What is a matrix? Traditional answer

Neo: What is the Matrix?
Trinity: The answer is out there, Neo, and it's looking for you, and it will find you if you want it to. *The Matrix*, 1999

Traditional notion of a matrix: two-dimensional array.

$$\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix}$$

- Two rows: $[1, 2, 3]$ and $[10, 20, 30]$.
- Three columns: $[1, 10]$, $[2, 20]$, and $[3, 30]$.
- A 2×3 matrix.

For a matrix A, the i, j element of A

- is the element in row i, column j
- is traditionally written $A_{i,j}$
- but we will use $A[i, j]$

List of row-lists, list of column-lists (Quiz)

- One obvious Python representation for a matrix: a list of row-lists:
  $\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix}$ represented by `[[1,2,3],[10,20,30]]`.
- Another: a list of column-lists:
  $\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix}$ represented by `[[1,10],[2,20],[3,30]]`.

List of row-lists, list of column-lists

Quiz: Write a nested comprehension whose value is list-of-*row*-list representation of a 3×4 matrix all of whose elements are zero:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hint: first write a comprehension for a typical row, then use that expression in a comprehension for the list of lists.

List of row-lists, list of column-lists

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$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hint: first write a comprehension for a typical row, then use that expression in a comprehension for the list of lists.

Answer:

```
>>> [[0 for j in range(4)] for i in range(3)]
[[0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0]]
```

List of row-lists, list of column-lists (Quiz)

Quiz: Write a nested comprehension whose value is list-of-*column*-lists representation of a 3×4 matrix whose i, j element is $i - j$:

$$\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

Hint: First write a comprension for column j, assuming j is bound to an integer. Then use that expression in a comprehension in which j is the control variable.

List of row-lists, list of column-lists (Quiz)

Quiz: Write a nested comprehension whose value is list-of-*column*-lists representation of a 3×4 matrix whose i, j element is $i - j$:

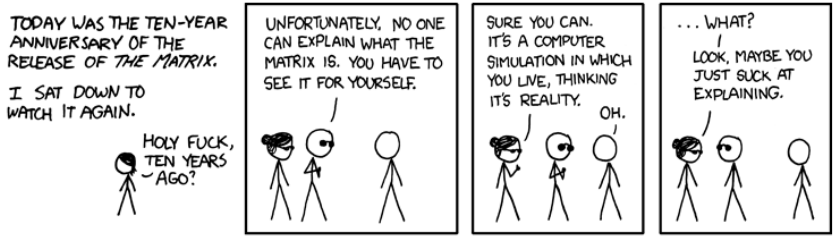
$$\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

Hint: First write a comprension for column j, assuming j is bound to an integer. Then use that expression in a comprehension in which j is the control variable.

Answer:

```
>>> [[i-j for i in range(3)] for j in range(4)]
[[0, 1, 2], [-1, 0, 1], [-2, -1, 0], [-3, -2, -1]]
```

The matrix revealed



The Matrix Revisited (excerpt) `http://xkcd.com/566/`

Definition: For finite sets R and C, an $R \times C$ *matrix over* $\mathbb{F}$ is a function from $R \times C$ to $\mathbb{F}$.

	@	#	?
a	1	2	3
b	10	20	30

- $R = \{\mathtt{a}, \mathtt{b}\}$ and $C = \{\mathtt{@}, \mathtt{\#}, \mathtt{?}\}$.
- R is set of row labels
- C is set of column labels

In Python, the function is represented by a dictionary:

```
{('a','@'):1, ('a','#'):2, ('a', '?'):3,
 ('b', '@'):10, ('b',  '#'):20, ('b','?'):30}
```

Rows, columns, and entries

	@	#	?
a	1	2	3
b	10	20	30

Rows and columns are vectors, e.g.

- Row `'a'` is the vector `Vec({'@', '#', '?'}, {'@':1, '#':2, '?':3})`
- Column `'#'` is the vector `Vec({'a','b'}, {'a':2, 'b':20})`

Dict-of-rows/dict-of-columns representations

	@	#	?
a	1	2	3
b	10	20	30

One representation: *dictionary of rows:*

```
{'a': Vec({'#', '@', '?'}, {'@':1, '#':2, '?':3}),
 'b': Vec({'#', '@', '?'}, {'@':10, '#':20, '?':30})}
```

Another representation: *dictionary of columns:*

```
{'@': Vec({'a','b'}, {'a':1, 'b':10}),
 '#': Vec({'a','b'}, {'a':2, 'b':20}),
 '?': Vec({'a','b'}, {'a':3, 'b':30})}
```

Our Python implementation

	@	#	?
a	1	2	3
b	10	20	30

```
>>> M=Mat(({'a','b'}, {'@', '#', '?'}),
          {('a','@'):1, ('a','#'):2,('a','?'):3,
           ('b','@'):10, ('b','#'):20, ('b','?'):30})
```

A class with two fields:

- `D`, a *pair* (R, C) of sets.
- `f`, a dictionary representing a function that maps pairs $(r, c) \in R \times C$ to field elements.

```
class Mat:
    def __init__(self, labels, function):
        self.D = labels
        self.f = function
```

We will later add lots of matrix operations to this class.

Identity matrix

```
    a b c
   -------
a  | 1 0 0
b  | 0 1 0
c  | 0 0 1
```

Definition: $D \times D$ *identity matrix* is the matrix $\mathbb{1}_D$ such that $\mathbb{1}_D[k, k] = 1$ for all $k \in D$ and zero elsewhere.

Usually we omit the subscript when D is clear from the context.
Often letter `I` (for "identity") is used instead of $\mathbb{1}$

```
Mat(({'a','b','c'},{'a','b','c'}),{('a','a'):1,('b','b'):1,('c','c'):1})
```

Quiz: Write procedure `identity(`D`)` that returns the $D \times D$ identity matrix represented as an instance of `Mat`.

Identity matrix

```
    a b c
   -------
a  | 1 0 0
b  | 0 1 0
c  | 0 0 1
```

Definition: $D \times D$ *identity matrix* is the matrix $\mathbb{1}_D$ such that $\mathbb{1}_D[k, k] = 1$ for all $k \in D$ and zero elsewhere.

Usually we omit the subscript when D is clear from the context.
Often letter `I` (for "identity") is used instead of $\mathbb{1}$

```
Mat(({'a','b','c'},{'a','b','c'}),{('a','a'):1,('b','b'):1,('c','c'):1})
```

Quiz: Write procedure `identity(D)` that returns the $D \times D$ identity matrix represented as an instance of `Mat`.

Answer:

```
>>> def identity(D): return Mat((D,D), {(k,k):1 for k in D})
```

Converting between representations

Converting an instance of `Mat` to a column-dictionary representation:

	@	#	?
a	1	2	3
b	10	20	30

```
Mat(({'a','b'}, {'@', '#', '?'}), {('a','@'):1, ('a','#'):2,
          ('a','?'):3, ('b','@'):10, ('b','#'):20, ('b','?'):30})
```

```
{'@': Vec({'a','b'}, {'a':1, 'b':10}),
 '#': Vec({'a','b'}, {'a':2, 'b':20}),
 '?': Vec({'a','b'}, {'a':3, 'b':30})}
```

Quiz: Write the procedure `mat2coldict(A)` that, given an instance of `Mat`, returns the column-dictionary representation of the same matrix.

Converting between representations

Converting an instance of `Mat` to a column-dictionary representation:

	@	#	?
a	1	2	3
b	10	20	30

```
Mat(({'a','b'}, {'@', '#', '?'}), {('a','@'):1, ('a','#'):2,
          ('a','?'):3, ('b','@'):10, ('b','#'):20, ('b','?'):30})
```

```
{'@': Vec({'a','b'}, {'a':1, 'b':10}),
 '#': Vec({'a','b'}, {'a':2, 'b':20}),
 '?': Vec({'a','b'}, {'a':3, 'b':30})}
```

Quiz: Write the procedure `mat2coldict(A)` that, given an instance of `Mat`, returns the column-dictionary representation of the same matrix.

Answer:

```
def mat2coldict(A):
 return {c:Vec(A.D[0],{r:A[r,c] for r in A.D[0]}) for c in A.D[1]}
```

Module `matutil`

We provide a module, `matutil`, that defines several conversion routines:

- `mat2coldict(A)`: from a `Mat` to a dictionary of columns represented as `Vec`s)
- `mat2rowdict(A)`: from a `Mat` to a dictionary of rows represented as `Vec`s
- `coldict2mat(coldict)` from a dictionary of columns (or a list of columns) to a `Mat`
- `rowdict2mat(rowdict)`: from a dictionary of rows (or a list of rows) to a `Mat`
- `listlist2mat(L)`: from a list of list of field elements to a `Mat` the inner lists turn into rows

and also:

- `identity(D)`: produce a `Mat` representing the $D \times D$ identity matrix

The Mat class

We gave the definition of a rudimentary matrix class:

```
class Mat:
    def __init__(self,
       labels, function):
        self.D = labels
        self.f = function
```

The more elaborate class definition allows for more concise vector code, e.g.

```
>>> M['a', 'B'] = 1.0
>>> b = M*v
>>> B = M*A
>>> print(B)
```

More elaborate version of this class definition allows operator overloading for element access, matrix-vector multiplication, etc.

operation	syntax
Matrix addition and subtraction	`A+B` and `A-B`
Matrix negative	`-A`
Scalar-matrix multiplication	`alpha*A`
Matrix equality test	`A == B`
Matrix transpose	`A.transpose()`
Getting a matrix entry	`A[r,c]`
Setting a matrix entry	`A[r,c] = alpha`
Matrix-vector multiplication	`A*v`
Vector-matrix multiplication	`v*A`
Matrix-matrix multiplication	`A*B`

You will code this class starting from a template we provide.

Using Mat

You will write the bodies of named procedures such as `setitem(M, k, val)` and `matrix_vector_mul(M, v)` and `transpose(M)`.

However, in actually using Mats in other code, you must use operators and methods instead of named procedures, e.g.

```
>>> M['a', 'b'] = 1.0
>>> v = M*u
>>> b_parallel =
      Q*Q.transpose()*b
```

instead of

```
>>> setitem(M, ('a','B'), 1.0)
>>> v = matrix_vector_mul(M, u)
>>> b_parallel =
    matrix_vector_mul(matrix_matrix_mul(Q,
                                        transpose(Q)), b)
```

In fact, in code outside the `mat` module that uses `Mat`, you will import just `Mat` from the `mat` module:

```
from mat import Mat
```

so the named procedures will not be imported into the namespace. Those named procedures in the `mat` module are intended to be used *only* inside the `mat` module itself.

In short: Use the operators [ ], +, *, - and the method `.transpose()` when working with Mats

Assertions in Mat

For each procedure you write, we will provide the stub of the procedure, e.g. for `matrix_vector_mul(M, v)`, we provide the stub

```
def matrix_vector_mul(M, v):
    "Returns the product of matrix M and vector v"
    assert M.D[1] == v.D
    pass
```

You are supposed to replace the pass statement with code for the procedure.

The first line in the body is a documentation string.

The second line is an assertion. It asserts that the second element of the pair `M.D`, the set of column-labels of `M`, must be equal to the domain of the vector `v`. If the procedure is called with arguments that violate this, Python reports an error.

The assertion is there to remind us of a rule about matrix-vector multiplication.

Please keep the assertions in your `mat` code while using it for this course.

Testing Mat

Because you will use `Mat` a lot, making sure your implementation is correct will save you from lots of pain later.

We have provided a file `test_mat.py` with lots of examples to test against.

You can test each of these examples while running Python in interactive mode by importing `Mat` from the module `mat` and then copying the example from `test_mat.py` and pasting:

```
>>> from vec import Mat
>>> M = Mat(({1,3,5}, {'a'}), {(1,'a'):4, (5,'a'): 2})
>>> M[1,'a']
4
```

You can also run all the tests at once from the console (outside the Python interpreter) using the following command:

```
python3 -m doctest test_mat.py
```

This will run the tests given in `test_mat.py`, including importing your `vec` module, and will print messages about any discrepancies that arise. If your code passes the tests, nothing will be printed.

