Simple verification of the parabolic shape of a rotating liquid and a boat on its surface

2010 Phys. Educ. 45 462

(http://iopscience.iop.org/0031-9120/45/5/001) www.iop.org/journals/physed

Simple verification of the parabolic shape of a rotating liquid and a boat on its surface

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Abstract
This article describes a simple and inexpensive way to create and to verify the parabolic surface of a rotating liquid. The liquid is water. The second part of the article deals with the problem of a boat on the surface of a rotating liquid.

Introduction

Parabolic surfaces belong to our everyday life. When we drive cars, we use headlights containing parabolic mirrors. A parabolic mirror is a part of Newton's telescope. In fact, there are reports on liquid mirror telescopes in which the reflective surface of a rotating liquid works as the telescope's mirror [1]. Students can find many other examples of paraboloids.

It is clear that objects that come from daily life and are also partly mysterious are attractive for students. We think that students can be surprised by the fact that a regularly rotating liquid has such a special shape. So it is worth measuring the shape of the surface and verifying the theory. We can also attract students' attention with the question: 'What will happen if I put a small boat on the parabolic surface of a rotating liquid?'.

Experiments with rotating liquid described in this article were also presented at the Physics Teachers' Inventions Fair 12 in Prague in 2007 and were published in the conference proceedings [2], which are in Czech. However, the problems are only briefly mentioned there, whereas details of experimental results as well as considerable theory are missing. Some new figures have been made for this article, too. These experiments also took place in a presentation by our department at Science on Stage Berlin 2008.

A simple theoretical explanation of the parabolic surface of a rotating liquid

The theory of liquid paraboloids has been described and verified in many articles, i.e. [3–5]. We will not review here all the possibilities of how to derive the shape of the surface of a rotating fluid but rather present an alternative explanation understandable at high school level (e.g. at special physics seminars for students interested in physics). It is an extended version of the work presented in Czech in [2].

We will describe the situation from the point of view of both inertial and rotating (i.e. non-inertial) reference frames. Though it is true that in many countries the concept of non-inertial frames is not discussed at high school level we think it is also worth presenting at least a short description in a rotating frame. For example, the discussion of the problem of a boat on the surface of a rotating liquid seems to be a bit easier and maybe more natural from the point of view of rotating frames—at least for people who know the concept

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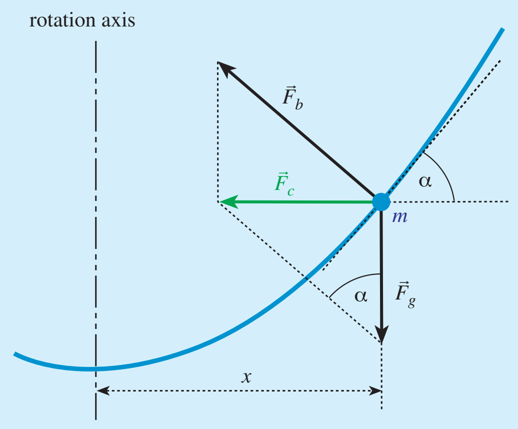


Figure 1. Forces acting on a liquid.

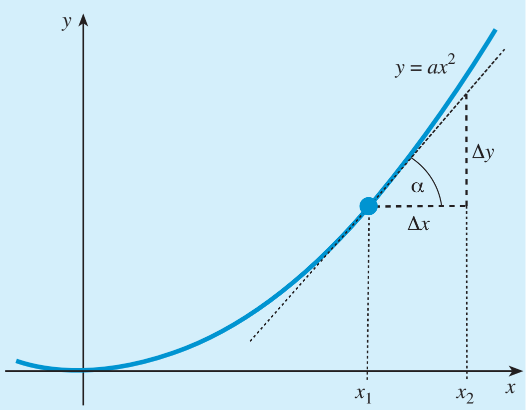


Figure 2. Supposed shape of the rotating water surface.

of centrifugal force[1]. We believe that to present the description in both inertial and rotating systems will be helpful for at least some teachers and students.

