Question 2.2.4: 4. Let x be a real number. Show that there exists a Cauchy sequence of rationals $x_1, x_2, ...$ representing x, such that $x_n < x_{n+l}, \forall n$

Proof. Suppose there is an increasing Cauchy sequence (x_n). Then for a case when $x \in \mathbb{Q}$ we know that it is represented by a Cauchy sequence of rationals. For case when $x \notin \mathbb{Q}$ we consider the Lemma: $\forall x, y \in \mathbb{Q}$, such that $x \neq y, \exists r \in \mathbb{Q}$, such that $x < r < y$. We can pick $x_1 \in \mathbb{Q}$, such that $x_1 < x$ and by the Lemma $\exists x_2 \in \mathbb{Q}$, such that $\frac{x_1+x}{2} < x_2 < x$ Therefore:

$$|x_2 - x| < 1/2|x_1 - x|$$

$$|x_{n+1} - x| < 1/2|x_n - x|$$

we then have sequence of rationals $(x_1, x_2, ..x_n)$, which represnts x and is clearly increasing. □

Question 2.2.4: 7. Prove $|x - y| > |x| - |y|$ for any real numbers x and y. (Hint: use the triangle inequality).

Proof. By Triangle inequality $|a + b| < |a| + |b|$, take $x = a + b$, and $y = b$:

$$|x| < |x - y| + |y|$$

$$|x| - |y| < |x - y|$$

□

Question 2.2.4: 10. Show that if a real number x can be represented by a Cauchy sequence of positive rationals, then $x \geq 0$. What does this tell you about real numbers that can be represented by two equivalent Cauchy sequences of rationals, one consisting of only positive rationals and the other consisting of only negative rationals.

Proof. Let x, y be real numbers represented by Cauchy sequences $(x_k), (y_k)$. Recounting the lemma: if $x_k > y_k, \forall k \geq m$, then $x \geq y$. We can take y_k to be the sequence $(0, 0, 0, 0, ...)$ and we can see that $x \geq 0$.
As a symmetric argument could be made for negative numbers we conclude that the only real number that can be represented by both a positive and a negative sequence is 0. □

Question 2.2.4: 11. Prove that no real number satisfies $x^2 = -1$.

Proof. From class we know that a real number can be either positive, negative or zero. From ordered field axiom we know that two positive numbers multiplied give a positive number, therefore $x \in Rp$ cannot satisfy the equation. Likewise $0^2 \neq -1$. Lastly suppose an x is negative, then:

$$(-x)^2 = (-x)(-x) = (-1)^2 x^2 = x^2$$

which still implies that square of two negatives is the same as square of two positives, which we already know does not satisfy the equation. □

Question 2.3.2: 1. Write out a proof that $\lim_{k\to\infty}(x_k + y_k) = x + y$ if $\lim_{k\to\infty} x_k = x, \lim_{k\to\infty} y_k = y$' for sequences of real numbers.

Proof. Let $(x_k), (y_k)$ be sequences of real numbers. Let $k > N$ for $|x_k - x| < 1/2N, |y_k - y| < 1/2N$. Then $|x_k + y_k - (x + y)| \leq |x_k - x| + |y_k - y| < 1/2N + 1/2N < 1/N$ □

Question 2.3.2: 8. Prove that if $\lim_{k\to\infty} x_k = x$ and $x_k \geq 0, \forall k$, then $\lim_{k\to\infty} \sqrt{x_k} = \sqrt{x}$.

Proof.

1. For $x = 0$: Let $\epsilon > 0$, then $\exists N$, such that $n > N$, then $|x_n - 0| < \epsilon^2$. Therefore $|\sqrt{x_n} - 0| < \epsilon$ and $\sqrt{x_n} \to 0$
2. For $x > 0$: Let $\epsilon > 0, \exists N : n > N$, then $|x_n - x| < \epsilon\sqrt{x}$. Therefore:

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} < \frac{|x_n - x|}{\sqrt{x}} < \frac{\epsilon\sqrt{x}}{\sqrt{x}} = \epsilon$$

and so $\sqrt{x_n} \to \sqrt{x}$.

□

Problem 1. Let $x \in \mathbb{R}$. Prove carefully from the definitions that if (x_j) is a rational Cauchy sequence representing x, then $(|x_j|)$ (i.e. the sequence whose j-th term is $|x_j|$) represents $|x|$.

Proof. A sequence is Cauchy $\iff$ it converges. If a sequence converges it has a limit, which is defined as $|x_j - x| < \epsilon$. So we need to prove that $||x_j| - |x|| \leq |x_j - x|$. We do this be using the triangle inequality from one of the previous exercises.

$$|x| \leq |x - y| + |y| \quad \& \quad |y| \leq |y - x| + |x|$$

$$|x| - |y| \leq |x - y| \quad \& \quad |y| - |x| \leq |y - x|$$

By absolute value property: $|x - y| = |y - x|$ and therefore we have $||x| - |y|| \leq |x - y|$ or $||x_j| - |x|| \leq |x_j - x|$. □

Problem 2. In 2.3.2 of the text theres a careful proof that any positive $S \in \mathbb{R}$ has a unique positive square root, denoted $\sqrt{S}$. In this problem were going to give an alternative proof of the existence of the square root using Problem 2 from HW2. (The uniqueness part of the argument in the text cant be improved upon and will be covered in lecture on Monday.) Throughout this problem you may freely use Theorems 2.3.1 (which we almost finished proving in class on Wednesday) and 2.3.2 (which will be stated in class on Monday) in the text, as well as everything from Problem 2 in HW2. Recall that in the aforementioned Problem 2, you constructed a certain Cauchy sequence (x_j) (this is Cauchy whether or not S is rational). It is also obvious (and implicitly used at various points in Problem 2) that $x_n \neq 0$ for all j, so that we may also consider the sequence (S/xj), i.e. the sequence whose j-th term is S/xj.

- Prove that $\lim_{j\to\infty} x_j = \lim_{j\to\infty} S/x_j$.

- Let x denote the common limit in part (a). Prove that $x^2 = S$.

Proof. • We know that $\lim_{n\to\infty} x_n = x$ and substitute $y = \lim_{n\to\infty} \frac{S}{x_n}$. Because if the limit is reached it does not matter if we reach another term of the Cauchy sequence we have $\lim_{n\to\infty} x_{n+1} = x$. We can now rewrite the initial formula as:

$$\lim_{n\to\infty} x_{n+1} = \frac{1}{2} \lim_{n\to\infty} \left(x_n + \frac{S}{x_n} \right) = \frac{1}{2} \lim_{n\to\infty} x_n + \frac{1}{2} \lim_{n\to\infty} (\frac{S}{x_n})$$

$$x = 1/2x + 1/2y = 1/2(x + y)$$

$$2x = (x + y) \Rightarrow x = y \Rightarrow \lim_{j\to\infty} x_j = \lim_{j\to\infty} \frac{S}{x_n} \quad \square$$

- Again we have $x_{n+1} = 1/2\left(x_n + \frac{S}{x_n}\right)$, by making the same substitutions as above we have $x = 1/2(x + S/x)$

$$2x^2 = x^2 + S$$

$$x^2 = S$$

$\square$