Column space and row space

One simple role for a matrix: packing together a bunch of columns or rows

Two vector spaces associated with a matrix M:

Definition:

- *column space* of M = Span {columns of M}
  Written Col M
- *row space* of M = Span {rows of M}
  Written Row M

Examples:

- Column space of $\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix}$ is Span $\{[1, 10], [2, 20], [3, 30]\}$.
  In this case, the span is equal to Span $\{[1, 10]\}$ since $[2, 20]$ and $[3, 30]$ are scalar multiples of $[1, 10]$.
- The row space of the same matrix is Span $\{[1, 2, 3], [10, 20, 30]\}$.
  In this case, the span is equal to Span $\{[1, 2, 3]\}$ since $[10, 20, 30]$ is a scalar multiple of $[1, 2, 3]$.

Transpose

Transpose swaps rows and columns.

```
      @  #  ?
    ---------
a   |  2  1  3
b   | 20 10 30
```

```
      a  b
    ------
@   |  2 20
#   |  1 10
?   |  3 30
```

Transpose (and Quiz)

Quiz: Write `transpose(M)`

Transpose (and Quiz)

Quiz: Write `transpose(M)`

Answer:

```
def transpose(M):
    return Mat((M.D[1], M.D[0]), {(q,p):v for (p,q),v in M.F.items()})
```

Matrices as vectors

Soon we study true matrix operations. But first....

A matrix can be interpreted as a vector:

- an $R \times S$ matrix is a function from $R \times S$ to $\mathbb{F}$,
- so it can be interpreted as an $R \times S$-vector:
  - *scalar-vector multiplication*
  - *vector addition*
- Our full implementation of `Mat` class will include these operations.

Matrix-vector and vector-matrix multiplication

Two ways to multiply a matrix by a vector:

- matrix-vector multiplication
- vector-matrix multiplication

For each of these, two *equivalent definitions*:

- in terms of linear combinations
- in terms of dot-products

Matrix-vector multiplication in terms of linear combinations

Linear-Combinations Definition of matrix-vector multiplication: Let M be an $R \times C$ matrix.

- If $\mathbf{v}$ is a C-vector then

$$M * \mathbf{v} = \sum_{c \in C} \mathbf{v}[c] \text{ (column } c \text{ of } M)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix} * [7,0,4] = 7[1,10] + 0[2,20] + 4[3,30]$$

- If $\mathbf{v}$ is *not* a C-vector then

$$M * \mathbf{v} = \text{ERROR!}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix} * [7,0] = \text{ERROR!}$$

Matrix-vector multiplication in terms of linear combinations

	@	#	?
a	2	1	3
b	20	10	30

$*$

@	#	?
0.5	5	-1

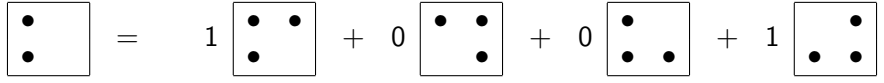
$=$

a	3
b	30

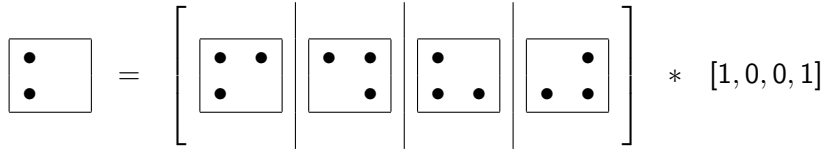
Matrix-vector multiplication in terms of linear combinations: *Lights Out*

A solution to a *Lights Out* configuration is a linear combination of "button vectors."

For example, the linear combination

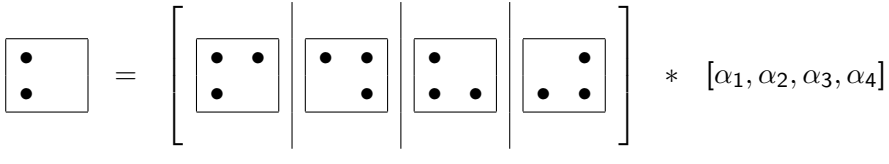


can be written as



Solving a matrix-vector equation: *Lights Out*

Solving an instance of *Lights Out* $\Rightarrow$ Solving a matrix-vector equation



Solving a matrix-vector equation

Fundamental Computational Problem: *Solving a matrix-vector equation*

- *input:* an $R \times C$ matrix A and an R-vector $\mathbf{b}$
- *output:* the C-vector $\mathbf{x}$ such that $A * \mathbf{x} = \mathbf{b}$

Solving a matrix-vector equation: 2×2 special case

Simple formula to solve

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} * [x, y] = [p, q]$$

if $ad \neq bc$:

$$x = \frac{dp - cq}{ad - bc} \text{ and } y = \frac{aq - bp}{ad - bc}$$

For example, to solve

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * [x, y] = [-1, 1]$$

we set

$$x = \frac{4 \cdot -1 - 2 \cdot 1}{1 \cdot 4 - 2 \cdot 3} = \frac{-6}{-2} = 3$$

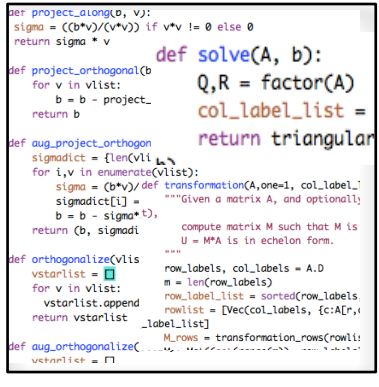
and

$$y = \frac{1 \cdot 1 - 3 \cdot -1}{1 \cdot 4 - 2 \cdot 3} = \frac{4}{-2} = -2$$

Later we study algorithms for more general cases.

The `solver` module

We provide a module `solver` that defines a procedure `solve(A, b)` that tries to find a solution to the matrix-vector equation $A\mathbf{x} = \mathbf{b}$
Currently `solve(A, b)` is a black box



but we will learn how to code it in the coming weeks.

Let's use it to solve this *Lights Out* instance...

Vector-matrix multiplication in terms of linear combinations

Vector-matrix multiplication is different from matrix-vector multiplication:

Let M be an $R \times C$ matrix.

Linear-Combinations Definition of matrix-vector multiplication: If $\mathbf{v}$ is a C-vector then

$$M * \mathbf{v} = \sum_{c \in C} \mathbf{v}[c] \text{ (column } c \text{ of } M)$$

Linear-Combinations Definition of vector-matrix multiplication: If $\mathbf{w}$ is an R-vector then

$$\mathbf{w} * M = \sum_{r \in R} \mathbf{w}[r] \text{ (row } r \text{ of } M)$$

$$[3,4] * \begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix} = 3\,[1,2,3] + 4\,[10,20,30]$$

Vector-matrix multiplication in terms of linear combinations: JunkCo





Let M =

	metal	concrete	plastic	water	electricity
garden gnome	0	1.3	.2	.8	.4
hula hoop	0	0	1.5	.4	.3
slinky	.25	0	0	.2	.7
silly putty	0	0	.3	.7	.5
salad shooter	.15	0	.5	.4	.8

$$\text{total resources used} \;=\; [\alpha_{\text{gnome}}, \alpha_{\text{hoop}}, \alpha_{\text{slinky}}, \alpha_{\text{putty}}, \alpha_{\text{shooter}}] \;*\; \text{M}$$

Suppose we know total resources used and we know M.

To find the values of $\alpha_{\text{gnome}}, \alpha_{\text{hoop}}, \alpha_{\text{slinky}}, \alpha_{\text{putty}}, \alpha_{\text{shooter}}$,

solve a *vector-matrix* equation $\mathbf{b} = \mathbf{x} * M$
where $\mathbf{b}$ is vector of total resources used.

Solving a matrix-vector equation

Fundamental Computational Problem: *Solving a matrix-vector equation*

- *input:* an $R \times C$ matrix A and an R-vector $\mathbf{b}$
- *output:* the C-vector $\mathbf{x}$ such that $A * \mathbf{x} = \mathbf{b}$

If we had an algorithm for solving a *matrix-vector* equation,
could also use it to solve a *vector-matrix* equation,
using transpose.

The solver module, and floating-point arithmetic

For arithmetic over $\mathbb{R}$, Python uses floats, so round-off errors occur:

```
>>> 10.0**16 + 1 == 10.0**16
True
```

Consequently algorithms such as that used in `solve(A, b)` do not find exactly correct solutions.

To see if solution $\mathbf{u}$ obtained is a reasonable solution to $A * \mathbf{x} = \mathbf{b}$, see if the vector $\mathbf{b} - A * \mathbf{u}$ has entries that are close to zero:

```
>>> A = listlist2mat([[1,3],[5,7]])
>>> u = solve(A, b)
>>> b - A*u
Vec({0, 1},{0: -4.440892098500626e-16, 1: -8.881784197001252e-16})
```

The vector $\mathbf{b} - A * \mathbf{u}$ is called the *residual*. Easy way to test if entries of the residual are close to zero: compute the dot-product of the residual with itself:

```
>>> res = b - A*u
>>> res * res
9.860761315262648e-31
```

Checking the output from `solve(A, b)`

For some matrix-vector equations $A * \mathbf{x} = \mathbf{b}$, there is no solution.
In this case, the vector returned by `solve(A, b)` gives rise to a largeish residual:

```
>>> A = listlist2mat([[1,2],[4,5],[-6,1]])
>>> b = list2vec([1,1,1])
>>> u = solve(A, b)
>>> res = b - A*u
>>> res * res
0.24287856071964012
```

Later in the course we will see that the residual is, in a sense, as small as possible.

Some matrix-vector equations are *ill-conditioned*, which can prevent an algorithm using floats from getting even approximate solutions, even when solutions exists:

```
>>> A = listlist2mat([[1e20,1],[1,0]])
>>> b = list2vec([1,1])
>>> u = solve(A, b)
>>> b - A*u
Vec({0, 1},{0: 0.0, 1: 1.0})
```

We will not study conditioning in this course.

Matrix-vector multiplication in terms of dot-products

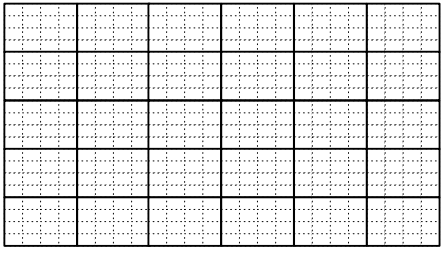
Let M be an $R \times C$ matrix.

Dot-Product Definition of matrix-vector multiplication: $M * \mathbf{u}$ is the R-vector $\mathbf{v}$ such that $\mathbf{v}[r]$ is the dot-product of row r of M with $\mathbf{u}$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 10 & 0 \end{bmatrix} * [3,-1] = [\ [1,2]\cdot[3,-1],\ [3,4]\cdot[3,-1],\ [10,0]\cdot[3,-1]\ ]$$

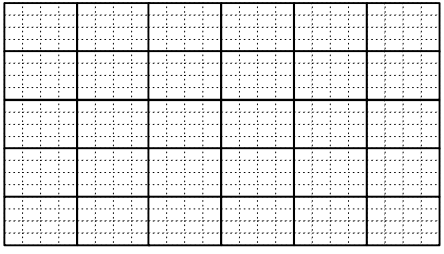
$$= [1,5,30]$$

Applications of dot-product definition of matrix-vector multiplication: Downsampling



- Each pixel of the low-res image corresponds to a little grid of pixels of the high-res image.
- The intensity value of a low-res pixel is the *average* of the intensity values of the corresponding high-res pixels.

