Question 3.2.3: 4. Let A be a set and x a number. Show that x is a limit-point of A if and only if there exists a sequence $x_1, x_2, ..$ of distinct points in A that converges to x.

Proof.

($\Rightarrow$) Suppose x is a limit of A, so $\forall m \in N, \exists x_m \in A \neq x,$ such that $|x_m - x| < 1/m$. So we have sequence $\{x_m\}$, of which we can pick a subsequence with distinct elements. So $\forall k, \exists j \in \mathbb{Z}_{>0},$ such that $|x - x_j| < |x - x_k|$ so $x_j \neq x_k$. $\square$

($\Leftarrow$) Suppose $\exists \{x_m\}$ of distinct points in A and $\{x_m\}$ converges to x. Then $\forall m \in \mathbb{Z}, \exists n \in \mathbb{Z}_{>0},$ such that $|x - x_j| < 1/m, \forall j \geq n$. Now for distinct x_j,if $x_n = x$ then there $\exists j \geq n,$ such that $x_j \neq x_n = x$. So $\forall m \in \mathbb{Z}, \exists x_j \neq x,$ such that $|x - x_j| < 1/m$. $\square$

Question 3.2.3: 7. Give an example of a set A that is not closed but such that every point of A is a limit-point.

Answer. Take $A = (1, 2)$ Take $x \in A$, then $\forall n \in \mathbb{Z}_{>0}, \exists y \in (x - 1/n, x)$. So $\forall n \in \mathbb{Z}_{>0} \exists y \neq x,$ such that $|x - y| < 1/n$. A is therefore a closed set of which every point is a limit point.

Question 3.2.3: 13. Define the derived set of a set A as the set of limit-points of A. Prove that the derived set is always closed. Give an example of a closed set A that is not equal to its derived set. Give an example of a set A such that the derived set of A is not equal to the derived set of the derived set of A. (Note: Cantor was originally led to study set theory in order to understand better the notion of derived set and to answer questions similar to the above.)

Proof. Let A' be a derived set of A. To show that A' is open is equivalent to show that its complement $(A')^c$ is open. Take $x \in (A')^c$, then $\exists n,$ such that $1/n > 0,$ such that $x \in (x - 1/n, x + 1/n)$ and this interval contains a finitely many points. Now suppose $y \in (x - 1/2n, x + 1/2n) \supset (x - 1/n, x + 1/n)$ so then $(x - 1/2n, x + 1/2n)$ also contains finitely many points. Therefore y is not a limit $\Rightarrow y \notin A' \Rightarrow x \in (A')^c \Rightarrow (A')^c$ is open. $\square$

Just take $A = \{0, 1, 1/2, ...1/n\}$, we immediately see that $A' = \{0\}$ so $A'' = \{\}$. $A \neq A' \neq A''$.

Question 3.3.1: 2. Show that the following finite intersection property for a set A is equivalent to compactness: if B is any collection of closed sets such that the intersection of any finite number of them contains a point of A, then the intersection of all of them contains a point of A. (Hint: consider the complements of the sets of B.)

Proof. Want to show A is compact $\iff$ for every collection of closed sets of which a finite subset $B_1 \cap B_2 \cap ...B_N \cap A \neq \emptyset$. Then $A \cap (\bigcap_{i=1}^{\infty} B_i) \neq \emptyset$.

$\Rightarrow$ Assume to the contrary that $A \cap (\bigcap_{i=1}^{\infty} B_i) = \emptyset$, which implies

$$A \subset (\bigcap_{i=1}^{\infty} B_i)^c \equiv \bigcup_{i=1}^{\infty} B_i^c$$

Then the set of all B^c is an open cover of A, but since A is compact it must have a finite subcover, which is a contradiction.

