subset X$ be compact. Since all metric spaces are Hausdorff spaces, and all compact subsets of Hausdorff spaces are closed, K is closed.

Now let $x \in K$, and let $\{B(x,n)\}_{n\in\mathbb{N}}$ be family of open balls centred in x, with radius $n \in \mathbb{N}$. It is obvious that $X = \bigcup_{n=1}^{\infty} B(x,n)$, and that $\bigcup_{n=1}^{\infty} B(x,n)$ is an open cover of K. Since K is compact we know that $\exists\{n_1, ..., n_r\} \subset \mathbb{N}$ such that $\bigcup_{i=1}^{r} B(x, n_i)$ covers K. Let $N = max\{n_1, ..., n_r\}$. Then we know that $K \subset B(x, N)$. This means that K is bounded.

b) Let $X = \mathbb{N}$, and define metric

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

The requirements:

(i) $\forall x, y \in X \ d(x,y) \geq 0 \wedge d(x,y) = 0 \iff x = y$
(ii) $\forall x, y \in X \ d(x,y) = d(y,x)$

for (X, d) to be a metric space, are trivially satisfied. The last requirement:
(iii) $\forall x, y, z \in X \ d(x,y) \leq d(x,z) + d(z,y)$

we can also show holds, if we separate the cases $x = y$ and $x \neq y$: First assume $x = y$:

$$d(x,y) = 0 \leq d(x,z) + d(z,y) = \begin{cases} 0 & z = x, y \\ 2 & z \neq x, y \end{cases}$$

And for $x \neq y$:

$$d(x,y) = 1 \leq d(x,z) + d(z,y) = \begin{cases} 1 & z = x \vee z = y \\ 2 & z \neq x, y \end{cases}$$

(i),(ii) and (iii) $\Rightarrow$ (X, d) is a metric space. Now notice that $X = \mathbb{N}$ is closed by definition, and that X is bounded since $\forall x \in X \ X \subset B(x, 2)$, where $B(x, 2)$ is the open ball centred in x with radius 2.

Now I want to find an open cover of X that does not contain a finite subcover. I claim that all points $x \in X$ are open in X. We see this by noticing that the open ball $B(x, \frac{1}{2})$ only contain x itself. Finaly we conclude that $A = \bigcup_{n=1}^{\infty} \{n\}$ is a disjoint open cover of X, and that there therefore exists no finite subcover of A that covers X. This means that X is not compact.

Exercise 3

a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real polynomial. This means that f is continuous. Since $\mathbb{R}$ is a Hausdorff space, the singleton set $\{0\}$ is closed in $\mathbb{R}$. This means that $A = f^{-1}(\{0\}) \subset \mathbb{R}^n$ the set in $\mathbb{R}^n$ of solutions to $f = 0$ is closed, since closedness is preserved through the inverse image of continuous functions.

b) We have the set $SL(2, \mathbb{R}) \subset \mathbb{R}^4$ defined as follows:

$$SL(2, \mathbb{R}) = \{A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} | det(A) = x_1x_4 - x_2x_3 = 1\}$$

We see that all $A \in SL(2, \mathbb{R})$ can be represented by a vector in $\mathbb{R}^4$:

$$A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$$

Now define a polynomilal $f : \mathbb{R}^4 \to \mathbb{R}$ by

$$f(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3 - 1$$

See that $SL(2, \mathbb{R}) = f^{-1}(\{0\})$. The previous exercise then tells us that $SL(2, \mathbb{R})$ is closed.

c) Want to show that $SL(2, \mathbb{R})$ is not compact. Since $\mathbb{R}^4$ is a metric space, it suffices to show that $SL(2, \mathbb{R})$ is not bounded.

To see that $SL(2,\mathbb{R})$ is not bounded, notice that $\forall x \in \mathbb{R}$ the matrix

$$A_x = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \in SL(2,\mathbb{R})$$

since $det(A) = 1$. We can then say that the sequence $\{A_n\}_{n=1}^{\infty}$ is contained in $SL(2,\mathbb{R})$. If we take the euclidean vector norm of A_n we get $||A_n|| = \sqrt{2+n^2}$, which means that $\{||A_n||\}_{n=1}^{\infty}$ is an unbounded sequence in $\mathbb{R}$. This means that there does not exists a number $M \in \mathbb{R}$ that bound the norms of elememts in $SL(2,\mathbb{R})$, and $SL(2,\mathbb{R})$ can therefore not be bounded.

