Johns Hopkins University

Lecture Notes

Analysis I

Lecturer: Brian Smithling

Andrzej Novak

April 4, 2016

This page is intentionally left blank.

Chapter 2

Real Numbers

2.1 Order properies of $\mathbb{R}$

Order properties of R

Def. $x \in \mathbb{R}$ is positive ($x > 0$), if $\exists N \in \mathbb{Z}$, such that $\forall$ Cauchy (x_j) of rational numbers representing $x, \exists m \in \mathbb{Z}_{>0}$, such that $x_j \geq \frac{1}{N}, \forall j \geq m$.

Def. $x \in R$ is negative ($x < 0$), if $-x$ is positive.

Thm. Each $x \in \mathbb{R}$ is either positive, negative or 0. The sum and product of positive real numbers is positive - $\mathbb{R}$ is an ordered field

Proof. $0 = [(0, 0, ...)]$ is clearly not positive. Then $-0 = 0$ is not positive, so 0 is not negative. Now suppose $x \neq 0$ Say $x = [(x_j)]$ Lemma from last time: $\exists N, m$ such that:

$$|x_j| \geq \frac{1}{N}, |x_j - x_k| \leq \frac{1}{2N}$$

$$\frac{-1}{2N} < x_j - x_k < \frac{1}{2N}$$

Let $j \geq m$.

1. If x_j is positive, then $1/N \leq x_j \Rightarrow 1/2N \ < x_k$ is positive $\forall k \geq m$
2. If x_j is negative, then $x_j \leq -1/N \Rightarrow x_k < -1/2N$ is negative $\forall k \geq m$

If (1), then x is a positive number. If (2), -x is a positive, so x is negative. - Mutually exclusive! Sum and products of lower bounds is a lower bound. $\square$

Upshot: Usual rules for inequalities apply to $\mathbb{R}$.

$$|x| = \left\{ \begin{array}{rl} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{array} \right\}$$

Lem. Let $x, y \in \mathbb{R}$ be represented by (x_j), (y_j) Then $x_j \leq y_j, \forall j \geq m \Rightarrow x \leq y$

Proof. Suppose $y - x$ is negative. Then $x - y$ is positive $\Rightarrow x_j - y_j > 1/n$ for some n, whenever j is large. $\Rightarrow x_j > y_j$ for j which is large. This proves the contrapostive! $\square$

Thm: Triangle inequality.

$$\forall x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$$

Alternate forms:

$$|x \pm y| \geq |x| - |y|$$

$$|x - z| \leq |x - y| + |y - z|$$

Proof. Use the triangle inequality in $\mathbb{Q}$ on representatives of x, y and apply the previous lemma. $\square$

Thm: Axiom of Archimedes. $\forall x \in \mathbb{R}_{>0}, \exists n \in \mathbb{Z}_{>0}$, such that $1/n \leq x$

Proof. By definition, x positive means $\exists n \in \mathbb{Z}_{>0}$, such that $1/n \leq x_j, \forall$ large j, where (x_j) represents x. So by the lemma $1/n \leq x$. $\square$

Corollary. $|x| \leq 1/n, \forall n \in \mathbb{Z}_{>0} \Rightarrow x = 0$

Thm. $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}_{>0}, \exists y \in \mathbb{Q}$, such that $|x - y| \leq 1/n$

Proof. Say $x = (x_j)$, then Cauchy criterion says : $|x_j - x_k| \leq 1/n, \forall j, k \geq$ some m. $\Rightarrow -1/n \leq x_j - x_m \leq 1/n, \forall j \geq m$. Apply the previous lemma $-1/n \leq x - x_m \leq 1/n | iff | x - x_m| \leq 1/n$. $\square$

2.2 Limits and completeness

Recall:

$$\lim_{j \to \infty} (x_j) = x$$

means $\forall n, m \in \mathbb{Z}_{>0}$, such that $j \geq m \Rightarrow |x - x_j| < 1/n$ makes sense for $x, x_j \in \mathbb{R}$.

Lem. Limits are unique, i.e. if x, y are limits of (x_j) then $x = y$

Proof. Let $n \in \mathbb{Z}_{>0}$. Then $\exists m$, such that $j \geq m \Rightarrow |x - x_j| < \frac{1}{2n}$ and $|y - y_j| < \frac{1}{2n}$. Then for $j \geq m, |x - y| \leq |x - x_j| + |y - x_j| < \frac{1}{2n} + \frac{1}{2n} = 1/n$. n is arbitrary so $|x - y| = 0$, i.e $x = y$ $\square$

Lem. If (x_j) is a rational Cauchy sequence representing $x \in \mathbb{R}$, then $\lim_{j \to \infty}(x_j) = x$

Proof. Let n be a positive integer. Want $|x - x_j| < 1/n$ for $j \gg 0 \iff 1/n - |x - x_j|$ is positive for a $j \gg 0 \iff \exists n' \in \mathbb{Z}_{>0}$, such that $1/n \leq 1/n - |x_k - x_j|$ for $k \gg 0, j \gg 0$. By the Cauchy condition $|x_k - x_j| < \frac{1}{2n}$ for $j, k \gg 0$ So take $n' = 2n$. $\square$

Def. Definition of Cauchy sequence of reals is the same as for rationals.

Thm. $\mathbb{R}$ is complete, i.e. a sequence (x_j) of real numbers has a limit $\iff$ it is Cauchy.

Proof.

$\Rightarrow$ 2 lectures ago.

$\Leftarrow$ Let (x_j) be a Cauchy sequence in $\mathbb{R}$. By density of $\mathbb{Q}, \forall j$ can choose $y_i \in \mathbb{Q}$, such that $|x_j - y_j| < 1/j$.

Claim: (x_j) is Cauchy. Well let $n \in \mathbb{Z}_{>0}$. (x_j) is Cauchy $\Rightarrow \exists m$, such that $|x_k - x_j| < \frac{1}{2n}, \forall j, k \geq m$. Then for $j, k \geq \max\{m, 4n\} : |y_j - y_k| \leq |y_j - x_j| + |x_j - x_k| < 1/j + 1/2n + 1/k \leq 1/4n + 1/2n + 1/4n = 1/n$

TBC

$\square$

17.2.2016 - Jesus

Thm. $\mathbb{R}$ is complete. In other words a sequence (x_j) in $\mathbb{R}$ has a limit $\iff$ (x_j) is Cauchy.

Proof.

$\Rightarrow$ Need to find a sequence (y_j) close to the (x_j). Take $y = \lim y_j$ and show $x = \lim x_j$. By density of $\mathbb{Q} \in \mathbb{R}$, we can choose given $j \in \mathbb{N}, y_j \in \mathbb{Q}$, such that $|x_j - y_j| < 1/j$.
From last time: (y_j) is Cauchy (b/c (x_j) is Cauchy). $y \equiv [(y_j)] \in R$ Suffices to show $y = \lim x_j$
Let $m > 0$:

$$|y - x_j| \leq |y - y_j| + |y_j - x_j| < |y - y_j| + 1/j < 1/2m + 1/2m = 1/n$$

$$\Rightarrow y = \lim x_j$$

$\square$

Thm. Let $(x_j), (y_j)$ be sequences in $\mathbb{R}$ with $\lim x_j = x, \lim y_j = y$ Then

- $\lim(x_j \pm y_j) = x \pm y$ likewise for products.
- If $y \neq 0$, then for $j \gg 0$, $\lim(x_j/y_j) = x/y$
- $x_j \geq y$ for $j \gg 0 \Rightarrow x \geq y$

Proof. Exercise... $\square$

Thm. Let $x \in \mathbb{R}_{>0}$ be a real positive number. Then $\exists! y \in \mathbb{R}_{>0}$, such that $y^2 = x \iff y = \sqrt{(x)}$

Proof - Existence: Homework $\square$

Proof - Uniqueness: Suppose $y, z \in \mathbb{R}_{>0}$, such that $y^2 = z^2 = x$, hence $0 = y^2 - z^2 = (y - z)(y + z) \Rightarrow y - z = 0 \Rightarrow y = z$ $\square$

Chapter 3

Topology of R

3.1 Limits

Def. Let $E \subset \mathbb{R}$ be an non-empty set.

1. An upper bound for E in $\mathbb{R}$ is $b \in \mathbb{R}$, such that $x \leq b, \forall x \in E$.
2. An lower bound for E in $\mathbb{R}$ is $b \in \mathbb{R}$, such that $x \geq b, \forall x \in E$.
3. Say E is bounded above or below, if it has an upper or lower bound.

   For example $\mathbb{Z}_{>0}$ is bounded by $b = -100$

Thm. Let non-empty $E \subset \mathbb{R}$ be bounded above. Then $\exists! x \in \mathbb{R} = \sup E$, such that :

1. $\sup E$ is an upper bound of E.
2. y is any upper bound for $E \Rightarrow \sup E \leq y$

Proof - Uniqueness: Suppose s, t satisfy (1) and (2). Then by (2) we have: $s \leq t, t \leq s \Rightarrow s = t$ □

Proof - Existence: Let y_1 be some upper bound, if y_1 is the smallest upper bound, then we are done $\sup E \equiv y_1$.

Suppose that is not the case, choose $x_1 \in E, y_1$ is upper bound, let $m_1 = \frac{x_1+y_1}{2}$. If m_1 is upper bound, then $y_2 := m_1, x_2 := x_1$. If m_1 is not an upper bound, then $x_2 := m_1, y_2 := y_1$. In both cases $x_1 \leq x_2, y_2 \leq y_1$

$$|y_2 - x_2| \leq 1/2|y_1 - x_1|$$

Continue: $x_1 \leq x_2 \leq x_3 \leq .... \leq y_3 \leq y_2 \leq y_1$. Using a similar trick as for HW3,problem 2, you see that $(y_k), (x_k)$ are equivalent Cauchy sequences so $\sup E = \lim x_k = \lim y_k$

$z \in E$ satisfies $z \leq y_m, \forall n$ because y_m is always an upper bound $\Rightarrow z \leq \lim y_m = \sup E \Rightarrow \sup E$ is upper bound. Let w be upper bound for $E, x_m \gg w, \forall v, \sup E = \lim x_m \leq w \Rightarrow$ (2) holds.

$\square$

Thm. Let non-empty $E \subset \mathbb{R}$ be bounded below then $\exists!$ real number $\inf e$ such that:

- $\inf E$ is a lower bound of E
- y is any lower bound for E, then $y \leq \inf E$

Def. For non-empty $E \subset \mathbb{R}$ bounded above, $\sup E$ is the supremum of E and for $\mathbb{R}$ bounded below $\inf E$ is the infimum of E. Supremum is lowest upper bound and infimum is greatest lower bound.

If E is not bounded above $\sup E = \infty$ and if E is not bounded below $infE = -\infty$

Def. A sequence (x_n) is **monotonic** or **weakly** increasing (decreasing) if $x_m \leq x_{m+1}(x_m \geq x_{m+1})$.

Thm. Every monotonic increasing or decreasing sequence bounded above or below has a limit in $\mathbb{R}$ equal to the $\sup x_m$ or$(\inf x_m), n \in \mathbb{N}$.