Applications of dot-product definition of matrix-vector multiplication: Downsampling



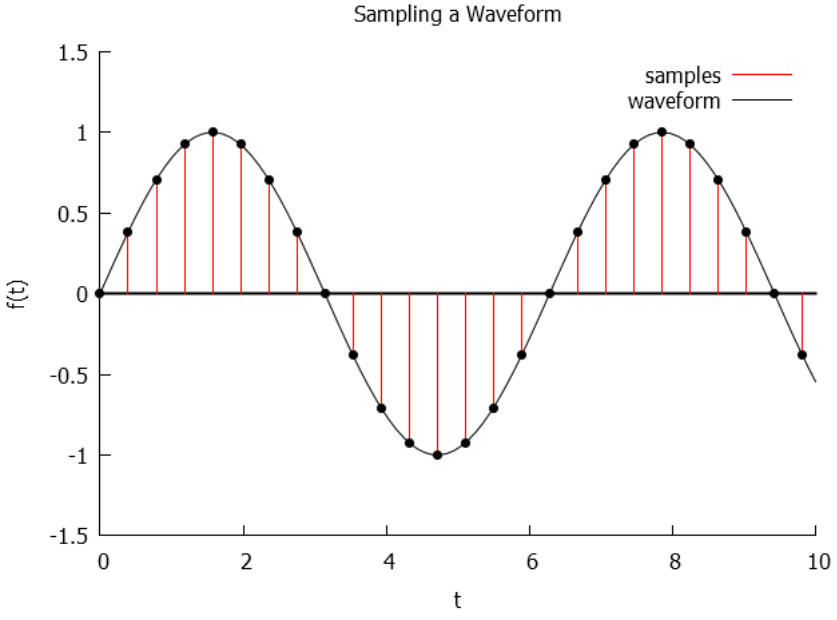
- Each pixel of the low-res image corresponds to a little grid of pixels of the high-res image.
- The intensity value of a low-res pixel is the *average* of the intensity values of the corresponding high-res pixels.

- Averaging can be expressed as dot-product.
- We want to compute a dot-product for each low-res pixel.
- Can be expressed as matrix-vector multiplication.

Applications of dot-product definition of matrix-vector multiplication: blurring

- To blur a face, replace each pixel in face with average of pixel intensities in its neighborhood.
- Average can be expressed as dot-product.
- By dot-product definition of matrix-vector multiplication, can express this image transformation as a matrix-vector product.
- Gaussian blur: a kind of weighted average

Applications of dot-product definition of matrix-vector multiplication: Audio search



Applications of dot-product definition of matrix-vector multiplication: Audio search

Lots of dot-products!

5	-6	9	-9	-5	-9	-5	5	-8	-5	-9	9	8	-5	-9	6	-2	-4	-9	-1	-1	-9	-3
2	7	4	-3	0	-1	-6	4	5	-8	-9												

5	-6	9	-9	-5	-9	-5	5	-8	-5	-9	9	8	-5	-9	6	-2	-4	-9	-1	-1	-9	-3
	2	7	4	-3	0	-1	-6	4	5	-8	-9											

5	-6	9	-9	-5	-9	-5	5	-8	-5	-9	9	8	-5	-9	6	-2	-4	-9	-1	-1	-9	-3
		2	7	4	-3	0	-1	-6	4	5	-8	-9										

5	-6	9	-9	-5	-9	-5	5	-8	-5	-9	9	8	-5	-9	6	-2	-4	-9	-1	-1	-9	-3
			2	7	4	-3	0	-1	-6	4	5	-8	-9									

5	-6	9	-9	-5	-9	-5	5	-8	-5	-9	9	8	-5	-9	6	-2	-4	-9	-1	-1	-9	-3
				2	7	4	-3	0	-1	-6	4	5	-8	-9								

5	-6	9	-9	-5	-9	-5	5	-8	-5	-9	9	8	-5	-9	6	-2	-4	-9	-1	-1	-9	-3
					2	7	4	-3	0	-1	-6	4	5	-8	-9							

5	-6	9	-9	-5	-9	-5	5	-8	-5	-9	9	8	-5	-9	6	-2	-4	-9	-1	-1	-9	-3
						2	7	4	-3	0	-1	-6	4	5	-8	-9						

Applications of dot-product definition of matrix-vector multiplication: Audio search

Lots of dot-products!

- Represent as a matrix-vector product.
- One row per dot-product.

To search for $[0, 1, -1]$ in $[0, 0, -1, 2, 3, -1, 0, 1, -1, -1]$:

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 3 \\ 2 & 3 & -1 \\ 3 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} * [0, 1, -1]$$

Formulating a system of linear equations as a matrix-vector equation

Recall the *sensor node* problem:

- In each of several test periods, measure total power consumed:

$$\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$$

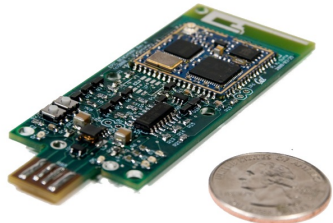
- For each test period, have a vector specifying how long each hardware component was operating during that period:

$$\mathbf{duration}_1, \mathbf{duration}_2, \mathbf{duration}_3, \mathbf{duration}_4, \mathbf{duration}_5$$

- Use measurements to calculate energy consumed per second by each hardware component.

Formulate as system of linear equations

$$\begin{aligned}
\mathbf{duration}_1 \cdot \mathbf{x} &= \beta_1 \\
\mathbf{duration}_2 \cdot \mathbf{x} &= \beta_2 \\
\mathbf{duration}_3 \cdot \mathbf{x} &= \beta_3 \\
\mathbf{duration}_4 \cdot \mathbf{x} &= \beta_4 \\
\mathbf{duration}_5 \cdot \mathbf{x} &= \beta_5
\end{aligned}$$



Formulating a system of linear equations as a matrix-vector equation

Linear equations

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= \beta_1 \\ \mathbf{a}_2 \cdot \mathbf{x} &= \beta_2 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= \beta_m \end{aligned}$$

Each equation specifies the value of a dot-product.

Rewrite as

$$\left[\begin{array}{c} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{array}\right] * \mathbf{x} = [\beta_1, \beta_2, \ldots, \beta_m]$$

Matrix-vector equation for sensor node

Define `D = {'radio', 'sensor', 'memory', 'CPU'}`.

Goal: Compute a `D`-vector $\mathbf{u}$ that, for each hardware component, gives the current drawn by that component.

Four test periods:

- total milliampere-seconds in these test periods $\mathbf{b} = [140, 170, 60, 170]$
- for each test period, vector specifying how long each hardware device was operating:
  - $\mathbf{duration}_1 =$ `Vec(D, 'radio':.1, 'CPU':.3)`
  - $\mathbf{duration}_2 =$ `Vec(D, 'sensor':.2, 'CPU':.4)`
  - $\mathbf{duration}_3 =$ `Vec(D, 'memory':.3, 'CPU':.1)`
  - $\mathbf{duration}_4 =$ `Vec(D, 'memory':.5, 'CPU':.4)`

To get $\mathbf{u}$, solve $A * \mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} \mathbf{duration}_1 \\ \hline \mathbf{duration}_2 \\ \hline \mathbf{duration}_3 \\ \hline \mathbf{duration}_4 \end{bmatrix}$

Triangular matrix

Recall: We considered *triangular* linear systems, e.g.

$$\begin{array}{rcl}
[\ 1, & 0.5, & -2, \quad 4\ ]\cdot\mathbf{x} = -8 \\
[\ 0, & 3, & 3, \quad 2\ ]\cdot\mathbf{x} = 3 \\
[\ 0, & 0, & 1, \quad 5\ ]\cdot\mathbf{x} = -4 \\
[\ 0, & 0, & 0, \quad 2\ ]\cdot\mathbf{x} = 6 \\
[\ 0, & 0, & 0, \quad 2\ ]\cdot\mathbf{x} = 6
\end{array}$$

We can rewrite this linear system as a matrix-vector equation:

$$\begin{bmatrix} 1 & 0.5 & -2 & 4 \\ 0 & 3 & 3 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix} * \mathbf{x} = [-8, 3, -4, 6]$$

The matrix is a *triangular* matrix.

Definition: An $n \times n$ *upper triangular* matrix A is a matrix with the property that $A_{ij} = 0$ for $j > i$. Note that the entries forming the triangle can be be zero or nonzero.

We can use backward substitution to solve such a matrix-vector equation.

Triangular matrices will play an important role later.

Computing sparse matrix-vector product

To compute matrix-vector or vector-matrix product,

- could use dot-product or linear-combinations definition.
  (You'll do that in homework.)
- However, using those definitions, it's not easy to exploit sparsity in the matrix.

"Ordinary" Definition of Matrix-Vector Multiplication: If M is an $R \times C$ matrix and $\mathbf{u}$ is a C-vector then $M * \mathbf{u}$ is the R-vector $\mathbf{v}$ such that, for each $r \in R$,

$$v[r] = \sum_{c \in C} M[r, c]u[c]$$

Computing sparse matrix-vector product

"Ordinary" Definition of Matrix-Vector Multiplication: If M is an $R \times C$ matrix and $\mathbf{u}$ is a C-vector then $M * \mathbf{u}$ is the R-vector $\mathbf{v}$ such that, for each $r \in R$,

$$v[r] = \sum_{c \in C} M[r, c]u[c]$$

Obvious method:

```
1 for i in R:
2     v[i] := sum_{j in C} M[i,j]u[j]
```

But this doesn't exploit sparsity!

Idea:

- Initialize output vector $\mathbf{v}$ to zero vector.
- Iterate over nonzero entries of M, adding terms according to ordinary definition.

```
1 initialize v to zero vector
2 for each pair (i,j) in sparse representation,
3     v[i] = v[i] + M[i,j]u[j]
```

Algebraic properties of matrix-vector multiplication

Proposition: Let A be an $R \times C$ matrix.

- For any C-vector $\mathbf{v}$ and any scalar α,

$$A * (\alpha \mathbf{v}) = \alpha (A * \mathbf{v})$$

- For any C-vectors $\mathbf{u}$ and $\mathbf{v}$,

$$A * (\mathbf{u} + \mathbf{v}) = A * \mathbf{u} + A * \mathbf{v}$$

Algebraic properties of matrix-vector multiplication

To prove

$$A * (\alpha \mathbf{v}) = \alpha (A * \mathbf{v})$$

we need to show corresponding entries are equal:

Need to show

$$\text{entry } i \text{ of } \quad A * (\alpha \mathbf{v}) = \text{entry } i \text{ of } \quad \alpha (A * \mathbf{v})$$

Proof: Write $A = \begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$.

By dot-product def. of matrix-vector mult,

$$\begin{aligned} \text{entry } i \text{ of } \quad A * (\alpha \mathbf{v}) &= \mathbf{a}_i \cdot \alpha \mathbf{v} \\ &= \alpha (\mathbf{a}_i \cdot \mathbf{v}) \end{aligned}$$

by homogeneity of dot-product

By definition of scalar-vector multiply,

$$\begin{aligned} \text{entry } i \text{ of } \alpha (A * \mathbf{v}) &= \alpha (\text{entry } i \text{ of } A * \mathbf{v}) \\ &= \alpha (\mathbf{a}_i \cdot \mathbf{v}) \end{aligned}$$

by dot-product definition of matrix-vector multiply

QED

Algebraic properties of matrix-vector multiplication

To prove

$$A * (\mathbf{u} + \mathbf{v}) = A * \mathbf{u} + A * \mathbf{v}$$

we need to show corresponding entries are equal:

Need to show

$$\text{entry } i \text{ of } \quad A * (\mathbf{u} + \mathbf{v}) = \text{entry } i \text{ of } \quad A * \mathbf{u} + A * \mathbf{v}$$

Proof: Write $A = \begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$.