Exercise 4

a) Want to show that the relation $\sim$ defined by :

$x, y \in X \; x \sim y \iff \exists\, \alpha : [0,1] \to X, \alpha \; contininious \; \wedge \alpha(0) = x \wedge \alpha(1) = y$

Is an equivalence relation on X.

(1): $x \sim x$ since the constant function $\alpha : [0,1] \to X$ given by $\alpha(r) = x$ is continuous, and obviously $\alpha(0) = \alpha(1) = x$

(2): Assume $x \sim y$. $\Rightarrow \exists\, \alpha : [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. Now define the function $f : [0,1] \to [0,1]$ by $f(t) = 1-t$. f is a polynomial, and is therefore continuous.

Now let $\beta : [0,1] \to X$ be the composition $\beta = \alpha \circ f$. Since β is a composition of continuous functions, β is continuous. Observe that $\beta(0) = \alpha(f(0)) = \alpha(1) = y$ and that$\beta(1) = \alpha(f(1)) = \alpha(0) = x$. This means that $y \sim x$.

(3): Assume $x \sim y$ and $y \sim z$. Then $\exists\, \alpha, \beta : [0,1] \to X$ such that α, β continuous, $\alpha(0) = x$, $\alpha(1) = \beta(0) = y$ and

$\beta(1) = z$.

Now define $f : [0, \frac{1}{2}] \to [0,1]$ and $g : [\frac{1}{2}, 1] \to [0,1]$ by $f(t) = 2t$ and $g(t) = 2t - 1$. Both g and f are continuous. Next let us define $\omega : [0,1] \to X$ by:

$$\omega(t) = \begin{cases} \alpha(f(t)) & t \in [0, \frac{1}{2}) \\ \beta(g(t)) & t \in [\frac{1}{2}, 1] \end{cases}$$

Notice that $\omega(0) = \alpha(0) = x$ and that $\omega(1) = \beta(1) = z$. We also see that $\forall t \neq \frac{1}{2}$ ω is continuous, by the same argument as in (2). Since $[0,1]$ a metric space, showing continuity at $t = \frac{1}{2}$ is the same as showing:

$$\{t_n\}_{n=1}^{\infty} \to \frac{1}{2} \Rightarrow \{\omega(t_n)\}_{n=1}^{\infty} \to y$$

I claim without proof that it is enough to show:

$$\lim_{t \to \frac{1}{2}^-} \omega(t) = \lim_{t \to \frac{1}{2}^+} \omega(t)$$

$\lim_{t \to \frac{1}{2}^-} \omega(t) = \lim_{t \to \frac{1}{2}^-} \alpha(f(t)) = \alpha(1) = y$ and $\lim_{t \to \frac{1}{2}^+} \omega(t) = \lim_{t \to \frac{1}{2}^+} \beta(g(t)) = \beta(0) = y$, by continuity of α, β, f, g. Finally we can conclude that ω is continuous, which means that $x \sim z$.

$(1) \wedge (2) \wedge (3) \Rightarrow \sim$ is an equivalence relation on X.

b) Assume $x, y \in \mathbb{R}^n$, $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. Then define $\alpha : [0,1] \to \mathbb{R}^n$ by

$$\alpha(t) = x + (y - x)t = (x_1 + (y_1 - x_1)t, x_2 + (y_2 - x_2)t, ..., x_n + (y_n - x_n)t)$$

Since α is a linear polynomial in all components, it is continuous in all components and therefore continuous. We also see that $\alpha(0) = x$ and $\alpha(1) = y$. this means that

$x \sim y$. We showed this for general x, y, which means that $\forall x, y \in \mathbb{R}^n x \sim y$. This implies that the equivalence class $[x]$ is equal to $\mathbb{R}^n$ $\forall x \in \mathbb{R}^n$.

