Question 3.1.3: 1. Compute the $\sup, \inf, \limsup, \liminf$, and all the limit points of the following sequences $x_1, x_2, \cdots$ where

a. $x_n = 1/n + (-1)^n$

b. $x_n = 1 + (-1)^n/n$

c. $x_n = (-1)^n + 1/n + 2\sin(n\pi/2)$.

Proof.

a. $\lim = \{1, -1\}, \limsup = 1, \liminf = -1, \sup = 3/2, \inf = -1$

b. $\lim = \{1\}, \limsup = 1, \liminf = 1, \sup = 3/2, \inf = 0$

c. $\lim = \{1, -3\}, \limsup = 1, \liminf = -3, \sup = 2, \inf = -3$

□

Question 3.1.3: 2. If a bounded sequence is the sum of a monotone increasing and a monotone decreasing sequence ($x_n = y_n + z_n$ where $\{y_n\}$ is monotone increasing and $\{z_n\}$ is monotone decreasing) does it follow that the sequence converges? What if $\{y_n\}$ and $\{z_n\}$ are bounded?

Proof. Take $(y_n) = 1/2(-1)^n + n$ and $(z_n) = 1/2(-1)^n - n$. Then $(x_n) = (-1)^n$, which is bounded but, not convergent.

Now if $\{y_n\}$ and $\{z_n\}$ are bounded and monotone, they converge and thus have a limit. Since sum of limits is a limit of sums we know that (x_n) will converge.

□

Question 3.1.3: 3. If E is a set and y a point that is the limit of two sequences, $\{x_n\}$ and $\{y_n\}$ such that x_n is in E and y_n is an upper bound for E, prove that $y = \sup E$. Is the converse true?

Proof. Proof by contradiction. Let's assume $x \in E,$ such that $x > y$. Since $\lim y_n = y$, where $x - y < 1/n$, then $\exists m,$ such that $|y_j - y| \leq 1/n$ for $k \geq m$. Then we would have $x > y_0 > y$, but since y_n is an upper bound of E, which implies $y_0 \geq x$, which is a contradiction. Therefore y is an upper bound.

Suppose another contradiction that there is another upper bound $z,$ such that $z < y$. So we have $x_n < z, \forall n$ But in the limit we have $x_n = y$ and therefore $y < z$, which is a contradiction. So $y = \sup E$.

Converse is not true as (x_n) does not have to be in E. □

Question 3.1.3: 4. Prove $\sup(A \cup B) \geq \sup A$ and $\sup(A \cap B) \leq \sup A$.

Proof. By the definition of supremum, there is a unique supremum for a given set. Therefore $\sup(A \cup B)$ can be re-written as $\max\{\sup A, \sup B\} \geq \sup A$. $\square$

Similarly, since an infimum is unique for a given set we can rewrite $\sup(A \cap B)$ as $\min\{\sup A, \sup B\}$. $\square$

Question 3.1.3: 8. Write out the proof that ∞ is a limit-point of $\{x_n\}$ if and only if there exists a subsequence whose limit is ∞.

Proof. Infinity is a limit point of a sequence, when a sequence is divergent. Therefore we can rewrite the statement as: A sequence is divergent $\iff$ there exists a subsequence which is divergent.

- $\Rightarrow$ Suppose a sequence is divergent, then by taking a subsequence (removing terms), the sequence will stay divergent.
- $\Leftarrow$ Suppose a subsequence is divergent, then any sequence, in which it is contained is divergent.

$\square$

Question 3.1.3: 9. Can there exist a sequence whose set of limit points is exactly $\{1, 1/2, 1/3, \cdots\}$? (Hint: what is the lim inf of the sequence?)

Proof. The $\liminf = 0$. Take a subsequence x_{j_r}, converging to $1/n_0$. Then for any $m, \exists n,$ such that $|x_{j_r} - 1/n_0| < 1/n$ for any $r \geq m$. Then given any $p, \exists n,$ such that $1/n + 1/n_0 \geq 1/p$. So for this $n \exists m,$ such that $|x_{j_r}| = |x_{j_r} - 1/n_0 + 1/n_0| \leq |x_{j_r} - 1/n_0| + |1/n_0| \leq 1/n + 1/n_0 \leq 1/p$ for any $r \geq m$. So the subsequence is convergent to 0, which therefore has to be limit. However 0 is not contained in the set and therefore the desired sequence cannot exist. $\square$

Question 3.1.3: 11. Consider a sequence obtained by diagonalizing a rectangular array. Prove that any limit-point of any row or column of the array is a limit-point of the sequence. Do you necessarily get all limit-points this way?

Proof. As a rectangular array is symmetric, only rows to columns need to be proved. Let's define a row as a sequence (a_i) Suppose we have $x = \lim a_{1i}$. By Theorem 3.1.3 x is also the limit of a subsequence a_{1i_r} Now this subsequence of a row is also a subsequence of the diagonal sequence $a_{11}, a_{21}, a_{12}, a_{31}$ of which x is a limit point by Theorem 3.1.3. $\square$

However we do not necessarily get all the limit points this way. For example a matrix of 1's, whose main diagonal is a sequence $1/n$.

Question 3.2.3: 1. Let A be an open set. Show that if a finite number of points are removed from A, the remaining set is still open. Is the same true if a countable number of points are removed?

Proof. The new set can be rewritten as:

$$A - \{x_1, x_2, x_3, \cdots, x_k\} = A \cap ((-\infty, x_1) \cup (x_1, x_2) \cup \cdots (x_k, \infty)) \equiv A \cap B$$

As B is an union of open sets and therefore open $\Rightarrow A \cap B$ is also open. However for a countable (infinite) number of elements we consider $\mathbb{R} - \mathbb{Q}$ and a point x in it. Then for every n we can find $y \in \mathbb{Q} \in (x - 1/n, x + 1/n)$ and so we cannot find a neighbourhood around x completely in $\mathbb{R} - \mathbb{Q}$ and therefore $\mathbb{R} - \mathbb{Q}$ is not open. $\square$