Linear Combinations

An expression

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

is a *linear combination* of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

The scalars $\alpha_1, \ldots, \alpha_n$ are the *coefficients* of the linear combination.

Example: One linear combination of $[2, 3.5]$ and $[4, 10]$ is

$$-5\,[2, 3.5] + 2\,[4, 10]$$

which is equal to $[-5 \cdot 2, -5 \cdot 3.5] + [2 \cdot 4, 2 \cdot 10]$

Another linear combination of the same vectors is

$$0\,[2, 3.5] + 0\,[4, 10]$$

which is equal to the zero vector $[0, 0]$.

Definition: A linear combination is *trivial* if the coefficients are all zero.

Linear Combinations: JunkCo

The JunkCo factory makes five products:





using various resources.

	metal	concrete	plastic	water	electricity
garden gnome	0	1.3	.2	.8	.4
hula hoop	0	0	1.5	.4	.3
slinky	.25	0	0	.2	.7
silly putty	0	0	.3	.7	.5
salad shooter	.15	0	.5	.4	.8

For each product, there is a vector specifying how much of each resource is used per unit of product.

For making one gnome:

$\mathbf{v}_1 = \{\texttt{metal:0, concrete:1.3, plastic:0.2, water:.8, electricity:.4}\}$

Linear Combinations: JunkCo

For making one gnome:
$\mathbf{v}_1 = \{\texttt{metal:0, concrete:1.3, plastic:0.2, water:.8, electricity:.4}\}$
For making one hula hoop:
$\mathbf{v}_2 = \{\texttt{metal:0, concrete:0, plastic:1.5, water:.4, electricity:.3}\}$
For making one slinky:
$\mathbf{v}_3 = \{\texttt{metal:.25, concrete:0, plastic:0, water:.2, electricity:.7}\}$
For making one silly putty:
$\mathbf{v}_4 = \{\texttt{metal:0, concrete:0, plastic:.3, water:.7, electricity:.5}\}$
For making one salad shooter:
$\mathbf{v}_5 = \{\texttt{metal:1.5, concrete:0, plastic:.5, water:.4, electricity:.8}\}$

Suppose the factory chooses to make α_1 gnomes, α_2 hula hoops, α_3 slinkies, α_4 silly putties, and α_5 salad shooters.

Total resource utilization is $\mathbf{b} = \alpha_1\,\mathbf{v}_1 + \alpha_2\,\mathbf{v}_2 + \alpha_3\,\mathbf{v}_3 + \alpha_4\,\mathbf{v}_4 + \alpha_5\,\mathbf{v}_5$

Linear Combinations: JunkCo: Industrial espionage

Total resource utilization is $\mathbf{b} = \alpha_1\,\mathbf{v}_1 + \alpha_2\,\mathbf{v}_2 + \alpha_3\,\mathbf{v}_3 + \alpha_4\,\mathbf{v}_4 + \alpha_5\,\mathbf{v}_5$

Suppose I am spying on JunkCo.

I find out how much metal, concrete, plastic, water, and electricity are consumed by the factory.
That is, I know the vector $\mathbf{b}$. Can I use this knowledge to figure out how many gnomes they are making?

Computational Problem: *Expressing a given vector as a linear combination of other given vectors*

- *input:* a vector $\mathbf{b}$ and a list $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ of vectors
- *output:* a list $[\alpha_1, \ldots, \alpha_n]$ of coefficients such that

$$\mathbf{b} = \alpha_1\,\mathbf{v}_1 + \cdots + \alpha_n\,\mathbf{v}_n$$

  or a report that none exists.

Question: Is the solution unique?

Lights Out

Button vectors for 2×2 *Lights Out:* $\boxed{\begin{matrix}\bullet & \bullet \\ \bullet & \end{matrix}}$ $\boxed{\begin{matrix}\bullet & \bullet \\ & \bullet\end{matrix}}$ $\boxed{\begin{matrix}\bullet & \\ \bullet & \bullet\end{matrix}}$ $\boxed{\begin{matrix} & \bullet \\ \bullet & \bullet\end{matrix}}$

For a given initial state vector $\mathbf{s} = \boxed{\begin{matrix}\bullet & \\ \bullet & \end{matrix}}$,

$\boxed{\text{Which subset of button vectors sum to } \mathbf{s}\text{?}}$

Reformulate in terms of linear combinations.
Write

$$\boxed{\begin{matrix}\bullet & \\ \bullet & \end{matrix}} = \alpha_1 \boxed{\begin{matrix}\bullet & \bullet \\ \bullet & \end{matrix}} + \alpha_2 \boxed{\begin{matrix}\bullet & \bullet \\ & \bullet\end{matrix}} + \alpha_3 \boxed{\begin{matrix}\bullet & \\ \bullet & \bullet\end{matrix}} + \alpha_4 \boxed{\begin{matrix} & \bullet \\ \bullet & \bullet\end{matrix}}$$

What values for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ make this equation true?