Proof. Suppose monotonic increasing sequence bound above. $x_j | j \in \mathbb{Z}_{>0}$ is non-empty, by previous theorem it has $\sup y_m < \infty$. So $x_j \leq x, \forall j$. Suppose $m > 0$, then $x - 1/n$ is not upper bound. So $\exists m$, such that $x - 1/n < x_m \leq x_j$ for $j \geq m$ by monotonicity. $\Rightarrow |x - x_j| < 1/m$ for $j \geq m$ for $j \gg 0 \Rightarrow limx_j = x$. Similar for decreasing. $\square$

Def. $x \in \mathbb{R}$ is a limit point of a sequence (x_j), if for all $n > 0$, there are infinitely many x_j, such that $|x - x_j| < 1/n$. EG: $x_n = (-1)^n$

Convention: Infinity is a limit point of a sequence not bounded above or below

Def. A subsequence of (x_j) is a collection of terms x_{j_r}, such that $j_1 \leq j_2 \leq j_3$

Thm. Let $x \in \mathbb{R}$ be a limit point of $(x_j) \iff \exists$ subsequence (x_{j_r}), such that $\lim_{r\to\infty} x_{j_r} = x$

19.2.2016

Proof.

$(\Leftarrow)$ Use jr's ?

($\Rightarrow$) Choose x_{j_r}, such that $|x - x_{j_r}| < 1$. Choose $j_2 > j_1$, such that $|x - x_{j_2}| < 1/2$ and j_r, such that $j_r > j_{r-1}, |x - x_{j_r}| < 1/r$. Clearly $\lim_{r\to\infty} x_{j_r} = x$

$\square$

Def. If (x_j) is bounded above, $\lim\limits_{j\to\infty} \sup(x_j) = \lim\limits_{k\to\infty} \sup\{x_j | j \geq k\}$.

If (x_j) is bounded below, $\lim\limits_{j\to\infty} \inf(x_j) = \lim\limits_{k\to\infty} \inf\{x_j | j \geq k\}$

E.g: $\{-2, 3/2, -4/3, 5/4...\} \equiv x_j = (-1)^j \frac{j+1}{j}$

$$\sup\{x_j | j \geq k\} = \left\{ \begin{array}{ll} \frac{k+1}{k} & \text{for k-even} \\ \frac{k+2}{k+1} & \text{for k-odd} \end{array} \right\} \Leftarrow \lim_{j\to\infty} \sup\{x_j | j \geq k\} = 1$$

Similarly $\liminf\limits_{j\to\infty} x_j = -1$

In general: Set $y_k \equiv \sup_{x_j | j \geq k}$. Then y_k is an upper bound for $\{x_j | j \geq k+1\} \Rightarrow y_k \geq y_{k=1} \Rightarrow y_1 \geq y_2 \geq ... \Rightarrow \limsup\limits_{j\to\infty} x_j = \lim y_k$ exists (possibly $\pm\infty$)

Similarly $\liminf\limits_{j\to\infty} x_j$ exists, possibly ($\pm\infty$)

Thm. $\limsup\limits_{j\to\infty} x_j$ is a limit point of (x_j) and $= \sup\{\text{limit points of } (x_j)\}$

Proof. Suppose $y = \limsup_{j\to\infty} x_j$ is finite $y = \lim_{k\to\infty} \sup_{j>k} x_j$ so given $n > 0, \exists m$, such that $|y - \sum_{j\geq m} x_j)| < 1/2n$, which is finite so $\exists$ infinitely many $l \geq m$, such that $|(\sup x_j) - x_j| < 1/2n \Rightarrow$ infinitely many l, such that $|y - x_l| < 1/2 \Rightarrow y$ is a limit point.

Now need to show y is the biggest limit. $y = \sup\{\text{limit points of } (x_j)\}$, since y is a limit point. It suffices to show that y is an upper bound for the set.

Let x be a limit point of (x_j) Let (x_{j_k}) be a subsequence with a limit x. Now $j_k \geq k, \forall k$, so $\sup_{j>k} x_j \geq x_{j_k}, \forall k$. Let $k \to \infty : y \geq x$ $\square$

Note: If $\limsup\limits_{j\to\infty} x_j = \pm\infty$ see text! Similarly:

Thm. $\liminf\limits_{j\to\infty} x_j$ is a limit point of (x_j) and $= \inf\{\text{limit points of } (x_j)\}$

Def. (x_j) is bounded, if it's bounded above and below $\iff$ $\exists N$, such that $|x_j| \leq N, \forall j$

Thm. Let (x_j) be a bounded sequence. TFAE (The following are equivalent):

1. (x_j) converges (it has a limit).
2. (x_j) has exactly one limit point
3. $\limsup\limits_{j\to\infty} x_j = \liminf x_j$

Proof.

(1) $\Longrightarrow$ (2) $x_k \to x \Rightarrow$ every subsequence $\to x \Rightarrow x$ only limit point.

(2) $\Longleftrightarrow$ (3) The sup and inf of a set E are equal $\Longleftrightarrow$ E has one point (obvious)! Apply the preceding theorems to $E = \{\text{limit points of } (x_j)\}$

(3) $\Longrightarrow$ (1) $x \equiv \limsup x_j = \liminf x_j$ & $y_k \equiv \sup_{j \geq k} x_j, z_k \equiv \inf_{j \geq k} x_j$
Then $\forall j \geq k, z_k \leq x_j \leq y_k$, and $y_k, z_k \to x$ as $k \to \infty$. Given $n > 0, \exists m$, such that

$$x - 1/n < z_m \leq x_j \leq y_m < x + 1/n$$

$\forall j \geq m \Rightarrow |x - x_j| < 1/n, \forall j \geq m$

□

-End of exam

3.2 Open and closed sets

Lem. For $a, b \in \mathbb{R}$ we define $(a, b) \equiv \{x \in \mathbb{R} | a < x < b\}$ as an open interval and $[a, b] \equiv \{x \in \mathbb{R} | a \leq x \leq b\}$ as a closed interval. Let $E \subset \mathbb{R}$. TFAE (The following are equivalent):

1. $\forall x \in E, \exists a, b$, such that $x \in (a, b) \subset E$.
2. $\forall x \in E, \exists n > 0$, such that $(x - 1/n, x + 1/n) \subset E$.

Proof.

1. $(2 \Rightarrow 1)$ is trivial
2. $(1 \Rightarrow 2)$ Let $x \in E$. Choose $n > 0$, such that $1/n < \min\{x - a, b - x\}$. Then $(x - 1/n, x + 1/n) \subset (a, b) \subset E$

□

Def. A set $E \subset \mathbb{R}$ satisfying the equivalent conditions in the lemma is open.

E.g: Empty set - $\varnothing$, $\mathbb{R}$, (a, b), $(-6, \infty)$ are all open set. $[-1, 0)$ is not open - doesn't contain an open interval around -1.

Thm.

1. Arbitrary unions of open sets are open!
2. Finite intersection of open sets are open.

Proof.

1. Let $U = \bigcup_{U_\alpha}$, where U_α is open in $\mathbb{R}$, $\forall \alpha$. Let $x \in U \Rightarrow x \in u_\alpha$, for some alpha $\Rightarrow \exists a, b$, such that $x \in (a, b) \subset U_\alpha \Rightarrow x \in (a, n) \subset U$. Open!

2. Let $U = \bigcap_{Ui}$, where U_i is open $\forall i$. Let $x \in U \Rightarrow x \in U_i$, for all $i \Rightarrow \exists a_i, b_i$, such that $x \in (a_i, b_i) \subset Ui$ for all i:
   Let $a \equiv max\{a_i, \cdots, a_n\}$
   Let $b \equiv min\{b_i, \cdots, b_n\}$
   Then $\forall i, j, x \in (a, b) \subset (a_i, b_i) \subset U_i \Rightarrow (a, b) \subset U$

$\square$

24.2.2016 Recall: $E \subset |R$ is open of $\forall x \in E, \exists a, b \in \mathbb{R}$, such that $x \in (a, b) \subset E$ ($\iff \forall x \in E, \exists n > 0$, such that $(x - 1/n, x + 1/n) \subset E$).

Def. A neighbourhood (nbd) of $x \in \mathbb{R}$ is an open $U \subset \mathbb{R}$, such that $x \in U$.

Note: $\lim\limits_{j \to \infty} x_j = x \iff \forall m, \exists m$, such that $j \geq m \Rightarrow x_j \in (x - 1/n, x + 1/n) \iff \forall$ neighbourhoods$U$of $x, \exists m$, such that $j \geq m \Rightarrow x_j \in U$

Def. x is a **limit point** of $A \subset \mathbb{R}$ if $\forall n > 0, \exists y_n \in A$, such that $0 < |x - y_n| < 1/n$
Equivalent: every neighbourhood of x contains (ctns) a point of $A \neq x$
Also equivalent every neighbourhood of x contains infinitely many points in A.

Warning: limit points of sequences and limit points of sets are not to be confused, but it is easy to do so.
Eg: $x_j = (-1)^j$, then 1 and -1 are limits points of the sequence. But the set of values $\{-1, 1\}$ has no limit points.

Lem. Let $B \subset \mathbb{R}, x \in \mathbb{R} \backslash B$ Then x is not a limit point of $B \iff \exists$ neighbourhood of x contained in $\mathbb{R} \setminus B$.

Proof. x not a limit point $\iff \exists$ neighbourhood U of x, such that $U \cap (B \setminus \{x\}) = \emptyset \iff \exists$ neighbourhood U of x, such that $U \cap B = \emptyset \iff \exists U$ of x, such that $U \subset \mathbb{R} \setminus B$ $\square$

Def. $B \subset \mathbb{R}$ is **closed** is $\{limitpointsofB\} \subset B$

Eg: $[a, b]$ is closed, since any finite set is trivially closed.
$[a, b)$ not closed , (fails to contain some limit point) (also not open)

Thm. $B \subset \mathbb{R}$ is closed $\iff \mathbb{R} \setminus B$ is open.

Proof. $B \subset \mathbb{R}$ is closed $\iff$ whenever $x \notin B \Rightarrow x$ is not a limit point of B. By the lemma $\iff$ ($x \in \mathbb{R} \setminus B \Rightarrow \exists$ neighbourhood U of x, such that $U \subset \mathbb{R} \setminus B$) $\iff \mathbb{R} \setminus B$ is open. $\square$

Thm.

1. Arbitrary intersections of closed subsets are closed
2. Finite unions of closed subsets are closed.

Proof.

1. Let $B = \bigcap\limits_{\alpha \in A} B_\alpha, B_\alpha$ closed $\forall \alpha$. $\mathbb{R} \setminus B = \bigcup\limits_{\alpha \in A} (\mathbb{R} \setminus B_\alpha)$ open $\Rightarrow B$ is closed $\square$.
2. Let $B = \bigcup\limits_{i=1}^{n} B_i, B_i$ closed $\forall i$ $\mathbb{R} \setminus B = \bigcap\limits_{i=1} (\mathbb{R} \setminus B_i)$ open $\Rightarrow B$ is closed .

Def. The **closure** of $A \subset \mathbb{R}$ is $\overline{A} \equiv A \cup \{limtpointsof A\}$

Thm. $\overline{A}$ is closed.

Proof. To show that this is closed, it suffices to show the complement is open. $\mathbb{R} \setminus \overline{A} = (\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus \{\text{limit points of } A\}) \equiv C$ is open. Well $x \in C \iff \exists x \subset U \subset \mathbb{R} \setminus A$ (every $y \in U$ will be $\notin A$ and not a limit point of A by the lemma) $\Rightarrow U \subset C$. $\square$

Remark: If B is a closed set containing A, then B contains limit points of $A \Rightarrow B \subset \overline{A} \Rightarrow \overline{A}$is the smallest closed set containing A.

Def. A subset B of A is **dense** in A if $A \subset \overline{B}$, i.e $B \subset A \subset \overline{B}$.