By dot-product def. of matrix-vector mult,

$$\begin{aligned} \text{entry } i \text{ of } \quad A * (\mathbf{u} + \mathbf{v}) &= \mathbf{a}_i \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{a}_i \cdot \mathbf{u} + \mathbf{a}_i \cdot \mathbf{v} \end{aligned}$$

by distributive property of dot-product

By dot-product def. of matrix-vector mult,

$$\begin{aligned} \text{entry } i \text{ of } \quad A * \mathbf{u} &= \mathbf{a}_i \cdot \mathbf{u} \\ \text{entry } i \text{ of } \quad A * \mathbf{v} &= \mathbf{a}_i \cdot \mathbf{v} \end{aligned}$$

so

$$\text{entry } i \text{ of } \quad A * \mathbf{u} + A * \mathbf{v} = \mathbf{a}_i \cdot \mathbf{u} + \mathbf{a}_i \cdot \mathbf{v}$$

QED

Null space of a matrix

Definition: *Null space* of a matrix A is $\{\mathbf{u} \ : \ A * \mathbf{u} = \mathbf{0}\}$. Written *Null A*

Null space of a matrix

Definition: *Null space* of a matrix A is $\{\mathbf{u} \ : \ A * \mathbf{u} = \mathbf{0}\}$. Written *Null A*

Example:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \end{bmatrix} * [0, 0, 0] = [0, 0]$$

so the null space includes $[0, 0, 0]$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \end{bmatrix} * [6, -1, -1] = [0, 0]$$

so the null space includes $[6, -1, -1]$

Null space of a matrix

Definition: *Null space* of a matrix A is $\{\mathbf{u} \ : \ A * \mathbf{u} = \mathbf{0}\}$. Written *Null A*

By dot-product definition,

$$\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix} * \mathbf{u} = [\mathbf{a}_1 \cdot \mathbf{u}, \ \ldots, \ \mathbf{a}_m \cdot \mathbf{u}]$$

Thus $\mathbf{u}$ is in null space of $\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$ if and only if $\mathbf{u}$ is a solution to the homogeneous linear system

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= 0 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= 0 \end{aligned}$$

Null space of a matrix

We just saw:

Null space of a matrix $\left[\begin{array}{c} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{array}\right]$

equals the solution set of the homogeneous linear system $\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & 0 \end{array}$

This shows: *Null space of a matrix is a vector space.*

Can also show it directly, using algebraic properties of matrix-vector multiplication:

Property V1: Since $A * \mathbf{0} = \mathbf{0}$,the null space of A contains $\mathbf{0}$

Property V2: if $\mathbf{u} \in$ Null A then $A * (\alpha\,\mathbf{u}) = \alpha\,(A * \mathbf{u}) = \alpha\,\mathbf{0} = \mathbf{0}$ so $\alpha\mathbf{u} \in$ Null A

Property V3: If $\mathbf{u} \in$ Null A and $\mathbf{v} \in$ Null A
then $A * (\mathbf{u} + \mathbf{v}) = A * \mathbf{u} + A * \mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$
so $\mathbf{u} + \mathbf{v} \in$ Null A

Null space of a matrix

Definition: *Null space* of a matrix A is $\{\mathbf{u} \ : \ A * \mathbf{u} = \mathbf{0}\}$. Written *Null A*

Proposition: Null space of a matrix is a vector space.

Example:

$$\text{Null} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \end{bmatrix} = \text{Span } \{[6, -1, -1]\}$$

Solution space of a matrix-vector equation

Earlier, we saw:

If $\mathbf{u}_1$ is a solution to the linear system $\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & \beta_1 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & \beta_m \end{array}$

then the solution set is $\mathbf{u}_1 + \mathcal{V}$,

where $\mathcal{V}$ = solution set of $\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & 0 \end{array}$

Restated: If $\mathbf{u}_1$ is a solution to $A * \mathbf{x} = \mathbf{b}$ then solution set is $\mathbf{u}_1 + \mathcal{V}$ where $\mathcal{V} = \text{Null } A$

Solution space of a matrix-vector equation

Proposition: If $\mathbf{u}_1$ is a solution to $A * \mathbf{x} = \mathbf{b}$ then solution set is $\mathbf{u}_1 + \mathcal{V}$ where $\mathcal{V} = \text{Null } A$

Example:

- Null space of $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \end{bmatrix}$ is Span $\{[6, -1, -1]\}$.
- One solution to $\boxed{\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \end{bmatrix} * \mathbf{x} = [1, 1]}$ is $x = [-1, 1, 0]$.
- Therefore solution set is $[-1, 1, 0] + \text{Span } \{[6, -1, -1]\}$
- For example, solutions include
  - $[-1, 1, 0] + [0, 0, 0]$
  - $[-1, 1, 0] + [6, -1, -1]$
  - $[-1, 1, 0] + 2\,[6, -1, -1]$

  $\vdots$

Solution space of a matrix-vector equation

Proposition: If $\mathbf{u}_1$ is a solution to $A * \mathbf{x} = \mathbf{b}$ then solution set is $\mathbf{u}_1 + \mathcal{V}$
where $\mathcal{V} = \text{Null } A$

- If $\mathcal{V}$ is a trivial vector space then $\mathbf{u}_1$ is the only solution.
- If $\mathcal{V}$ is not trivial then $\mathbf{u}_1$ is *not* the only solution.

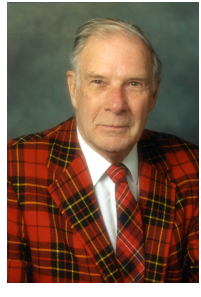
Corollary: $A * \mathbf{x} = \mathbf{b}$ has at most one solution iff Null A is a trivial vector space.

Question: How can we tell if the null space of a matrix is trivial?

Answer comes later...

Error-correcting codes

- Originally inspired by errors in reading programs on punched cards
- Now used in WiFi, cell phones, communication with satellites and spacecraft, digital television, RAM, disk drives, flash memory, CDs, and DVDs

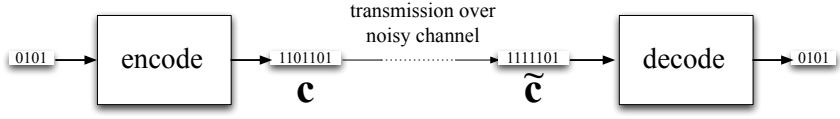


Richard Hamming

Hamming code is a *linear binary block code*:

- *linear* because it is based on linear algebra,
- *binary* because the input and output are assumed to be in binary, and
- *block* because the code involves a fixed-length sequence of bits.

Error-correcting codes: Block codes



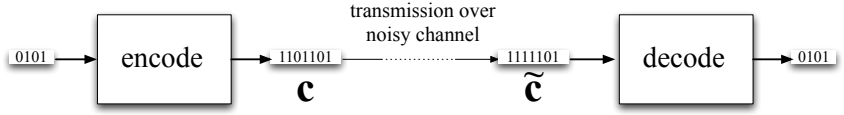
To protect an 4-bit block:

- Sender *encodes* 4-bit block as a 7-bit block $\mathbf{c}$
- Sender transmits $\mathbf{c}$
- $\mathbf{c}$ passes through noisy channel—errors might be introduced.
- Receiver receives 7-bit block $\tilde{\mathbf{c}}$
- Receiver tries to figure out original 4-bit block

The 7-bit encodings are called *codewords*.

$\mathcal{C}$ = set of permitted codewords

Error-correcting codes: Linear binary block codes



Hamming's first code is a *linear* code:

- Represent 4-bit and 7-bit blocks as 4-vectors and 7-vectors over $GF(2)$.
- 7-bit block received is $\tilde{\mathbf{c}} = \mathbf{c} + \mathbf{e}$
- $\mathbf{e}$ has 1's in positions where noisy channel flipped a bit
  ($\mathbf{e}$ is the *error vector*)
- *Key idea:* set $\mathcal{C}$ of codewords is the null space of a matrix H.

This makes Receiver's job easier:

- Receiver has $\tilde{\mathbf{c}}$, needs to figure out $\mathbf{e}$.
- Receiver multiplies $\tilde{\mathbf{c}}$ by H.

$$H * \tilde{\mathbf{c}} = H * (\mathbf{c} + \mathbf{e}) = H * \mathbf{c} + H * \mathbf{e} = \mathbf{0} + H * \mathbf{e} = H * \mathbf{e}$$

- Receiver must calculate $\mathbf{e}$ from the value of $H * \mathbf{e}$. *How?*

Hamming Code

In the Hamming code, the codewords are 7-vectors, and

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Notice anything special about the columns and their order?

- Suppose that the noisy channel introduces at most one bit error.
- Then $\mathbf{e}$ has only one 1.
- Can you determine the position of the bit error from the matrix-vector product $H * \mathbf{e}$?

Example: Suppose $\mathbf{e}$ has a 1 in its third position, $\mathbf{e} = [0, 0, 1, 0, 0, 0, 0]$.

Then $H * \mathbf{e}$ is the third column of H, which is $[0, 1, 1]$.

As long as $\mathbf{e}$ has at most one bit error, the position of the bit can be determined from $H * \mathbf{e}$. This shows that the Hamming code allows the recipient to correct one-bit errors.

Hamming code

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Quiz: Show that the Hamming code does not allow the recipient to correct two-bit errors: give two different error vectors, $\mathbf{e}_1$ and $\mathbf{e}_2$, each with at most two 1's, such that $H * \mathbf{e}_1 = H * \mathbf{e}_2$.

Hamming code

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Quiz: Show that the Hamming code does not allow the recipient to correct two-bit errors: give two different error vectors, $\mathbf{e}_1$ and $\mathbf{e}_2$, each with at most two 1's, such that $H * \mathbf{e}_1 = H * \mathbf{e}_2$.

Answer: There are many acceptable answers. For example, $\mathbf{e}_1 = [1, 1, 0, 0, 0, 0, 0]$ and $\mathbf{e}_2 = [0, 0, 1, 0, 0, 0, 0]$ or $\mathbf{e}_1 = [0, 0, 1, 0, 0, 1, 0]$ and $\mathbf{e}_2 = [0, 1, 0, 0, 0, 0, 1]$.

Matrices and their functions

Now we study the relationship between a matrix M and the function $\mathbf{x} \mapsto M * \mathbf{x}$

- *Easy:* Going from a matrix M to the function $\mathbf{x} \mapsto M * \mathbf{x}$
- *A little harder:* Going from the function $\mathbf{x} \mapsto M * \mathbf{x}$ to the matrix M.

In studying this relationship, we come up with the fundamental notion of a *linear function*.

From matrix to function

Starting with a M, define the function $f(\mathbf{x}) = M * x$.

Domain and co-domain?
If M is an $R \times C$ matrix over $\mathbb{F}$ then

- domain of f is $\mathbb{F}^C$
- co-domain of f is $\mathbb{F}^R$

Example: Let M be the matrix

	#	@	?
a	1	2	3
b	10	20	30

and define $f(\mathbf{x}) = M * \mathbf{x}$

- Domain of f is $\mathbb{R}^{\{\#,@,?\}}$.
- Co-domain of f is $\mathbb{R}^{\{a,b\}}$.

f maps

#	@	?
2	2	-2

to

a	b
0	0

Example: Define $f(\mathbf{x}) = \begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix} * \mathbf{x}$.

- Domain of f is $\mathbb{R}^3$
- Co-domain of f is $\mathbb{R}^2$

f maps $[2, 2, -2]$ to $[0, 0]$

From function to matrix

We have a function $f : \mathbb{F}^A \longrightarrow \mathbb{F}^B$

We want to compute matrix M such that $f(\mathbf{x}) = M * \mathbf{x}$.

- Since the domain is $\mathbb{F}^A$, we know that the input $\mathbf{x}$ is an A-vector.
- For the product $M * \mathbf{x}$ to be legal, we need the column-label set of M to be A.
- Since the co-domain is $\mathbb{F}^B$, we know that the output $f(\mathbf{x}) = M * \mathbf{x}$ is B-vector.
- To achieve that, we need row-label set of M to be B.

Now we know that M must be a $B \times A$ matrix....

... but what about its entries?

From function to matrix

- We have a function $f : \mathbb{F}^n \longrightarrow \mathbb{F}^m$
- We think there is an $m \times n$ matrix M such that $f(\mathbf{x}) = M * \mathbf{x}$

How to go from the function f to the entries of M?