Solution: $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 0, \alpha_4 = 0$

$\boxed{\text{Solve an instance of } \textit{Lights Out}} \Rightarrow \boxed{\text{Which set of button vectors sum to } \mathbf{s}\text{?}}$

$\Rightarrow \boxed{\text{Find subset of } GF(2) \text{ vectors } \mathbf{v}_1, \ldots, \mathbf{v}_n \text{ whose sum equals } \mathbf{s}} \Rightarrow \boxed{\text{Express } \mathbf{s} \text{ as a linear combination of } \mathbf{v}_1, \ldots, \mathbf{v}_n}$

Lights Out

We can solve the puzzle if we have an algorithm for

Computational Problem: *Expressing a given vector as a linear combination of other given vectors*

Span

Definition: The set of all linear combinations of some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is called the *span* of these vectors

Written Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Span: Attacking the authentication scheme

If Eve knows the password satisfies

$$\begin{aligned}
\mathbf{a}_1 \cdot \mathbf{x} &= \beta_1 \\
&\vdots \\
\mathbf{a}_m \cdot \mathbf{x} &= \beta_m
\end{aligned}$$

Then she can calculate right response to any challenge in Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$:

Proof: Suppose $\mathbf{a} = \alpha_1\,\mathbf{a}_1 + \cdots + \alpha_m\,\mathbf{a}_m$. Then

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{x} &= (\alpha_1\,\mathbf{a}_1 + \cdots + \alpha_m\,\mathbf{a}_m) \cdot \mathbf{x} \\
&= \alpha_1\,\mathbf{a}_1 \cdot \mathbf{x} + \cdots + \alpha_m\,\mathbf{a}_m \cdot \mathbf{x} && \text{by distributivity} \\
&= \alpha_1\,(\mathbf{a}_1 \cdot \mathbf{x}) + \cdots + \alpha_m\,(\mathbf{a}_m \cdot \mathbf{x}) && \text{by homogeneity} \\
&= \alpha_1\,\beta_1 + \cdots + \alpha_m\,\beta_m
\end{aligned}$$

Question: Any others? Answer will come later.

Span: $GF(2)$ vectors

Quiz: How many vectors are in Span $\{[1,1],[0,1]\}$ over the field $GF(2)$?

Span: $GF(2)$ vectors

Quiz: How many vectors are in Span $\{[1,1],[0,1]\}$ over the field $GF(2)$?

Answer: The linear combinations are

$$\begin{aligned} 0\,[1,1] + 0\,[0,1] &= [0,0] \\ 0\,[1,1] + 1\,[0,1] &= [0,1] \\ 1\,[1,1] + 0\,[0,1] &= [1,1] \\ 1\,[1,1] + 1\,[0,1] &= [1,0] \end{aligned}$$

Thus there are four vectors in the span.

Span: $GF(2)$ vectors

Question: How many vectors in Span $\{[1,1]\}$ over $GF(2)$?

Answer: The linear combinations are

$$0\,[1,1] = [0,0]$$
$$1\,[1,1] = [1,1]$$

Thus there are two vectors in the span.

Question: How many vectors in Span $\{\}$?

Answer: Only one: the zero vector

Question: How many vectors in Span $\{[2,3]\}$ over $\mathbb{R}$?

Answer: An infinite number: $\{\alpha\,[2,3]\ :\ \alpha \in \mathbb{R}\}$
Forms the line through the origin and $(2,3)$.

Generators

Definition: Let $\mathcal{V}$ be a set of vectors. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are vectors such that $\mathcal{V} = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ then

- we say $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a *generating set* for $\mathcal{V}$;
- we refer to the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as *generators* for $\mathcal{V}$.

Example: $\{[3, 0, 0], [0, 2, 0], [0, 0, 1]\}$ is a generating set for $\mathbb{R}^3$.

Proof: Must show two things:

1. Every linear combination is a vector in $\mathbb{R}^3$.
2. Every vector in $\mathbb{R}^3$ is a linear combination.

First statement is easy: every linear combination of 3-vectors over $\mathbb{R}$ is a 3-vector over $\mathbb{R}$, and $\mathbb{R}^3$ contains all 3-vectors over $\mathbb{R}$.

Proof of second statement: Let $[x, y, z]$ be any vector in $\mathbb{R}^3$. I must show it is a linear combination of my three vectors....

$$[x, y, z] = (x/3)\,[3, 0, 0] + (y/2)\,[0, 2, 0] + z\,[0, 0, 1]$$

Generators

Claim: Another generating set for $\mathbb{R}^3$ is $\{[1,0,0],[1,1,0],[1,1,1]\}$

Another way to prove that every vector in $\mathbb{R}^3$ is in the span:

- We already know $\mathbb{R}^3 = \text{Span } \{[3,0,0],[0,2,0],[0,0,1]\}$,
- so just show $[3,0,0]$, $[0,2,0]$, and $[0,0,1]$ are in Span $\{[1,0,0],[1,1,0],[1,1,1]\}$

$$
\begin{aligned}
[3,0,0] &= 3\,[1,0,0] \\
[0,2,0] &= -2\,[1,0,0] + 2\,[1,1,0] \\
[0,0,1] &= -1\,[1,0,0] - 1\,[1,1,0] + 1\,[1,1,1]
\end{aligned}
$$

Why is that sufficient?

- We already know any vector in $\mathbb{R}^3$ can be written as a linear combination of the old vectors.
- We know each old vector can be written as a linear combination of the new vectors.
- We can convert *a linear combination of linear combination of new vectors* into *a linear combination of new vectors*.

Generators

We can convert *a linear combination of linear combination of new vectors* into *a linear combination of new vectors*.