Eg: B is dense in $\overline{B}$.
(a, b) is dense in $[a, b]$
$\mathbb{Q}$ is dense in $\mathbb{R} \iff \overline{Q} = \mathbb{R}$

3.2.1 Compact Sets

Def. $A \subset \mathbb{R}$ is **compact (cpt)** if every sequence (x_j) with $x_j \in A \forall j$ has a limit point in A.

Thm. $A \subset \mathbb{R}$ is compact $\iff$ A is closed a bounded.

Proof.

$\Rightarrow$

- A is not closed $\Rightarrow \exists$ limit point $y \notin A. \forall n, \exists y_n \in A$, such that $|y - y_n| < 1/n \Rightarrow \lim\limits_{n \to \infty} y_n = y$. Every subsequence of (y_n) converges to $y \notin A$, so A is not compact.
- A is not bounded $\Rightarrow \exists$ sequence (x_n) in A, such that $x_n > n$ or $x_n < -n, \forall n \Rightarrow (x_n)$ has no finite limit points $\Rightarrow$ A is not compact

$\Leftarrow$ Let (x_j) be a sequence in $A. A$ is bounded $\Rightarrow (x_j)$ is bounded $\Rightarrow (x_j)$ has a limit point $y \in \mathbb{R}$.

- If y is a limit point of A, then $y \in A$ since A is closed and we are done.
- If y is not a limit point of A, then $\exists n$, such that $\nexists y_n \in A$, such that $0 < |y - y_n| < 1/n$. But if $x_{j_k} \to y$, this can only happen if $0 = |y - x_{j_k}|$ for $k \gg 0 \Rightarrow y = x_{j_k} \in A$.

$\square$

Def. *A* **cover** of a set B is a collections of sets $\{U_\alpha\}_{\alpha \in A}$, such that $B \subset \bigcup_{\alpha \in A} U_\alpha$. If U_α's are open subset of $\mathbb{R}, \forall \alpha$, we say $\{U_\alpha\}_{\alpha \in A}$ is an open cover of B.

Eg. The sets $(1/2, 3/2)$ and $(1/n, 1), \forall n \geq 2$, form open cover of $(0, 1]$.

Def. If $\{U_\alpha\}_{\alpha \in A}$ cover B, a **subcover** is a subcollection of U_α's which still covers B. A **finite subcover** means finitely many U_α's.

Eg. In the previous example $\nexists$ a finite subcover of $(0, 1]$

Thm. $A \in \mathbb{R}$ is compact $\iff$ every open cover of A has a finite subcover.

Review

Def. $U \subset \mathbb{R}$ is open, if $\forall x \in U, \exists I = (a, b)$for c, such that $I \subset U$.

$U \subset \mathbb{R}$ is closed if $V = \overline{(V)}$

Thm. Duality: $U \subset \mathbb{R}$ is open $\iff$ U^0 is closed.

If $\{U_\alpha : \alpha \in A\}$ open sets, then $\cup = \cup_{\alpha \sin A}$

2.1.2016

Thm.

- $A \in \mathbb{R}$ is compact if every sequence in A has a limit point in A($\iff$ every sequence in A has a convergent subsequence in A)
- A is compact $\iff$ A is closed and bounded.

Thm. $A \in \mathbb{R}$ is compact $\iff$ every open cover of A has a finite subcover.

Proof.

$\Leftarrow$ Want to show: A is closed and bounded.

Closed Suppose that $y \notin A$. Consider $\{(-\infty, y-1/n) \cup (y+1/n, \infty)\}_{n\in\mathbb{Z}_{>0}} =$ open cover of $\mathbb{R} \setminus \{y\}$, hence of A. Since there exists a finite subcover $\Rightarrow A \subset (-\infty, y-1/n) \cup (y+1/n, \infty)$ for some n. $\Rightarrow \nexists x \in A$ with $|y-x| < 1/n$, which is a negation of a limit point $\Rightarrow y$ is not a limit point of $A \Rightarrow A$ is closed.

Bounded $\bigcup\limits_{n\in\mathbb{Z}_{>0}} (-n,n) = \mathbb{R} \supset A \Rightarrow A \subset (-n,n)$ for some $n \Rightarrow A$ is bounded.

$\Rightarrow$ Suppose A is compact. Let $\{U_\beta\}_{\beta\in B}$ be an open cover.

(a) $\forall x \in A, \exists \beta$ & (a,b), such that $x \in (a,b) \subset U_\beta$. Then $\exists p,q \in \mathbb{Q}$, such that (p,q) - there exist countably many such intervals (p,q). $\Rightarrow A$ can be covered by countably many (p,q)'s, hence by Axiom of Choice $U_{\beta 1}, U_{\beta 2}, U_{\beta 3}...$

(b) Suppose $\forall n, U_{\beta 1}, U_{\beta 2}, U_{\beta 3}...U_{\beta n}$ does not cover A. Then $\forall n, \exists x_n \in A \setminus \bigcup\limits_{i=1} U_{\beta i}$. Assuming A is compact $\Rightarrow (x_n)$ has a limit point $x \in A$. $A \subset \bigcup\limits_{j=1} U_{\beta j} \Rightarrow x \in U_{\beta m}$ for some m. But $x_n \notin U_{\beta m}, \forall n \geq m$. $\Rightarrow$ no subsequence of (x_n) can converge to x, which is a contradiction! $\square$

Def. A sequence of sets $A_1, A_2, ...$ is **nested** if $A_1 \supset A_2 \supset A_3 \supset ...$

Eg. $(0,1) \supset (0,1/2) \supset (0,1/3) \supset ... \Rightarrow \bigcap(n, 1/n) = \emptyset$

Thm. A nested sequence of non-empty compact sets always has a non-empty intersection.

Proof. Text/HW $\square$

Eg. If we replace the sets by their closures $[0,1] \supset [0,1/2] \supset ... \Rightarrow \bigcap([0,1/n] = \{0\}$

Chapter 4

Continuous functions

Def. Let $D \subset \mathbb{R}$ and function f defined on $D, x_0 \in D$. We say f is **continuous** at x_0 if $\forall 1/m > 0, \exists 1/n$(depending on $f, D, x_0, 1/m$), such that $\forall x \in D, |x - x_0| < 1/n \Rightarrow |f(x) - f(x_0)| < 1/m$. We say f is **continuous** if it is continuous at every $x_0 \in D$. We say f is **uniformly continuous** (on D) if $\forall 1/m > 0, \exists 1/n$(dependet on f, m, D) $, such that \forall x, x_0 \in D, |x - x_0| < 1/n \Rightarrow |f(x) - f(x_0)| < 1/n$

Remark. If x_0 is **not** a limit point of D (call x_0 an isolated point), then f is automatically continuous at x_0.

E.g. Any function on $(0, 1) \cup \{2\}$ is continuous at $x_0 = 2$

Def. Let f be a function on D and x_0 a limit point of D. We say that $y \in \mathbb{R}$ is a **limit point of** f **at** x_0 and write $\lim_{x \to x_0} f(x)$ if $\forall 1/m > 0, \exists 1/n$, such that $\forall x \in D$:

$$0 < |x - x_0| < 1/n \Rightarrow |f(x) - y| < 1/m$$

Remark.

1. A limit if unique if it exists (same proof as for sequence).
2. Value of f at x_0 (which ma not even exist) is irrelevant.
3. If $x_0 \in D$ then f is continuus at $x_0 \iff \lim\limits_{x \to x_0} f(x) = f(x_0)$

Thm. Let $f : D \to \mathbb{R}$ function, x_0 is a limit point of D. Then $\lim\limits_{x \to x_0} f(x)$ exists $\iff \forall$ sequences $(x_j) \in D \setminus \{x_0\}$, such that $x_j \to x_0$, $\lim\limits_{j \to \infty} f(x_j)$ exists.

Remark. Not necessary to assume that all such limits are equal, but it turns out that they are.

Proof.

($\Rightarrow$) Let $(*) y \equiv \lim\limits_{x \to x_0} f(x)$ and let $x_1, x_2 ... \in D \setminus \{x_0\}$ and the limit converges to x_0

Claim: $y \equiv \lim_{j\to\infty} f(x_j)$. To prove claim, let $m > 0$. By $(*), \exists 1/n$, such that $0 < |x - x_0| < 1/n \Rightarrow |f(x) - y| < 1/m$. Since $x_j \to x_0, \exists M$, such that $j \geq M \Rightarrow |x_j - x_0| < 1/n$, where $|x_j - x_0|$ is always positive. Combine these: $j \geq M|f(x_j) - y| < 1/m$. TBC ☐

s

Review - 4.3.2016

Thm.
- A is compact
- A is closed and bounded.
- Every open cover of A has a finite subcover.
- Every sequence $x_n \in A$ has a convergent subsequence

Question: 3.3.1.4. A is compact. $A \in B_1 \cup B_2$. Where $B - i$ is open and $B_1 \cup B_2 = \emptyset$. Prove $A \cap B_i$ is compact

Proof. Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of $A \cap B_i$. Because $B_1 \cap B_2 = \emptyset$ we can write $A = (A \cap B_1) \cup (A \cap B_2)$. If we consider $\mathcal{U}' = \{U_i \cup B_2 : i \in I\}. \forall i, U_i \cup B_2 \supset (A \cap B_1) \cup (A \cap B_2) = A$. Observe that $\mathcal{U}'$ is an open cover of A and because A is compact, $\exists U_{i1} \cup B_2, U_{i2} \cup B_2 ... U_{ir} \cup B_2$, such that $\mathcal{U}_{j=1}^{r}(U_{ij} \cup B_2) \supset A$.

Then $\forall 1 \leq j \leq r, (U_{ij} \cup B_2) \cap B_1 = (U_{ij} \cap B_1) \cup (B_2 \cap B_1)$ where $(B_2 \cap B_1)$ is an empty set $\Rightarrow \mathcal{U}_{j=1}^{r}(U_{ij} \cap B_1) \supset (A \cap B_1)$ So that $\mathcal{U}_{j=1}^{r} U_{ij} \Rightarrow \mathcal{U}'' = \{U_{i1}, ... U_{ij}\}$ ☐

Question: 3.3.1:6. $A, b \subset \mathbb{R}$. and $A + B \equiv \{a + b : a \in A, b \in B\}$ Prove: A, B is open $\Rightarrow A + N$ is open and A, B is compact $\Rightarrow A + B$ is compact.

Proof. Let $U = \{U_i : i \in I\}$ be open cover of $A + B$. Notice that $\forall x \in \mathbb{R}$. A is compact $\iff A + \{x\}$ is compact. $\forall b \in B, \mathcal{U}$ is still an open cover of $A + \{b\}$. By compactness of $A, \exists \mathcal{U}_b = \{U_{bi} ...\}$ finite subcover of $A + \{b\}$. $\mathcal{V}_b \equiv \mathcal{U}_{i=1}^{kb} U_{bi}$ is open, $\mathcal{V}_b \in b$. Further $\mathcal{V} = \{\mathcal{U}_b : b \in B\}$ is an open cover of B. By compactness of $B, \exists b_i ... b_n \in \{\}$, such that $B \supset \mathcal{U}_{j=1}^{n} V_{b_i}$.

Each of $V_{bi} \equiv \mathcal{U}_{l=1}^{k_{bi}} U_{bi,;}$. $\mathcal{U}'' = \{U_{i,j}, i \in \{b_1, ... b_n\}, j \in \{b_1, ... b_n\}\}$ ☐

Def. M is a **metric space** is $\forall x, y, z \in M, d : M \times M \to M$ where d is the distance.

