MATH 587 NOTES

Test

October 2022

Lectures 1-14

Events

An essential result: $\mathbb{P}(\liminf E_n) \leq \liminf \mathbb{P}(E_n) \leq \limsup \mathbb{P}(E_n) \leq \mathbb{P}(\limsup E_n)$

Examples seen in class

Travelling block

Eventual behaviour of i.i.d. exponential random variables

Material for Quiz 1

algebra

Contains the whole space, closed under complements and finite unions

sigma algebra

An algebra that is closed under countable unions

π-system

d-system

Dynkin's π-d lemma

Measure

countlabe addiative

Key definitions

Convergence in probability

A sequence of random variables $\{X_n\}$ converges in probability to a random variable X if $\forall \epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0$

> Let $\delta > 0$. For any positive ϵ, you can find a large enough $N \in \mathbb{N}$ such that for all $n \geq N$, the measure of the subset where X_n and X differ by more than ϵ is capped by δ.

Convergence almost surely

A sequence of random variables $\{X_n\}$ converges to X *almost surely* if $\mathbb{P}(\lim_{n\to\infty} X_n = X) = 1$
NB:
$X_n \to X$ almost surely
$\longleftrightarrow \forall \epsilon > 0, \mathbb{P}(|X_n - X| > \epsilon \text{ i.o }) = 0$
$\longleftrightarrow \forall \epsilon > 0, \mathbb{P}(|X_n - X| < \epsilon \text{ eventually always }) = 1$

> In other words, the probability measure of the set where $\lim_{n\to\infty} X_n = X$ is 1.
>
> If X_n converges to X almost surely, then the difference between X_n and X will eventually sink to levels and never rise above them. This does not happen for convergence in probability
>
> Convergence almost surely implies convergence in probability BUT NOT VICE VERSA:

Convergence in L^P

We need to first define that if $X \in \mathbb{P}$ A sequence $\{X_n\}$ converges in $L^P(\mathbb{P})$ if

Convergence almost everywhere

Uniform integrability

We say a sequence of random variables $\{X_n\}$ is **uniformly integrable** if $\lim_{M\to\infty} \sum_{m\geq 1} \mathbb{E}(|X_n|; |X_n| > M) = 0$

> If $\{X_n\}$ is uniformly integrable, then $\{X_n\}$ is bounded in $L^1(\mathbb{P})$

If $\{X_n\}$ is bounded in L^P for some $P > 1$ STRICTLY GREATER THAN 1, then $\{X_n\}$ is uniformly integrable.

Given random variables $\{X_n\}, X$, the following are equivalent:
(i) All X_n's and X are in $L^1(\mathbb{P})$, and X_n converges to X in L^1
(ii) $X_n \to X$ in probability and $\{X_n\}$ is uniformly integrable

Convergence theorems

Monotone convergence theorem

Let $f_n, f \subset m\Sigma$ such that $f_n \nearrow f$ almost everywhere AND $\mu(f_1^-) < \infty$. Then $\mu(f_n) \nearrow \mu(f)$.

We can use this to "bring out" limits: Say $f_n \nearrow f$ almost everywhere, and $\mu(f_1^-) < \infty$. From the first statement, we have $\mu(\lim_{n\to\infty} f_n \neq f) = 0$. Using this theorem, we can say $\mu(f_n) \nearrow \mu(f) \to \lim_{n\to\infty} \mu(f_n) = \mu(f)$

Dominated convergence theorem

Let $f_n, f \subset m\Sigma$. If $f_n \longrightarrow f$ almost everywhere and there exists a function $g \in L^1$ such that $|f_n| \leq |g|\ \forall n$, then $f_n \longrightarrow f$ in L^1.

This theorem can be helpful when we want to show convergence in L^1 but we only have almost-everywhere convergence. If our sequence of functions f_n converges almost everywhere to f AND we know that there is a function $g \in L^1$ that satisfies $|g| \geq |f_n|$ for all n, Then we can say f_n converges to f in L^1.

Scheffe's Lemma

Let $f_n, f \subset L^1$ and $f_n \longrightarrow f$ almost everywhere. Then:
$f_n \longrightarrow f$ in $L^1 \longleftrightarrow \lim_{n\to\infty} \mu(|f_n|) = \mu(|f|)$

In simpler words: Say we have a sequence of functions f_n in L^1, and also a function f in L^1. Suppose f_n converges to f almost everywhere. What if we wanted the stronger convergence in L^1? Scheffe's lemma says that in this case, f_n converging to f in L^1 is equivalent to saying the limit of $\mu(|f_n|)$ is $\mu(|f|)$.

Fatou's Lemma

Suppose $f_n, g \subset m\Sigma$ such that $\mu(g^-) < \infty$ and $f_n \geq g \ \forall n$.
Then $\mu(\liminf f_n) \leq \liminf \mu(f_n)$

Say we have general measurable functions $f_n \subseteq m\Sigma$ and $g \in m\Sigma$ such that g lower-bounds every f_n. If the negative part of g has a finite integral, then $\mu(liminf f_n) \leq \liminf \mu(f_n)$
Fatou's lemma assumes little beforehand, making it handy.

Question 1

(i)

Let $f = \sum_{n=1}^{\infty} f_n$. We claim $\sum_{n=1}^{\infty} f_n$ converges to f almost everywhere. First, note that $\sum_{n=1}^{\infty} \mu(|f_n|) < \infty \longrightarrow \mu(|f_n|) < \infty \ \forall n \in \mathbb{N}$. In other words, $f_n \in L^1(\mu) \ \forall n \in \mathbb{N}$.
let $g_k = \sum_{n=1}^{k} f_n$. and $h = \sum_{k=1}^{\infty} |f_k|$.
We have $g_k \longrightarrow \sum_{n=1}^{\infty} f_n$ a.e.
di $h \in L^1(\mu)$ since $\mu(|h|) = \mu(|\sum_{k=1}^{\infty} |f_k||) = \mu(\sum_{k=1}^{\infty} |f_k|) < \infty$ by assumption.

$\forall k \in \mathbb{N}$, $|g_k| = |\sum_{n=1}^{k} f_n| \leq \sum_{n=1}^{k} |f_n| \leq \sum_{n=1}^{\infty} |f_n| = |h|$. So by the dominated convergence theorem, we have $g_k \longrightarrow f$ in $L^1 \longrightarrow lim g_k = f \longrightarrow \sum_{n=1}^{\infty} f_n = f$ in L^1 $\square$

(ii)

Minor notes

"Almost surely" implies we are working with probability, while "almost everywhere" means a general measure space

Although L_X is a measure on the entire real line when X is a random variable, we still have $L_X(\mathbb{R}) = 1$

Theorem

Probability inequalities

Markov's inequality

Let us have a probability space $(\Omega, F, \mathbb{P})$ and a random variable $X : \Omega \to \mathbb{R}$.
Also let $g : \mathbb{R} \to \mathbb{R}$ be non-negative, non-decreasing, and Borel.
Then $\forall c \in \mathbb{R}$, $\mathbb{E}(g(X)) \geq \mathbb{E}(g(X); X \geq c) \geq \mathbb{E}(g(c); X \geq c) = g(c)\mathbb{P}(X \geq c)$

Cheybyshev's inequality

Jensen's inequality

Uniform integrability

To-do

Caratheodory's extension theorem

Everything related to Radon-Nikodym

Completion of a measure space

Probabilistic versions of convergence theorems

Dyadic rationals

Common pitfall: Accidentally having $\infty - \infty$ at a point