- Write $[x, y, z]$ as a linear combination of the old vectors:

$$[x, y, z] = (x/3)\,[3, 0, 0] + (y/2)\,[0, 2, 0] + z\,[0, 0, 1]$$

- Replace each old vector with an equivalent linear combination of the new vectors:

$$\begin{aligned}[x, y, z] = (x/3)\Big(3\,[1, 0, 0]\Big) \quad &+ \quad (y/2)\Big(-2\,[1, 0, 0] + 2\,[1, 1, 0]\Big) \\ &+ \quad z\,\Big(-1\,[1, 0, 0] - 1\,[1, 1, 0] + 1\,[1, 1, 1]\Big)\end{aligned}$$

- Multiply through, using distributivity and associativity:

$$[x, y, z] = x\,[1, 0, 0] - y\,[1, 0, 0] + y\,[1, 1, 0] - z\,[1, 0, 0] - z\,[1, 1, 0] + z\,[1, 1, 1]$$

- Collect like terms, using distributivity:

$$[x, y, z] = (x - y - z)\,[1, 0, 0] + (y - z)\,[1, 1, 0] + z\,[1, 1, 1]$$

Generators

Question: How to write each of the old vectors $[3, 0, 0]$, $[0, 2, 0]$, and $[0, 0, 1]$ as a linear combination of new vectors $[2, 0, 1]$, $[1, 0, 2]$, $[2, 2, 2]$, and $[0, 1, 0]$?

Answer:

$$\begin{aligned}
[3, 0, 0] &= 2\,[2, 0, 1] - 1\,[1, 0, 2] + 0\,[2, 2, 2] \\
[0, 2, 0] &= -\frac{2}{3}\,[2, 0, 1] - \frac{2}{3}\,[1, 0, 2] + 1\,[2, 2, 2] \\
[0, 0, 1] &= -\frac{1}{3}\,[2, 0, 1] + \frac{2}{3}\,[1, 0, 2] + 0\,[2, 2, 2]
\end{aligned}$$

Standard generators

Writing $[x, y, z]$ as a linear combination of the vectors $[3, 0, 0]$, $[0, 2, 0]$, and $[0, 0, 1]$ is simple.

$$[x, y, z] = (x/3)\,[3, 0, 0] + (y/2)\,[0, 2, 0] + z\,[0, 0, 1]$$

Even simpler if instead we use $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$:

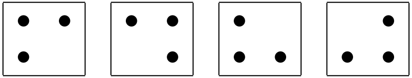
$$[x, y, z] = x\,[1, 0, 0] + y\,[0, 1, 0] + z\,[0, 0, 1]$$

These are called *standard generators* for $\mathbb{R}^3$.
Written $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

Standard generators

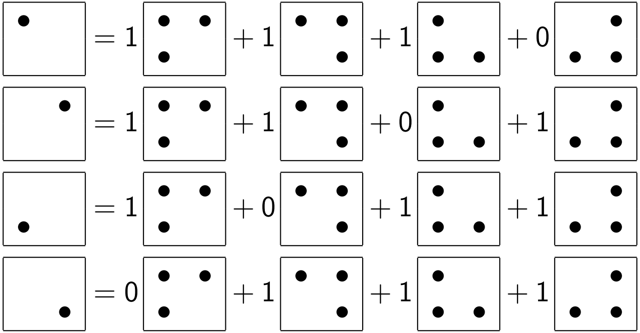
Question: Can 2×2 *Lights Out* be solved from every starting configuration?

Equivalent to asking whether the 2×2 button vectors



are generators for $GF(2)^D$, where $D = \{(0,0),(0,1),(1,0),(1,1)\}$.

Yes! For proof, we show that each standard generator can be written as a linear combination of the button vectors:



Geometry of sets of vectors: span of vectors over $\mathbb{R}$

Span of a single nonzero vector $\mathbf{v}$:

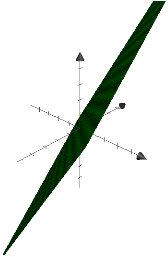
$$\text{Span } \{\mathbf{v}\} = \{\alpha\, \mathbf{v}\ :\ \alpha \in \mathbb{R}\}$$

This is the line through the origin and $\mathbf{v}$. *One-dimensional*

Span of the empty set:just the origin. *Zero-dimensional*

Span $\{[1, 2], [3, 4]\}$: all points in the plane. *Two-dimensional*

Span of two 3-vectors? Span $\{[1, 0, 1.65], [0, 1, 1]\}$ is a plane in three dimensions:



Two-dimensional

Geometry of sets of vectors: span of vectors over $\mathbb{R}$

Is the span of k vectors always k-dimensional?
No.

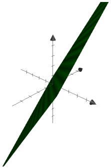
- Span $\{[0, 0]\}$ is 0-dimensional.
- Span $\{[1, 3], [2, 6]\}$ is 1-dimensional.
- Span $\{[1, 0, 0], [0, 1, 0], [1, 1, 0]\}$ is 2-dimensional.

Fundamental Question: How can we predict the dimensionality of the span of some vectors?