We have now shown that $(\pi_0(\mathbb{R}^n) = \{\mathbb{R}^n\}) \Rightarrow |\pi_0(\mathbb{R}^n)| = 1$.

c) Let $x, y \in \mathbb{R} \setminus \{0\}$, and assume $x, y < 0$. Then $\alpha(t) = x + (y - x)t$ satisfies the conditions for $x \sim y$. The same argument work for $x, y > 0$. This means that
$\forall x, y \in (-\infty, 0)$ $x \sim y$ and $\forall x, y \in (0, \infty)$ $x \sim y$.

Want to show that for $x < 0$ and $y > 0$ x and y are not equivalent in our relation $\sim$. Assume for contradiction that $x \sim y \Rightarrow \exists\ \alpha : [0, 1] \to \mathbb{R} \setminus \{0\}$ such that α continuous, $\alpha(0) = x$ and $\alpha(1) = y$. However, since α is continuous, $[0, 1]$ is connected and $\mathbb{R} \setminus \{0\}$ is unconnected, the image $\alpha([0, 1])$ is either entirely contained in $(-\infty, 0)$ or in $(0, \infty)$. This is a contradiction because $\alpha(0) \in (-\infty, 0)$ and $\alpha(1) \in (0, \infty)$. This means $x \sim y$ is not true. This means that $\pi_0(\mathbb{R} \setminus \{0\}) = \{(-\infty, 0), (0, \infty)\} \Rightarrow |\pi_0(\mathbb{R} \setminus \{0\})| = 2$.

d) Let $z \in \mathbb{R}^n$ and $x, y \in \mathbb{R}^n \setminus \{z\}$. Define $\alpha : [0, 1] \to \mathbb{R}^n$ as in b. There are now two cases:

(I): $z \notin \alpha([0, 1]) \Rightarrow x \sim y$
(II): $z \in \alpha([0, 1])$

Lets look at case (II). Define

$$L = \{x_0 \in \mathbb{R}^n \mid \exists t \in \mathbb{R} \text{ s.t. } x_0 = x + t(y - x)\}$$

as the line in $\mathbb{R}^n$ that contains $\alpha([0, 1])$. This means that z also is contained in L. Now choose any point $a \in \mathbb{R}^n \setminus L$. It is clear that the line segments between x and a and between a and y does not contain z. We can then construct continuous functions as we did in b, such that $x \sim a$ and $a \sim y$. Transitivity property of $\sim$ then gives us $x \sim y$.

This shows that $\forall x, y \in \mathbb{R}^n \setminus \{z\}$ we have $x \sim y$
$\Rightarrow \pi_0(\mathbb{R}^n \setminus \{z\}) = \{\mathbb{R}^n \setminus \{z\}\} \Rightarrow |\pi_0(\mathbb{R}^n \setminus \{z\})| = 1$

e) Exercise 4b) and 4d) show that the sets $\{\mathbb{R}^n\}_{n=1}^{\infty}$ and $\{\mathbb{R}^n \setminus \{z\}\}_{n=2}^{\infty}$ are path connected. A result from the book tells us that path connectedness implies connectedness. This means that $\{\mathbb{R}^n\}_{n=1}^{\infty}$ and $\{\mathbb{R}^n \setminus \{z\}\}_{n=2}^{\infty}$ are connected. It is also trivial to show that $\mathbb{R} \setminus \{0\}$ is not connected. Lets use this to show $\mathbb{R}^n$ and $\mathbb{R}$ are non-homomorphic.

Assume for contradiction that $\mathbb{R}^n$ and $\mathbb{R}$ are homomorphic. Then $\exists\ f : \mathbb{R}^n \to \mathbb{R}$ where f is a homomorphism. Now remove $\{0\}$ from $\mathbb{R}$ and $z = f^{-1}(\{0\})$ from $\mathbb{R}^n$. If we restrict f to $\mathbb{R}^n \setminus \{z\}$, f should now be a homomorphism between $\mathbb{R}^n \setminus \{z\}$ and $\mathbb{R} \setminus \{0\}$. However, $\mathbb{R}^n \setminus \{z\}$ is connected while $\mathbb{R} \setminus \{0\}$ is not. This is a contradiction since $\mathbb{R} \setminus \{0\}$ is the image of $\mathbb{R}^n \setminus \{z\}$ under f, and connectedness is preserved through the image of a continuous function. Therefore $\mathbb{R}^n$ and $\mathbb{R}$ are non-homomorphic.