The main result we will need in our derivation is the fact that a centripetal (or centrifugal) force is proportional to the distance from the axis of rotation.

Derivation in an inertial reference frame

Let us imagine a certain small amount of water of mass m at the water surface. We can treat it as a point mass. Being a part of the rotating water it moves uniformly in a circle around the axis of rotation. So there must be a centripetal force $\vec{F}_c$ causing our bit of water to move in this way. Its value is given by the well known formula

$$F_c = mR\omega^2. \tag{1}$$

Here R is the distance from the axis of rotation (denoted also by x in the following derivation), m the mass of our piece of water and ω the angular velocity of rotating water.

The centripetal force is the sum of all forces acting at the mass, which are the gravitational force $\vec{F}_g = m\vec{g}$ and the buoyancy $\vec{F}_b$, which is perpendicular to the water surface.

From figure 1 we see that the tangent of the angle α can be expressed as

$$\tan\alpha = \frac{F_c}{F_g} = \frac{mx\omega^2}{mg} = x\frac{\omega^2}{g}, \tag{2}$$

[1] We should note that concepts of both centripetal and centrifugal force are introduced in Czech high school physics textbooks. Also, a person on a merry-go-round clearly feels the centrifugal force.

where g is the acceleration of gravity and x is the distance of the piece of water from the rotation axis.

Now we should find the shape which fulfils equation (2). We can start with a qualitative observation: the larger x, the steeper is the profile of the liquid surface, as equation (2) suggests and figure 1 shows. The profile looks like a parabola (the simplest curve of a similar shape students know). But is it really so?

To make the derivation simple we will *suppose* the shape to be parabolic and we will *prove* that this is true. So, assume that the vertical cut of the surface, which includes the rotation axis has the form

$$y = ax^2, \tag{3}$$

where y is the height above the lowest point of the surface, a is some positive constant and x is the distance from the rotation axis.

According to figure 2 we can now calculate $\tan\alpha$:

$$\tan\alpha = \frac{\Delta y}{\Delta x} = \frac{ax_2^2 - ax_1^2}{x_2 - x_1} = \frac{a(x_2+x_1)(x_2-x_1)}{x_2 - x_1} = a(x_2+x_1) \doteq 2ax, \tag{4}$$

where we take $x_2 \doteq x_1 = x$.

So, for our supposed shape $\tan\alpha$ is proportional to x as is required by (2). We have proved that the profile of the rotating water is really parabolic.

Moreover, we can easily find the value of the constant a. Comparing equations (2) and (4), we see that

$$a = \frac{\omega^2}{2g}. \tag{5}$$

This means that equation (2) can be rewritten as $y = \omega^2/(2g)x^2$. Usually the height above the lowest point of the surface is denoted by h (instead of y), the distance from the vertical axis (for which we used the symbol x) being denoted by R. So we arrive at the familiar form of equation for the surface of the rotating liquid:

$$h = \frac{\omega^2}{2g}R^2, \tag{6}$$

where h is the height above the lowest point of the surface and R is the distance from the rotation axis.

Derivation in a non-inertial reference frame.

The qualitative discussion from the point of view of the rotating frame is simple for students understanding the concept of centrifugal force—and, in fact, for anybody who has ever perceived it on a merry-go-round or similar attraction in an amusement park. If a person sitting on a merry-go-round tries to describe the shape of the water in a vessel that he holds in his hands using the same physical laws as in a non-inertial frame, he has to add in his description a force, called centrifugal force, which is pulling away from the axis of rotation with a magnitude that is proportional to the distance from the axis. But there is also gravity which pulls the water down. The final shape of the surface is the result of the 'struggle' between these two forces.

To make the treatment quantitative, let us imagine again a certain small amount of water at the water surface. The gravity $\vec{F}_g$ influencing this water is the same at all places. On the other hand, the centrifugal force $\vec{F}_{cf}$ is directly proportional to the distance from the rotation axis. (Its value is given by the same formula as for the centripetal force, $F_{cf} = mR\omega^2$.) The resultant of these forces deviates from the vertical direction (as shown at figure 3) and the surface has to be perpendicular to it[2]. Of course, it has the same value as the buoyancy and the net force acting on a piece of water equals zero—it must be so because the piece of water is still in the rotating frame.