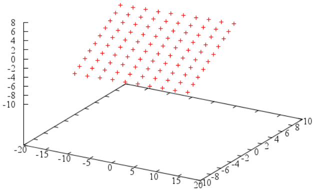
Geometry of sets of vectors: span of vectors over $\mathbb{R}$

Span of two 3-vectors? Span $\{[1,0,1.65],[0,1,1]\}$ is a plane in three dimensions:

Two-dimensional



Useful for plotting the plane

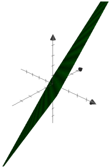


$\{\alpha\,[1,0.1.65]+\beta\,[0,1,1]\ :$
$\alpha \in \{-5,-4,\ldots,3,4\},$
$\beta \in \{-5,-4,\ldots,3,4\}\}$

Geometry of sets of vectors: span of vectors over $\mathbb{R}$

Span of two 3-vectors? Span $\{[1, 0, 1.65], [0, 1, 1]\}$ is a plane in three dimensions:

Two-dimensional



Perhaps a more familiar way to specify a plane:

$$\{(x, y, z) \ : \ ax + by + cz = 0\}$$

Using dot-product, we could rewrite as

$$\{[x, y, z] \ : \ [a, b, c] \cdot [x, y, z] = 0\}$$

Set of vectors satisfying a linear equation with right-hand side *zero*.

We can similarly specify a line in three dimensions:

$$\{[x, y, z] \ : \ \mathbf{a}_1 \cdot [x, y, z] = 0, \mathbf{a}_2 \cdot [x, y, z] = 0\}$$

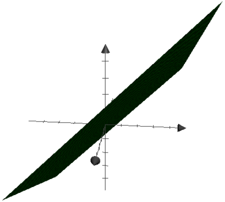
Two ways to represent a geometric object (line, plane, etc.) containing the origin:

- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides

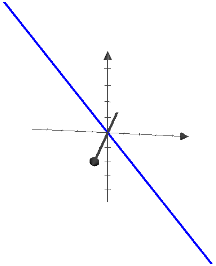
Geometry of sets of vectors: Two representations

Two ways to represent a geometric object (line, plane, etc.) containing the origin:

- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides



Span $\{[4, -1, 1], [0, 1, 1]\}$ $\quad\quad$ $\{[x, y, z] \; : \; [1, 2, -2] \cdot [x, y, z] = 0\}$



Span $\{[1, 2, -2]\}$ $\quad\quad$ $\{[x, y, z] \; :$
$[4, -1, 1] \cdot [x, y, z] = 0,$
$[0, 1, 1] \cdot [x, y, z] = 0\}$

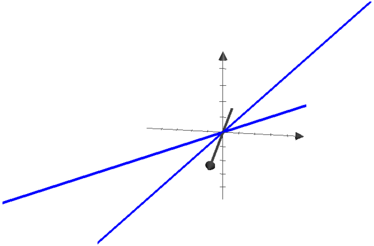
Geometry of sets of vectors: Two representations

Two ways to represent a geometric object (line, plane, etc.) containing the origin:

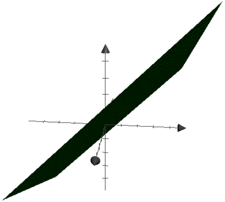
- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides

Each representation has its uses.

Suppose you want to find the plane containing two given lines



- First line is Span $\{[4, -1, 1]\}$.
- Second line is Span $\{[0, 1, 1]\}$.



- The plane containing these two lines is Span $\{[4, -1, 1], [0, 1, 1]\}$

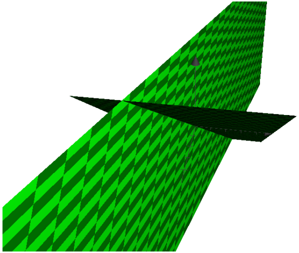
Geometry of sets of vectors: Two representations

Two ways to represent a geometric object (line, plane, etc.) containing the origin:

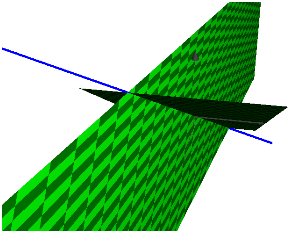
- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides

Each representation has its uses.

Suppose you want to find the intersection of two given planes:



- First plane is $\{[x, y, z] \ : \ [4, -1, 1] \cdot [x, y, z] = 0\}$.
- Second plane is $\{[x, y, z] \ : \ [0, 1, 1] \cdot [x, y, z] = 0\}$.



- The intersection is $\{[x, y, z] \ : \ [4, -1, 1] \cdot [x, y, z] = 0, [0, 1, 1] \cdot [x, y, z] = 0\}$

Two representations: What's common?

Subset of $\mathbb{F}^D$ that satisfies three properties:

Property V1 Subset contains the zero vector $\mathbf{0}$

Property V2 If subset contains $\mathbf{v}$ then it contains $\alpha\,\mathbf{v}$ for every scalar α

Property V3 If subset contains $\mathbf{u}$ and $\mathbf{v}$ then it contains $\mathbf{u}+\mathbf{v}$

Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ satisfies

- Property V1 because

$$0\,\mathbf{v}_1+\cdots+0\,\mathbf{v}_n$$

- Property V2 because

$$\text{if } \mathbf{v}=\beta_1\,\mathbf{v}_1+\cdots+\beta_n\,\mathbf{v}_n \text{ then } \alpha\,\mathbf{v}=\alpha\,\beta_1\mathbf{v}_1+\cdots+\alpha\,\beta_n\,\mathbf{v}_n$$

- Property V3 because

$$\text{if } \mathbf{u}=\alpha_1\,\mathbf{v}_1+\cdots+\alpha_n\,\mathbf{v}_1$$
$$\text{and } \mathbf{v}=\beta_1\,\mathbf{v}_1+\cdots+\beta_n\,\mathbf{v}_n$$
$$\text{then } \mathbf{u}+\mathbf{v}=(\alpha_1+\beta_1)\mathbf{v}_1+\cdots+(\alpha_n+\beta_n)\,\mathbf{v}_n$$

Two representations: What's common?