- $d(x, y) \geq 0, (d(x, y) = 0 \iff x = y)$
- $d(x, y) = d(y, x)(|x - y| = |y - x|)$
- $d(x, z) \leq d(x, y) + d(y, z)$

E.g.

1. $\mathbb{R}, d(x,y) = |x-y|$
2. $\mathbb{R}^n$ $\{(x_1, ...x_n) : x_i \in \mathbb{R}\}$
3. $C^0([a,b]) = \{f : [a,b] \to \mathbb{R}\}$ and $d(f,a) = sup|f(x0 - g(x)|$ for $x \in [a,b]$

Def.

- $\{x_n\} \subset M$ is Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$
- $\{x_n\} \to x$ if $\forall \epsilon > 0, \exists K$, such that $n \geq K \Rightarrow d(x_n, x) < \epsilon$

Def: Completion of M.

1. $M*$ = set of Cauchy sequence in M.
2. $\sim$ on $M^* \iff \{x_n\} \sim \{y_n\}$ if $d(x_n, y_n) \to 0$, as $n \to \infty$
3. $\hat{M} = M^* \setminus \sim, P \in \hat{M}, P = [\{x_n\}]$
4. Define $d : \hat{M} \times \hat{M} \to \hat{M}$, $P = [\{x_n\}], \hat{d}(P,Q) = \lim d(x_n - y_n)$, $Q = [\{y_n\}]$. Verify that $(\hat{M}, \hat{d})$ is a metric space

E.g. $(M,d) = (Q$,1-1$), \hat{\mathbb{Q}} = \mathbb{R}$

Def. A continuous $f : M \to N(M,N$ is a metric space$)$, $x_n \to x, f(x_n) \to f(x)$

Def: Back to $\mathbb{R}$. $A, B, \subset \mathbb{R}, f : A \to B$ continuous at $x_0.\forall \epsilon = 1/N > 0, \exists \delta > 0$, such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

Thm. f is continuous $\iff$ $f^{-1}(u)$ is open for every $U \subset \mathbb{R}$ open. $f(x) \in (f(x_0) - \epsilon f(x_0) + \epsilon) \iff x \in f^{-1}((f(x_0) - \epsilon f(x_0) + \epsilon))$

Thm. Suppose $f : K \to \mathbb{R}$ is continuous. K is compact $\Rightarrow f(K)$ is also compact. In particular f achieves max and min.

Proof. Let $\mathcal{U}_i \equiv \{U_i : i \in I\}$ be an open cover of $f(K)$ of M.

$$f^{-1}(f(K)) \subset f^{-1}(\mathcal{U}_{i \in I} U_i) = \mathcal{U}_{i \in I} f^{-1}(U_i)$$

By continuity $\mathcal{U}_i' = \{f^{-1}(U_i) : i \in I\}$ is an open cover of K. So we can extract$U_i...U_n$, such that $K \subset \mathcal{U}_{i=1}^n f^{-1}(u_i) \Rightarrow f(K) \subset \mathcal{U}_{i=1}^n U_i$ □

7.3.2016

Thm. Let $f : D \to \mathbb{R}$ function, x_0 is a limit point of D. Then $\lim_{x \to x_0} f(x)$ exists $\iff \forall$ sequences $(x_j) \in D \setminus \{x_0\}$, such that $x_j \to x_0$, $\lim_{j \to \infty} f(x_j)$ exists.

Proof. (continued)

($\Leftarrow$) First show that if $x_j \to x_0$ and $y_j \to x_0$, then $\lim\limits_{j\to\infty} f(x_j) = \lim\limits_{j\to\infty} f(y_j) \equiv a = b$. Well $x_1, y_1, x_2, y_2...$ converges to x_0 so $f(x_1), f(y_1), f(x_2)...$ converges, hence it has a unique limit point. a, b are both limit points, so $a = b$.

Now let $a \equiv \lim_{j\to\infty} f(x_j)$ for some (hence any) sequence $x_j \to x_0$. Suppose by contradiction that $\lim_{x\to x_0} f(x) \neq a$. Then $\exists m$, such that $\forall n, \exists x_n \in D$ with $0 < |x_n - x_0| < 1/n$, but $|f(x_n) - a| < 1/m$. Clearly $x_n \to x_0$, but $lim_{n\to\infty} x_n \neq a$, which is a contradiction.

$\square$

Thm. Let $f : D \to \mathbb{R}$. Then f is continuous on $D \iff \forall$ sequences (x_j) in D with a limit in D, the sequence $(f(x_j))$ is convergent.

Remark. As before, it's not necessary to assume that $\lim_{j\to\infty} f(x) = f(\lim_{j\to\infty} x_j)$, but this will always be the case. Continuous functions commute with limits.

Proof.

($\Rightarrow$) Same argument as in ($\Rightarrow$) part of previous proof

($\Leftarrow$) Let $x_0 \in D$. Use a "shuffle sequence" argument as in previous proof to show that the $\lim_{x\to\infty} f(x_j)$ is the same $\forall$ sequences $x_k \to x_0 \in D$.

Note that $x_0, x_0, x_0...$ converges to x_0, so this common limit is $f(x_0)$.

If x_0 is not a limit point of D, then f is trivially continuous at x_0. If x_0 is a limit point of D, then previous theorem: $\lim_{x\to x_0} f(x) = f(x_0)$.

$\square$

Recall: $f : S \to T, T' \subset T$,then$f^{-1}(T') = \{s \in S | f(s) \in T'\}$, which is called inverse image.

Thm. Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$ is open. Then f is continuous $\iff \forall$ open $U \subset \mathbb{R}, f^{-1}(U)$ is open.

Proof. !!!!

($\Rightarrow$) Let $U \subset \mathbb{R}$ be open. Let $x_0 \in f^{-1}(U)$. So, when U is open $f(x - 0) \in U \Rightarrow \exists 1/m > 0$, such that $(f(x_0) - 1/, f(x_0) + 1/m) \subset U$. When f is continuous and D is open $\Rightarrow \exists$ single $1/n$, such that $(x_0 - 1/n, x_0 + 1/n) \subset D$ and $|x - x_0| < 1/n \Rightarrow |f(x) - f(x_0)| < 1/m$. In other words $x \in J \Rightarrow f(x) \in I \subset U \Rightarrow x \in f^{-1}(U)$. So $x_0 \in T \subset f^{-1}(U)$. Therefore $f^{-1}(U)$ is open.

($\Leftarrow$) Let $x_0 \in D$ and $1/m > 0$. $U \equiv (f(x_0) - 1/m, f(x_0) + 1/m) = \{y \in \mathbb{R} || y - f(x_0)| < 1/m\}$. Obviously $x_0 \in f^{-1}(U)$, which is open, so $\exists 1/nst(x_0 - 1/n, x_0 + 1/n) \subset f^{-1}(U)$. Hence $|x - x_0| < 1/n \Rightarrow x \in J \Rightarrow f(x) \in U \Rightarrow |f(x) - f(x_0)| < 1/m$.

□

Remark. Similarly, if f is defined on some neighbourhood of x_0, then f is continuous at $x0 \iff \forall$ neighbourhood U of $f(x_0), \exists$ neighbourhood V of x_0, such that $f(V) \subset U \iff$ ....$f^{-1}(U)$ is continuous in the neighbourhood of x_0.

Def. $f : D \to \mathbb{R}$ satisfies a **Lipschitz condition** if $\exists M > 0$, such that $\forall x, x_0 \in D$, then $|f(x) - f(x_0)| \leq M|x - x_0|$.

Thm. If f satisfies a **Lipschitz condition** on D, then f is uniformly continuous on D.

Proof. Homework □

Def. Let $f : D \to \mathbb{R}$ and x_0 be a limit point of D. We say:

$\lim_{x \to x_0^+} f(x) = y$ if $\forall 1/n > 0, \exists 1/m$, such that $x_0 < x < x_0 + 1/m, x \in D \Rightarrow |f(x) - y| < 1/n$.

$\lim_{x \to x_0^-} f(x) = y$ if $\forall 1/n > 0, \exists 1/m$, such that $x_0 - 1/m < x < x_0, x \in D \Rightarrow |f(x) - y| < 1/n$.

f is **continuous from the right at** x_0 if $\lim_{x \to x_0^+} f(x) = f(x_0)$.

f is **continuous from the left at** x_0 if $\lim_{x \to x_0^-} f(x) = f(x_0)$.

If $\lim_{x \to x_0^+} f(x) = f(x_0)$ and $\lim_{x \to x_0^-} f(x) = f(x_0)$ exist, but are not equal, we say that f has a **jump discontinuity** at x_0.

4.1 Properties of Continuous functions

Let $f, g : D \to \mathbb{R}$, we define $f + g : D \to \mathbb{R}, x \rightarrowtail f(x) + g(x)$. $f + g$ is continuous at $x_0 \in D$ if f, g are. (Limits respect sums.)

Similarly for $f - g$, fg, f/g. Easy: constant function and identity function are continuous. $\Rightarrow$ all rational functions $x \rightarrowtail p(x)/g(x0)$, where polynomials p, g are continuous on $\{x \in \mathbb{R} | g(x) \neq 0\}$.

$\max(f, g) : D \to \mathbb{R} : x \rightarrowtail \max\{f(x), g(x)\}$, likewise for $\min$.

Thm. Whenever f, g are continuous at $x_0 \Rightarrow \max(f, g), \min(f, g)$ are also continuous at x_0.

Proof. Just do $\max(f, g) \equiv h$. $h(x_0) = f(x_0)$ or $g(x_0)$. Without loss of generality (WOLOG) assume $f(x_0)$. Then $f(x_0) - g(x_0) \geq 0$. Let $1/m > 0$.

Case 1: $f(x_0) - g(x_0) = (f - g)(x_0 > 0)$. f is continuous: $\exists 1/n_1$, such that $|x - x_0| < 1/n_1, x \in D \Rightarrow |f(x) - f(x_0)| < 1/m$.

$f - g$ is continuous: $\exists 1/n_2$, such that $|x - x_0| < 1/n_2, x \in D \Rightarrow (f - g)(x) \in (0, \infty) \Rightarrow h(x) = f(x)$. Hence $|x - x_0| < \min\{1/n_1, 1/n_2\}, x \in D \Rightarrow |h(x) - h(x_0)| = |f(x) - f(x_0)| < 1/m$.

Case 2: $f(x_0) = g(x_0)$
f is continuous: $\exists 1/n_1$ as before

g is continuous: $\exists 1/n_2$, such that $|x - x_0| < 1/n_2, x \in D \Rightarrow |g(x) - g(x_0)| < 1/m$.

Hence $|x - x_0| < \min\{1/n_1, 1/n_2\}, x \in D \Rightarrow |h(x) - h(x_0)| = |f(x) - f(x_0)| or |g(x) - g(x_0)|$, both $1/m$.

□

9.3.2016

Thm: Intermediate Value Theorem. Let $f : [a, b] \leftarrow \mathbb{R}$ be continuous, say with $f(a) < f(b)$. Then $[f(a), f(b)] \subset f([a, b])$ (or analogously if $f(b) < f(a)$).

