Guelph-Waterloo Physics Institute

Quantum Mechanics – Physics 701/7010 – Fall 2016 – Problem Set 2

Due Thursday, September 29th at the lecture in Waterloo, or main office in Guelph.

Question numbers are from Sakurai 2nd (most recent) edition (the grey cover book, **not** the red cover *Revised* edition).

(1) Problem 1.18 from Sakurai and Napolitano, *Modern Quantum Mechanics*, 2nd ed.

(a) The simplest way to derive the Schwarz inequality goes as follows. First, observe

$$(\langle\alpha| + \lambda^* \langle\beta|) \cdot (|\alpha\rangle + \lambda |\beta\rangle \geq 0$$

for any complex number λ; then choose λ in such a way that the preceding inequality reduces to the Schwarz inequality.

(b) Show that the equality sign in the generalized uncertainty relation holds if the state in question satisfies

$$\Delta A |\alpha\rangle = \lambda \Delta B |\alpha\rangle$$

with λ purely imaginary.

(c) Explicit calculations using the usual rules of wave mechanics show that the wave function for a Gaussian wave packet given by

$$\langle x'|\alpha\rangle = (2\pi d^2)^{-1/4} \exp\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x' - \langle x\rangle)^2}{4d^2}\right]$$

satisfies the minimum uncertainty relation

$$\sqrt{\langle(\Delta x)^2\rangle}\sqrt{\langle(\Delta p)^2\rangle} = \frac{\hbar}{2}.$$

Prove that the requirement

$$\langle x'| \Delta x |\alpha\rangle = (\text{imaginary number}) \langle x'| \Delta p |\alpha\rangle$$

is indeed satisfied for such a Gaussian wave packet, in agreement with (b).

(2) Problem 1.20 from Sakurai and Napolitano, *Modern Quantum Mechanics*, 2nd ed.
Find the linear combination of $|+\rangle$ and $|-\rangle$ kets that maximizes the uncertainty product

$$\langle(\Delta S_x)^2\rangle \langle(\Delta S_y)^2\rangle.$$

Verify explicitly that for the linear combination you found, the uncertainty relation for S_x and S_y is not violated.

(3) Problem 1.28 from Sakurai and Napolitano, *Modern Quantum Mechanics*, 2nd ed.

(a) Let x and p_x be the coordinate momentum and the linear momentum in one dimension. Evaluate the classical Poisson bracket

$$[x, F(p_x)]_{\text{classical}}.$$

(b) Let x and p_x be the corresponding quantum-mechanical operators this time. Evaluate the commutator

$$\left[x, \exp\left(\frac{ip_x a}{\hbar}\right)\right].$$

(c) Using the result obtained in (b), prove that

$$\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle, \quad (x\,|x'\rangle = x'\,|x'\rangle)$$

is an eigenstate of the coordinate operator x. What is the corresponding eigenvalue?

(4) Problem 1.29 from Sakurai and Napolitano, *Modern Quantum Mechanics*, 2nd ed.

(a) On page 247, Gottfried (1966) states that

$$[x_i, G(\mathbf{p})] = i\hbar\frac{\partial G}{\partial p_i}, \quad [p_i, F(\mathbf{x})] = -i\hbar\frac{\partial F}{\partial x_i}$$

can be "easily derived" from the fundamental commutation relations for all functions of F and G that can be expressed as power series in their arguments. Verify this statement.

(b) Evaluate $[x^2, p^2]$. Compare your result with the classical Poisson bracket $[x^2, p^2]_{\text{classical}}$

(5) Problem 1.33 from Sakurai and Napolitano, *Modern Quantum Mechanics*, 2nd ed.

(a) Prove the following:

(i) $\langle p'|\,x\,|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\,\langle p'|\alpha\rangle$,

(ii) $\langle\beta|\,x\,|\alpha\rangle = \int dp'\phi_\beta^*(p')i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')$,

where $\phi_\alpha(p') = \langle p'|\alpha\rangle$ and $\phi_\beta(p') = \langle p'|\beta\rangle$ are momentum-space wave functions.

(b) What is the physical significance of

$$\exp\left(\frac{ix\Xi}{\hbar}\right),$$

where x is the position operator and Ξ is some number with the dimension of momentum? Justify your answer.