Subset of $\mathbb{F}^D$ that satisfies three properties:

Property V1 Subset contains the zero vector $\mathbf{0}$

Property V2 If subset contains $\mathbf{v}$ then it contains $\alpha\,\mathbf{v}$ for every scalar α

Property V3 If subset contains $\mathbf{u}$ and $\mathbf{v}$ then it contains $\mathbf{u}+\mathbf{v}$

Solution set $\{\mathbf{x} \;:\; \mathbf{a}_1 \cdot \mathbf{x} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{x} = 0\}$ satisfies

- Property V1 because

$$\mathbf{a}_1 \cdot \mathbf{0} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{0} = 0$$

- Property V2 because

$$\text{if } \mathbf{a}_1 \cdot \mathbf{v} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{v} = 0$$
$$\text{then } \mathbf{a}_1 \cdot (\alpha\,\mathbf{v}) = \alpha\,(\mathbf{a}_1 \cdot \mathbf{v}) = 0, \cdots, \mathbf{a}_m \cdot (\alpha\,\mathbf{v}) = \alpha\,(\mathbf{a}_m \cdot \mathbf{v}) = 0$$

- Property V3 because

$$\text{if } \mathbf{a}_1 \cdot \mathbf{u} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{u} = 0$$
$$\text{and } \mathbf{a}_1 \cdot \mathbf{v} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{v} = 0$$
$$\text{then } \mathbf{a}_1 \cdot (\mathbf{u}+\mathbf{v}) = \mathbf{a}_1 \cdot \mathbf{u} + \mathbf{a}_1 \cdot \mathbf{v} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot (\mathbf{u}+\mathbf{v}) = \mathbf{a}_m \cdot \mathbf{u} + \mathbf{a}_m \cdot \mathbf{v} = 0$$

Two representations: What's common?

Subset of $\mathbb{F}^D$ that satisfies three properties:

Property V1 Subset contains the zero vector $\mathbf{0}$

Property V2 If subset contains $\mathbf{v}$ then it contains $\alpha\,\mathbf{v}$ for every scalar α

Property V3 If subset contains $\mathbf{u}$ and $\mathbf{v}$ then it contains $\mathbf{u}+\mathbf{v}$

Any subset $\mathcal{V}$ of $\mathbb{F}^D$ satisfying the three properties is called a *vector space.*

Example: Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{x} \;:\; \mathbf{a}_1 \cdot \mathbf{x} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{x} = 0\}$ are vector spaces.

If $\mathcal{U}$ is also a vector space and $\mathcal{U}$ is a subset of $\mathcal{V}$ then $\mathcal{U}$ is called a *subspace* of $\mathcal{V}$.

Example: Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{x} \;:\; \mathbf{a}_1 \cdot \mathbf{x} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{x} = 0\}$ are *subspaces* of $\mathbb{R}^D$

Possibly profound fact we will learn later: Every subspace of $\mathbb{R}^D$

- can be written in the form Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$
- can be written in the form $\{\mathbf{x} \;:\; \mathbf{a}_1 \cdot \mathbf{x} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{x} = 0\}$

Abstract vector spaces

In traditional, abstract approach to linear algebra:

- We don't define vectors as sequences [1,2,3] or even functions $\{$`a:1, b:2, c:3`$\}$.
- We define a vector space over a field $\mathbb{F}$ to be any set $\mathcal{V}$ that is equipped with
  - an *addition* operation, and
  - a *scalar-multiplication* operation

  satisfying certain axioms (e.g. commutate and distributive laws) and Properties V1, V2, V3.

Abstract approach has the advantage that it avoids committing to specific structure for vectors.

I avoid abstract approach in this class because more concrete notion of vectors is helpful in developing intuition.

Geometry of sets of vectors: convex hull

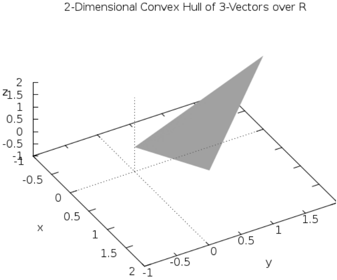
Earlier, we saw: The **u**-to-**v** line segment is

$$\{\alpha\,\mathbf{u} + \beta\,\mathbf{v} \;:\; \alpha \in \mathbb{R}, \beta \in \mathbb{R}, \alpha \geq 0, \beta \geq 0, \alpha + \beta = 1\}$$

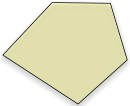
Definition: For vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ over $\mathbb{R}$, a linear combination

$$\alpha_1\,\mathbf{v}_1 + \cdots + \alpha_n\,\mathbf{v}_n$$

is a *convex combination* if the coefficients are all nonnegative and they sum to 1.



