Andrzej Novak

Math 405 Homework 2

Question 1.2.3: 3. Prove that the rational numbers are countable. (Hint: they can be written as the union over $k \in N$ of the sets $Q_k = j/k : j \in N$.)

Proof. Cartesian product of countable sets is countable and $Q_k = \{j/k : j \in \mathbb{Z}, k \in N\}$. $\forall q \in \mathbb{Q}, \exists (j,k) \in \mathbb{Z} \times \mathbb{N}$ such that $q = j/k$. Therefore there exists an injection from $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{N}$. And since both $\mathbb{Z}$ and $\mathbb{N}$ are countable, their Cartesian product is countable and so is $\mathbb{Q}$. □

Question 1.2.3: 6. Let A be a set for which there exists a function $f : A \longrightarrow \mathbb{N}$ with the property that $\forall k \in \mathbb{N}$, the subset of A given by the solutions to f(a) = k is finite. Show that A is finite or countable.

Proof. If $\forall k \in N, \exists B \subset A$, which is finite. Then A is the union of countable number of finite sets, which makes it countable. Since by axiom of choice a countable union of finite sets is countable. □

Question 2.1.3: 3. What kinds of real numbers are representable by Cauchy sequences of integers?

Proof. Assume there is a Cauchy sequence of integers that corresponds to a $q \in \mathbb{Q}$. Then from the definition $\forall n, \exists m$ such that $|x_j - x_k| \leq 1/n, \forall j,k \geq m$. However, since $x_j - x_k \in \mathbb{Z}$ and $|x_j - x_k| \leq 1/n \leq 1$ then $x_j - x_k = 0$ for $j,k \geq m$. So the sequence eventually converges to a constant, which is an integer. Therefore Cauchy sequences of integers can only represent real numbers that are integers. □

Question 2.1.3: 5. Prove that if a Cauchy sequence x_1, x_2, ... of rationals is modified by changing a finite number of terms, the result is an equivalent Cauchy sequence.

Proof. Let there be a modified sequence $y_1, y_2, \ldots.$ Suppose that l is the largest index of the modified elements so that for $n \geq l+1, x_n = y_n$. Now because $\forall n, \exists m = l+1$, s.t. $|x_k - y_k| = 0 \leq 1/n$ for $k \geq m$, the sequences are equivalent. Now by transitive property if the first sequence is Cauchy, then the second is too. □

Question 2.1.3: 7. Show that the Cauchy sequence .9, .99, .999, . . . is equivalent to 1,1,1, ...

Proof. If x_n and y_n are two Cauchy sequences, then they are defined to be equal as real numbers if the sequence $(x_n - y_n)$ has the limit of 0. Since $(x_n - y_n) = (1 - 0.9, 1 - 0.99, 1 - 0.999, \ldots) = (0.1, 0.01, 0.001, \ldots) = (\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots) = \lim_{x \to \infty} \frac{1}{10^n} = 0$ the two sequences are equivalent. □

Question 2.1.3: 8. Can a Cauchy sequence of positive rational numbers be equivalent to a Cauchy sequence of negative rational numbers?

Proof. Suppose we have a positive sequence {1/x} and a negative sequence {-1/x}, which both have a limit of 0. To show that they are Cauchy we need to satisfy $\forall n > 0, \exists m$ such that $\forall j, k \geq m, |1/j - 1/k| < 1/n$. Taking $m \geq 2n$, by triangle inequality we have $|1/j - 1/k| \leq |1/j| - |1/k| \leq 1/2n + 1/2n = 1/n$. This can be likewise shown for {-1/x}. To show that these are equivalent we take $\forall n, \exists m \geq 2n$ such that $|1/k - (-1/k)| = 2/k$ Now since $k >\geq mandk \geq 2n$ and $2/k \geq 2/2n = 1/n$ we have a positive Cauchy sequence which is equivalent to a negative one. □

Problem 1. Let $f : S \to T$ be a function. A section of f is a function $s : T \to S$ such that $f \circ s = id_T$.

a Show that if f admits a section, then f is surjective.

b Show that, conversely, if f is surjective, then it admits a section. (HINT: You will need the axiom of choice for this.)

c Give an example of sets S and T and a function $f : S \to T$ such that f admits more than one section.

