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1) $A x = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = b$

$$Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$s = \frac{x_2}{\sqrt{x_1^2 + y_2^2}} = \frac{1}{\sqrt{2}} \qquad c = \frac{x_1}{\sqrt{x_1^2 + y_2^2}} = \frac{1}{\sqrt{2}}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = QR \qquad R = Q^T A$$

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$$

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$$

$x = ?$

$$A x = b \qquad Q^T A x = Q^T b = y$$

$$y = Q^T b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$R x = y$

$$\cancel{\frac{1}{\sqrt{2}}} \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \cancel{\frac{1}{\sqrt{2}}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$x_2 = 1$

$x_1 = -1$

$$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

2) $Ax = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = b$

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$r_{11} = \|v_1\|_2 = \sqrt{2}$

$q_1 = \frac{v_1}{r_{11}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$r_{12} = \langle q_1 ; v_2 \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{5}{\sqrt{2}}$

$\hat{q}_2 = v_2 - q_1 r_{12} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{5}{\sqrt{2}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$r_{22} = \|\hat{q}\|_2 = \sqrt{\frac{1}{4} + \frac{1}{4}} = \cancel{\sqrt{\frac{58}{16}}} = \cancel{\frac{\sqrt{58}}{4}} \ \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$q_2 = \frac{\hat{q}_2}{r_{22}} = \cancel{\begin{bmatrix} \frac{3}{4} \\ \frac{7}{4} \end{bmatrix} \cdot \frac{4}{\sqrt{58}} = \frac{1}{\sqrt{58}} \begin{bmatrix} 3 \\ 7 \end{bmatrix}} \ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \cdot \sqrt{2} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \frac{5}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

x tak jak w 1)

3) $Ax = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = b$

$A = QR \Rightarrow R = Q^T A$

$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\tau = \|x\|_2 \qquad \tau = \sqrt{1^2+1^2} = \sqrt{2} \qquad x_1 = 1$

$\gamma = \frac{x_1 + \tau}{\tau} = \frac{1+\sqrt{2}}{\sqrt{2}}$

$Y = Qx = Q\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\tau \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}$

$U = \frac{x - Y}{x_1 + \tau} = \frac{1}{1+\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{1+\sqrt{2}} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{1+\sqrt{2}} \end{bmatrix}$

$Q = I - \gamma U U^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1+\sqrt{2}}{\sqrt{2}} \begin{bmatrix} 1 \\ \frac{1}{1+\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{1+\sqrt{2}} \end{bmatrix} =$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1+\sqrt{2}}{\sqrt{2}} \begin{bmatrix} 1 & \frac{1}{1+\sqrt{2}} \\ \frac{1}{1+\sqrt{2}} & \frac{1}{1+\sqrt{2}} \cdot \frac{1}{1+\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{2}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}(1+\sqrt{2})} \end{bmatrix}$

$= \begin{bmatrix} \frac{\sqrt{2} - 1 - \sqrt{2}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}+2-1}{\sqrt{2}(1+\sqrt{2})} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}+1}{\sqrt{2}(1+\sqrt{2})} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$