- Convex hull of a single vector is a point.
- Convex hull of two vectors is a line segment.
- Convex hull of three vectors is a triangle

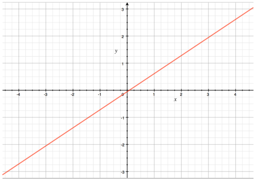


Convex hull of more vectors? Could be higher-dimensional... but not necessarily.

For example, a convex polygon is the convex hull of its vertices

Geometric objects that exclude the origin

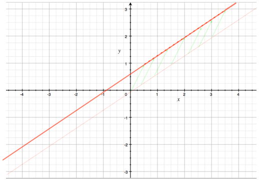
How to represent a line that does *not* contain the origin?



Start with a line that *does* contain the origin.

We know that points of such a line form a vector space $\mathcal{V}$.

Translate the line by adding a vector $\mathbf{c}$ to every vector in $\mathcal{V}$:



$$\{\mathbf{c} + \mathbf{v} \ : \ \mathbf{v} \in \mathcal{V}\}$$

(abbreviated $\mathbf{c} + \mathcal{V}$)

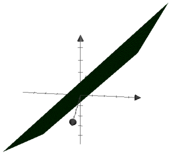
Result is line through $\mathbf{c}$ instead of through origin.

Geometric objects that exclude the origin

How to represent a plane that does *not* contain the origin?

- Start with a plane that *does* contain the origin.

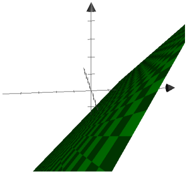
  We know that points of such a plane form a vector space $\mathcal{V}$.



- Translate it by adding a vector $\mathbf{c}$ to every vector in $\mathcal{V}$

$$\{\mathbf{c} + \mathbf{v} \; : \; \mathbf{v} \in \mathcal{V}\}$$

  (abbreviated $\mathbf{c} + \mathcal{V}$)



- Result is plane containing $\mathbf{c}$.

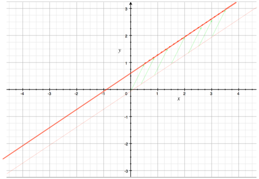
Affine space

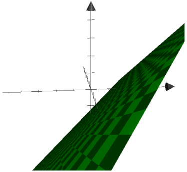
Definition: If $\mathbf{c}$ is a vector and $\mathcal{V}$ is a vector space then

$$\mathbf{c} + \mathcal{V}$$

is called an *affine space*.

Examples: A plane or a line not necessarily containing the origin.

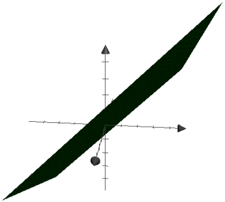




Affine space and affine combination

Example: The plane containing $\mathbf{u}_1 = [3, 0, 0]$, $\mathbf{u}_2 = [-3, 1, -1]$, and $\mathbf{u}_3 = [1, -1, 1]$.

Want to express this plane as $\mathbf{u}_1 + \mathcal{V}$
where $\mathcal{V}$ is the span of two vectors
(a plane containing the origin)

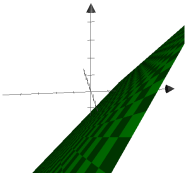


Let $\mathcal{V} = \text{Span } \{\mathbf{a}, \mathbf{b}\}$ where

$$\mathbf{a} = \mathbf{u}_2 - \mathbf{u}_1 \text{ and } \mathbf{b} = \mathbf{u}_3 - \mathbf{u}_1$$

Since $\mathbf{u}_1 + \mathcal{V}$ is a translation of a plane, it is also a plane.

- Span $\{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{0}$, so $\mathbf{u}_1 + \text{Span } \{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_1$.
- Span $\{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_2 - \mathbf{u}_1$ so $\mathbf{u}_1 + \text{Span } \{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_2$.
- Span $\{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_3 - \mathbf{u}_1$ so $\mathbf{u}_1 + \text{Span } \{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_3$.



Thus the plane $\mathbf{u}_1 + \text{Span } \{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.
Only one plane contains those three points, so this is that one.

Affine space and affine combination

Example: The plane containing $\mathbf{u}_1 = [3, 0, 0]$, $\mathbf{u}_2 = [-3, 1, -1]$, and $\mathbf{u}_1 = [1, -1, 1]$:

$$\mathbf{u}_1 + \text{Span } \{\mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_3 - \mathbf{u}_1\}$$

Cleaner way to write it?

$$\begin{aligned}
\mathbf{u}_1 + \text{Span } \{\mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_3 - \mathbf{u}_1\} &= \{\mathbf{u}_1 + \alpha\,(\mathbf{u}_2 - \mathbf{u}_1) + \beta\,(\mathbf{u}_3 - \mathbf{u}_1) : \alpha, \beta \in \mathbb{R}\} \\
&= \{\mathbf{u}_1 + \alpha\,\mathbf{u}_2 - \alpha\,\mathbf{u}_1 + \beta\,\mathbf{u}_3 - \beta\,\mathbf{u}_1 \;:\; \alpha, \beta \in \mathbb{R}\} \\
&= \{(1 - \alpha - \beta)\,\mathbf{u}_1 + \alpha\,\mathbf{u}_2 + \beta\,\mathbf{u}_3 \;:\; \alpha, \beta \in \mathbb{R}\} \\
&= \{\gamma\,\mathbf{u}_1 + \alpha\,\mathbf{u}_2 + \beta\,\mathbf{u}_3 \;:\; \gamma + \alpha + \beta = 1\}
\end{aligned}$$

Definition: A linear combination $\gamma\,\mathbf{u}_1 + \alpha\,\mathbf{u}_2 + \beta\,\mathbf{u}_3$ where $\gamma + \alpha + \beta = 1$ is an *affine combination*.