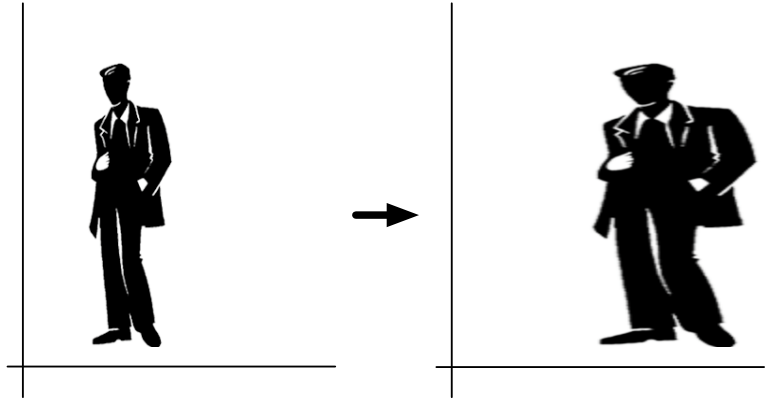
- Write mystery matrix in terms of its columns: $M = \left[\begin{array}{c|c|c} & & \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ & & \end{array}\right]$
- Use standard generators $\mathbf{e}_1 = [1, 0, \ldots, 0, 0], \ldots, \mathbf{e}_n = [0, \ldots, 0, 1]$ with *linear-combinations* definition of matrix-vector multiplication:

$$f(\mathbf{e}_1) = \left[\begin{array}{c|c|c} & & \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ & & \end{array}\right] * [1, 0, \ldots, 0, 0] = \mathbf{v}_1$$

$$\vdots$$

$$f(\mathbf{e}_n) = \left[\begin{array}{c|c|c} & & \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ & & \end{array}\right] * [0, 0, \ldots, 0, 1] = \mathbf{v}_n$$

From function to matrix: horizontal scaling

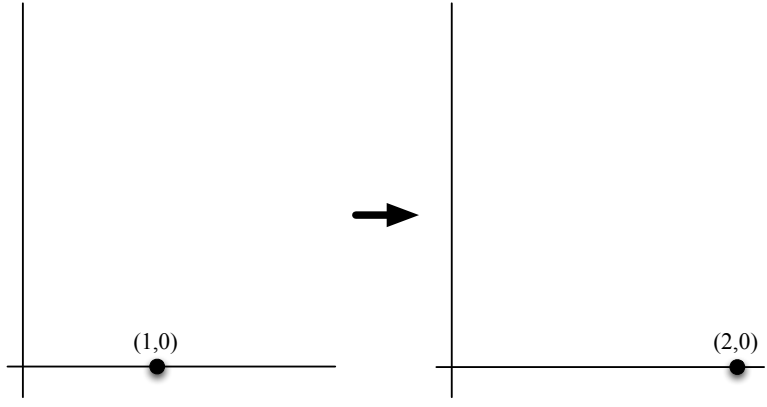


Define $s([x, y]) =$ stretching by two in horizontal direction
Assume $s([x, y]) = M * [x, y]$ for some matrix M.

- We know $s([1, 0]) = [2, 0]$ because we are stretching by two in horizontal direction
- We know $s([0, 1]) = [0, 1]$ because no change in vertical direction.

Therefore $M = \left[\begin{array}{c|c} 2 & 0 \\ 0 & 1 \end{array}\right]$

From function to matrix: horizontal scaling



Define $s([x, y]) =$ stretching by two in horizontal direction
Assume $s([x, y]) = M * [x, y]$ for some matrix M.

- We know $s([1, 0]) = [2, 0]$ because we are stretching by two in horizontal direction
- We know $s([0, 1]) = [0, 1]$ because no change in vertical direction.

Therefore $M = \left[\begin{array}{c|c} 2 & 0 \\ 0 & 1 \end{array}\right]$

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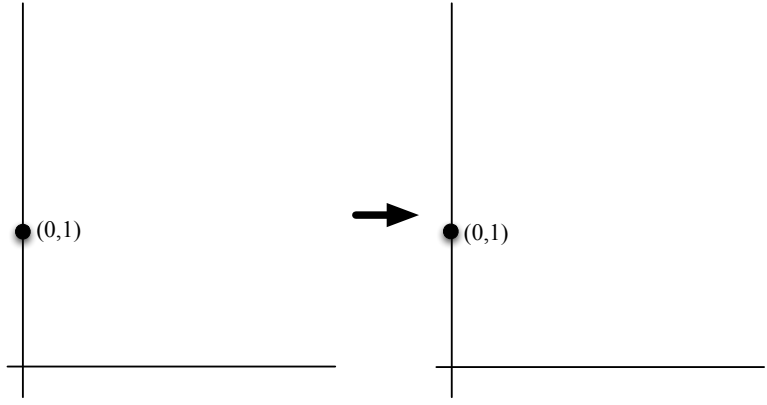

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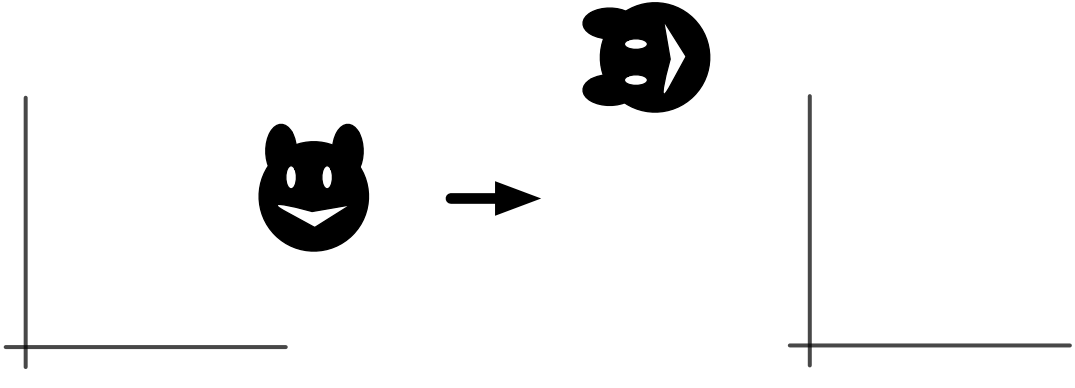
From function to matrix: rotation by 90 degrees

Define $r([x, y])$ = rotation by 90 degrees
Assume $r([x, y]) = M * [x, y]$ for some matrix M.

- We know rotating $[1, 0]$ should give $[0, 1]$ so $r([1, 0]) = [0, 1]$
- We know rotating $[0, 1]$ should give $[-1, 0]$ so $r([0, 1]) = [-1, 0]$

Therefore $M = \left[\begin{array}{c|c} 0 & -1 \\ 1 & 0 \end{array}\right]$



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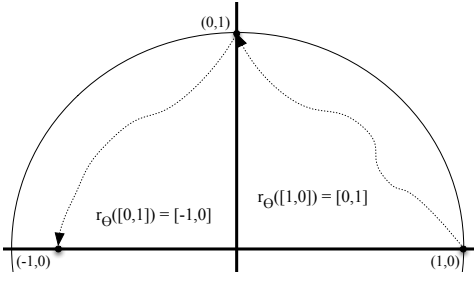
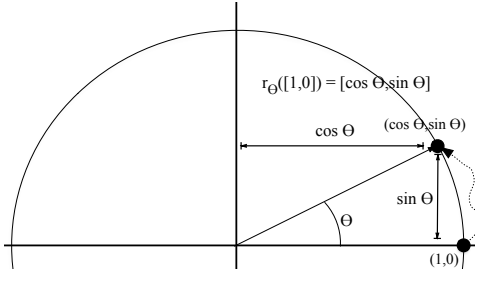


From function to matrix: rotation by θ degrees

Define $r([x, y])$ = rotation by θ.
Assume $r([x, y]) = M * [x, y]$ for some matrix M.

- We know $r([1, 0]) = [\cos\theta, \sin\theta]$ so column 1 is $[\cos\theta, \sin\theta]$
- We know $r([0, 1]) = [-\sin\theta, \cos\theta]$ so column 2 is $[-\sin\theta, \cos\theta]$

Therefore $M = \left[\begin{array}{c|c} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$



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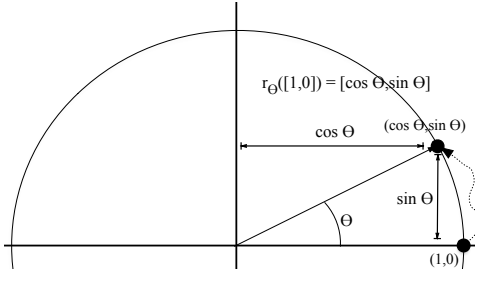

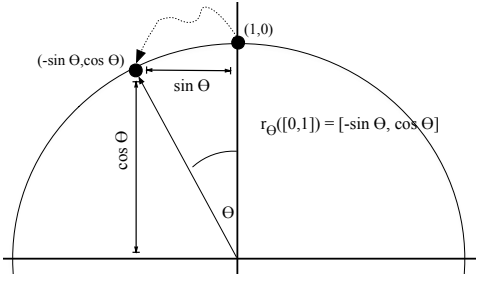

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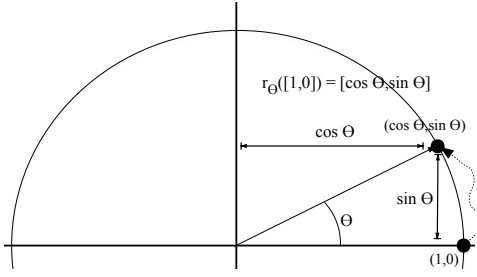

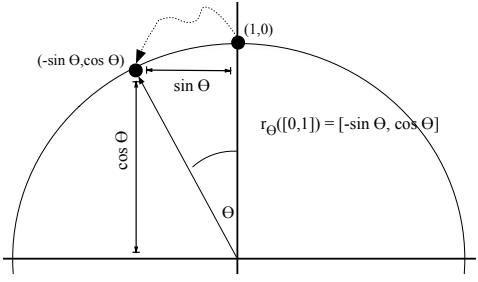

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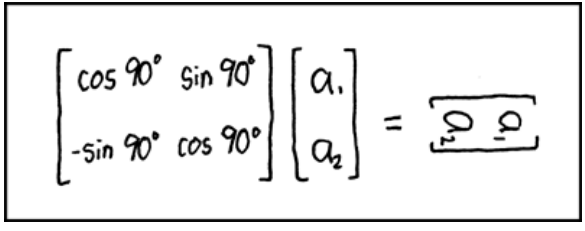
For clockwise rotation by 90 degrees, plug in θ = -90 degrees...



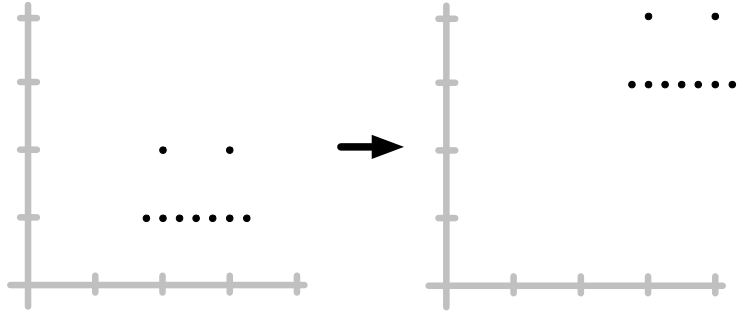
Matrix Transform (`http://xkcd.com/824`)

From function to matrix: translation

$t([x, y])$ = translation by $[1, 2]$. Assume $t([x, y]) = M * [x, y]$ for some matrix M.

- We know $t([1, 0]) = [2, 2]$ so column 1 is $[2, 2]$.
- We know $t([0, 1]) = [1, 3]$ so column 2 is $[1, 3]$.

Therefore $M = \left[\begin{array}{c|c} 2 & 1 \\ 2 & 3 \end{array}\right]$



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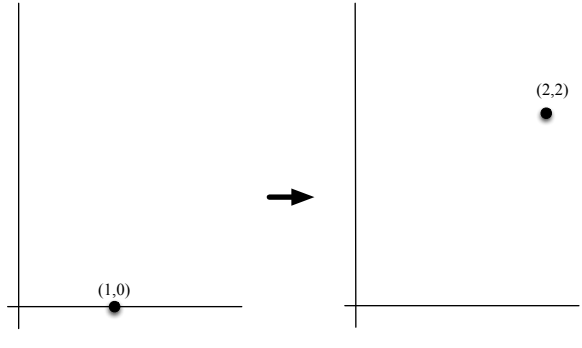

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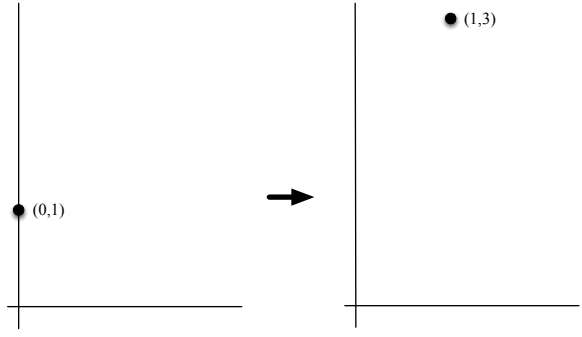


From function to matrix: identity function

Consider the function $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined by $f(\mathbf{x}) = \mathbf{x}$
This is the identity function on $\mathbb{R}^4$.

Assume $f(\mathbf{x}) = M * \mathbf{x}$ for some matrix M.

