Confidence Intervals for Binomial Proportions

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In []: import scipy.stats.distributions as dist
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Consider a discrete random variable X which indicates either "success" ($X = 1$) or "failure" ($X = 0$) as the outcome of a random experiment. Such an experiment is called a *Bernoulli trial*. The probability of success is $p = P\{X = 1\}$, and the probability of failure is $P\{X = 0\} = 1 - p$. Repeating a Bernoulli trial n times means drawing a sample of n independent and identically distributed random variables X_i. The probability mass function of observing k success is the binomial distribution

$$\binom{n}{k} p^k (1-p)^{n-k}.$$

The expected number of successes is np with variance $np(1-p)$. Hence, the success probability p is the expected *proportion* of successes $\frac{k}{n}$, with variance $p(1-p)/n$.

Let $\hat{p} := \frac{k}{n}$ denote the sample proportion which is the unbiased (and maximum likelihood) estimator for the success probability p, and let $\hat{\sigma} := \hat{p}(1-\hat{p})/n$ denote the sample variance. Then the normal $1-\alpha$ confidence interval for the binomial proportion $\hat{p}$ is

$$\hat{p} \pm z_{\alpha/2}\hat{\sigma},$$

where $z_{\alpha/2}$ is the $1 - \frac{\alpha}{2}$ quantile of the standard normal distribution. This normal confidence interval is also called the *Wald confidence interval*.

As Cameron [1] puts it, this normal approximation "suffers a *systematic* decline in performance both for small n and towards extreme values of p near 0 and 1, generating binomial [confidence intervals] with effective coverage far below the desired level." (see also [2] and [3])

A different approach to quantifying uncertainty is Bayesian inference. The normal (frequentist) $1-\alpha$ confidence interval derives from a procedure that produces $1-\alpha$ confidence intervals that contain the true parameter value $100(1-\alpha)\%$ of the times. The $1-\alpha$ credible interval of Bayesian inference is the interval in which the parameter lies with probability $1-\alpha$. [4]

Specifically, Bayesian inference employes Bayes' theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

Associate A with the parameter and B with the outcome from an experiment (the data). Then $P(A)$ is the *prior* probability of the parameter event A, with the *likelihood* $P(B|A)$ of the outcome event B given the parameter event A. The *posterior* $P(A|B)$ is the probability of the parameter event A given the outcome event B.

For probability density functions, this reads

$$f(\theta|x) = \frac{f(x|\theta)f(\theta)}{\int d\theta f(\theta|x)f(\theta)}$$

with parameter θ and data x. A Bayesian interval estimate is the $1-\alpha$ *posterior interval* or *credible interval* (l, u) with $\int_{-\infty}^{l} d\theta f(\theta|x) = \int_{u}^{+\infty} d\theta f(\theta|x) = \alpha/2$ such that $P(\theta \in (l, u)|x) = 1 - \alpha$.

For n independent Bernoulli trials with common success probability p, the *likelihood* to have k successes given p is the binomial distribution

$$P(k|p) = \binom{n}{k} p^k (1-p)^{n-k} \equiv B(a, b),$$

where $B(a, b)$ is the *Beta distribution* with parameters $a = k+1$ and $b = n-k+1$. Assuming a uniform prior $P(p) = 1$, the *posterior* is [4]

$$P(p|k) = P(k|p) = B(a, b).$$

A point estimate is the posterior mean

$$\bar{p} = \frac{k+1}{n+2}$$

with $1-\alpha$ credible interval (p_l, p_u) given by

$$\int_0^{p_l} dp B(a, b) = \int_{p_u}^1 dp B(a, b) = \frac{\alpha}{2}.$$

References

[1] E. Cameron, Publications of the Astronomical Society of Australia **28**, 128 (2011).

[2] A. Agresti and B. A. Coull, The American Statistician **52**, 119 (1998).

[3] A. DasGupta, T. T. Cai, and L. D. Brown, Statistical Science **16**, 101 (2001).

[4] L. Wasserman, *All of Statistics* (Springer New York, 2004).