Question 5.1.3: 1. Show that $f(x) = O(|x - x_0|^2)$ as $x \to x_0$ implies $f(x) = o(|x - x_0|)$ as $x \to x_0$, but give an example to show that the converse is not true.

Proof. From $f(x) = O(|x - x_0|^2)$ we have $|f(x)| \leq C|x - x_0|^2$ for $|x - x_0| < \delta$ for some $C, \delta > 0$, then:

$$0 \leq \frac{|f(x)}{|x - x_0|} \leq C|x - x_0|$$

As $x \to x_0, |x - x_0| \to 0$, we have (almost like a squeeze thm):

$$\lim_{x \to x_0} \frac{|f(x)|}{|x - x_0|} = 0 \Rightarrow f(x) = o(|x - x_0|) \qquad \square$$

To show that converse is not true take $|x|^{3/2}$ which satisfies $f(x) = o(|x|)$ as $x \to x_0$, but doesn't $f(x) = O(|x^2|)$

Question 5.1.3: 5. Show that if $f(x_0) = 0$ and $f(x) = o(|x - x_0|)$ as $x \to x_0$, then $f'(x_0)$ exits. What is $f'(x_0)$? What does this tell you about $x^2 \sin(1/x^{1000})$?

Proof. Since f is differentiable at $x_0 \iff \exists k \in R$, such that $f(x) = f(x_0) + k(x - x_0) + o(x - x_0)$ as $x \to x_0$ and we know $f(x_0) = 0$, $f(x) = o(|x - x_0|)$ so we have:

$$|f(x)| \leq |f(x_0)| + |k(x - x_0)| + |o(x - x_0)|$$

$$|f(x)| \leq |k(x - x_0)| + \frac{|f(x)|}{|x - x_0|}$$

$$|f(x)||x - x_0| \leq |k||(x - x_0)|^2 + |f(x)|$$

when $x \to x_0$ this reduces to $0 \leq |f(x)|$, which is true and therefore f' is differentiable at x_0. $\square$

We have that $f'(x) = 0$ and in case of $x^2 \sin(1/x^{1000})$ we get $f(0) = 0, f(x) = o(|x|)$ as $x \to 0$ and $f'(x) = 0$.

Question 5.2.4: 1. Let f and g be continuous functions on $[a, b]$ and differentiable at every point in the interior, with $g(a) \neq g(b)$. Prove that there exists a point x_0 in (a, b) such that:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

(Hint: apply the mean value theorem to the function $(f(b) - f(a))g(x) - (g(b) - g(a))f(x)$.) This is sometimes called the second mean value theorem.

Proof. Define $y(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$ and evaluate $y(b), y(a)$.

$$y(b) = [f(b) - f(a)]g(b) - [g(b) - g(a)]f(b)$$

$$y(a) = [f(b) - f(a)]g(a) - [g(b) - g(a)]f(a)$$

$$y(b) = f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)f(b)$$
$$y(a) = f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a)$$
$$y(b) = -f(a)g(b) + g(a)f(b)$$
$$y(a) = f(b)g(a) - g(b)f(a)$$
$$y(a) = y(b)$$

From Rolle's theorem we know that since $y(a) = y(b)$, $\exists c \in (a, b)$ where $y'(c) = 0$. We can then take the derivative of y.

$$y'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x) = 0$$
$$y'(x) = [f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$
$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}$$ □

Question 5.2.4: 2. If f is a function satisfying

$$|f(x) - f(y)| \leq M|x - y|^\alpha$$

for all x and y and some fixed M and $\alpha > 1$, prove that f is constant. (Hint: what is f'?) It is rumored that a graduate student once wrote a whole thesis on the class of functions satisfying this condition!

Proof. From the definition of derivative we have:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Since f has to satisfy $|f(x) - f(y)| \leq M|x - y|^\alpha$ we have:

$$0 \leq |\frac{f(x + h) - f(x)}{|h|} \leq M\frac{|h|^\alpha}{|h|} = M|h|^{\alpha - 1|}$$

Now as $h \to 0$, $M|h|^{\alpha-1|} \to 0$ the derivative is bounded from both sides by 0. and therefore is also equal to 0. $f'(x) = 0 \Rightarrow f(x) = c$. □

Question 5.2.4: 6. Show that if f is differentiable and $f'(x) > 0$ on (a, b), then f is strictly increasing provided there is no subinterval (c, d) with $c < d$ on which f' is identically zero.

Proof. Since $f'(x) > 0$ and from the definition of derivative we have

$$0 < f'(x_0) = \frac{f(x) - f(x_0)}{x - x_0} > 0$$

As x is from the real numbers we have $c < d$ and $d - c > 0$. Therefore since $f'(x)$ is always greater than 0:

$$f(d) - f(c) > 0 \Rightarrow f(d) > f(c)$$ □

Question 5.2.4: 8. Suppose f is continuously differentiable on an interval (a, b). Prove that on any closed subinterval $[c, d]$ the function is uniformly differentiable in the sense that given any $1/n$ there exists $1/m$ (independent of x_0) such that:

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq |x - x_0|/n$$

whenever $|x - x_0| < 1/m$. (Hint: use the mean value theorem and the uniform continuity of f' on $[c, d]$.)

Proof. If f is continuously differentiable on (a, b), then f' is continuous on (a, b) and thus it is uniformly continuous on any closed interval $[c, d] \in (a, b)$. So we have that for any $1/n > 0, \exists m > 0,$ such that $|f'(x) - f'(x_0)| < 1/n$ as long as $x, x_0 \in [c, d]$ and $|x - x_0| < 1/m$. From the mean value theorem we have:

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| = f'(a)(x - x_0) - f'(x_0)(x - x_0)| =$$

$$= |f'(a) - f'(x_0)||x - x_0| \leq \frac{|x - x_0|}{n} \Rightarrow |a - x_0| < 1/m$$

Therefore f is uniformly differentiable. □

Question 5.2.4: 11. Prove that if f' is constant, then f is an affine function.

Proof. f' constant $\Rightarrow f'(x) = c$ so by MVT:

$$f(x) = f(x_0) + f'(y)(x - x_0) = f(x_0) + c(x - x_0)$$

$\Rightarrow f$ is affine. □

Question 5.3.4: 8. For any rational number r give a definition of $f(x) = x^r$ for $x > 0$ and show $f'(x) = rx^{r-1}$.

Proof. A rational number r can be represented as $r = \frac{p}{q}$, where $p, q \in \mathbb{Z}, q \neq 0$. So we have $f(x) = x^{\frac{p}{q}} = (x^p)^{1/q}$. Let's first take the case, where $p = 1$ Then by the inverse function theorem we have:

$$f'(x) = \frac{1}{(x^q)'} = \frac{1}{qx^{q-1}} = \frac{1}{q(x)^{1/q})^{q-1}} = \frac{1}{q(x)^{1-1/q}}$$

$$f'(x) = \frac{1}{q}(x)^{1/q-1}$$

So we have obtained the power-law for any rational $1/q$. We can now consider the the general case $f(x) = x^{\frac{p}{q}}$. By the chain rule we have:

$$f'(x) = (x^p)'\frac{1}{q}(x^p)^{1/q-1} = \frac{p}{q}(x)^{p/q-1} \Rightarrow f'(x) = rx^{r-1}$$ □