Plug in the standard generators
$\mathbf{e}_1 = [1, 0, 0, 0], \mathbf{e}_2 = [0, 1, 0, 0], \mathbf{e}_3 = [0, 0, 1, 0], \mathbf{e}_4 = [0, 0, 0, 1]$

- $f(\mathbf{e}_1) = \mathbf{e}_1$ so first column is $\mathbf{e}_1$
- $f(\mathbf{e}_2) = \mathbf{e}_2$ so second column is $\mathbf{e}_2$
- $f(\mathbf{e}_3) = \mathbf{e}_3$ so third column is $\mathbf{e}_3$
- $f(\mathbf{e}_4) = \mathbf{e}_4$ so fourth column is $\mathbf{e}_4$

So $M = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Identity function $f(\mathbf{x})$ corresponds to identity matrix $\mathbb{1}$

Diagonal matrices

Let $d_1, \ldots, d_n$ be real numbers. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the function such that $f([x_1, \ldots, x_n]) = [d_1 x_1, \ldots, d_n x_n]$. The matrix corresponding to this function is

$$\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

Such a matrix is called a *diagonal* matrix because the only entries allowed to be nonzero form a diagonal.

Definition: For a domain D, a $D \times D$ matrix M is a *diagonal* matrix if $M[r, c] = 0$ for every pair $r, c \in D$ such that $r \neq c$.

Special case: $d_1 = \cdots = d_n = 1$. In this case, $f(\mathbf{x}) = \mathbf{x}$ (*identity function*)

The matrix $\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$ is an identity matrix.

Linear functions: Which functions can be expressed as a matrix-vector product?

In each example, we *assumed* the function could be expressed as a matrix-vector product.

How can we verify that assumption?

We'll state two algebraic properties.

- If a function can be expressed as a matrix-vector product $\mathbf{x} \mapsto M * \mathbf{x}$, it has these properties.
- If the function from $\mathbb{F}^C$ to $\mathbb{F}^R$ has these properties, it can be expressed as a matrix-vector product.

Linear functions: Which functions can be expressed as a matrix-vector product?

Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over a field $\mathbb{F}$.

Suppose a function $f : \mathcal{V} \longrightarrow \mathcal{W}$ satisfies two properties:

Property L1: For every vector $\mathbf{v}$ in $\mathcal{V}$ and every scalar α in $\mathbb{F}$,

$$f(\alpha\,\mathbf{v}) = \alpha\, f(\mathbf{v})$$

Property L2: For every two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{V}$,

$$f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$$

We then call f a *linear function*.

Proposition: Let M be an $R \times C$ matrix, and suppose $f : \mathbb{F}^C \mapsto \mathbb{F}^R$ is defined by $f(\mathbf{x}) = M * \mathbf{x}$. Then f is a linear function.

Proof: Certainly $\mathbb{F}^C$ and $\mathbb{F}^R$ are vector spaces.

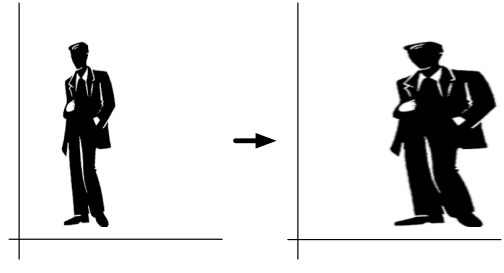
We showed that $M * (\alpha\,\mathbf{v}) = \alpha\, M * \mathbf{v}$. This proves that f satisfies Property L1.

We showed that $M * (\mathbf{u}+\mathbf{v}) = M * \mathbf{u} + M * \mathbf{v}$. This proves that f satisfies Property L2.

QED

Which functions are linear?

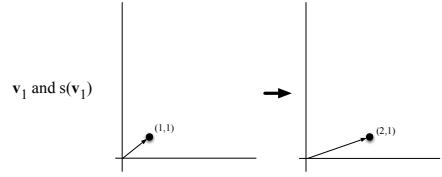
Define $s([x, y]) =$ stretching by two in horizontal direction



Which functions are linear?

Define $s([x, y]) =$ stretching by two in horizontal direction

Property L1: $s(\mathbf{v}_1 + \mathbf{v}_2) = s(\mathbf{v}_1) + s(\mathbf{v}_2)$



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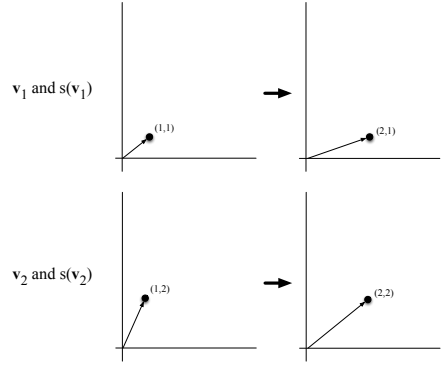

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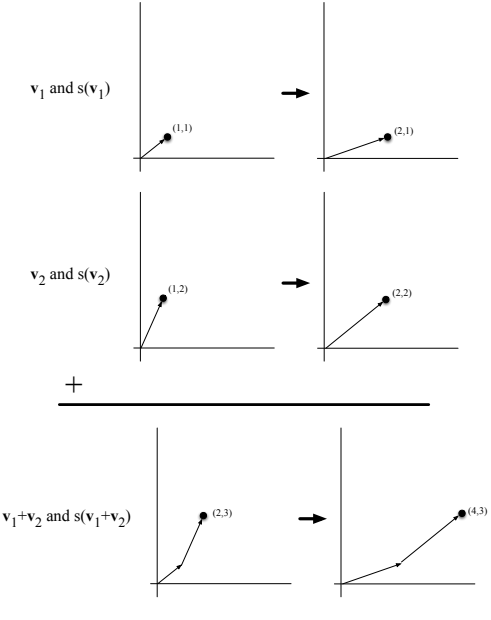

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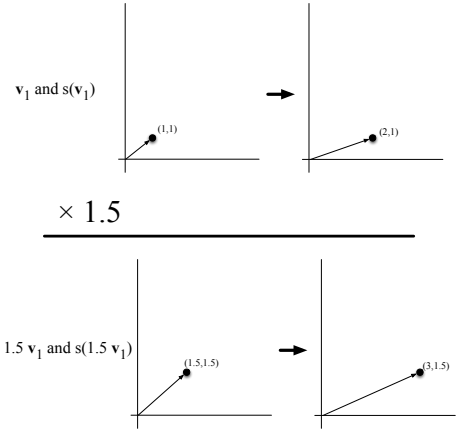
Define $s([x, y]) =$ stretching by two in horizontal direction

Property L1: $s(\mathbf{v}_1 + \mathbf{v}_2) = s(\mathbf{v}_1) + s(\mathbf{v}_2)$

Property L2: $s(\alpha\, \mathbf{v}) = \alpha\, s(\mathbf{v})$



Since the function $s(\cdot)$ satisfies Properties L1 and L2, it is a linear function.

Similarly can show rotation by θ degrees is a linear function.

What about translation?

$$t([x, y]) = [x, y] + [1, 2]$$

This function violates Property L1. For example:

$$t([4, 5] + [2, -1]) = t([6, 4]) = [7, 6]$$

but

$$t([4, 5]) + t([2, -1]) = [5, 7] + [3, 1] = [8, 8]$$

Since $t(\cdot)$ violates Property L1 for at least one input, it is not a linear function.

Can similarly show that $t(\cdot)$ does not satisfy Property L2.

A linear function maps zero vector to zero vector

Lemma: If $f : \mathcal{U} \longrightarrow \mathcal{V}$ is a linear function then f maps the zero vector of $\mathcal{U}$ to the zero vector of $\mathcal{V}$.

Proof: Let $\mathbf{0}$ denote the zero vector of $\mathcal{U}$, and let $\mathbf{0}_{\mathcal{V}}$ denote the zero vector of $\mathcal{V}$.

$$f(\mathbf{0}) = f(\mathbf{0} + \mathbf{0}) = f(\mathbf{0}) + f(\mathbf{0})$$

Subtracting $f(\mathbf{0})$ from both sides, we obtain

$$\mathbf{0}_{\mathcal{V}} = f(\mathbf{0})$$

QED

Linear functions: Pushing linear combinations through the function

Defining properties of linear functions:

Property L1: $f(\alpha\,\mathbf{v}) = \alpha\, f(\mathbf{v})$

Property L2: $f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$

Proposition: For a linear function f,
for any vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in the domain of f and any scalars $\alpha_1, \ldots, \alpha_n$,

$$f(\alpha_1\,\mathbf{v}_1 + \cdots + \alpha_n\,\mathbf{v}_n) = \alpha_1\, f(\mathbf{v}_1) + \cdots + \alpha_n\, f(\mathbf{v}_n)$$

Proof: Consider the case of $n = 2$.

$$\begin{aligned} f(\alpha_1\,\mathbf{v}_1 + \alpha_2\,\mathbf{v}_2) &= f(\alpha_1\,\mathbf{v}_1) + f(\alpha_2\,\mathbf{v}_2) && \text{by Property L2} \\ &= \alpha_1\, f(\mathbf{v}_1) + \alpha_2\, f(\mathbf{v}_2) && \text{by Property L1} \end{aligned}$$

Proof for general n is similar. QED

Linear functions: Pushing linear combinations through the function

Proposition: For a linear function f,
$f(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n) = \alpha_1 f(\mathbf{v}_1) + \cdots + \alpha_n f(\mathbf{v}_n)$

Example: $f(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \mathbf{x}$

Verify that $f(10\,[1,-1] + 20\,[1,0]) = 10\,f([1,-1]) + 20\,f([1,0])$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left(10\,[1,-1] + 20\,[1,0] \right)$$
$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left([10,-10] + [20,0] \right)$$
$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} [30,-10]$$
$$= 30\,[1,3] - 10[2,4]$$
$$= [30,90] - [20,40]$$
$$= [10,50]$$

$$10 \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * [1,-1] \right) + 20 \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * [1,0] \right)$$
$$= 10\,([1,3] - [2,4]) + 20\,(1[1,3])$$
$$= 10\,[-1,-1] + 20\,[1,3]$$
$$= [-10,-10] + [20,60]$$
$$= [10,50]$$

From function to matrix, revisited

We saw a method to derive a matrix from a function:
Given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, we want a matrix M such that $f(\mathbf{x}) = M * \mathbf{x}$....

- Plug in the standard generators $\mathbf{e}_1 = [1, 0, \ldots, 0, 0], \ldots, \mathbf{e}_n = [0, \ldots, 0, 1]$
- Column i of M is $f(\mathbf{e}_i)$.

This works correctly whenever such a matrix M really exists:

Proof: If there is such a matrix then f is linear:

- (Property L1) $f(\alpha\,\mathbf{v}) = \alpha\, f(\mathbf{v})$ and
- (Property L2) $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$

Let $\mathbf{v} = [\alpha_1, \ldots, \alpha_n]$ be any vector in $\mathbb{R}^n$.
We can write $\mathbf{v}$ in terms of the standard generators.

$$
\begin{aligned}
\mathbf{v} &= \alpha_1\,\mathbf{e}_1 + \cdots + \alpha_n\,\mathbf{e}_n \\
\text{so} & \\
f(\mathbf{v}) &= f(\alpha_1\,\mathbf{e}_1 + \cdots + \alpha_n\,\mathbf{e}_n) \\
&= \alpha_1\, f(\mathbf{e}_1) + \cdots + \alpha_n\, f(\mathbf{e}_n) \\
&= \alpha_1\,(\text{column 1 of } M) + \cdots + \alpha_n\,(\text{column} n \text{ of } M) \\
&= M * \mathbf{v} \textit{QED}
\end{aligned}
$$

Linear functions and zero vectors: Kernel

Definition: *Kernel* of a linear function f is $\{\mathbf{v} \ : \ f(\mathbf{v}) = \mathbf{0}\}$

Written Ker f

For a function $f(\mathbf{x}) = M * \mathbf{x}$,

$$\text{Ker } f = \text{Null } M$$

Kernel and one-to-one

One-to-One Lemma: A linear function is one-to-one if and only if its kernel is a trivial vector space.

Proof: Let $f : \mathcal{U} \longrightarrow \mathcal{V}$ be a linear function. We prove two directions.