[2] We can remind students that the surface of non-rotating (static) water is perpendicular to the force of gravity acting on it. If the surface of the water is not perpendicular to the force, the water would flow until equilibrium is reached.

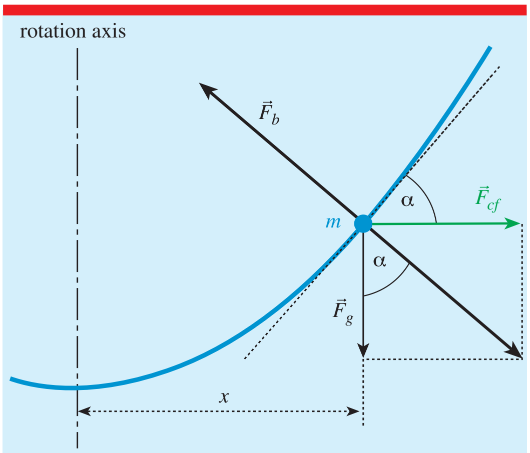


Figure 3. Centrifugal force ($\vec{F}_{cf}$), gravity ($\vec{F}_g$) and buoyancy ($\vec{F}_b$) acting on a liquid in a rotating reference frame.

The further derivation would be the same as in the case of the inertial reference frame, just with F_c being replaced by F_{cf}. So we would again arrive at equation (6).

Apparatus for measuring the surface of a rotating liquid

Our apparatus is inspired by the apparatus presented in Basta's article [5]. They were able to measure small *solid* paraboloids. The apparatus we present here is able to measure the shape of a liquid surface in motion. This method may be preferable because it is very difficult to let something solidify.

Of course, there are other methods of how to measure the shape of a rotating liquid, e.g. using reflection of light at the water surface. The advantage of our approach is that it enables us to measure the shape *directly* and it also clearly visualizes it.

The measuring device consists of 31 thin bars, the positions of which can be set up so their lower ends touch the surface of the water, as shown in figure 4. In our case we made the water paraboloid in an ordinary plastic basin. A wooden desk with a plastic basin is placed on a variable-speed motor (figure 5). Each vertical metal bar has a spike at its lower end. When the water rotates with constant angular velocity ω (it is necessary to wait for a while before the surface 'stabilizes' itself) we should adjust the bars one by one. We start with all bars up. Our experience is that it is

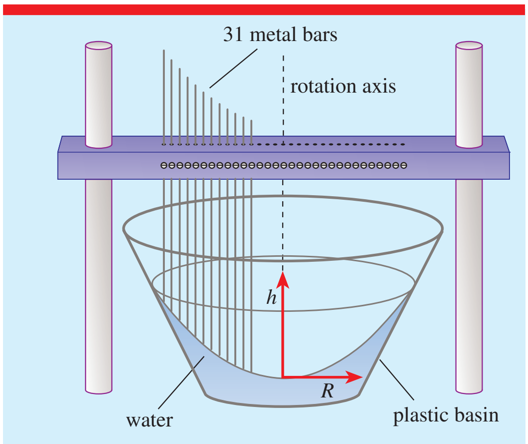


Figure 4. Bars used for measuring the surface of the rotating liquid.

Figure 5. The final apparatus with metal bars.

better to move the bar down until it touches the surface and then move it up until there's no contact between the bar and the water. The ends of the bars now replicate the surface and they create a clearly visible parabola.

Even simpler and cheaper constructions can be used, e.g. in a student project. Wooden skewers can be used instead of metal bars and their fixing can also be simpler—see figure 6. In fact, our first device was constructed so and it worked

Figure 6. A prototype of the apparatus with wooden bars.

reasonably well. The problem is that skewers tend to be too flexible and bend.

For verification it is necessary to know the period T (or frequency or angular velocity). There can be a problem with older motors because of their speed oscillation and missing speedometers, so we have measured the frequency using a computer and a phototransistor connected to the microphone input.