Proof. Let $y \in (f(a), f(b)$. Divide and conquer. If $f(\frac{a+b}{2}) = y$, then done. If $f(\frac{a+b}{2}) < y$, then set $a_1 = \frac{a+b}{2}, b_1 = b$. If $f(\frac{a+b}{2}) > y$, then set $a_1 = a, b_1 = \frac{a+b}{2}$. Either way $f(a_1) < y < f(b_1)$. Either at some stage we find x, such that $f(x) = y$ and we are done or we obtain sequences: $a_1 \leq a_2 \leq a_3 ...$ and $b_1, b_2, b_3 ....$, such that $f(a_k) < y < f(b_k)$. So $b_j - a_k = \frac{b-a}{2^k} \forall k \implies \lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k \equiv x \in [a, b]$ Since f is continuous, $f(x) = \lim_{k\to\infty} f(a_k) \leq \lim_{k\to\infty} f(b_k) = f(x) \Rightarrow f(x) = y \Rightarrow y \in f([a, b])$. □

Thm. Let $f : D \to \mathbb{R}$ be a continuous function. If D is compact, then $f(D)$ is compact.

Proof. Let (y_k) be a sequence in $f(D)$. $\forall k$, choose any $\tilde{y_k} \in D$, such that $f(\tilde{y_k}) = y_k \Rightarrow (\tilde{y_k})$ has a subsequence $(\tilde{y_{k_r}})$ converging to some $x \in D$.

Hence $\lim r \to \infty y_{k_r} = \lim r \to \infty f(\tilde{y_{k_r}}) = f(x) \in f(D)$. Therefore y_{k_r} is a convergent subsequence of (y_k). $\square$

Corollary: Extreme Value Theorem. let D be compact and $f : D \to R$ be continuous, then f is bounded (i.e $f(D)$ is bounded) and $f(D)$ contains its sup and inf.

Proof. By previous theorem: $f(D)$ ic compact $\Rightarrow f(D)$ os cpunded and contains sup and inf by HW. $\square$

[Uniform continuity theorem]

Thm. Let $f : D \to \mathbb{R}$ be continuous and D be compact. The f is uniformly continuous on D.

Proof. Let $1/m > 0$. $\forall x_0 \in D, \exists 1/n_{x_0}$, such that $x \in (x_0 - 1/n_{x_0}, x_0 + 1/n_{x_0}) \equiv J_{x_0} \Rightarrow |f(x) - f(x_0)| < 1/2m$.
$I_{x_0} \equiv (x_0 - 1/2n_{x_0}, x_0 + 1/2n_{x_0}), D \subset \bigcup_{x_0 \in D} I_{x_0}$.

Since D is compact $\implies \exists x_i, ...x_r$, such that $D \subset \bigcup_{i=1}^{r} I_{x_0}$. Now suppose $|x - y| < min\{1/2n_{x_1}, ...1/2n_{x_r}\}, x, y \in D$.

$x \in I_{x_i} \subset J_{x_i}$ for some i.

$|x_i - y| \leq |x_i - x| + |x - y| < 1/2n_{x_i} + 1/2n_{x_i} = 1/n_{x_i} \Rightarrow y \in J_{x_i}$.

Hence $|f(x) - f(y)| \leq |f(x) - f(x_i))| + |f(x_i) + f(y)| < 1/2m + 1/2m = 1/m$ $\square$

Def. $f : D \to \mathbb{R}$ is **monotone increasing** if $x < y, x, y \in D \Rightarrow f(x) \leq f(y)$.

Def. $f : D \to \mathbb{R}$ is **monotone decreasing** if $x < y, x, y \in D \Rightarrow f(x) \geq f(y)$.

Thm: Monotone function theorem. Let f be a monotone function defined on interval I. Then $\forall x_0$ in interior of $I, \lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ exist and are finite. Appropriate one-sided limits exist (possibly $\pm\infty$) at endpoints too.

Proof. WOLOG f is monotone increasing. Let x_0 be an interior point in I. $E \equiv \{f(x) | x < x_0, x \in I\} \neq \emptyset$, bounded above by $f(x_0)$. ClaimL if we take $\sup E = \lim_{x \to x_0^-} f(x)$. Given $/1m > 0, \exists x_m < x_0 \in I$, such that $y - 1/m < f(x_m) \leq y$. -By midterm problem 4. hence $x \in (x_m, x_0) \Rightarrow y - 1/m < f(x_m) \leq f(x) \leq y \Rightarrow |y - f(x)| < 1/m$. Then $\lim_{x \to x_0^+} f(x)$ is similar. $\square$

Corollary. Let $f : (a, b) \to \mathbb{R}$ be monotone. Then f is continuous except at countably many points, where there exist a jump discontinuityy.

Proof. By previous theorem STS (suffices to show) that at most countably many points where f is discontinuous. WOLOG let's assume that f is increasing function. $\forall x_0 \in (a,b), j(x_0) \equiv \lim_{x \to x_0^+} f(x) - \lim_{x \to x_0^-} f(x)$ is a non-negative function ("jump of f at x_0).

f is discontinuous at $x_0 \iff j(x_0) > 0$. For $m \in \mathbb{Z}_{>0}, [c,d] \subset (a,b)$. Define $S_{m,[c,d]} \equiv \{x_0 \in (c,d) | j(x_0) \geq 1/m\}$.

$\{x_0 \in (a,b) |$f discontinuous at $x_0\} = \bigcup_{m \in \mathbb{Z}_{>0}, [c,d] \subset (a,b)} S_{m,[c,d]}$, which is a countable union. It suffice so show (STS) that $S_{m,[c,d]} \equiv S$ is always finite.

Suppose $x_1 < x_2 < ... x_n \in S$ Choose $y_i, ... y_{n-1} \in (c,d)$, such that $y_0 \equiv c < x_1 < y_1 < x_2 < y_2 < ... < y_{n-1} < x_n < d \equiv y_n$. Then $\forall i = 1, ...n, \lim_{x \to x_0^-} f(x0 \geq f(y_{i-1})$ and $\lim_{x \to x_0^+} f(x0 \leq f(y_i) \implies j(x_i) \leq f(y_i)$ or $f(y_{i-1}) \implies j(x_1) + j(x_2) + ... j(x_n) \leq [f(y_1) - f(y_0)] + [f(y_2) - f(y_1)] + ... + [f(y_n) - f(y_{n-1})] = f(d) - f(c) \implies n \leq m(f(d) - f(c))$. ☐

11.3.2016

Def. $f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0 if $\forall \epsilon, \exists \delta > 0$, such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.

Def. $f : \mathbb{R} \to \mathbb{R}$ is uniform continuous on $U \subset \mathbb{R}$ if $\forall \epsilon, \exists \delta$, such that $\forall x, y \in U$ and $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.

Def. $f : \mathbb{R} \to \mathbb{R}$ is monotone.

increasing if $x < y \Rightarrow f(x) \leq f(y)$

decreasing if $x < y \Rightarrow f(x) \geq f(y)$

E.g. A function that is continuous, but not uniform, $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$). We know that $|f(x) - f(x)| = |x - y|.|x + y|$. Formally we want to show that $\exists \epsilon > 0$, such that $\forall \delta > 0, \exists x, y \in \mathbb{R}$, such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

Take $\epsilon = 1$, let $\delta > 0$ be given, so $x = \delta/2 + 1/\delta, y = \delta + 1/\delta$, then $|x - y| = \delta/2 < \delta$

E.g. $f : (0, \infty), f(x) = 1/x) \Rightarrow |f(x) - f(y)| = |1/x - 1/y| = \frac{|x-y}{|xy|}$

Thm: 1. $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $K \subset \mathbb{R}$ is compact $\Rightarrow f(K) \subset \mathbb{R}$ is also compact.

Thm: 2. $f : K \to \mathbb{R}$ is continuous function and $K \subset \mathbb{R}$ is compact $\Rightarrow f$ is uniformly continuous.

Thm: Intermediate value theorem - IvT. Ff $f : (a,b) \to \mathbb{R}, f(a) < f(b)$ is continuous. Then $\forall c \in (f(a), f(b)), \exists x_0 \in (a,b)$, such that $f(x_0) = c$

Thm: 3. Suppose f, g are continuous at $x_0(f, g : \mathbb{R} \to \mathbb{R})$

1. $f \pm g$ is continuous
2. $\forall c \in \mathbb{R}, cf$ is continuous
3. $f.g$ is continuous at x_0
4. If $g(x_0) \neq 0 \Rightarrow f/g$ is also continuous at x_0
5. $\max\{f, g\}, \min\{f, g\}$ are continuous at x_0

   Remark. Let $C^0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f$ is continuous $\}$,

- $(1), (2) \Rightarrow C^0(\mathbb{R})$ is a vector space
- $(2), (3) \Rightarrow C^0(\mathbb{R})$ is a ring - addition, multiplication component-wise.
- By Thm 1 $\Rightarrow f : K \to \mathbb{R}$ continuous K compact, then f attains max and min

Thm 1. Show $f(K)$ is compact. Let$\mathcal{U} = \{U_i, i \in I\}$ be an open cover of $f(K)$. then $K \subset f^{-1}(f(K)) \subset f^{-1}(U_{i \in I} U_i) = U_{i \in I} f^{-1}(U_i)$

Because K is compact $\Rightarrow K \subset U_{i=1}^n f^{-1}(U_i))$... □

THm2. Since $f : K \to \mathbb{R}$ is continous, let $\epsilon > 0, \forall x \in K, \exists \delta_x > 0, |x' - x| < \delta_x \Rightarrow |f(x) - f(x')| < \epsilon$. Let $\gamma_x = \delta_x/3 > 0$. Consider $\mathcal{U} = \{B_{\gamma x} : x \in K\}, N_{\gamma x}(x) = (x - \gamma_x, x + \gamma_x)$.
content...
By compactness extract $B_i = B_{\gamma i}(x_i), 1 \leq i \leq n$, such that $K \subset N_1 \cup B_2, \cup .... B_n (*)$

$.(*) \Rightarrow \forall x \in K, |x - x_i| < \delta_x/2 for some 1 \geq i \geq n..$
Goal: $\exists \delta > 0,$, such that $\forall x, y \in K, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

$\delta = min\{\gamma_{x1}, \gamma_{x2} ... \gamma_{xn}\} > 0$

Let $x, y \in K, |y - x_i| \leq |x - x_i| + |x - y| < \delta_x i/2 + \gamma_{xi} = \delta_{xi} \Rightarrow |f(y) - f(x_i) < \epsilon$.

By $|x - x_i| < \delta_x/2 \Rightarrow |f(x) - f(x_i)| < \epsilon..$
Together $|f(x) - f(y) \leq |f(x) - f(x_i) + |f(y) - f(x_i)| < 2\epsilon$. □

Chapter 5

Differential Calculus

21.3.2016

5.1 bla?

Def. Let f be a function defined on some neighbourhood of $x_0 \in \mathbb{R}$. We say that f is **differentiable at** x_0 if $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists (in $\mathbb{R}$) and we call this limit the **derivative** $f'(x_0)$.

Equivalently substitute $h = x - x_0$ then:

$$f'(x_0 = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

If f is defined on an open set, we say f is **differentiable** if it is differentiable at every point. Geometrically $f'(x_0)$ is slope of tangent line to the graph of f at $(x_0, f(x_0))$

$f \rightarrowtail f(x_0) + f'(x_0)(x - x_0)$ is best approximating affine (linear) function to f near x_0.

Def. Let f, g be function defined near x_0. We say $f(x) = O(g(x))$ as $x \to x_0$ if $\exists 1/n$ & $c > 0$, such that $|x - x_0| < 1/n \Rightarrow |f(x)| \geq c|g(x)|$. Almost equivalently f/g is bounded for $|x - x_0| < 1/n$.

We say $f(x) = o(g(x))$ as $x \to x_0$ if $\forall 1/m > 0 \exists 1/n$, such that $|x - x_0| < 1/n \Rightarrow |f(x)| \leq 1/m|g(x)|$. Almost equivalently $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$

Remark. $o(g(x)) \Rightarrow O(g(x))$, but not conversely.