- Suppose Ker f contains some nonzero vector $\mathbf{u}$, so $f(\mathbf{u}) = \mathbf{0}_{\mathcal{V}}$.
  Because a linear function maps zero to zero, $f(\mathbf{0}) = \mathbf{0}_{\mathcal{V}}$ as well,
  so f is not one-to-one.

- Suppose Ker $f = \{\mathbf{0}\}$.
  Let $\mathbf{v}_1, \mathbf{v}_2$ be any vectors such that $f(\mathbf{v}_1) = f(\mathbf{v}_2)$.
  Then $f(\mathbf{v}_1) - f(\mathbf{v}_2) = \mathbf{0}_{\mathcal{V}}$
  so, by linearity, $f(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}_{\mathcal{V}}$,
  so $\mathbf{v}_1 - \mathbf{v}_2 \in$ Ker f.
  Since Ker f consists solely of $\mathbf{0}$,
  it follows that $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$, so $\mathbf{v}_1 = \mathbf{v}_2$.

QED

Kernel and one-to-one

One-to-One Lemma A linear function is one-to-one if and only if its kernel is a trivial vector space.

Define the function $f(\mathbf{x}) = A * \mathbf{x}$.

If Ker f is trivial (i.e. if Null A is trivial)

then a vector $\mathbf{b}$ is the image under f of at most one vector.

That is, at most one vector $\mathbf{u}$ such that $A * \mathbf{u} = \mathbf{b}$
That is, the solution set of $A * \mathbf{x} = \mathbf{b}$ has at most one vector.

Linear functions that are onto?

Question: How can we tell if a linear function is onto?

Recall: for a function $f : \mathcal{V} \longrightarrow \mathcal{W}$, the *image* of f is the set of all images of elements of the domain:

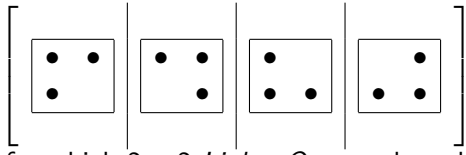
$$\{f(\mathbf{v}) \ : \mathbf{v} \in \mathcal{V}\}$$

(You might know it as the "range" but we avoid that word here.)

The image of function f is written Im f

"Is function f is onto?" same as "is Im f = co-domain of f?"

Example: *Lights Out*

Define $f([\alpha_1, \alpha_2, \alpha_3, \alpha_4]) =$  $* [\alpha_1, \alpha_2, \alpha_3, \alpha_4]$

Im f is set of configurations for which 2×2 *Lights Out* can be solved,
so "f is onto" means "2×2 *Lights Out* can be solved for every configuration"

Can 2×2 *Lights Out* be solved for every configuration? What about 5×5?
Each of these questions amounts to asking whether a certain function is onto.

Linear functions that are onto?

"Is function f is onto?" same as "is Im f = co-domain of f?"

First step in understanding how to tell if a linear function f is onto:

- study the image of f

Proposition: The image of a linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$ is a vector space

Matrix-matrix multiplication

If

- A is a $R \times S$ matrix, and
- B is a $S \times T$ matrix

then it is legal to multiply A times B.

- In Mathese, written AB
- In our `Mat` class, written `A*B`

AB is different from BA.

In fact, one product might be legal while the other is illegal.

Matrix-matrix multiplication

We'll see two equivalent definitions:

- one in terms of vector-matrix multiplication,
- one in terms of matrix-vector multiplication.

Matrix-matrix multiplication: vector-matrix definition

Vector-matrix definition of matrix-matrix multiplication:
For each row-label r of A,

$$\text{row } r \text{ of } AB = \underbrace{(\text{row } r \text{ of } A)}_{\text{vector}} * B$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 2 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \\ B \\ \\ \end{array}\right] = \left[\begin{array}{c} [1,0,0] * B \\ \hline [2,1,0] * B \\ \hline [0,0,1] * B \end{array}\right]$$

How to interpret $[1,0,0] * B$?

- *Linear combinations* definition of vector-matrix multiplication?
- *Dot-product* definition of vector-matrix multiplication?

Each is correct.

Matrix-matrix multiplication: vector-matrix interpretation

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 2 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \\ \mathrm{B} \\ \\ \end{array}\right] = \left[\begin{array}{c} [1,0,0] * B \\ \hline [2,1,0] * B \\ \hline [0,0,1] * B \end{array}\right]$$

How to interpret $[1,0,0] * B$? *Linear combinations* definition:

$$[1,0,0] * \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right] = \mathbf{b}_1 \qquad [0,0,1] * \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right] = \mathbf{b}_3$$

$$[2,1,0] * \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right] = 2\,\mathbf{b}_1 + \mathbf{b}_2$$

Conclusion:

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 2 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right] = \left[\begin{array}{c} \mathbf{b}_1 \\ \hline 2\,\mathbf{b}_1 + \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right]$$

Matrix-matrix multiplication: vector-matrix interpretation

Conclusion:

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 2 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right] = \left[\begin{array}{c} \mathbf{b}_1 \\ \hline 2\,\mathbf{b}_1 + \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right]$$

We call $\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 2 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$ an *elementary row-addition matrix*.

Matrix-matrix multiplication: matrix-vector definition

Matrix-vector definition of matrix-matrix multiplication:
For each column-label s of B,

$$\text{column } s \text{ of } AB = A * (\text{column } s \text{ of } B)$$

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $B =$ matrix with columns $[4,3]$, $[2,1]$, and $[0,-1]$

$$B = \left[\begin{array}{c|c|c} 4 & 2 & 0 \\ 3 & 1 & -1 \end{array}\right]$$

AB is the matrix with column $i = A * (\text{ column } i \text{ of } B)$

$A * [4,3] = [10,-1]$ $\quad$ $A * [2,1] = [4,-1]$ $\quad$ $A * [0,-1] = [-2,-1]$

$$AB = \left[\begin{array}{c|c|c} 10 & 4 & -2 \\ -1 & -1 & -1 \end{array}\right]$$

Matrix-matrix multiplication: Dot-product definition

Combine

- *matrix-vector* definition of matrix-matrix multiplication, and
- *dot-product* definition of matrix-vector multiplication

to get...

Dot-product definition of matrix-matrix multiplication:
Entry rc of AB is the dot-product of row r of A with column c of B.

Example:

$$\left[\begin{array}{ccc} 1 & 0 & 2 \\ \hline 3 & 1 & 0 \\ \hline 2 & 0 & 1 \end{array}\right] \left[\begin{array}{c|c} 2 & 1 \\ 5 & 0 \\ 1 & 3 \end{array}\right] = \begin{bmatrix} [1,0,2]\cdot[2,5,1] & [1,0,2]\cdot[1,0,3] \\ [3,1,0]\cdot[2,5,1] & [3,1,0]\cdot[1,0,3] \\ [2,0,1]\cdot[2,5,1] & [2,0,1]\cdot[1,0,3] \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 11 & 3 \\ 5 & 5 \end{bmatrix}$$

Matrix-matrix multiplication: transpose

$$(AB)^T = B^T A^T$$

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 19 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 19 \\ 4 & 8 \end{bmatrix}$$

You might think "$(AB)^T = A^T B^T$" but this is **false**.
In fact, doesn't even make sense!

- For AB to be legal, A's column labels $=$ B's row labels.
- For $A^T B^T$ to be legal, A's row labels $=$ B's column labels.

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$ is legal but $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ is not.

Matrix-matrix multiplication: Column vectors

Multiplying a matrix A by a one-column matrix B

$$\begin{bmatrix} & A & \\ & & \end{bmatrix} \begin{bmatrix} \\ \mathbf{b} \\ \\ \end{bmatrix}$$

By matrix-vector definition of matrix-matrix multiplication, result is matrix with one column: $A * \mathbf{b}$

This shows that matrix-vector multiplication is subsumed by matrix-matrix multiplication.

Convention: Interpret a vector $\mathbf{b}$ as a one-column matrix ("column vector")

- Write vector $[1, 2, 3]$ as $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
- Write $A * [1, 2, 3]$ as $\begin{bmatrix} & A & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ or $A\,\mathbf{b}$

Matrix-matrix multiplication: Row vectors

If we interpret vectors as one-column matrices.... what about vector-matrix multiplication?

Use transpose to turn a column vector into a row vector: Suppose $\mathbf{b} = [1, 2, 3]$.

$$[1, 2, 3] * A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} & A & \\ & & \end{bmatrix} = \mathbf{b}^T A$$

Inner product

Let $\mathbf{u}$ and $\mathbf{v}$ be two D-vectors interpreted as matrices (column vectors).
Matrix-matrix product $\mathbf{u}^T\mathbf{v}$.

Example: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \end{bmatrix}$

- First "matrix" has one row.
- Second "matrix" has one column.
- Therefore product "matrix" has one entry.

By dot-product definition of matrix-matrix multiplication,
that one entry is the dot-product of $\mathbf{u}$ and $\mathbf{v}$.

Sometimes called *inner product* of matrices.
However, that term has taken on another meaning, which we study later.

Outer product

Another way to multiply vectors as matrices.
For any $\mathbf{u}$ and $\mathbf{v}$, consider $\mathbf{u}\mathbf{v}^T$.

Example:
$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 & u_1 v_4 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 & u_2 v_4 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 & u_3 v_4 \end{bmatrix}$$

For each element s of the domain of $\mathbf{u}$ and each element t of the domain of $\mathbf{v}$, the s, t element of $\mathbf{u}\mathbf{v}^T$ is $\mathbf{u}[s]\ \mathbf{v}[t]$.

Called *outer product* of $\mathbf{u}$ and $\mathbf{v}$.

Matrix-matrix multiplication and function composition

Corresponding to an $R \times C$ matrix A over a field $\mathbb{F}$, there is a function

$$f : \mathbb{F}^C \longrightarrow \mathbb{F}^R$$

namely the function defined by $f(\mathbf{y}) = A * \mathbf{y}$

Matrix-matrix multiplication and function composition

Matrices A and $B \Rightarrow$ functions $f(\mathbf{y}) = A * \mathbf{y}$ and $g(\mathbf{x}) = B * \mathbf{x}$ and $h(\mathbf{x}) = (AB) * \mathbf{x}$

Matrix-Multiplication Lemma $f \circ g = h$

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix}$$

product $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

corresponds to function $h\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2, x_1 + x_2 \end{bmatrix}$

$$f \circ g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} \text{ so } f \circ g = h$$

Matrix-matrix multiplication and function composition

Matrices A and $B \Rightarrow$ functions $f(\mathbf{y}) = A * \mathbf{y}$ and $g(\mathbf{x}) = B * \mathbf{x}$ and $h(\mathbf{x}) = (AB) * \mathbf{x}$

Matrix-Multiplication Lemma $f \circ g = h$

Proof: Let columns of B be $\mathbf{b}_1, \ldots, \mathbf{b}_n$. By the matrix-vector definition of matrix-matrix multiplication, column j of AB is $A *$ (column j of B).
For any n-vector $\mathbf{x} = [x_1, \ldots, x_n]$,

$$\begin{aligned} g(\mathbf{x}) &= B * \mathbf{x} && \text{by definition of } g \\ &= x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n && \text{by linear combinations definition} \end{aligned}$$

Therefore

$$\begin{aligned} f(g(\mathbf{x})) &= f(x_1\mathbf{b}_1 + \cdots x_n\mathbf{b}_n) \\ &= x_1(f(\mathbf{b}_1)) + \cdots + x_n(f(\mathbf{b}_n)) && \text{by linearity of } f \\ &= x_1(A * \mathbf{b}_1) + \cdots + x_n(A * \mathbf{b}_n) && \text{by definition of } f \\ &= x_1(\text{column 1 of } AB) + \cdots + x_n(\text{column } n \text{ of } AB) && \text{by matrix-vector def.} \\ &= (AB) * \mathbf{x} && \text{by linear-combinations def.} \\ &= h(\mathbf{x}) && \text{by definition of } h \end{aligned}$$

QED

Associativity of matrix-matrix multiplication

Matrices A and $B \Rightarrow$ functions $f(\mathbf{y}) = A * \mathbf{y}$ and $g(\mathbf{x}) = B * \mathbf{x}$ and $h(\mathbf{x}) = (AB) * \mathbf{x}$

Matrix-Multiplication Lemma $f \circ g = h$

Matrix-matrix multiplication corresponds to function composition.