Comparing theory and experiment

We will compare the theoretical coefficient from equation (6), which we denote as c_t,

$$c_t = \frac{\omega^2}{2g} = \frac{2\pi^2}{g}\frac{1}{T^2} \qquad (7)$$

with the experimental coefficient obtained from fitting the ends of the bars with a parabola. The experimental coefficient is denoted by c_e.

Some experimental results are given in table 1 and in figure 7[3].

[3] In two measurements (table 1, figures 6(a) and (b)—symbols $+$ and Δ) the angular speed was very high and there was not enough water in the basin for the whole parabola to be created. So, there was a dry place in the centre of the basin and the shape of the rotating water was a truncated paraboloid. We could not add more water, because it would have spilled out.

Table 1. Experimental results. S—symbols in graphs, T—period, c_t—theoretical coefficient, c_e—experimental coefficient, δc—relative deviation of c_e and c_t.

Bars	S	T (s)	c_t (m^{-1})	c_e (m^{-1})	δc
Wooden	+	0.635	4.99	4.94	0.9%
Metal	Δ	0.643	4.86	4.67	4.0%
	□	0.722	3.86	3.83	0.8%
	○	0.873	2.64	2.65	−0.6%
	+	1.212	1.37	1.35	2.2%

The experimental data match the theoretical model very well. The maximum relative deviation is 4%, which is a good result considering the simplicity of the experimental setup. Even the apparatus with skewers (figure 7(a)) shows good results.

A boat on a rotating liquid

Looking at the shape of the surface of rotating liquid, a question may arise: what will happen if we put a small object (a 'boat') on the surface of rotating water?

A parallel to this situation can be found in Edgar Allan Poe's short story 'A Descent into the Maelström' [6]. The hero solves a very similar problem. The problem is how to get out of a huge whirlpool (maelstrom). The maelstrom and the circular paraboloid are of course different but there are certain similarities. And this story makes the problem interdisciplinary.

Qualitative grounds for a boat's movement

The problem is too complicated to be treated quantitatively at the high school level. (In fact, we are not aware of any published quantitative solution.) But even qualitative discussion will lead us to interesting results, which can be compared with experiment.

Let us look at a very simple model of a boat—a small ball or a sphere. We will assume its centre of mass is just at the water surface. The situation is the same as that shown in figure 1 or 3. The boat just floats on the water's surface keeping the same position in relation to the surrounding water.

Now let us take a boat which has some mass well below the water. It can be a heavy anchor hanging on a long string from the boat. We shall see that such a boat will be pulled *upwards*, farther from the rotational axis.

It is simpler to discuss the relevant effects in the rotating frame using the concept of centrifugal

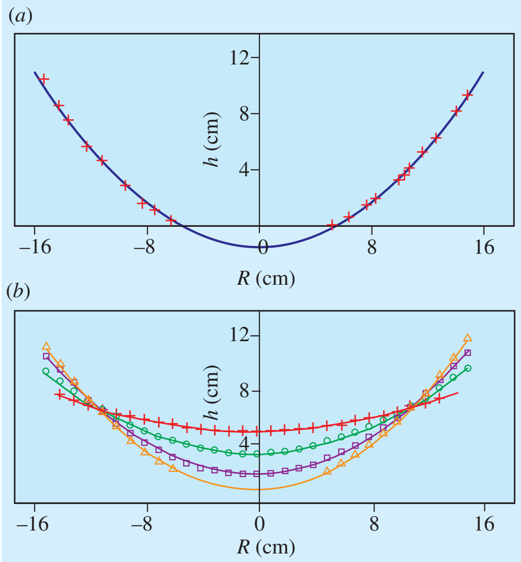


Figure 7. (*a*) Points on the surface of the rotating liquid and the corresponding interpolated curve (parabola), measured with wooden skewers. (*b*) Points on the surface of the rotating liquid and the corresponding interpolated curves (parabolas).

force. The anchor is *farther* from the axis of rotation than the boat. So the centrifugal force $F_{cf} = mR\omega^2$ acting on the anchor is greater than when its mass was in the boat. Therefore the resulting force $\vec{F}^{anchor} = \vec{F}^{anchor}_{cf} + \vec{F}^{anchor}_{g}$ acting on the anchor is no longer perpendicular to the water surface—see figure 8[4]. Clearly it *pulls the whole boat upwards*, farther from the rotation axis[5].