Affine combination

Definition: A linear combination

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n$$

where

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$

is an *affine combination*.

Definition: The set of all affine combinations of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is called the *affine hull* of those vectors.

$$\text{Affine hull of } \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n = \mathbf{u}_1 + \text{Span } \{\mathbf{u}_2 - \mathbf{u}_1, \ldots, \mathbf{u}_n - \mathbf{u}_1\}$$

This shows that the affine hull of some vectors is an affine space..

Geometric objects not containing the origin: equations

Can express a plane as $\mathbf{u}_1 + \mathcal{V}$ or affine hull of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.
More familiar way to express a plane:
The solution set of an equation $ax + by + cz = d$

In vector terms,

$$\{[x, y, z] \; : \; [a, b, c] \cdot [x, y, z] = d\}$$

In general, a geometric object (point, line, plane, ...) can be expressed as the solution set of a system of linear equations.

$$\{\mathbf{x} \; : \; \mathbf{a}_1 \cdot \mathbf{x} = \beta_1, \ldots, \mathbf{a}_m \cdot \mathbf{x} = \beta_m\}$$

Conversely, is the solution set an affine space?

Consider solution set of a contradictory system of equations, e.g. $1\,x = 1, 2\,x = 1$:

- Solution set is empty....
- ...but a vector space $\mathcal{V}$ always contains the zero vector,
- ...so an affine space $\mathbf{u}_1 + \mathcal{V}$ always contains at least one vector.

Turns out this the only exception:

Theorem: The solution set of a linear system is either empty or an affine space.

Affine spaces and linear systems

Theorem: The solution set of a linear system is either empty or an affine space.

Each linear system corresponds to a linear system with zero right-hand sides:

$$\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & \beta_1 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & \beta_m \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & 0 \end{array}$$

Definition:
A linear equation $\mathbf{a} \cdot \mathbf{x} = 0$ with zero right-hand side is a *homogeneous* linear equation.
A system of homogeneous linear equations is called a *homogeneous* linear system.

We already know: The solution set of a homogeneous linear system is a vector space.

Lemma: Let $\mathbf{u}_1$ be a solution to a linear system. Then, for any other vector $\mathbf{u}_2$,

$\mathbf{u}_2$ is also a solution

if and only if

$\mathbf{u}_2 - \mathbf{u}_1$ is a solution to the corresponding homogeneous linear system.

Affine spaces and linear systems

$$\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & \beta_1 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & \beta_m \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & 0 \end{array}$$

Lemma: Let $\mathbf{u}_1$ be a solution to a linear system. Then, for any other vector $\mathbf{u}_2$,
$\mathbf{u}_2$ is also a solution
if and only if
$\mathbf{u}_2 - \mathbf{u}_1$ is a solution to the corresponding homogeneous linear system.

Proof: We assume $\mathbf{a}_1 \cdot \mathbf{u}_1 = \beta_1, \ldots, \mathbf{a}_m \cdot \mathbf{u}_1 = \beta_m$, so

$$\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{u}_2 & = & \beta_1 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{u}_2 & = & \beta_m \end{array} \quad \text{iff} \quad \begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{u}_2 - \mathbf{a}_1 \cdot \mathbf{u}_1 & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{u}_2 - \mathbf{a}_m \cdot \mathbf{u}_1 & = & 0 \end{array} \quad \text{iff} \quad \begin{array}{rcl} \mathbf{a}_1 \cdot (\mathbf{u}_2 - \mathbf{u}_1) & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot (\mathbf{u}_2 - \mathbf{u}_2) & = & 0 \end{array}$$

QED

Lemma: Let $\mathbf{u}_1$ be a solution to a linear system. Then, for any other vector $\mathbf{u}_2$,
$\mathbf{u}_2$ is also a solution
if and only if
$\mathbf{u}_2 - \mathbf{u}_1$ is a solution to the corresponding homogeneous linear system.

We use this lemma to prove the theorem:

Theorem: The solution set of a linear system is either empty or an affine space.

- Let $\mathcal{V}$ = set of solutions to corresponding homogeneous linear system.
- If the linear system has no solution, its solution set is empty.
- If it does has a solution $\mathbf{u}_1$ then

$$\begin{aligned} \{\text{solutions to linear system}\} &= \{\mathbf{u}_2 \ : \ \mathbf{u}_2 - \mathbf{u}_1 \in \mathcal{V}\} \\ (\text{substitute } \mathbf{v} = \mathbf{u}_2 - \mathbf{u}_1) & \\ &= \{\mathbf{u}_1 + \mathbf{v} : \mathbf{v} \in \mathcal{V}\} \end{aligned}$$

QED

Number of solutions to a linear system

We just proved:

If $\mathbf{u}_1$ is a solution to a linear system then

$$\{\text{solutions to linear system}\} = \{\mathbf{u}_1 + \mathbf{v} : \mathbf{v} \in \mathcal{V}\}$$

where $\mathcal{V} = \{\text{solutions to corresponding homogeneous linear system}\}$

Implications:

Long ago we asked: *How can we tell if a linear system has only one solution?*

Now we know: If a linear system has a solution $\mathbf{u}_1$ then that solution is unique if the only solution to the corresponding homogeneous linear system is $\mathbf{0}$.