Answer 1.a. If $f : S \to T$, then f is surjective if $\forall t \in T, \exists s \in S, f(s) = t$ and a function s is a section of f if $f \circ s : T \to T$, which is an identity. Now if we take an element $s \in S$ and $t \in T$ such that $f(s) = t$ and $s(t) = s$, we can take $f(s(t)) = t$ and therefore f is surjective.

Answer 1.b. If f is surjective, we know that $\forall t \in T, f^{-1}(t)$ is non-empty and by axiom of choice $\forall t \in T$ we can choose $s_t \in f^{-1}(t)$. If we now define a function $s : T \to S; s(t) = s_t$, we can see that $f(g(t)) = t$. Therefore a surjective function admits section.

Answer 1.c. $\{S\} = \{a, b\}, \{T\} = \{c\}$, $f(a) = c, f(b) = c, s_1(c) = (a), s_2(c) = b$

Problem 2. In this problem we will generalize the sequence of approximations to 2 in 2.1 of the text to $\sqrt{S}$ for any positive S. (For now we will limit ourselves to $S \in Q$, but everything you do should generalize later to any positive $S \in R$.) Let x_1 be any positive rational, and recursively define, for $n \geq 1$.

$$x_{n+1} := \frac{1}{2}(x_n + \frac{S}{x_n})$$

a Show that $S \leq x_{n+1}^2, \forall n \geq 1$, and deduce from this that $x_{n+1} \leq x_n$ for all $n \geq 2$. It follows that the closed interval:

$$I_n := [\frac{S}{x_n}, x_n] := \{x \in \mathbb{Q} | \frac{S}{x_n} \leq x \leq x_n\}$$

contains $\sqrt{S}$ in the sense that $\frac{S^2}{x_n^2} \geq S \geq x_n^2$, and these form a decreasing nested sequence $I_2 \supset I_3 \supset I_4 \supset$.

b Show that $|I_{n+1}| \leq 1/2|I_n|$, where $|I_n|$ denotes the length of I_n, i.e. the distance between its endpoints.

c Show that x_n is a Cauchy sequence.

Answer 2.a.

$$S \leq \frac{S}{2} + \frac{x_n^2}{4} + \frac{S^2}{4x_n^2}$$
$$2Sx_n^2 \leq x_n^4 + S^2$$
$$0 \leq x_n^4 - 2Sx_n^2 + S^2$$
$$0 \leq (x_n^2 - S)^2$$

Since A^2 is always positive, the inequality holds. Then $\sqrt{S} \leq x_{n+1}$ and by differentiating the original formula we have $f'(x) = 1/2(1 - S/x^2) = 0$ from which we can see that the formula has a minimum at $\sqrt{S}$, which means that $x_n \geq \sqrt{S}$ and $x_n^2 \geq S$. To have $x_{n+1} \leq x_n$ we take:

$$0 \leq x_n - x_{n+1} = x_n - \frac{x_n}{2} - \frac{S}{2x_n}$$
$$0 \leq \frac{x_n}{2} - \frac{S}{2x_n} = \frac{x_n^2 - S}{2x_n}$$

which is positive and therefore $x_n \geq x_{n+1}$.

Answer 2.b.

$$|I_{n+1}| \leq 1/2|I_n| \iff x_{n+1} - \frac{S}{x_{n+1}} \leq 1/2\left(x_n - \frac{S}{x_n}\right)$$

Since $x_n \geq x_{n+1}$

Answer 2.c. To show that a sequence is Cauchy it is necessary to show that it converges to a specific number. Seeing from b we know that the interval $|I_{n+1}| \leq 1/2|I_n|$ is smaller with every iteration and eventually goes to 0 and therefore the sequence is convergent and Cauchy.