We omit here the discussion of the situation in an inertial frame. (It is a bit less straightforward and the diagrams are more complicated.) What we should note is the fact that the effect of pulling the boat farther from the rotational axis is not limited to the boat with an anchor. It is the same for the boat with a keel and, in general, for any boat with the centre of mass below the water surface. (At least this is our qualified judgement. It may be difficult to prove it for a general shape of the boat

[4] To make our discussion simple we neglect the buoyancy acting at the anchor and the mass of the string. We have also exaggerated the mass of the anchor in figure 8—the forces shown in this figure correspond to the same masses of the anchor and of the boat.

[5] To be more precise: it is the component of $\vec{F}^{anchor}$ tangential to the water surface that pulls the boat upwards. The component perpendicular to the surface tries to sink the boat into the water. This force is balanced by increased buoyancy (for example, we can increase the volume of the boat).

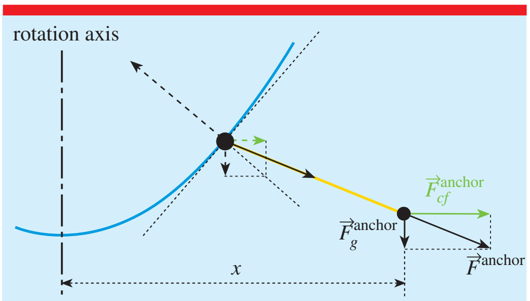


Figure 8. The anchor will pull the boat upwards.

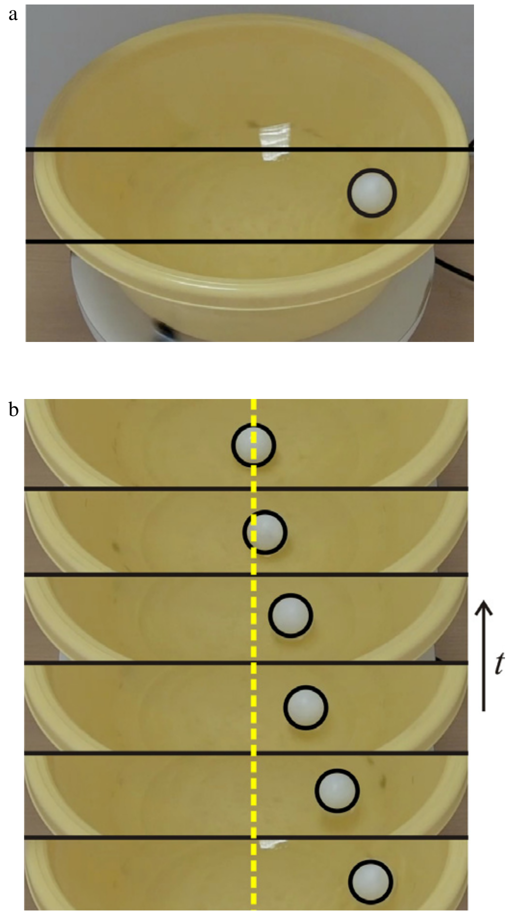


Figure 9. (a) The experiment with a ping pong ball (the boat is circled in the picture). This frame was captured shortly after placing the ball on the surface of the rotating water. The part of the picture between the two black lines matches the lowest part of (b). The pictures are ordered by time—the time is increasing from bottom to top. The yellow dashed line shows the approximate centre of the basin. In agreement with the theory, the ping pong ball is pulled to the centre.

and just a slight shift of the centre of mass, but the general tendency is clear.)