Long ago we asked: How can we find the number of solutions to a linear system over $GF(2)$?

Now we know: Number of solutions either is zero or is equal to the number of solutions to the corresponding *homogeneous* linear system.

Number of solutions: checksum function

MD5 checksums and sizes of the released files:

3c63a6d97333f4da35976b6a0755eb67	12732276	Python-3.2.2.tgz
9d763097a13a59ff53428c9e4d098a05	10743647	Python-3.2.2.tar.bz2
3720ce9460597e49264bbb63b48b946d	8923224	Python-3.2.2.tar.xz
f6001a9b2be57ecfbefa865e50698cdf	19519332	python-3.2.2-macosx10.3.dmg
8fe82d14dbb2e96a84fd6fa1985b6f73	16226426	python-3.2.2-macosx10.6.dmg
cccb03e14146f7ef82907cf12bf5883c	18241506	python-3.2.2-pdb.zip
72d11475c986182bcb0e5c91acec45bc	19940424	python-3.2.2.amd64-pdb.zip
ddeb3e3fb93ab5a900adb6f04edab21e	18542592	python-3.2.2.amd64.msi
8afb1b01e8fab738e7b234eb4fe3955c	18034688	python-3.2.2.msi

A *checksum function* maps long files to short sequences.

Idea:

- Web page shows the checksum of each file to be downloaded.
- Download the file and run the checksum function on it.
- If result does not match checksum on web page, you know the file has been corrupted.
- If random corruption occurs, how likely are you to detect it?

Impractical but instructive checksum function:

- *input:* an n-vector $\mathbf{x}$ over $GF(2)$
- *output:* $[\mathbf{a}_1 \cdot \mathbf{x}, \mathbf{a}_2 \cdot \mathbf{x}, \ \ldots, \ \mathbf{a}_{64} \cdot \mathbf{x}]$

where $\mathbf{a}_1, \mathbf{a}_2, \ \ldots, \ \mathbf{a}_{64}$ are sixty-four n-vectors.

Number of solutions: checksum function

Our checksum function:

- *input:* an n-vector $\mathbf{x}$ over $GF(2)$
- *output:* $[\mathbf{a}_1 \cdot \mathbf{x}, \mathbf{a}_2 \cdot \mathbf{x}, \ \ldots, \ \mathbf{a}_{64} \cdot \mathbf{x}]$

where $\mathbf{a}_1, \mathbf{a}_2, \ \ldots, \ \mathbf{a}_{64}$ are sixty-four n-vectors.

Suppose $\mathbf{p}$ is the original file, and it is randomly corrupted during download.

What is the probability that the corruption is undetected?

The checksum of the original file is $[\beta_1, \ldots, \beta_{64}] = [\mathbf{a}_1 \cdot \mathbf{p}, \ldots, \mathbf{a}_{64} \cdot \mathbf{p}]$.

Suppose corrupted version is $\mathbf{p} + \mathbf{e}$.

Then checksum of corrupted file matches checkum of original if and only if

$$\begin{array}{rcl} \mathbf{a}_1 \cdot (\mathbf{p} + \mathbf{e}) & = & \beta_1 \\ & \vdots & \\ \mathbf{a}_{64} \cdot (\mathbf{p} + \mathbf{e}) & = & \beta_{64} \end{array} \quad \text{iff} \quad \begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{p} - \mathbf{a}_1 \cdot (\mathbf{p} + \mathbf{e}) & = & 0 \\ & \vdots & \\ \mathbf{a}_{64} \cdot \mathbf{p} - \mathbf{a}_{64} \cdot (\mathbf{p} + \mathbf{e}) & = & 0 \end{array} \quad \text{iff} \quad \begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{e} & = & 0 \\ & \vdots & \\ \mathbf{a}_{64} \cdot \mathbf{e} & = & 0 \end{array}$$

iff $\mathbf{e}$ is a solution to the homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ \ldots \ \mathbf{a}_{64} \cdot \mathbf{x} = 0$.

Number of solutions: checksum function

Suppose corrupted version is $\mathbf{p} + \mathbf{e}$.
Then checksum of corrupted file matches checkum of original if and only if $\mathbf{e}$ is a solution to homogeneous linear system

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= 0 \\ &\vdots \\ \mathbf{a}_{64} \cdot \mathbf{x} &= 0 \end{aligned}$$

If $\mathbf{e}$ is chosen according to the uniform distribution,

$$\begin{aligned} &\text{Probability}\,(\mathbf{p} + \mathbf{e} \text{ has same checksum as } \mathbf{p}) \\ &= \text{Probability}\,(\mathbf{e} \text{ is a solution to homogeneous linear system}) \\ &= \frac{\text{number of solutions to homogeneous linear system}}{\text{number of } n\text{-vectors}} \\ &= \frac{\text{number of solutions to homogeneous linear system}}{2^n} \end{aligned}$$

Question:
How to find out number of solutions to a homogeneous linear system over $GF(2)$?