Fall 2016

AMATH 732: Assignment 2.
Due Friday, Oct. 7

1. Read section 6.2 in Lin & Segel and do problems 5–6 on pages 204–205.
2. Read section 6.3 in Lin & Segel and do problems 2–4 on page 223.
3. Use dimensional analysis to prove the Pythagorean Theorem.
4. Reconsider the projectile problem when air resistance is taken into account but changing gravitational acceleration is not. The equation and initial conditions are

$$m\frac{d^2x^*}{dt^{*2}} + k\frac{dx^*}{dt^*} + mg = 0, \qquad x^*(0) = 0, \qquad \frac{dx^*}{dt^*}(0) = V,$$

where the parameter $k > 0$ is associated with air resistance. This is a linear drag law and is really only appropriate for objects moving 'slowly' through a fluid (e.g., think of a grain of pollen falling through the air).

   (a) When V is sufficiently small, the scalings of x and t are the same as for the projectile problem discussed in class. Why?

   (b) Introduce scaled dimensionless variables (without the $*$) and show that the problem becomes one in which the only parameter is $\beta = kV/mg$. Give a physical interpretation of β.

   (c) Find a power series expansion for $x(t,\beta)$ assuming small β. Find approximations to the maximum height x_m and the corresponding time t_m to $O(\beta^3)$.

5. This problem considers the harmonics of a piano wire.

   (a) The displacement of an ideal piano wire, which has no bending stiffness, is described by the wave equation

$$\rho A\frac{\partial^2 y}{\partial t^2} - T\frac{\partial^2 y}{\partial x^2} = 0 \qquad 0 < x < L, \tag{1}$$

with boundary conditions $y(0) = y(L) = 0$. Here ρ is the mass density of the wire, A the cross-sectional area and T the tension in the wire. The eigenmodes are solutions of the form

$$y = e^{-i\omega_n t}\sin\frac{n\pi x}{L}. \tag{2}$$

What are the eigenvalues ω_n? The lowest, or fundamental, frequency ω_1 determines the note played while the higher frequencies (overtones) determine the quality of the sound made by the piano wire. The second harmonic ω_2 is one octave above the fundamental ($\omega_2/\omega_1 = 2$), the third harmonic ω_3 is a fifth above the second harmonic ($\omega_3/\omega_2 = 3/2$), the fourth harmonic is a fourth above the third ($\omega_4/\omega_3 = 4/3$), etc. If all musical instruments had the same frequencies they would sound pretty much the same. In reality, the overtones are different for different instruments.

(b) A real piano wire has a small bending stiffness which modifies the overtones. Incorporating a bending stiffness term into the equation for the piano wire's displacement leads to the equation

$$\rho A \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} + EAk^2 \frac{\partial^4 y}{\partial x^4} = 0, \tag{3}$$

where E is the Young's modulus and k is the radius of gyration. Nondimensionalize this equation using the lengthscale L and the time scale L/c where $c^2 = T/(A\rho)$ (c is the propagation speed in the ideal equation (1)) to obtain

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + \epsilon \frac{\partial^4 y}{\partial x^4} = 0, \qquad 0 < x < 1, \tag{4}$$

where y, x and t are now dimensionless scaled variables. What is ϵ? For a piano wire a typical value for ϵ is 10^{-4}, so it is very small.

(c) Using perturbation methods find the normal modes and eigenfrequencies of the piano wire up to and including the $O(\epsilon)$ contribution. Comment on the ratio of consecutive harmonics. Are they perfect octaves, fifths, fourths, etc.?

6. Stability of a satellite in orbit.

In the classical Newtonian model for the motion of a satellite in orbit around the earth the location of the satellite is governed by

$$\frac{d^2 r}{dt^2} - r\frac{d\theta}{dt}^2 = -\frac{GM}{r^2}, \tag{5}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0. \tag{6}$$

Here G is the universal gravitational constant, M is the mass of the earth, $(r(t), \theta(t))$ are the polar coordinates of the satellite, with origin at the centre of the earth, and the second equation states that the angular moment per unit mass, $r^2 \frac{d\theta}{dt}$, is constant (Kepler's second law).

(a) Show that a circular orbit, $r = a$, $\frac{d\theta}{dt} = \omega$ where a and ω are constant, is a solution of these equations. Find a relationship between a and ω (this is Kepler's third law).

(b) Is the orbit stable? Imagine a booster rocket on the satellite gives it a small initial radial velocity ϵv (radial so the angular momentum does not change). Expand $r(t, \epsilon)$ and $\frac{d\theta}{dt}(t, \epsilon)$ in a regular perturbation expansion and determine whether or not the orbit is stable (based on the first correction to the unperturbed orbit).

7. (a) Consider the following sequences of functions. Are they asymptotic sequences as $x \to 0$ or as $x \to \infty$?

$$\phi_n(x) = (1 + x^2)^{-n}, \quad n = 1, 2, 3, \dots,$$

$$f_n(x) = e^{(6-n)x}, \quad n = 1, 2, 3, \dots,$$

$$\psi_n(x) = \tanh^n(1/x), \quad n = 1, 2, 3, \dots,$$

$$q_n(x) = \frac{\sin(nx + \pi/n)}{x^n}, \quad n = 1, 2, 3, \dots.$$

(b) Find the asymptotic expansion for $h(x) = 1/x$ (1st three nonzero terms), if it exists, as $x \to \infty$ in terms of each of the sequences from part (a) which form an asymptotic sequence as $x \to \infty$.

8. Explain, using the definitions of an asymptotic sequence and an asymptotic expansion, why it is correct to write

$$Ei(x) \sim \frac{e^{-x}}{x}\left(1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots + (-1)^n \frac{n!}{x^n} + \cdots\right) \quad \text{as } x \to \infty. \tag{7}$$

Would you use this expansion to find $Ei(2.0)$? Explain.

9. (a) Find the asymptotic expansion for the function $Ei_{a-1}(x)$ (see notes - fill in details). Show that the series is divergent.

   (b) Plot $S_n(x, a)$ as a function of n for $x = 5$ and 10 for $a = 2.5$ and $a = 10.5$. For a given x and a compute the optimal number of terms to use to approximate the value of $Ei_{a-1}(x)$.