Question 5.3.4: 13. Let f_n denote the n-th iterate of f, $f_1 = f, f_2(x) = f(f_1(x)), ...f_n(x) = f(f_{n-l}(x))$. Express f_n' in terms of f'. Show that if $a \leq |f'(x)| \leq b$ for all x, then $a^n \leq |f_n'(x)| < b^n$.

Proof. By Chain rule:

$$f_n'(x) = f'(f_{n-1}(x))f'(f_{n-2}(x))...f'(x) = f'(x)\prod_{i=1}^{n-1} f'(f_i(x))$$

$$|f_n'(x)| = |f'(x)|\prod_{i=1}^{n-1} |f'(f_i(x))|$$

$$a \leq |f_n'(x)| \leq b_n$$

□

Question 5.4.6: 1. Suppose f is a C^2 function on an interval (a, b) and the graph of f lies above every secant line. Prove that $f''(x) \leq 0$ on the interval.

Proof. We have for some $x, y, z \in (a, b)$, $x < y < z$:

$$f(y) \geq \frac{f(z) - f(x)}{z - x}(y - x) + f(x)$$

$$\frac{f(y) - f(x)}{y - x} \geq \frac{f(z) - f(x)}{z - x}$$

Take $x = y - h, z = y + h$:

$$\frac{f(y) - f(y - h)}{h} \geq \frac{f(y + h) - f(y - h)}{2h}$$

$$0 \geq \frac{f(y + h) + f(y - h) - 2f(y)}{2h^2}$$

We take $h \to 0$ and by the L'Hospital rule, we know that $f''(y) \leq 0$. □

Question 5.4.6: 3. If f is C^2 on an interval prove that:

$$\lim_{h\to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x)$$

The expression $f(x + h) - 2f(x) + f(x - h)$ is called the symmetric second difference.

Proof. If we assume f to be C^2 we have $f(x + h) = f(x) + hf'(x) + 1/2h^2f''(x) + o(h^2)$ and $f(x - h) = f(x) - hf'(x) + 1/2h^2f''(x) + o(h^2)$. Therefore:

$$\lim_{h\to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = \lim_{h\to 0} \frac{h^2f''(x) + o(h^2)}{h^2}$$

$$= f''(x) + \lim_{h\to 0} \frac{o(h^2)}{h^2} = f''(x)$$ □

Question 5.4.6: 9. Suppose f is a C^n function on an interval and $T_n(x_0, x)$ is the same function of x for all x_0 in the interval. What can you say about f?

Proof. f is constant on the interval. $\square$

Question 5.4.6: 13. Suppose f is a C^n function in a neighbourhood of x_0 and g is C^n in a neighbourhood of $f(x_0)$. Let $\sum_{k=0}^{n} a_k(x-x_0)^k$ be $T_n(f, x_0, x)$, and let $\sum_{j=0}^{n} b_j(y-y_0)^j$ be $T_n(g, y_0, 0)$ where $y_0 = f(x_0)$. Show that $T_n(g \circ f, x_0, x)$ is obtained from:

$$b_0 + \sum_{j=1}^{n} b_j \left(\sum_{k=1}^{n} a_k(x - x_0)^k \right)^j$$

by retaining only the powers $(x - x_0)$ up to n.

Proof.

$$T_n = b_0 + \sum_{j=1}^{b} b_j \left(a_0 + a_1(x - x_0) + a_2(x - x_0)^2 ... + a_n(x - x_0)^n \right)^j$$

$$T_n = b_0 + b_1 \left(a_0 + a_1(x - x_0) + a_2(x - x_0)^2 ... a_n(x - x_0)^n \right) + b_2 \Big(.... \Big)^2 ... + \Big(... \Big)^n$$

$$T_n = b_0 + b_1 \left(a_0 + ... + a_n(x - x_0)^n \right)^1 + b_2 \left(...(x - x_0)^{n/2} \right)^2 ... + \Big(a_0 + a_1(x - x_0) \Big)^n$$

$\square$

Question 5.4.6: 14. If $f(x) = \frac{1}{1+x}$ show that $T_n(f, 0, x) = \sum_{k=0}^{n}(-1)^k x^k$.

Proof.

$$f(x) = (1 + x)^{-1} \Rightarrow f'(x) = (-1)(1 + x)^{-2}, f''(x) = 2(1 + x)^{-3}$$
$$\Rightarrow f^{(n)} = n!(-1)^n(1 + x)^{-1-n} \Rightarrow f^{(n)}(0) = n!(-1)^n$$

Therefore the zeroth order Taylor's polynomial is $T_n(f, 0, x) = \sum_{k=0}^{n}(-1)^k x^k$. $\square$

Question 5.4.6: 23 (a,b,c).

1. If f is C^n on an interval and has $n + 1$ distinct zeros, prove that $f^{(n)}$ has at least one zero on the interval.
2. If f is C^n on an interval and $f^{(n)}$ never vanishes, then f has at most n zeros on the interval.
3. A polynomial of degree n has at most n real zero.

Proof.

1. We have from Rolle's theorem that between two zeros of f will be at least one zero of f'. If we assume f has $n+1$ distinct zeros, than f' has n distinct zeros and $f^{(n)}$ has at least one zero.

2. Assuming to the contrary: from above we know that if f has $n+1$ zeros $f^{(n)}$ vanishes at least once, which contradicts the condition of non-vanishing. Therefore we have that f with at most n zeros.

3. A polynomial has the form of $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$. So we have $f^n(x) = a_n n! \neq 0$. Therefore a polynomial never vanishes and by (2.) it has at most n zeros.

□

Question 5.4.6: 24. If f is C^2, prove that f cannot have a local maximum or minimum at an inflection point (note that an inflection point is defined as a point where f'' changes sign; it is not enough that f'' vanish at the point).

Proof. If f is C^2, we have f', f'' exist and are continuous. An inflection point of f, at $f''(x_0) = 0$ is also an local extremum of f'. Now let's assume to the contrary that $f(x_0)$ is also the extremum of f. If it is a maximum we have that $f'(x-\epsilon) > 0$ and $f'(x+\epsilon) < 0$ and by theorem 5.4.2 $f'' < 0$. If it is a minimum we have $f'(x-\epsilon) < 0$ and $f'(x+\epsilon) > 0$ and by theorem 5.4.2 $f'' > 0$. In either case we arrive at contradiction and therefore a C^2 function cannot have a local max or min at an inflection point.

□