

sectioning-and-bootstrapping

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1 Sectioning and bootstrapping

The following presentation is based on the excellent textbook by Asmussen and Glynn [1].

Given a random element X , its distribution F , and some real-valued functional ψ , we would like to estimate $\psi(F)$ and its $1 - \alpha$ confidence interval without further assumptions. For example, the mean is the functional $\psi(F) = \int xF(dx)$, where dx is the probability to find X in dx .

Given R independent samples X_1, \dots, X_R from F , an estimate for $\psi(F)$ is $\psi(\hat{F}_R)$, where

$$\hat{F}_R(dx) := \frac{1}{R} \sum_{r=1}^R \delta_{X_r}(dx)$$

is the *empirical distribution* and $\delta_{X_r}(A) = 1 \Leftrightarrow X_r \in A$.

For real-valued random variables, the empirical cumulative distribution function is

$$\hat{F}_R(x) := \frac{1}{R} \sum_{r=1}^R \mathbb{1}_{\{X_r \leq x\}}.$$

As $R \rightarrow \infty$, we have $\psi(\hat{F}_R) \rightarrow \psi(F)$ almost surely [1]. Furthermore, we have a central limit theorem such that $\psi(\hat{F}_R)$ is distributed as $\psi(F) + Y$, where $Y \sim \mathcal{N}(0, \sigma/\sqrt{R})$.

1.1 Sectioning

Sectioning means splitting the sample into N subsamples (sections) of size K . The empirical distribution of the n -th section is

$$\hat{F}_{n,K}(dx) := \frac{1}{K} \sum_{r=(n-1)K+1}^{nK} \delta_{X_r}(dx).$$

The $1 - \alpha$ confidence interval for the estimator $\psi(\hat{F})$ is

$$\psi(\hat{F}) \pm t_{1-\alpha/2} \frac{\hat{\sigma}}{\sqrt{N}},$$

where $t_{1-\alpha/2}$ is the critical value of the Student t distribution with $N - 1$ degrees of freedom and the estimator for the variance

$$\hat{\sigma}^2 := \frac{1}{N-1} \sum_{n=1}^N \left(\psi(\hat{F}_{n,K}) - \psi(\hat{F}_R) \right)^2.$$

The number of sections needs to be sufficiently large in order for the central limit theorem to approximately hold.

1.2 Bootstrapping

When a model for the distribution F is lacking, or too complicated for statistical inference, bootstrapping methods provide alternatives. Bootstrapping takes the empirical distribution \hat{F}_R as a surrogate for the true distribution F . Instead of drawing more samples from F , bootstrapping involves resampling from \hat{F}_R .

The true $1 - \alpha$ confidence interval of the estimator $\psi(\hat{F}_R)$ is

$$(\psi(\hat{F}_R) - z_2, \psi(\hat{F}_R) - z_1)$$

with the $\alpha/2$ and $1 - \alpha/2$ quantiles z_1, z_2

$$P(\psi(\hat{F}_R) - \psi(F) < z_1) = P(\psi(\hat{F}_R) - \psi(F) > z_2) = \frac{\alpha}{2}$$

such that

$$P(\psi(F) \in (\psi(\hat{F}_R) - z_2, \psi(\hat{F}_R) - z_1)) = 1 - \alpha.$$

Assuming that $\hat{F}_R \approx F$, the empirical quantiles z_1^*, z_2^* satisfy [1]

$$P_{\hat{F}_R}(\psi(\hat{F}_R) - \psi(F) < z_1^*) = P_{\hat{F}_R}(\psi(\hat{F}_R) - \psi(F) > z_2^*) = \frac{\alpha}{2}$$

We draw B bootstrap samples of size R from \hat{F}_R . The b -th bootstrap sample is $X_{1,b}^*, \dots, X_{R,b}^*$ with each random variable $X_{r,b}^*$ drawn independently from X_1, \dots, X_R with equal probabilities $P(X_{r,b}^* = X_{r'}) = \frac{1}{R}$. The empirical distribution of the b -th bootstrap sample is

$$\hat{F}_{R,b}^*(dx) := \frac{1}{R} \sum_{r=1}^R \delta_{X_{r,b}^*}(dx).$$

Then the empirical quantiles z_1^*, z_2^* are the $\lfloor \frac{\alpha}{2}(B+1) \rfloor$ -th and $\lfloor (1 - \frac{\alpha}{2})(B+1) \rfloor$ -th order statistic, respectively, of the B independent and identically distributed random variables

$$\left(\psi(\hat{F}_{R,b}^*) - \psi(\hat{F}_R) \right)_{b=1}^B.$$

These quantiles z_1^*, z_2^* approximate the quantiles z_1, z_2 of the true distribution F , and hence, yield the approximate $1 - \alpha$ confidence interval [1]

$$\left(\psi(\hat{F}_R) - z_2^*, \psi(\hat{F}_R) - z_1^* \right).$$

References

- [1] S. Asmussen and P. W. Glynn, *Stochastic Simulation: Algorithms and Analysis*, Stochastic Modelling and Applied Probability, Vol. 57 (Springer New York, 2007).