Corollary: Matrix-matrix multiplication is associative:

$$(AB)C = A(BC)$$

Proof: Function composition is associative. QED

Example:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 1 & 7 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 1 & 7 \end{bmatrix}$$

From function inverse to matrix inverse

Matrices A and $B \Rightarrow$ functions $f(\mathbf{y}) = A * \mathbf{y}$ and $g(\mathbf{x}) = B * \mathbf{x}$ and $h(\mathbf{x}) = (AB) * \mathbf{x}$

Definition If f and g are functional inverses of each other, we say A and B are matrix inverses of each other.

Example: An elementary row-addition matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{function } f([x_1, x_2, x_3]) = [x_1, x_2 + 2x_1, x_3])$$

Function adds twice the first entry to the second entry.

Functional inverse: subtracts the twice the first entry from the second entry:

$$f^{-1}([x_1, x_2, x_3]) = [x_1, x_2 - 2x_1, x_3]$$

Thus the inverse of A is

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix is also an elementary row-addition matrix.

Matrix inverse

If A and B are matrix inverses of each other, we say A and B are *invertible* matrices.

Can show that a matrix has at most one inverse.

We denote the inverse of matrix A by A^{-1}.

(A matrix that is not invertible is sometimes called a *singular* matrix, and an invertible matrix is called a *nonsingular* matrix.)

Invertible matrices: why care?

Reason 1: Existence and uniqueness of solution to matrix-vector equations.
Let A be an $m \times n$ matrix, and define $f : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ by $f(\mathbf{x}) = A\mathbf{x}$
Suppose A is an invertible matrix. Then f is an invertible function. Then f is one-to-one and onto:

- Since f is onto, for any m-vector $\mathbf{b}$ there is *some* vector $\mathbf{u}$ such that $f(\mathbf{u}) = \mathbf{b}$. That is, there is at least one solution to the matrix-vector equation $A\mathbf{x} = \mathbf{b}$.
- Since f is one-to-one, for any m-vector $\mathbf{b}$ there is *at most one* vector $\mathbf{u}$ such that $f(\mathbf{u}) = \mathbf{b}$. That is, there is at most one solution to $A\mathbf{x} = \mathbf{b}$.

If A is invertible then, for every right-hand side vector $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has *exactly one solution*.

Example 1: Industrial espionage. Given the vector $\mathbf{b}$ specifying the amount of each resource consumed, figure out quantity of each product JunkCo has made.
Solve vector-matrix equation $\mathbf{x}^T M = \mathbf{b}$ where

$M =$

	metal	concrete	plastic	water	electricity
garden gnome	0	1.3	.2	.8	.4
hula hoop	0	0	1.5	.4	.3
slinky	.25	0	0	.2	.7
silly putty	0	0	.3	.7	.5
salad shooter	.15	0	.5	.4	.8

Invertible matrices: why care?

Reason 1: Existence and uniqueness of solution to matrix-vector equations.
If A is invertible then, for every right-hand side vector $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has *exactly one solution*.

Example 1: Industrial espionage. Given the vector $\mathbf{b}$ specifying the amount of each resource consumed, figure out quantity of each product JunkCo has made.
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silly putty	0	0	.3	.7	.5
salad shooter	.15	0	.5	.4	.8

Will this work for every vector $\mathbf{b}$?

- Is there a unique solution? If multiple solutions then we cannot be certain we have calculated true quantities.

Since M^T is an invertible matrix, the function $f(\mathbf{x}) = \mathbf{x} * M$ is an invertible function, so there is a unique solution for every vector $\mathbf{b}$.

Example 2: Sensor node with hardware components {radio,sensor,CPU,memory}. Use three test periods

Invertible matrices: why care?

Reason 1: Existence and uniqueness of solution to matrix-vector equations.
If A is invertible then, for every right-hand side vector $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has *exactly one solution*.

Example 2: Sensor node with hardware components {radio,sensor,CPU,memory}.
Use three test periods

- total power consumed in these test periods $\mathbf{b} = [140, 170, 60]$
- for each test period, vector says how long each hardware component worked:
  - $\mathbf{duration}_1 =$ `Vec(D, 'radio':0.1, 'CPU':0.3)`
  - $\mathbf{duration}_2 =$ `Vec(D, 'sensor':0.2, 'CPU':0.4)`
  - $\mathbf{duration}_3 =$ `Vec(D, 'memory':0.3, 'CPU':0.1)`

To get $\mathbf{u}$, solve $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} \mathbf{duration}_1 \\ \hline \mathbf{duration}_2 \\ \hline \mathbf{duration}_3 \end{bmatrix}$$

Does this yield current draw for each hardware component?

- The matrix A is *not* invertible.
- In particular, the function $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one.
- Therefore no guarantee that solution to $A\mathbf{x} = \mathbf{b}$ is unique
  $\Rightarrow$ we can't derive power per hardware component.
- Need to add more test periods....

Use **four** test periods

- total power consumed in these test periods $\mathbf{b} = [140, 170, 60, 170, 250]$

Invertible matrices: why care?

Reason 1: Existence and uniqueness of solution to matrix-vector equations.
If A is invertible then, for every right-hand side vector $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has *exactly one solution*.
Example 2: Sensor node with hardware components {radio,sensor,CPU,memory}.
Use **four** test periods

- total power consumed in these test periods $\mathbf{b} = [140, 170, 60, 170, 250]$
- for each test period, vector says how long each hardware component worked:
  - $\mathbf{duration}_1 =$ `Vec(D, 'radio':0.1, 'CPU':0.3)`
  - $\mathbf{duration}_2 =$ `Vec(D, 'sensor':0.2, 'CPU':0.4)`
  - $\mathbf{duration}_3 =$ `Vec(D, 'memory':0.3, 'CPU':0.1)`
  - $\mathbf{duration}_4 =$ `Vec(D, 'memory':0.5, 'CPU':0.4)`

To get $\mathbf{u}$, solve $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} \mathbf{duration}_1 \\ \hline \mathbf{duration}_2 \\ \hline \mathbf{duration}_3 \\ \hline \mathbf{duration}_4 \end{bmatrix}$$

Does this yield current draw for each hardware component?

- This time the matrix A *is* invertible...
- so equation has exactly one solution.
- We can in principle find power consumption per component.

Invertible matrices: why care?

Reason 2: Algorithms for solving matrix-vector equation $A\mathbf{x} = \mathbf{b}$ are simpler if we can assume A is invertible.

Later we learn two such algorithms.

We also learn how to cope if A is not invertible.

Reason 3:
Invertible matrices play a key role in *change of basis*.

Change of basis is important part of linear algebra

- used e.g. in image compression;
- we will see it used in adding/removing perspective from an image.

Product of invertible matrices is an invertible matrix

Proposition: Suppose the matrix product AB is defined. Then AB is an invertible matrix if and only both A and B are invertible matrices.

Example:

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ correspond to functions

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$

f is an invertible function.

$$g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix}$$

g is an invertible function.

The functions f and g are invertible so the function $f \circ g$ is invertible.

By the Matrix-Multiplication Lemma, the function $f \circ g$ corresponds to the matrix product $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ so that matrix is invertible.

Example:

Product of invertible matrices is an invertible matrix

Proposition: Suppose the matrix product AB is defined. Then AB is an invertible matrix if and only both A and B are invertible matrices.

Example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$$

Multiplication by matrix A adds 4 times first element to second element:

$f([x_1, x_2, x_3]) = [x_1, x_2 + 4x_1, x_3])$

This function is invertible.

Multiplication by matrix B adds 5 times first element to third element:

$g([x_1, x_2, x_3]) = [x_1, x_2, x_3 + 5x_1]$

This function is invertible

By Matrix Multiplication Lemma, multiplication by matrix AB corresponds to composition of functions $f \circ g$: $(f \circ g)([x_1, x_2, x_3]) = [x_1, x_2 + 4x_1, x_3 + 5x_1]$

The function $f \circ g$ is also an invertible function.... so AB is an invertible matrix.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Product of invertible matrices is an invertible matrix

Proposition: Suppose the matrix product AB is defined. Then AB is an invertible matrix if and only both A and B are invertible matrices.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

The product is $AB = \begin{bmatrix} 4 & 5 & 1 \\ 10 & 11 & 4 \\ 16 & 17 & 7 \end{bmatrix}$

which is *not* invertible

so at least one of A and B is not invertible

and in fact $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is *not* invertible.

Proof: Define the functions f and g by $f(\mathbf{x}) = A\mathbf{x}$ and $g(\mathbf{x}) = B\mathbf{x}$.

- Suppose A and B are invertible matrices.
  Then the corresponding functions f and g are invertible.
  Therefore $f \circ g$ is invertible
  so the matrix corresponding to $f \circ g$ (which is AB) is an invertible matrix.

Product of invertible matrices is an invertible matrix

Proposition: Suppose the matrix product AB is defined. Then AB is an invertible matrix if and only both A and B are invertible matrices.

Proof: Define the functions f and g by $f(\mathbf{x}) = A\mathbf{x}$ and $g(\mathbf{x}) = B\mathbf{x}$.

- Suppose A and B are invertible matrices.
  Then the corresponding functions f and g are invertible.
  Therefore $f \circ g$ is invertible
  so the matrix corresponding to $f \circ g$ (which is AB) is an invertible matrix.

- Conversely, suppose AB is an invertible matrix.
  Then the corresponding function $f \circ g$ is an invertible function.
  It follows that f and g must be invertible functions,
  so the corresponding matrices A and B must be invertible matrices.

QED

Matrix inverse

Lemma: If the $R \times C$ matrix A has an inverse A^{-1} then AA^{-1} is the $R \times R$ identity matrix.

Proof: Let $B = A^{-1}$. Define $f(\mathbf{x}) = A\mathbf{x}$ and $g(\mathbf{y}) = B\mathbf{y}$.

- By the Matrix-Multiplication Lemma, $f \circ g$ satisfies $(f \circ g)(\mathbf{x}) = AB\mathbf{x}$.
- On the other hand, $f \circ g$ is the identity function,
- so AB is the $R \times R$ identity matrix.

QED

Matrix inverse

Lemma: If the $R \times C$ matrix A has an inverse A^{-1} then AA^{-1} is identity matrix.

What about the converse?

Conjecture: If AB is an indentity matrix then A and B are inverses...?

Counterexample:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A * [0, 0, 1]$ and $A * [0, 0, 0]$ both equal $[0, 0]$, so null space of A is not trivial,
so the function $f(\mathbf{x}) = A\mathbf{x}$ is not one-to-one,
so f is not an invertible function.

Shows: $AB = I$ is *not* sufficient to ensure that A and B are inverses.

Matrix inverse

Lemma: If the $R \times C$ matrix A has an inverse A^{-1} then AA^{-1} is identity matrix.

What about the converse?

FALSE Conjecture: If AB is an indentity matrix then A and B are inverses...?

Corollary: Matrices A and B are inverses of each other if and only if both AB and BA are identity matrices.

Matrix inverse

Lemma: If the $R \times C$ matrix A has an inverse A^{-1} then AA^{-1} is identity matrix.

Corollary: A and B are inverses of each other iff both AB and BA are identity matrices.

Proof:

- Suppose A and B are inverses of each other. By lemma, AB and BA are identity matrices.
- Suppose AB and BA are both identity matrices.
  Define $f(\mathbf{y}) = A * \mathbf{y}$ and $g(\mathbf{x}) = B * \mathbf{x}$
  - Because AB is identity matrix, by Matrix-Multiplication Lemma, $f \circ g$ is the identity function.
  - Because BA is identity matrix, by Matrix-Multiplication Lemma, $g \circ f$ is the identity function.
  - This proves that f and g are functional inverses of each other, so A and B are matrix inverses of each other.

QED

Matrix inverse

Question: How can we tell if a matrix M is invertible?

Partial Answer: By definition, M is an invertible matrix if the function $f(\mathbf{x}) = M\mathbf{x}$ is an invertible function, i.e. if the function is one-to-one and onto.

- *One-to-one:* Since the function is linear, we know by the One-to-One Lemma that the function is one-to-one if its kernel is trivial, i.e. if the null space of M is trivial.
- *Onto:* We haven't yet answered the question *how we can tell if a linear function is onto?*

If we knew how to tell if a linear function is onto, therefore, we would know how to tell if a matrix is invertible.