Guelph-Waterloo Physics Institute

Quantum Mechanics – Physics 701/7010 – Fall 2016 – Problem Set 3

Due Thursday, October 27th, at the lecture in Waterloo, or at Guelph, in Kiley Rider's mailbox in the copy room.

(1) This problem concerns a special case of the Baker-Campbell-Hausdorff (BCH) theorem: If A and B are operators that do not necessarily commute with each other, but which both commute with $[A, B]$ they satisfy

$$e^{A+B} = e^A\, e^B\, e^{-\frac{1}{2}[A,B]}$$

To show this, first show that $[B, e^{xA}] = e^{xA}[B, A]x$. Next, define $G(x) = e^{xA}\, e^{xB}$, and show that

$$\frac{dG}{dx} = (A + B + [A, B]x)G.$$

Integrate this to obtain the desired result.
(Based on Problem 13, Chapter 2 of Gottfried and Yan's "Quantum Mechanics: Fundamentals")

(2) Coherent states (These results are useful when discussing quantization of the electromagnetic field. You may need to review the creation and annihilation operator treatment of the harmonic oscillator.)
Problem 2.19 from Sakurai and Napolitano, *Modern Quantum Mechanics*, 2nd ed.
A coherent state of a one-dimensional simple harmonic oscillator is defined to be an eigenstate of the (non-Hermitian) annihilation operator a:

$$a\,|\lambda\rangle = \lambda\,|\lambda\rangle$$

where λ is, in general, a complex number.

(a) Prove that

$$|\lambda\rangle = e^{-|\lambda|^2/2} e^{\lambda a^\dagger}\,|0\rangle$$

is a normalized coherent state.

(b) Prove the minimum uncertainty relation for such a state.

(c) Write $|\lambda\rangle$ as

$$|\lambda\rangle = \sum_{n=0}^{\infty} f(n)\,|n\rangle\,.$$

Show that the distribution of $|f(n)|^2$ with respect to n is of the Poisson form. Find the most probable value of n, and hence of E.

(d) Show that a coherent state can also be obtained by applying the translation (finite-displacement) operator $e^{-ipl/\hbar}$ (where p is the momentum operator and l is the displacement distance) to the ground state. (See also Gottfried 1966, 262-64.)

(3) (a) Write down an expression for the classical action for a simple harmonic oscillator for a finite time interval.

(b) Comment on the relationship between your result for the classical action and the propagator for the harmonic oscillator given by Sakurai and Napolitano in Eq. 2.6.18. More specifically, is this relationship something that you generally expect to be true?

Based on Problem 2.34 of Sakurai and Napolitano's *Modern Quantum Mechanics*, 2nd ed.

(4) For the following question, please see the posted notes: *Equivalence of the path integral approach to the Schrödinger equation.*
When we showed that the path integral and Schrödinger approaches were equivalent in the lecture, we mentioned the so-called *mid-point prescription*, where V must be evaluated at $(x' + x'')/2$. It was argued that in our derivation this was not necessary.

Consider however the case of a vector potential which will introduce a factor of (using SI electromagnetic units):

$$\exp\left[\frac{iq\epsilon}{\hbar}\frac{\eta}{\epsilon}A_x(x' + \alpha\eta)\right]$$

into the propagator $\langle x'', t''|x', t'\rangle$ where $\eta = x'' - x'$, $\epsilon = t'' - t'$ and ϵ is considered to be infinitesimal. This factor arises from the last term in the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} - q\phi + q\mathbf{v}\cdot\mathbf{A}$$

We simplify this problem slightly by considering one-dimension only — this point is not essential. The quantity $0 \le \alpha \le 1$ is used to characterize where in the x'', x' interval the vector potential should be evaluated; the mid-point case is $\alpha = 1/2$. Note that ϵ now gets cancelled, in contrast to the scalar potential case. Thus going to order ϵ to derive the Schrödinger equation means going to order η^2 in expanding the exponential. This will not only bring in a A_x^2 term, but will also make the answer sensitive to the argument of A_x in the linear term. Show that $\alpha = 1/2$ is necessary to obtain the one-dimensional version of:

$$H = \frac{1}{2m}\left(\mathbf{p}\cdot\mathbf{p} - q\,\mathbf{p}\cdot\mathbf{A} - q\,\mathbf{A}\cdot\mathbf{p} + q^2\,\mathbf{A}\cdot\mathbf{A}\right) + q\,\phi$$

in $H\psi = i\hbar\partial\psi/\partial t$. Note that we derived this H by more conventional means in the lecture. This again, shows the equivalence of the conventional and path integral approachs, but with the requirement for use of the mid-point prescription.