$\Leftarrow$ Assume to the contrary that A is not compact (we have an open cover and no finite subcover). Then we have an open cover, which is a set of all B^c. But we have just established that:

$$A \subset (\bigcap_{i=1}^{\infty} B_i)^c \equiv \bigcup_{i=1}^{\infty} B_i^c$$

which means that $A \cap (\bigcap_{i=1}^{\infty} B_i) = \emptyset$, but this is in contradiction with the requirement that $A \cap (\bigcap_{i=1}^{\infty} B_i) \neq \emptyset$ and therefore we have the other contradiction. □

Question 3.3.1: 4. If $A \subseteq B_1 \cup B_2$, where B_1 and B_2 are disjoint open sets and A is compact, show that $A \cap B_1$is compact. Is the same true if B_1 and B_2 are not disjoint?

Proof. A is compact, so A is closed and bounded. Therefore $A \cap B_1$ is bounded. Suppose we have a sequence $\{x_j\} \subset A \cap B_1$ converging to the limit x. Since A is closed $x \in A \subset B_1 \cup B_2$. $\Rightarrow x \in B_1$ or $x \in B_2$. To have a contradiction take $x \in B_2$., which is open and therefore contains an open interval around x then B_2 contains infinitely many points of $\{x_j\}$, but $x_j \in B_1$ and $B_1 \cap B_2 = \{\}, x_j$ cannot be in B_2, which is a contradiction. So $x \in B_1 \Rightarrow x \in A \cap B_1 \Rightarrow A \cap B_1$ is closed $\Rightarrow A \cap B_1$ is bounded $\Rightarrow A \cap B_1$ is compact. □

In case $B_1 \cap B_2 \neq \{\}$ we can take $A \subset B_1 \cup B_2 = [1, 2] \subset (0, 2) \cup (1, 3)$, but $A \cap B_1$ $[1, 2)$ is not compact.

Question 3.3.1: 6. For two non-empty sets of numbers A and B, define $A + B$ to be the set of all sums $a + b$ where $a \in A$ and $b \in B$. Show that if A and B are compact, then $A + B$ is compact. Give an example where A and B are closed but $A + B$ is not.

Proof. Suppose A, B are compact. Then for some N, $A, B \subset [-N, N] \Rightarrow A + B \subset [-2N, 2N]$. So $A+B$ is **bounded**. Now take $x_n \in A$ and $y_n \in B$ and define $z_n = x_n + y_n$. Since A, B are compact, x_n, y_n have convergent subsequences, with limits x, y. Since A, B are compact and therefore closed $x \in A, y \in B$. Therefore the subsequence of z_n converges to $z = x + y \in A + B$ and so z_n also converges to $z \in A + B$. So $A + B$ is **closed** and therefore compact. □

Take $A = \{(-1)^n\}$ and $\{B = (-1)^{n+1} + 1/n\}$, then $A + B = \{1/n\}$, which doesn't contain its limit and therefore is not closed.

Question 3.3.1: 8. If A is compact, show that $\sup A$ and $\inf A$ belong to A. Give an example of a non-compact set A such that both $\sup A$ and $\inf A$ belong to A.

Proof. If A is compact it is bounded and closed. As it is bounded it has to have a sup and inf and because it is closed, these have to be contained in A. $\square$

To have an example of a non-compact set containing its sup and inf take $A = [1, 3] \setminus 2$. Here $\sup = 3$, $\inf = 1$, but A isn't closed.

Problem 1. Use the result of Exercise 2 in 3.3.1 to give an alternative proof of Theorem 3.3.3 (to be clear, your proof should not in any way depend on the proof given in the text).

Theorem 3.3.3. A nested sequence of non-empty compact sets has a non-empty intersection.

Proof. From 3.3.1:2 we know that A is compact $\iff$ for every collection of closed sets of which a finite subset $B_1 \cap B_2 \cap ...B_N \cap A \neq \emptyset$. Then $A \cap (\bigcap_{i=1}^{\infty} B_i) \neq \emptyset$.

A nested sequence of non-empty compact sets like $[a_1, b_1] \subset [a_2, b_2] \subset ...[a_n, b_n]$ where $a_{n+1} > a_n$ and$b_{n+1} > b_n$ is always contained in some compact set A. By exercise 2 then if A is compact, then it follows directly that $A \cap (\bigcap_{i=1}^{\infty} B_i) \neq \emptyset$ and therefore the nested sequence has a non-empty intersection. $\square$