

# Sectioning and bootstrapping

Andreas Sorge

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The following presentation is based on the excellent textbook by Asmussen and Glynn [1].

Given a random element  $X$ , its distribution  $F$ , and some real-valued functional  $\psi$ , we would like to estimate  $\psi(F)$  and its  $1 - \alpha$  confidence interval without further assumptions. For example, the mean is the functional  $\psi(F) = \int xF(dx)$ , where  $dx$  is the probability to find  $X$  in  $dx$ .

Given  $R$  independent samples  $X_1, \dots, X_R$  from  $F$ , an estimate for  $\psi(F)$  is  $\psi(\hat{F}_R)$ , where

$$\hat{F}_R(dx) := \frac{1}{R} \sum_{r=1}^R \delta_{X_r}(dx)$$

is the *empirical distribution* and  $\delta_{X_r}(A) = 1 \Leftrightarrow X_r \in A$ .

For real-valued random variables, the empirical cumulative distribution function is

$$\hat{F}_R(x) := \frac{1}{R} \sum_{r=1}^R \mathbb{1}_{\{X_r \leq x\}}.$$

As  $R \rightarrow \infty$ , we have  $\psi(\hat{F}_R) \rightarrow \psi(F)$  almost surely [1]. Furthermore, we have a central limit theorem such that  $\psi(\hat{F}_R)$  is distributed as  $\psi(F) + Y$ , where  $Y \sim \mathcal{N}(0, \sigma/\sqrt{R})$ .

## 0.1 Sectioning

*Sectioning* means splitting the sample into  $N$  subsamples (sections) of size  $K$ . The empirical distribution of the  $n$ -th section is

$$\hat{F}_{n,K}(dx) := \frac{1}{K} \sum_{r=(n-1)K+1}^{nK} \delta_{X_r}(dx).$$

The  $1 - \alpha$  confidence interval for the estimator  $\psi(\hat{F})$  is

$$\psi(\hat{F}) \pm t_{1-\alpha/2} \frac{\hat{\sigma}}{\sqrt{N}},$$

where  $t_{1-\alpha/2}$  is the critical value of the Student t distribution with  $N - 1$  degrees of freedom and the estimator for the variance

$$\hat{\sigma}^2 := \frac{1}{N-1} \sum_{n=1}^N \left( \psi(\hat{F}_{n,K}) - \psi(\hat{F}_R) \right)^2.$$

The number of sections needs to be sufficiently large in order for the central limit theorem to approximately hold.

## 0.2 Bootstrapping

When a model for the distribution  $F$  is lacking, or too complicated for statistical inference, bootstrapping methods provide alternatives. Bootstrapping takes the empirical distribution  $\hat{F}_R$  as a surrogate for the true distribution  $F$ . Instead of drawing more samples from  $F$ , bootstrapping involves resampling from  $\hat{F}_R$ .

The true  $1 - \alpha$  confidence interval of the estimator  $\psi(\hat{F}_R)$  is

$$(\psi(\hat{F}_R) - z_2, \psi(\hat{F}_R) - z_1)$$

with the  $\alpha/2$  and  $1 - \alpha/2$  quantiles  $z_1, z_2$

$$P(\psi(\hat{F}_R) - \psi(F) < z_1) = P(\psi(\hat{F}_R) - \psi(F) > z_2) = \frac{\alpha}{2}$$

such that

$$P(\psi(F) \in (\psi(\hat{F}_R) - z_2, \psi(\hat{F}_R) - z_1)) = 1 - \alpha.$$

Assuming that  $\hat{F}_R \approx F$ , the empirical quantiles  $z_1^*, z_2^*$  satisfy [1]

$$P_{\hat{F}_R}(\psi(\hat{F}_R) - \psi(F) < z_1^*) = P_{\hat{F}_R}(\psi(\hat{F}_R) - \psi(F) > z_2^*) = \frac{\alpha}{2}$$

We draw  $B$  bootstrap samples of size  $R$  from  $\hat{F}_R$ . The  $b$ -th bootstrap sample is  $X_{1,b}^*, \dots, X_{R,b}^*$  with each random variable  $X_{r,b}^*$  drawn independently from  $X_1, \dots, X_R$  with equal probabilities  $P(X_{r,b}^* = X_{r'}) = \frac{1}{R}$ . The empirical distribution of the  $b$ -th bootstrap sample is

$$\hat{F}_{R,b}^*(dx) := \frac{1}{R} \sum_{r=1}^R \delta_{X_{r,b}^*}(dx).$$

Then the empirical quantiles  $z_1^*, z_2^*$  are the  $\lfloor \frac{\alpha}{2}(B+1) \rfloor$ -th and  $\lfloor (1 - \frac{\alpha}{2})(B+1) \rfloor$ -th order statistic, respectively, of the  $B$  independent and identically distributed random variables

$$\left( \psi(\hat{F}_{R,b}^*) - \psi(\hat{F}_R) \right)_{b=1}^B.$$

These quantiles  $z_1^*, z_2^*$  approximate the quantiles  $z_1, z_2$  of the true distribution  $F$ , and hence, yield the approximate  $1 - \alpha$  confidence interval [1]

$$\left( \psi(\hat{F}_R) - z_2^*, \psi(\hat{F}_R) - z_1^* \right).$$

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## References

- [1] S. Asmussen and P. W. Glynn, *Stochastic Simulation: Algorithms and Analysis*, Stochastic Modelling and Applied Probability, Vol. 57 (Springer New York, 2007).