Question 4.1.5: 1. Let f be a function defined on a closed domain. Show that f is continuous if and only if the inverse image of every closed set is a closed set.

Proof.

$\Rightarrow$ Suppose f is continuous. We take a closed set D and to show that it's inverse image is a closed set we take a sequence (x_n) converging to x_0 in $f^{-1}(D)$. Now since f is continuous we have a convergent sequence $\lim_{n\to\infty} f(x_n) = f(x_0) = y$, which must be in D as D is closed.

$\Leftarrow$ Suppose that the inverse image of every closed set is a closed set. We take a sequence x_n, converging to x_0 in D, since D is closed. We then take to the contrary a sequence $f(x_n)$, which does not converge to $f(x_0)$ and some subsequence of it. We can then take the closure of the subsequence $f(\bar{x}_{n_r}$, which does not include $f(x_0)$. Then the limit point x_0 of $f^{-1}(\bar{x}_{n_r})$ is not contained in it, which is in contradiction to the assumption that $f^{-1}(\bar{x}_{n_r})$ is closed. Therefore $f(x_n)$ converges to $f(x_0)$ and is continuous. □

Question 4.1.5: 3. Let A be the set defined by the inequalities $f_1(x) \geq 0, f_2(x) \geq 0 ... f_n(x) \geq 0$ where $f_1, ..., f_n$ are continuous functions defined on the whole line. Show that A is closed. Show that the set defined by $f_1(x) > 0, ..., f_n(x) > 0$ is open.

Proof. f_i are continuous and $f_i^{-1}([0,\infty))$ are closed because $[0,\infty) = \mathbb{R} \setminus (-\infty, 0)$ is closed. Then because a finite intersection of closed sets is closed we have $A = \cap_{1\geq i\geq n} f_i^{-1}([0,\infty))$ is closed. On the contrary $A = \cap_{1\geq i\geq n} f_i^{-1}((0,\infty))$ is open because a finite intersection of open sets is open. □

Question 4.1.5: 4. Give a definition of $\lim_{x\to\infty} f(x) = y$. Show that this is true if and only if for every sequence $x_l, x_2, ...$ of points in the domain of f such that $\lim_{x\to\infty} x_n = +\infty$, we have $\lim_{n\to\infty} f(x_n) = y$.

Proof. The limit of f is $\lim_{x\to\infty} f(x) = y$ such that for every $1/m$ there exists $1/n$ such that $|f(x) - y| < 1/m$ for all $x \neq x_0 \in D$ satisfying $|x - x_0| < 1/n$.

$\Rightarrow$ Assuming we have a limit we want to show that for any sequence in the domain such that $\lim_{n\to\infty} x_n = +\infty$, we have $\lim_{n\to\infty} f(x_n) = y$. So we can pick $m \in f(D)$ such that $|f(x) - y| < 1/m$ and $n \in D$ such that $|x - x_0| < 1/n$.

$\Leftarrow$ If we assume $\lim_{x\to\infty} x_n = +\infty$, we have $\lim_{n\to\infty} f(x_n) = y$. We by default have a limit y.

□

Question 4.1.5: 10. Show that a function that satisfies a Lipschitz condition is uniformly continuous.

Proof. Suppose the function f satisfies the Lipschitz condition, then $\exists N,$ such that $\forall x, y, |f(x)- f(y)| \leq N|x-y|$. So for any $m \in \mathbb{Z}_{>0}, \exists n,$ such that $n > mN$. So when $|x-y| < 1/n$ we have:

$$|f(x) - f(y) \geq N|x-y| < N/n < N/mN = 1/m$$

And therefore f is continuous. $\square$

Question 4.2.4: 1. If f is monotone increasing on an interval and has a jump discontinuity at x_0 in the interior of the domain, show that the jump is bounded above by $f(x_2) - f(x_1)$ for any two points x_1, x_2 of the domain surrounding $x_0, x_1 < x_0 < x_2$.

Proof. A function f with a jump discontinuity has at x_0 if both one-sided limits exist $\lim_{x \to x_0^+} f(x) \equiv y_1 \neq \lim_{x \to x_0^-} f(x) \equiv y_2$. Since f is monotone increasing we have $y_2 - y_1 > 0$ or $f(x_2) - f(x_1) > 0$. $\square$

Question 4.2.4: 4. If a continuous function on an interval takes only a finite set of values, show that the function is constant.

Proof. Since the function is continuous and the domain connected the range is also connected. Then if the domain is a finite set of values, then the range is also a finite set of values, but in order for it to be connected it cannot contain more than one element. Therefore the function is constant. $\square$

Question 4.2.4: 7. Let f be a monotone function on an interval. Show that if the image of f is an interval, then f is continuous. Give an example of a non-monotone function on an interval whose image is an interval but that is not continuous.

Proof. Suppose to the contrary that f with a domain D is not continuous, then by theorem 4.2.6, $\exists x_0 \in D,$ such that $\lim_{x \to x_0^-} f(x) < \lim_{x \to x_0^+} f(x)$. Then since f is monotone the image of f cannot be a single interval, which is a contradiction. Therefore f is continuous. $\square$

For example a function $f(x) = x$ for $1 \leq x \leq 2$ and $f(x) = 2$ everywhere else. Then the image of $[0, 2]$ is the interval $[1, 2]$, but it isn't continuous.

Question 4.2.4: 11. If f is a continuous function on a compact set, show that either f has a zero or f is bounded away from zero. ($|f(x)| > 1/n$ for all x in the domain, for some $1/n$).

Proof. Suppose a function is continuous on a compact set D. Now if f is not bounded away from zero, then for any $n \in \mathbb{Z}_{>0}, \exists x_n \in D,$ such that $|f(x_n)| < 1/n$. So that a sequence $\{x_n\}$, such that $\lim_{n\to\infty} f(x_n) = 0$. Now because D is compact, we have a subsequence $\{x'_n\}$, which converges to x_0. Then since f is continuous, $f(x_0) = \lim_{n\to\infty} f(x_n) = 0$. Therefore $x_0 \in D$ is a zero of f.

If f is bounded away from zero, that means that the range of f does not contain a zero and therefore the function does not have a zero. $\square$