INF5620 - Compulsory project 1

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Exercise 2.2

Want to discretizise the equation (1):

$$\frac{\partial^2 u}{\partial^2 t} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(q(x,y)\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(q(x,y)\frac{\partial u}{\partial y}) + f(x,y,t)$$

on the domain $\Omega = [0, L_x] \times [0, L_y]$, and time in $t \in [0, T]$.

Divide $[0, L_x]$ into N_x intervals $[x_i, x_{i+1}]$, with $i \in \{0, ..., N_x - 1\}$, with $x_0 = 0$ and $x_{N_x} = L_x$. Do the same for $[0, L_y]$, so you get $\{y_j\}_{j=0}^{N_y}$, and $[0, T]$ so you get $\{t_n\}_{n=0}^{N_t}$. From now on let $u(x_i, y_j, t_n) = u_{i,j}^n$.

I discretize with finite differences term by term. For $\frac{\partial^2 u}{\partial^2 t}$ I use centred difference:

$$\frac{\partial^2 u}{\partial^2 t}(x_i, y_j, t_n) \approx \frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2}$$

For $b\frac{\partial u}{\partial t}$ I use forward difference:

$$b\frac{\partial u}{\partial t}(x_i, y_j, t_n) \approx b\frac{u_{i,j}^{n+1} - u_{i,j}^n +}{\Delta t}$$

For $\frac{\partial}{\partial x}(q(x,y)\frac{\partial u}{\partial x})$ and $\frac{\partial}{\partial y}(q(x,y)\frac{\partial u}{\partial y})$ I use the normal centred difference for variable coeffichent:

$$\frac{\partial}{\partial x}(q(x,y)\frac{\partial u}{\partial x}(x_i, y_j, t_n)) \approx \frac{1}{\Delta x^2}(q_{i+\frac{1}{2},j}(u_{i+1,j}^n - u_{i,j}^n) - q_{i-\frac{1}{2},j}(u_{i,j}^n - u_{i-1,j}^n))$$

$$\frac{\partial}{\partial y}(q(x,y)\frac{\partial u}{\partial y}(x_i, y_j, t_n)) \approx \frac{1}{\Delta y^2}(q_{i,j+\frac{1}{2}}(u_{i,j+1}^n - u_{i,j}^n) - q_{i,j-\frac{1}{2}}(u_{i,j}^n - u_{i,j-1}^n))$$

Want to find expression for $u_{i,j}^{n+1}$, for calculating inner points. Set $C = \frac{\partial}{\partial x}(q(x,y)\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(q(x,y)\frac{\partial u}{\partial y}) + f(x,y,t)$. Then :

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} + b\frac{u_{i,j}^{n+1} - u_{i,j}^n +}{\Delta t} = C$$

$$\iff u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1} + \Delta t b(u_{i,j}^{n+1} - u_{i,j}^n) = \Delta t^2 C$$

$$\iff u_{i,j}^{n+1}(1 + \Delta t b) = u_{i,j}^n(2 + \Delta t b) - u_{i,j}^{n-1} + \Delta t^2 C$$

$$\iff u_{i,j}^{n+1} = \frac{u_{i,j}^{n}(2+\Delta t b) - u_{i,j}^{n-1} + \Delta t^2 C}{(1+\Delta t b)}$$

If we set in expression for C we get (2):

$$u_{i,j}^{n+1} = \frac{u_{i,j}^{n}(2+\Delta t b) - u_{i,j}^{n-1} + \Delta t^2 f_{i,j}^{n}}{(1+\Delta t b)}$$

$$+\frac{\Delta t^2}{\Delta x^2}\frac{(q_{i+\frac{1}{2},j}(u_{i+1,j}^{n} - u_{i,j}^{n}) - q_{i-\frac{1}{2},j}(u_{i,j}^{n} - u_{i-1,j}^{n}))}{(1+\Delta t b)}$$

$$+\frac{\Delta t^2}{\Delta y^2}\frac{(q_{i,j+\frac{1}{2}}(u_{i,j+1}^{n} - u_{i,j}^{n}) - q_{i,j-\frac{1}{2}}(u_{i,j}^{n} - u_{i,j-1}^{n}))}{(1+\Delta t b)}$$

For the first step we so not know $u_{i,j}^{-1}$, but the initial condition $u(x, y.0) = I(x, y)$ gives us $u_{i,j}^{0}$. The second initial condition is $u_t(x, y, 0) = V(x, y)$. To find an expression for $u_{i,j}^{-1}$, we discretize $u_t(x, y, 0) = V(x, y)$:

$$u_t(x_i, y_j, 0) \approx \frac{u_{i,j}^{1} - u_{i,j}^{-1}}{2\Delta t}$$

Set this equal to $V_{i,j}$, and we get an expression for $u_{i,j}^{-1}$:

$$\frac{u_{i,j}^{1} - u_{i,j}^{-1}}{2\Delta t} = V_{i,j} \iff u_{i,j}^{-1} = u_{i,j}^{1} - 2\Delta t V_{i,j}$$

