

### Exercise 1

a) We have  $u'(0) \approx \frac{u^1 - u^{-1}}{2\Delta t}$  with initial condition  $u'(0) = V$ , we get following expression for  $u^{-1}$ :

$$u^{-1} = u^1 - 2\Delta t V$$

Then we set this expression into  $[D_t D_t u + \omega^2 u = f]^0$ . This gives us:

$$\frac{u^1 - 2u^0 + u^{-1}}{\Delta t^2} + \omega^2 u^0 = f(0) \iff u^1 = \frac{1}{2}(u^0(2 - (\Delta t \omega)^2) + \Delta t^2 f(0) + 2V \Delta t)$$

b) Let  $u(t) = ct + d$ .

$$u(0) = I, u'(0) = V \Rightarrow d = I, c = V$$

Then I calculate discrete double derivative of u by term:

$$c[D_t D_t t]^n = c \frac{n + \Delta t - 2n + n - \Delta t}{\Delta t^2} = 0$$

For the constant term:

$$[D_t D_t d]^n = \frac{d - 2d + d}{\Delta t^2} = 0$$

We are also asked to calculate source term. We know  $u''(t) = 0$ . When we put u into our equation we therefore get

$$f(t) = u''(t) + \omega^2 u(t) = \omega^2 u(t) = \omega^2 (Vt + I)$$

If we use this f, we see that  $u(t) = ct + d$  solves  $[D_t D_t u + \omega^2 u = f]^n$

c) Everything in code.

d) Everything in code, notice that I make an additional symbolic variable b that is the coefficient of  $x^2$ , since this is not determined by the ODE.

e) I check make a function cubic in code, and use the symbolic functions to check if cubic polynomial solves discrete equation. The result is that the cubic polynomial gives zero residual on  $[D_t D_t u + \omega^2 u = f]^n$ , but on the initial condition we get a residual  $r = 2a$ , where  $a$  is the coefficient to  $x^3$ . This means that cubic polynomial does not solve discrete equation.

f and g explained in code.

### Exercise 21

a) Everything in code. b) Everything in code. c) We want an equation that is vertical only. This means  $x(t) = 0$ . This gives us the equation :

$$y'' = -\frac{\beta}{1-\beta} \left(1 - \frac{\beta}{|y-1|}\right) (y-1) - \beta$$

If we assume  $y < 1$  we get:

$$y'' = -\frac{\beta}{1-\beta} (y-1+\beta) - \beta = -\frac{\beta}{1-\beta} y$$

This gives us equation exercise was asking for. For initial conditions, we see:

$$x(0) = 0 \Rightarrow \Theta = 0 \Rightarrow y(0) = 1 - (\epsilon + 1)\cos(\Theta) = -\epsilon$$

This means that the exact solution to the vertical motion is

$$y(t) = -\epsilon \cos\left(\sqrt{\frac{\beta}{1-\beta}}t\right)$$

With requirement  $|\epsilon| < 1$ . Rest of exercise is in code. d) Everything in code.