E.g. $g(x) \equiv 1$ Then $f(x) = O(1)$ near $x_0 \iff f$ is bounded near x_0 and $f(x) = o(1)$near $x_0 \iff \lim f(x) = 0$

$f(x) - f(x_0) = o(1)$ near $x_0 \iff f$ is continuous at x_0.

$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ is equivalent to $\forall 1/m > 0 \exists 1/n s.t O < |x - x_0| < 1.n \Rightarrow |\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)| < 1/m \iff |f(x)-f(x_0)-f'(x_0)(x-x_0)| < 1/m|x-x_0|$
Set $h(x) = f(x_0) = f'(x_0)(x-x_0)$. f is differentiable at $x_0 \iff f(x)-h(x) = o(|x-x_0|)$ as $x \to x_0$.

Remark. If f is differentiable on an open set, then $x \rightarrowtail f'(x)$ is itself a function.

Thm. If f is differentiable at $x_0 \Rightarrow f$ is continuous at x_0.

Proof. $\lim_{x\to x_0}[f(x) - f(x_0)] = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}(x - x_0) = f'(x_0) \times 0 = 0$ □

Subtle point: Even if f' exists on some open set A, f' may not be continuous.

E.g. $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $(x) = 0$ at $x = 0$.

Def. We say that f is **continuously differentiable** or **C'** on A if f' exists and is continuous on A.

5.2 Properties of derivatives

Def. Let f be defined in some neighbourhood of x_0 we say that:

1. f is monotone increasing at x_0 if there exists a neighbourhood U of x_0 such that $\forall x_1 < x_0, x_2 \in U$, $f(x_1) \leq f(x_0) \leq f(x_2)$
2. f is monotone increasing on set A if $\forall x < y \in A, f(x) \geq f(y)$
3. f has a **local maximum** at x_0 if $\exists$ a neighbourhood U of x_0, such that $\forall x \in U, f(x) < f(x_0)$
4. Analogous for decreasing and minimum.

Thm. Let f be defined in neighbourhood at x_0 and differentiable at x_0

1. $f'(x_0) > 0 \Rightarrow f$ is strictly increasing at x_0
   $f'(x_0) > 0 \Rightarrow f$ is strictly decreasing at x_0
2. f is monotone increasing at $x_0 \Rightarrow f'(x_0) \geq 0$
3. f has a local maximum or minimum at $x_0 \Rightarrow f'(x_0) = 0$

Proof.

1.
$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}>0\Rightarrow\frac{f(x)-f(x_0)}{x-x_0}>0,\forall x\neq x_0$$
hence for $x\neq x_0$ in U, if $x>x_0$, then $f(x)-f(x_0)>0\Rightarrow f(x)>f(x_0)$

2. f is monotone inxreasing at $x_0\Rightarrow\frac{f(x)-f(x_0)}{x-x_0}\geq 0$ for $x\neq x_0$ near $x_0\Rightarrow f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\geq 0$

3. Local maximum case: $\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}\geq 0$ and $\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}\leq 0\Longrightarrow f'(x_0)=0$

□

Thm: Intermediate value Theorem for derivatives. Let f be differentiable on some open interval (a,b) and $x_1<x_2\in(a,b)$. Then f' assumes all values between $f'(x_1)$ and $f'(x_2)$.

Remark. 1. This follows from IvT if f' is contiuous on $[x_1,x_2]$, but we needn't assume this.

2. Theorem says f' can't have a jump discontinuity.

Proof. s

Step 1 Suppose O between $f'(x_1)$ and $f'(x_2)$ we say $f'(x_1)<0<f'(x_2)$. If f is differentiable on $(a,b)\Rightarrow f$ is continuous on $[x_1,x_2]\Rightarrow f$ attains its inf on $[x_1,x_2]$, say at x_0 By previous theorem 1. $f'(x_1)<0\Rightarrow f$ stricly decreases at $x_1\Rightarrow x_0\neq x_1$, $f'(x_2)>0\Rightarrow f$ strictly increases at $x_2\Rightarrow x_0\in(x_1,x_2$ is a local min, by previous theorem $f'(x_0)=0$

Step 2 General case: Suppose y_0 is between $f'(x_1)$ and $f'(x_2)$, WOLOG $f'(x_1)<y_0<f'(x_2)$ and $g(x)\equiv y_0x$.
$$F\equiv f-g\Rightarrow F'=f'-g'=f'-y_0$$
$$F'(x_1)=f'(x_1)-y_0<0$$
$$F'(x_2)=f'(x_2)-y_0>0$$
Step 1: $\exists x_0\in(x_1,x_2)$, such that $F'(x_0)=0\Rightarrow f'(x_0)-y_0\Rightarrow f'(x_0)=y_0$

□

23.3.2016

Thm: Mean Value Theorem. Let f be continuous function on $[a,b]$ and differentiable on (a,b). Then $\exists x_0\in(a,b)$, such that $f'(x_0)=\frac{f(b)-f(a)}{b-a}$.

Proof.

Step 1 Assume $f(a) = f(b)$, i.e. slope is 0. Since f is continuous on $[a, b] \Rightarrow f$ attains its sup and inf on $[a, b]$. If either is attained in (a, b), we get local max or min and we win by earlier theorem. If sup and inf are both attained at endpoints, then since $f(a) = f(b)$ the function f is constant on $[a, b] \Rightarrow f' = 0$.

Step 2 General case: $g(x) = m(x - a) + f(a)$. Define $F = f - g$ to be continuous on $[a, b]$ and differentiable on (a, b).

$$F(a) = f(a) - g(a) = f(a) - f(a) = 0$$
$$F(b) = f(b) - (\frac{f(b) - f(a)}{b - a} \times (b - a) + f(a)) = 0 = F(a)$$

By Case 1, $\exists x_0 \in (a, b)$, such that $F'(x_0) = 0 \Rightarrow f'(x_0) - g'(x_0) = f'(x_0) - m \Rightarrow f'(x_0) = m$

□

Thm. Let f be differentiable on (a, b)

1. f is monotone increasing on $(a, b) \iff f'(x) \geq 0, \forall x \in (a, b)$.
2. $f'(x) > 0, \forall x \in (a, b) \Rightarrow f$ is strictly increasing on the interval.
3. $f'(x) = 0, \forall x \in (a, b) \Rightarrow f$ is constant.

Proof. 1. Monotone increasing/decreasing on $(a, b) \Rightarrow$ monotone increasing/decreasing at every point in (a, b). So we win by earlier theorem.

Conversely suppose $f'(x) \geq 0, \forall x$. Let $x_1 < x_2 \in (a, b)$. Apply MVT to f on $[x_1, x_2]. \exists x_0 \in (x_1, x_2)$, such that $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x_0) \geq 0 \Rightarrow f(x_2) \geq f(x_0)$.

2. Same argument...
3. Same argument (i.e. conclude $f(x_1) = f(x_1), \forall x_1, x_2 \in (a, b)$)

□

5.3 Differentiation Rules

$$\Delta_h f(x) \equiv f(x + h) = f(x)$$

Thm. f, g are differentiable at $x_0 \Rightarrow f \pm g$ & fg are differentiable at x_0.

$$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$$
$$(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$$

If $g(x_0) \neq 0$ thenf/g is differentiable at x_0:

$$\left(\frac{f}{g}\right)(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Proof. Just quotient rule. g is differentiable at $x_0 \Rightarrow g$ is continuous at $x_0 \Rightarrow g(x_0) \neq 0, g(x_0+h) \neq 0$ for h small.

$$\lim_{h\to 0} \frac{\Delta_h(f/g)(x_0)}{h} = \lim_{h\to 0} 1/h \left[\frac{f(x_0+h)}{g(x_0+h)} - \frac{f(x_0)}{g(x_0)}\right]$$

$$= \lim_{h\to 0} 1/h \left[\frac{g(x_0)f(x_0+h) - f(x_0)g(x_0+h) \pm g(x_0)g(x_0)}{g(x_0+h)g(x_0)}\right]$$

$$= \lim_{h\to 0} \frac{g(x_0)\frac{\Delta_h f(x_0)}{h} - f(x_0)\frac{\Delta_h g(x_0)}{h}}{g(x_0+h)g(x_0)}$$

$$= \frac{g(x_0)f'(x_0) - g'(x_0)f(x_0)}{g(x_0)^2}$$

□

Thm: Chain Rule. f, g is differentiable in neighbourhood of $x_0, f(x_0)$ and differentiable at $x_0, f(x_0) \Rightarrow g \circ f$ is differentiable at x_0

$$(g \circ f)(x_0) = g'(g(x_0))f'(x_0) = \frac{dg}{dx} = \frac{dg}{dy}\frac{dy}{dx}$$

Proof. It suffices to show that $g(f(x)) - g(f(x_0)) - s(x-x_0) = o(x-x_0)$ as $x \to x_0$. I.e. given $/1m > 0$, it suffices to show that $|LHS| \leq \frac{|x-x_0}{m}$ for x sufficiently close to x_0.

$$|LHS| \leq |g(f(x)) - g(f(x_0)) - g'(f(x_0))(f(x) - f(x_0)| + ...(1)$$
$$+|g'(f(x_0))(f(x) - f(x_0) - g'(f(x_0))f(x_0)(x-x_0)|...(2)$$

For (2) $= |g'(f(x_0))||f(x) - f(x_0) - f'(x_0)(x-x_0)|$ If $g'(f(x_0)) = 0$, then $(2) = 0$. If $(2) \neq 0$: f is differentiable at $x_0 \Rightarrow$ can make:

$$|f(x) - f(x_0) - f'(x_0)(x-x_0)| \leq \frac{|x-x_0}{2m|g'(f(x_0))}$$

for some x sufficiently close to x_0. $\Rightarrow (2) \leq \frac{|x-x_0|}{2m}$.

For (1) First since f is differentiable, we can make $|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq |x - x_0|$ for $x \sim x_0 \Rightarrow |f(x) - f(x_0)| \leq (1 + |f'(x_0)|)|x - x_0| \equiv M|x - x_0|$.
g is differentiable at $f(x_0)$ for $f(x) \sim f(x_0)$:

$$|g(f(x)) - f(f(x_0)) - g'(f(x_0))(f(x) - f(x_0))| \leq \frac{|f(x) - f(x_0)|}{2mM} \leq \frac{M|x - x_0|}{2mM}$$

for $x \sim x_0$. Conclude: for $x \sim x_0, (1) + (2) \leq \frac{|x-x_0|}{2m} + \frac{|x-x_0}{2m} = \frac{|x-x_0|}{m}$

□

Thm: Inverse function theorem. Let f be a C^1 function on (a, b) with image (c, d) & $f' < 0$ or $f' > 0$ on (a, b). Then $f^{-1} : (c, d) \longrightarrow (a, b)$ exists and is C^1 and $(f^{-1})'(y) = \frac{1}{f'(x)}$ where $y = f(x)$.