Just the opposite is true if the centre of mass of a boat is (well) above the water surface. The centrifugal force on the mass above water (i.e. closer to the rotation axis) is lower so the resulting force (sum of centrifugal and gravitational) is again not perpendicular to the water surface. In this case it is tilted *towards* the rotational axis, pulling the whole boat towards it.

So, our theoretical discussion leads us to the conclusion that the boat with its centre of mass above water will be pulled towards the rotational axis, the boat with its centre of mass below the water surface will be pulled farther from it. Will the experiment confirm this prediction?

Apparatus to demonstrate the behaviour of a boat on a rotating liquid

The apparatus is almost the same as in the previous experiment, just without the bars measuring the shape of the water surface. As boat models with centre of mass well above the water, we used ping pong balls or pieces of styrofoam. For a simple boat with centre of mass below the water surface we took a piece of styrofoam with a small screw acting as a heavy keel.

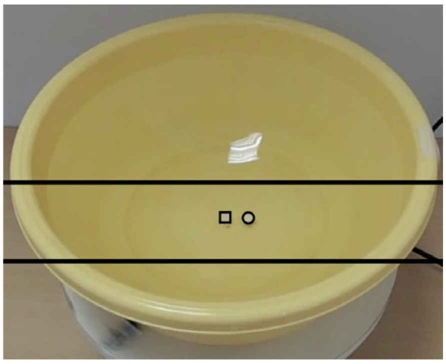
Comparing theory and the real movement of the boat model on the rotating liquid

Because there are no equations to verify, the experiment is only qualitative. We watch what happens to the boat models that were put on the surface of the rotating water. You can see the results in figures 9 and 10. They show that, in agreement with theory, the styrofoam pieces with screws really go up, farther from the rotation axis, and the ping pong balls down, towards it.

Note that figures 9(b) and 10(b) are composites of images taken at successive times. The time interval between them is not equal. The overall time of each experiment (figures 9(b) and 10(b)) is approximately 20 s. There are also short videos of these situations available on the author's webpage [7].

Conclusions

As we have stated already, the problem of the shape of a rotating liquid is an old one, described and solved in many textbooks and articles. We hope that we might persuade at least some readers a



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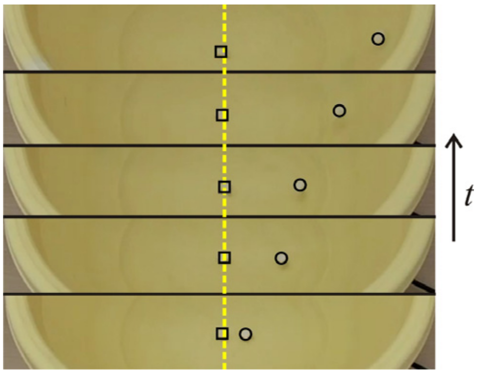

Figure 10. The experiment with boats with and without a keel. A piece of styrofoam denoted by a square is the boat without a keel; the other one (denoted by a circle) uses a metal screw as a keel. This image was taken shortly after placing the piece with the screw near the centre where the piece without the screw was already floating. The motion of boats with and without a keel is shown as images taken at successive times. The boat with a keel (marked by a circle) goes to the edge of the basin. The boat without a keel (marked by a square) stays in the centre.

that it still offers some new possibilities, both from experimental and theoretical points of view.

The direct measurement of the shape described here may be used at high school level either in the laboratory or as a small project. The apparatus can be modified; for example, more or fewer rods or skewers can be used. Also, even a relatively simple instrument can give results in very good agreement with theory.

The second, qualitative experiment, described here is even simpler to prepare and present. The problem can be first presented to students theoretically (perhaps motivated by Poe's short story mentioned here). The prediction students make can then be confronted with the real experiment. Of course, it would be interesting to solve this problem quantitatively, at least to some approximation. We hope to return to this problem in some future work.

Acknowledgments

We would like to thank Professor Gorazd Planinšič for helpful discussions on how to keep this article sufficiently clear and not to make the explanations too complicated.

Received 18 December 2009, in final form 9 April 2010
doi:10.1088/0031-9120/45/5/001

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