This is a slightly reworded version of Exercise 8.6.4 of Shankar's Principles of Quantum Mechanics, 2nd ed. [With the use of the SI electromagnetic unit system, factors of c in Shankar's statement of the problem are not present here. I also use the opposite sign for η.]

(5) Problem 2.28 from Sakurai and Napolitano, *Modern Quantum Mechanics*, 2nd ed.
Consider an electron confined to the *interior* of a hollow cylindrical shell whose axis coincides with the z-axis. The wave function is required to vanish on the inner and outer walls, $\rho = \rho_a$ and ρ_b and also at the top and bottom $z = 0$ and L.

(a) Find the energy eigenfunctions. (Do not bother with normalization.) Show that the energy eigenvalues are given by

$$E_{lmn} = \left(\frac{\hbar^2}{2m_e}\right)\left[k_{m,n}^2 + \left(\frac{l\pi}{L}\right)^2\right] \quad (l = 1, 2, 3, \ldots, m = 0, 1, 2, \ldots),$$

where k_{mn} is the nth root of the transcendental equation:

$$J_m(k_{mn}\rho_b)N_m(k_{mn}\rho_a) - N_m(k_{mn}\rho_b)J_m(k_{mn}\rho_a) = 0.$$

(b) Repeat the same problem when there is a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ for $0 < \rho < \rho_a$. Note that the energy eigenvalues are influenced by the magnetic field even though the electron never "touches" the magnetic field.

(c) Compare, in particular, the ground state of the $B = 0$ problem with that of the $B \neq 0$ problem. Show that if we require the ground-state energy to be unchanged in the presence of B, we obtain "flux quantization"

$$\pi\rho_a^2 B = \frac{2\pi N\hbar}{e}, \quad (N = 0, \pm1, \pm2, \ldots).$$

Note: I have restated this problem in the SI electromagnetic unit system. (Sakurai and Napolitano use the Heaviside-Lorentz system.)

(6) Spinless particle in a uniform magnetic field — diamagnetism
Problem 2.38 from Sakurai and Napolitano, *Modern Quantum Mechanics*, 2nd ed.
Consider the Hamiltonian of a spinless particle of charge e. In the presence of a static magnetic field, the interaction terms can be generated by

$$p_{\text{operator}} \to p_{\text{operator}} - e\mathbf{A}$$

where $\mathbf{A}$ is the appropriate vector potential. Suppose, for simplicity, that the magnetic field $\mathbf{B}$ is uniform in the positive z-direction. Prove that the above prescription indeed leads to the correct expression for the interaction of the orbital magnetic moment $(e/(2m))\mathbf{L}$ with the magnetic field $\mathbf{B}$. Show that there is also an extra term proportional to $B^2(x^2 + y^2)$, and comment briefly on its physical significance.

Note: I have restated this problem in the SI electromagnetic unit system. (Sakurai and Napolitano use the Heaviside-Lorentz system.)

(7) A particle in a uniform magnetic field — analogy with simple harmonic oscillator
Problem 2.39 from Sakurai and Napolitano, *Modern Quantum Mechanics*, 2nd ed.
An electron moves in the presence of a uniform magnetic field in the z-direction ($\mathbf{B} = B\hat{\mathbf{z}}$).

(a) Evaluate

$$[\Pi_x, \Pi_y]$$

where

$$\Pi_x \equiv p_x - eA_x, \quad \Pi_y \equiv p_y - eA_y.$$

(b) By comparing the Hamiltonian and the commutation relation obtained in (a) with those of the one-dimensional oscillator problem, show how we can immediately write the energy eigenvalues as

$$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \left(\frac{|qB|\hbar}{m}\right)\left(n + \frac{1}{2}\right),$$

where $\hbar k$ is the continuous eigenvalue of the p_z operator and n is a nonnegative integer including zero.

Note: I have restated this problem in the SI electromagnetic unit system. (Sakurai and Napolitano use the Heaviside-Lorentz system.)