Heuristic: if we assume f^{-1} is differentiable then $f^{-1}(f(x)) = x, \forall x \in (a, b) \Rightarrow (f^{-1})'(f(x))f'(x) = 1 \Rightarrow (f^{-1})'(y) = \frac{1}{f'(x)}$

28.3.2016 Might be on a final a continuous fcn on an interval, which is one-to-one is eitehr increasing or decreasing.

Proof. WOLOG $f' > 0 \Rightarrow f$ is strictly increasing on (a, b)

Step 1 f^{-1} is continuous. Let $y_0 \in (c, d)$ we say $y_0 = f(x_0)$. Fix $1/n > 0$, f' is continuous and $f'(x_0) \neq 0 \Rightarrow \exists N > 0$ and neighbourhood (α, β) of x_0, such that $|f'(x_1) \geq 1/N, \forall x \in (\alpha, \beta)$ Then $\forall x \neq x_0 \in (\alpha, \beta)$, $\frac{f(x)-f(x_0)}{x-x_0} = f'(x_1)$ for some $x < x_1 < x_0$ and in (α, β) by the MVT $\Rightarrow |f(x) - f(x_0)| \geq \frac{|x-x_0|}{N}$

Then if $|y - y_0| \leq 1/Nn$ and $y \in (f(\alpha), f(\beta))$

$$|f^{-1}(y) - f^{-1}(y_0)| = |f^{-1}(y) - x_0|$$

$$\leq N|f(f^{-1}(y)) - f(x_0)| = N|y - y_0| \leq N1/Nn = 1/n$$

Step 2 Proof of theorem:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \Rightarrow \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$$

On the other hand (OTOH)

$$(f^{-1})'(f(x_0)) = \lim_{y \to f(x_0)} \frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} = \lim_{y \to f(x_0} \frac{f^{-1}(y) - x_0}{y - f(x_0)}$$

$$= \lim_{y\to f(x_0)} \frac{x-x_0}{f(x)-f(x_0)}$$

Since f^{-1} is continuous, $y \to f(x_0) \Rightarrow f^{-1}(y) \to f^{-1}(f(x_0)) = x_0$. So:

$$\lim_{y\to f(x_0)} \frac{x-x_0}{f(x)-f(x_0)} = \lim_{x\to x_0} \frac{x-x_0}{f(x)-f(x_0)} = \frac{1}{f'(x_0)}$$

□

Thm: Local Inverse Function Theorem. Let f be C^1 on neighoburhood of x_0 and suppose that $f'(x_0) \neq 0$. Then $\exists$ a neighbourhood (a,b) of x_0, such that $f|_{a,b}$ has C^1 inverse on $f((a,b))$.

Proof. Since f' is continuous and non-zero at $x_0 \Rightarrow \exists(a,b) \ni x_0$, such that $f'|_{(a,b)} > 0$ or < 0. Win by previous theorem. □

5.4 Higher derivatives and Taylor's theorem

Thm. Suppose f is differentiable in the neighbourhood of x_0 and $f''(x_0)$ exists. Define $g(x) \equiv f(x_0) = f'(x_0)(x-x_0)$

1. $f''(x_0) > 0 \Rightarrow \exists$ neighbourhood $U \ni x_0$, such that $f(x) > g(x), \forall x \in U\backslash\{x_0\}$.
2. $f''(x_0) > 0 \Rightarrow \exists$ neighbourhood $U \ni x_0$, such that $f(x) < g(x), \forall x \in U\backslash\{x_0\}$.
3. $f(x) \geq g(x)$ in neighbourhood of $x_0 \Rightarrow f''(x_0) \geq 0$.
4. Suppose also $f'(x_0) = 0$ Then $f''(x_0) > 0 \Rightarrow x_0$ is strictly a local min.
5. If x_0 is a local mi $\Rightarrow f''(x_0) \geq 0$.

Proof.

4. WOLOG $f'(x_0) > 0 \Rightarrow f'$ stirctly increasing at x_0. Since $f'(x_0) = 0$ we have for x sufficiently close to x_0 we say $|x-x_0| < 1/n$. So if $x < x_0 \Rightarrow f'(x) < f'(x_0) = 0$ and if $x > x_0 \Rightarrow f'(x) > 0$. So we have f strictly increasing or decreasing on $(x_0 - 1/n), x_0$ or $(x_0, x_0 + 1/n) \Rightarrow x_0$ is a strict local min.
5. Contrapositive version follows from 4.
1. Assume $f''(x_0) > 0$. Take $g(x) = f(x_0) + f'(x_0)(x-x_0)$. Define $h \equiv f - g$. We have $h(x_0) = h'(x_0) = 0$, $h''(x_0) = f''(x_0) > 0$. Part 4. now applies to h, which has a strict local minimum at $x_0 \Rightarrow h(x) > 0 \forall x \neq x_0$ in the neighbourhood of x_0 i.e. $f(x) > g(x)$ for such x. $f''(x_0) < 0$ is similar.
3. Follows similarly form [5.]

□

E.g. $f(x) = x^4$ has a strict local minimum at x=0, but $f''(0) = 0$

Thm. Let f be C^2 on (a, b). Let g be an affine function, such that $g(x_1) = f(x_1)$ and $g(x_2) = f(x_2)$ for some $x_1 < x_2 \in (a, b)$. Then $f''(x) > 0 \forall x \in (x_1, x_2) \Rightarrow f(x) < g(x) \forall x \in (x_1, x_2)$.

Proof. WOLOG $f'' > 0$ on (x_1, x_2). Define $h \equiv f - g$. Since $g'' \equiv O, h'' > 0$ on (x_1, x_2) We want $h < 0$on (x_1, x_2) Suppose not, then since $h(x_1) = h(x_2) = 0$ h would have local maximum at some $x_0 \in (x_1, x_2)$. By previous theorem(5.) $\Rightarrow f''(x_0) \leq 0$, which is a contradiction as $h'' > 0$ on (x_1, x_2). □

Recall that $\Delta_h f(x) = f(x + h) - f(x), f'(x) = \frac{\Delta_h f(x)}{h}$

$$\Delta_h^2 f(x) = \Delta_h(\Delta_h f(x))(x) = \Delta_h(f(x + h) - f(x)) =$$

$$= f(x + 2h) - f(x + h) - (f(x + h) - f(x)) = f(x + 2h) - 2f(x + h) + f(x)$$

Thm. Suppose f is C^2 on (a, b) Then $f''(x) = \lim_{h\to\infty} \frac{\Delta_h f(x)}{h^2} \forall x \in (a, b)$.

30.3.2016

Proof. Assume $h > 0$. Consider $g(t) \equiv f(t + h) - f(t), t \in [x, x + h]$

$$g'(t) = f'(t + h) - f'(t)$$

Apply the Mean value theorem:

$$\frac{g(x + h) - g(x)}{h} = g'(x_0)$$

for some $x_0 \in (x, x + h)$.

$$\frac{\Delta_h^2 f(x)}{h} = f'(x_0 + h) - f'(x_0) \Rightarrow \frac{\Delta_h^2 f(x)}{h^2} = \frac{f'(x_0 + h) - f'(x_0)}{h} = f''(x_1)$$

for some $x_1 \in (x_0, x_0 + h)$. Apply MVT again: $x_1 \in x(x, x + 2h)$ and f'' is continuous $\Rightarrow \lim_{h\to o^+} \frac{\Delta_h^2 f(x)}{h^2} = f''(x)$ and $\lim_{h\to o^-} \frac{\Delta_h^2 f(x)}{h^2} = f''(x)$ □

Thm. Let f be a C^2 function on neighbourhood of x_0.

$$g_2(x) \equiv f(x_0) + f'(x_0)(x - x_0) + 1/2 f''(x_0)(x - x_0)^2$$

Then $f - g_2 = o(|x - x_0|^2)$ as $x \to x0$.

Proof.
$$g_2(x_0) = f(x_0), g_2'(x_0) = f'(x_0), g''(x_0) = f''(x_0)$$
$\Rightarrow F \equiv f - g_2$ is C^2, and $F(x_0) = F'(x_0) = F''(x_0) = 0$. Let $1/m > 0$. We want $1/n$, such that $|x - x_0| < 1/m \Rightarrow |F(x)| < \frac{|x-x_0|^2}{m}$. By MVT: given x, $\frac{F(x)-F(x_0)}{x-x_0} = F'(x_1)$ for some x_1 between x, x_0. $\Rightarrow F(x) = F'(x_1)(x - x_0)$

Similarly $F'(x_1) = F''(x_2)(x_1 - x_0)$ for some $x_2 \in (x_1, x_0) \Rightarrow |F(x)| = |F''(x_2)(x_1 - x_0)(x - x_0)| \leq |F''(x_0)||x - x_0|^2$, which can be made $< 1/m$ for x sufficiently close to x_0 because F'' is continuous, $F''(x_0) = 0$ and $x_2 \in (x, x_0)$. □

Def. f is C^n on (a, b) if $f^{(n)}$ exists and is continuous on (a, b). If $f^{(n)}(x_0)$ exists, we define Taylor polynomial:

$$T_n(f, x_0, x) = f(x_0) + f'(x_0)(x - x_0) + 1/2f''(x_0)(x - x_0)^2 + ... + 1/nf^{(n)}(x - x_0)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Thm: Taylor's theorem. Let f be a C^n function on neighbourhood of x_0, $T_n(x) = T_n(f, x_0, x)$ Then $f - T_n = o(|x - x_0|^n)$ as $x \to x_0$.

Proof. Same as before, $F \equiv f - T_n$, $F^{(k)}(x_0) = 0, k = 0, 1, 2, 3...n$ We apply the MVT n-times to get $x_1, x_2, x_3...x_n$ all between x_0, x, such that $F(x) = F'(x_1)(x - x_0)$, $F''(x_1) = F''(x_2)(x - x_0)...F^{(n-1)}(x_{n-1}) = F^{(n)}(x_n)(x_{n-1} - x_0)$

$$\Rightarrow |F(x)| = |(x - x_0)(x_1 - x_2)(x_2 - x_3)....(x_{n-1} - x_0)F^{(n)}(x_n)| \leq |x - x_0|^m |F^{(n)}(x_n)|$$

where $F^{(n)}(x_n)|$ can be made $< 1/m$ for x sufficiently close to x_0 as before. □

Remark. T_n is **uniquely determined** among def $\leq n$ polynomials by condition $f - T_n = o(|x - x_0|^n)$ Indeed if g is another, then $(f - T) - (f - g) = g - T = o(|x - x_0|^n)$ as $x \to x_0$

Proof.
$$f - T = \sum a_i(x - x_0)^i \Rightarrow \lim_{x \to x_0} \frac{\sum_{i=0}^n a_i(x - x_0)^i}{(x - x_0)^n} = 0$$
$$0 = \lim_{x \to x_0} \frac{\sum_{i=0}^n a_i(x - x_0)^i}{(x - x_0)^n}(x - x_0)^n = a_0$$
$$0 = \lim_{x \to x_0} \frac{\sum_{i=0}^n a_i(x - x_0)^i}{(x - x_0)^n}(x - x_0)^{n-1} = a_1$$
$$0 = \lim_{x \to x_0} \frac{a_n(x - x_0)^n}{(x - x_0)^n} = a_n \Rightarrow g - T = 0$$

□

Thm: L'Hopital's Rule. Suppose f, g are C^n functions on the neighbourhood of x_0 and that $f^{(k)}(x_0) = g^{(k)}(x_0) = 0 \forall k = 0, 1, 2, ...n-1$ and that $g^{(n)}(x_0) \neq 0$. Then:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f^{(n)}(x_0)}{g^{(n)}(x_0)}$$

Proof. From Taylor's thm:

$$\frac{f(x)}{g(x)} = \frac{1/n! f^{(n)}(x_0)(x - x_0^n) + o(|x - x_0|^n)}{1/n! g^{(n)}(x_0)(x - x_0^n) + o(|x - x_0|^n)}$$

$$\frac{f(x)}{g(x)} = \frac{f^{(n)}(x_0) + o(1}{g^{(n)}(x_0) + o(1)}, x \to x_0 \Rightarrow \frac{f^{(n)}(x_0)}{g^{(n)}(x_0)}$$

$\square$

Chapter 6

Integration

6.1 Integrals of continuous functions

Goal is to define $\int_a^b f(x)dx$ for f continuous on $[a, b]$.