Set this into the above result for inner points we get:

$$u_{i,j}^{1}(1+\Delta t b) = u_{i,j}^{0}(2+\Delta t b) - u_{i,j}^{1} + 2\Delta t V_{i,j} + \Delta t^2 C$$

$$\iff u_{i,j}^{1}(2+\Delta t b) = u_{i,j}^{0}(2+\Delta t b) + 2\Delta t V_{i,j} + \Delta t^2 C$$

$$\iff u_{i,j}^{1} = \frac{u_{i,j}^{0}(2+\Delta t b) + 2\Delta t V_{i,j} + \Delta t^2 C}{2+\Delta t b}$$

Here C is the decrete version of:

$$C = \frac{\partial}{\partial x}(q(x_i, y_j)\frac{\partial u(x_i, y_j, 0)}{\partial x}) + \frac{\partial}{\partial y}(q(x_i, y_j)\frac{\partial u(x_i, y_j, 0)}{\partial y}) + f(x_i, y_j, 0)$$

Lastly we need to handle the boundary, with homogeneous Neumann conditions. There are three cases:

(i): $(x_i, y_j) \in (0, L_x) \times \{0\} \cup (0, L_x) \times \{L_y\}$.

(ii): $(x_i, y_j) \in \{0\} \times (0, L_y) \cup \{L_x\} \times (0, L_y)$.

(iii): $(x_i, y_j) \in \{(0, 0), (0, L_y), (L_x, 0), (L_x, L_y)\}$.

For case (i) I only look at points (x_i, y_0). The Neumann condition gives us $u_y(x_i, y_0, t_n) = 0$. With the normal discretization of this we get:

$$\frac{u^n_{i,1} - u^n_{i,-1}}{2\Delta y} = 0 \iff u^n_{i,-1} = u^n_{i,1}$$

The only part of the scheme that is affected is $\frac{\partial}{\partial y}(q(x,y)\frac{\partial u}{\partial y})$, we get:

$$\frac{1}{\Delta y^2}(q_{i,\frac{1}{2}}(u^n_{i,1}-u^n_{i,0})-q_{i,-\frac{1}{2}}(u^n_{i,0}-u^n_{i,-1})) = \frac{1}{\Delta y^2}(q_{i,\frac{1}{2}}(u^n_{i,1}-u^n_{i,0})+q_{i,-\frac{1}{2}}(u^n_{i,1}-u^n_{i,0}))$$

$$= \frac{1}{\Delta y^2}(u^n_{i,1} - u^n_{i,0})(q_{i,\frac{1}{2}} + q_{i,-\frac{1}{2}})$$

Since $q_{i,-\frac{1}{2}}$ is evaluating q outside of our domain, we make the following ansatz: $q_{i,-\frac{1}{2}} \approx q_{i,\frac{1}{2}} \approx q_{i,0}$. This gives:

$$\frac{\partial}{\partial y}(q(x_i, 0)\frac{\partial u(x_i, 0, t_n)}{\partial y}) \approx \frac{2(u^n_{i,1} - u^n_{i,0})q_{i,0}}{\Delta y^2}$$

For $y = L_y$, we get the following by same arguments:

$$\frac{\partial}{\partial y}(q(x_i, L_y)\frac{\partial u(x_i, y_{N_y}, t_n)}{\partial y}) \approx \frac{2(u^n_{i,N_y-1} - u^n_{i,N_y})q_{i,N_y}}{\Delta y^2}$$

For case (ii), we get the same, but now we have:

$$u_x(x_0, y_j, t_n) = u_x(x_{N_x}, y_j, t_n) = 0$$

By the same approach as for case (i), we get

$$\frac{\partial}{\partial x}(q(0, y_j)\frac{\partial u(0, y_j, t_n)}{\partial x}) \approx \frac{2(u^n_{1,j} - u^n_{0,j})q_{0,j}}{\Delta x^2}$$

and

$$\frac{\partial}{\partial x}(q(L_x, y_j)\frac{\partial u(x_{N_x}, y_j, t_n)}{\partial x}) \approx \frac{2(u^n_{N_x-1,j} - u^n_{N_x,j})q_{N_x,j}}{\Delta x^2}$$

For case (iii), we get both case (i) and (ii). This means we already have the expressions needed to explicitly find u for points in (iii). If you want the formulas for the boundary points, you need to substitute the expressions for the x and y derivatives in (2) with the ones we have found above.