Def. A **pointed partition** P of $[a, b]$ consists of

1. $a = x_0 < x_1 < .... < x_n = b$
2. $\forall k = 1, 2, 3..n$ a point $a_k \in [x_{k-1}, x_k]$

For P a pointed partition we have a **Cauchy sum**:

$$S(f, P) = \sum_{k=1}^{n} f(a_k)(x_k - x_{k-1})$$

Special cases: $M_k \equiv$ sup of f on $[x_{k-1}, x_k]$ and $d_k \equiv$ inf of f on $[x_{k-1}, x_k]$

$$S^+(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

$$S^-(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

Def. For any point in partition P, the **maximum interval length** is the maximum length of the sub-intervals of P. Having fixed f we say that the limit of $S(f, P)$ exists and equals $\int_a^b f(x)dx$ if $\forall 1/N > 0, \exists 1/m$, such that for any pointed partition P with a maximum interval length $< 1/m$, we have:

$$S(f, P) - \int_a^b f(x)dx < 1/N$$

Section - 1.4.2016

Thm: Inverse function theorem. $f : (a,b) \to \mathbb{R}$ satisfies $f'(x) \neq 0$ in neighbourhood of x.

$$f^{-1}(f(x)) = x$$

$$\frac{d}{dx}(f^{-1}(f(x))) = 1$$

$$(f^{-1})'(f(x))f'(x) = 1 \Rightarrow (f^{-1})(y) = \frac{1}{f'(x)} \Rightarrow y = f(x)$$

Def. $f \in C^n([a,b]) \iff f^{(n)}(x)$ exists $\forall x \in [a,b]$ and $f^{(n)}$ is continuous. We can associate $T_n(f) = T_n(f,\alpha,t) = \sum_{k=0}^{n} a_k(t-\alpha)^k$, where $a_k = f^{(k)}(\alpha)/k!$. Observe that:

$$T_n'(t) = a_1 + 2a_2(t-\alpha) + ... + n \times a_n(t - \alpha^{n-1})$$

$$T_n''(t) = 2!a_2 + 3 \times 2a_3(t-\alpha) + ... + n \times (n-1)a_n(t-\alpha)^{n-2}$$

$$T_n^{(k)}(t) = k!a_k + (k+1)!a_{k+1}(t-\alpha) + ...$$

$$\Rightarrow T_n^{(k)}(\alpha) = k!a_k = f^{(k)}(\alpha)$$

E.g.

$$f(x) = \frac{1}{1-x}$$

Calculate $T_n(t) = T_n(f,0,t)$. Need to compute $f^{(k)}(0)$

$$f''(x) = \frac{(-1)}{(1-x^3)^3} \Rightarrow f^{(k)}(x) = \frac{k!}{(1-x)^k} \Rightarrow f^{(k)} = 0$$

$$T_n(t) = 1 + t + t^2 + ... + t^n$$

Thm: Lagrange's Remainders formula. $f : [a,b] \to \mathbb{R}$ if $f^{(n+1)}$ exists $f^{(n)}$ is continuous on $[a,b]$

$\exists \xi \in (\alpha,\beta)$, such that $f(\beta) - T_n(f,\alpha,t)(\beta) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(\beta-\alpha)^{n+1}$ content...

Corollary. If $f^{-1} \in C^{n+1}([a,b]) \Rightarrow f - T_n = o(|x-\alpha|^n)$

Proof. By the previous theorem $f(x) - T_n(t) = \frac{f^{(n+1)}(\xi)}{n+1}(t-\alpha)^{n+1}$. Because $f^{(n+1)}$ is continuous:

$$\lim_{f\to\alpha} \frac{(f(t) - T_n(t))}{|t-\alpha|^{n+1}} = \lim_{t\to\alpha} \frac{f^{n+1}(\xi)}{(n+1)!}(t-\alpha) = 0$$

$\square$

LRF.

$$M \equiv \frac{f(\beta) - T_n(\beta)}{(\beta - \alpha)^{n+1}}$$

It suffices to show $M(n+1)! = f^{n+1}(\xi), \xi \in (\alpha, \beta)$.

$$g(t) = f(t) - T_n(t) - M(t-\alpha)^{n+1}$$

$$g^{(n+1)}(t) = f^{(n+1)}(t) - 0 - (n+1)!M$$

We are close if we can find $\xi \in (\alpha, \beta)$, such that $g^{(n+1)}(\xi) = 0$

Claim:

$$g(\alpha) = g'(\alpha) = ... = g^{(n)}(\alpha) = 0$$

Combine $g(\beta) = g(\alpha) = 0$. Then by Mean Value Theorem $\Rightarrow \exists \xi_1 \in (\alpha, \beta)$, such that $g'(\xi_1) = 0$ and $g'(\alpha) = 0$. Then again by MVT: $\exists \xi_2 \in (\alpha, \xi_1)$, such that $g''(\xi_2) = 0$. Continue to the n-th state $g^{(n)}(\xi_n) = 0, \exists \xi_n \in (\alpha, \xi_{n+1}) \Rightarrow g^{(n+1)}(\xi_{n+1}) = 0$

To check claim: $g^{(R)}(\alpha)$ with $0 \le k \le n$. From definition we have $f^{(k)}(\alpha) = T_n^{(k)}(\alpha)$.

$$g(t) = f(t) - T_n(t) - M(t-\alpha)^{n+1}$$
$$g^{(n+1)}(t) = f^{(n+1)}(t) - 0 - (n+1)!M$$

We are done if we can find $\xi \in (\alpha, \beta)$, such that $g^{(n+1)}(\xi) = 0$.

□

Problem. $f \in C^2(a,b), \forall x \in (a,b)$

$$f''(x) = \lim_{h \to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

Hint:

$$\frac{f(x-h) - 2f(x) + f(x+h)}{h}\frac{1}{h} = \frac{1}{h}\left(- \frac{f(x) - f(x-h)}{h} + \frac{f(x_h) - f(x)}{h}\right)$$

By MVT:

$$= \frac{1}{h} = (f'(C^2) - f(C^1)$$

By MVT:

$$= \frac{C_2 - C - 1}{h} f''(c)$$

$c_1 \in (x+h, x), c_2 \in (x, x+h), x \in (c_1, c_2)$

4.4.2016

Thm. f is continuous on $[a,b] \Rightarrow$ lim at $S(f,P)$ exists, and $\int_a^b f(x)dx = \inf_p S^+(f,P) = \sup_p S^-(f,P)$. Where P varies over all partitions of $[a,b]$.

Proof.

$$U \equiv \{S^+(f,P)|P \in [a,b]\}$$
$$L \equiv \{S^-(f,P)|...\}$$

If P_1, P_2 are partitions and P_3 the union of P_1, P_2, then:

$$S^-(f,P_1) \leq S^-(f,P_3) \leq S^+(f,P_3) \leq S^+(f,P_2)$$

$$\Rightarrow \sup L \leq \inf U$$

Now f is continuous on $[a,b] \Rightarrow f$ is uniformly continuous on $[a,b]$. So given $1/N, \exists 1/m$, such that $|x-y| < 1/m \Rightarrow |f(x)-f(y)| < \frac{1}{N(b-a)}$. If P is a pointed partition of $[a,b]$ of max $< 1/m$, then:

1. $S^+(f,P) - S^-(f,P) = \sum_{k=1}^n (M_k - m_k)(x_k - x_{k-1})$, where $(M_k - m_k) < \frac{1}{N(b-a)}$ So:

$$\frac{1}{N(b-a)} \sum_{k=1}^n (x_k - x_{k-1}) = \frac{1}{N(b-a)}(b-a) = 1/n$$

$$\Rightarrow 0 \leq \inf U - \sup L \leq S^+(f,P) - S^-(f,P) < 1/N, \forall N$$

$$\Rightarrow \sup L - \inf U \equiv \int_a^b f(x)dx$$

2.

$$S(f,P), \int_a^b f(x)dx \in [S^-(f,P), S^+(f,P)] < 1/N$$ □

Properties

1.

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

2.

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)$$

Def. 1.

$$\int_a^a f(x)dx = 0$$

2.
$$\int_a^b f(x)dx = -\int_b^a f(x)dx, b < a$$

3.
$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

where $M = \sup, n = \inf$.

4. **Average of f** on $[a, b]$ is
$$\frac{\int_a^b f(x)dx}{b-a}$$

Fundamental theorem(s) of calculus

Thm: optional. Let f be continuous on $[a, b]$.

$$F(x) \equiv \int_a^x f(t)dt, x \in [a, b]$$

Then F is a C^1 on $[a, b], F' = f$.

$$\frac{F(x_0+h) - F(x_0)}{h} = \frac{\int_a^{x_0+h} f(t)dt - \int_a^x f(x)dt}{h} = \frac{\int_{x_0}^{x_0+h} f(t)dt}{h}$$

If $h > 0$. Let M, m, be respectively sup and inf of f on $[x_0, x_0 + h] \Rightarrow m \leq 1/h \int_{x_0}^{x_0+h} f(t)dt \leq M$.

If $h > 0$. Let M, m, be respectively sup and inf of f on $[x_0 + h, x_0] \Rightarrow m \leq 1/-h \int_{x_0+h}^{x_0} f(t)dt = 1/h \int_{x_0}^{x_0+h} f(t)dt \leq M$.

As $h \to 0, M, m \to f(x_0)$ since f is continuous $\Rightarrow F'(x_0) = \lim_{h \to 0} 1/h \int_{x_0}^{x_0+h} f(t)dt = f'(x_0)$. F' is continuous because f is continuous.

Remark. If G is any function , such that $G' = f$ we call G an **antiderivative** or an **indefinite integral** or a **primitive** of f. The theorem says $F(x) = \int_a^x f(t)dt$ is antiderivative of f. If G is another, then $(G - F)' = G' - F' = f - f = 0 \Rightarrow G - F = c \Rightarrow G = F + c$

Thm: optional. Let f, C^1 on $[a, b]$. Then $\int_a^b f'(x)dx = f(b) - f(a)$.

Proof. Let $a = x_0 < x_1 < ... < x_n = b$ be a partition of $[a, b]. \forall k = 1, ...n$ by the mean value theorem on $[x_{k-1}, x_k]$ we can choose $a_k \in [x_{k-1}, x_k]$, such that $f'(a_k)(x_k - x_{k-1}) = f(x_k) - f(x_{k-1})$.

For a pointed partition P with these a_k's we have:

$$S(f, P) = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) = f(b) - f(a)$$

Since $\int_a^b f'(x)dx$ is a limit over all $S(f', P)$'s $\square$

Thm: Integration by parts. let f, g be C^1 function on $[a, b]$. Then:

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

Proof.

$$(fg)' = f'g + fg' \Rightarrow fg' = (fg)' - f'g$$

Apply previous theorem. $\square$

Thm: Change of variables / u-substitution. Let g be C^1 and increasing on $[a, b]$. Then $\forall$ continuous f on $[g(a), g(b)]$:

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g'(x)dx$$

Proof. Let F be the antiderivative of f. Then $F' = f$

$$(F \circ g)' = (F' \circ g)g' = (f \circ g)g'$$

$$\Rightarrow \int_{g(a)}^{g(b)} f(x)dx = F(g(b)) - F(g(a))$$

$$\int_a^b f(g(x))g'(x)dx = (F \circ g)(b) - (F \circ g)(a)$$

$\square$