Statistical Physics – PHYS 704
Course summary

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Contents

1 Some Thermodynamic relations

- Energy: E, $\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N$. Min. in equilibrium when S and V are const.
- Helmholtz free energy: $F = E - TS$, $\mathrm{d}F = -S\mathrm{d}T - P\mathrm{d}V + \mu\mathrm{d}N$. Min. in equil. when T and V are const.
- Enthalph: $W = E + PV$, $\mathrm{d}W = T\mathrm{d}S + V\mathrm{d}P + \mu\mathrm{d}N$. Min. in equil. when S (adiabatic) and P are const.
- Gibbs free energy: $\Phi = E + PV - TS$, $\mathrm{d}\Phi = -S\mathrm{d}T + V\mathrm{d}P + \mu\mathrm{d}N$. Min. in equil. when T and P are const.
- Grand potential: $\Omega = -PV$, $\mathrm{d}\Omega = -S\mathrm{d}T + P\mathrm{d}V + N\mathrm{d}\mu$.

Derivative relations

$$\begin{aligned}
+\left(\frac{\partial T}{\partial V}\right)_S &= -\left(\frac{\partial P}{\partial S}\right)_T = +\frac{\partial^2 E}{\partial S \partial V} \\
+\left(\frac{\partial T}{\partial P}\right)_S &= +\left(\frac{\partial V}{\partial S}\right)_P = +\frac{\partial^2 W}{\partial S \partial P} \\
+\left(\frac{\partial S}{\partial V}\right)_T &= +\left(\frac{\partial P}{\partial T}\right)_V = -\frac{\partial^2 F}{\partial T \partial V} \\
-\left(\frac{\partial S}{\partial P}\right)_T &= +\left(\frac{\partial V}{\partial T}\right)_P = +\frac{\partial^2 \Phi}{\partial T \partial P}
\end{aligned} \tag{1}$$

Temperature

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N} \tag{2}$$

Heat-capacity

$$\begin{aligned}
C_V &= \left(\frac{\mathrm{d}E}{\mathrm{d}T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V \\
C_P &= \left(\frac{\mathrm{d}W}{\mathrm{d}T}\right)_P = \left(\frac{\partial E}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P
\end{aligned} \tag{3}$$

Compressibility

$$\kappa_X = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_X \tag{4}$$

1.1 Some def. in stat. mech.

Definition of entropy

$$S := -\sum_{n,(N)} \rho_{n,(N)} \ln\Big(\rho_{n,(N)}\Big), \tag{5}$$

where ρ is the density function or distribution fuction.

Canonical distribution (N constant)

$$\rho_n = \frac{1}{Z}\mathrm{e}^{-E_n/T}, \quad Z = \sum_n \mathrm{e}^{-E_n/T}. \tag{6}$$

$$F = -T\ln Z \tag{7}$$

Grand canonical distribution

$$\rho_{n,N} = \frac{1}{\mathcal{Z}}\mathrm{e}^{-(E_n-\mu N)/T}, \quad \mathcal{Z} = \sum_n \mathrm{e}^{-(E_n-\mu N)/T}. \tag{8}$$

$$\Omega = -T\ln\mathcal{Z} \tag{9}$$

2 Theory of ideal gases

Ideal gas means that there is no interaction between particles, $\varepsilon = \hbar^2k^2/(2m)$. In 3 dim.

$$\frac{1}{V}\sum_k \to \int\frac{\mathrm{d}^3k}{(2\pi)^3} = \int \mathrm{d}\varepsilon\, g(\varepsilon), \tag{10}$$

$$\frac{\mathrm{d}^3k}{(2\pi)^3} = \mathrm{d}\varepsilon\, g(\varepsilon), \quad g(\varepsilon) = \frac{m^{3/2}}{\sqrt{2}\pi\hbar^3}\sqrt{\varepsilon}. \tag{11}$$

See assignment 3 for other dim.

2.1 Ideal Fermi gases

$$n^{(\mathrm{F.D.})} = \frac{1}{\mathrm{e}^{(\varepsilon-\mu)/T}+1} \tag{12}$$

$$-\Omega = PV = \frac{2}{3}E = \frac{Vg_\mathrm{s}T}{\lambda^3}f_{5/2}(z) \tag{13}$$

$$N = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V} = \frac{Vg_\mathrm{s}}{\lambda^3}f_{3/2}(z) \tag{14}$$

$$\frac{PV}{NT} = \frac{f_{5/2}(z)}{f_{3/2}(z)} \tag{15}$$

Thermal wavelength $\lambda = h/\sqrt{2\pi mT} =: \Lambda/\sqrt{T}$, Λ is a constant.

Fermi functions

$$f_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \mathrm{d}x\frac{x^{\nu-1}}{z^{-1}\mathrm{e}^x+1} \tag{16}$$

Fugacity $z = \mathrm{e}^{\mu/T}$.

$$z\frac{\partial f_\nu(z)}{\partial z} = \frac{\partial f_\nu(z)}{\partial(\ln z)} = f_{\nu-1}(z) \tag{17}$$

Fermi energy As $T \to 0$, the ccamical potential will go to

$$\mu(T\to 0) =: \varepsilon_\mathrm{F} = \frac{\hbar^2}{2m}\left(\frac{6}{g_\mathrm{s}}\pi^2 n\right)^{2/3}, \tag{18}$$

where g_s is the spin deganareacy, and $n = N/V$.

In a regular metal, $\varepsilon_\mathrm{F} \sim 10^4\,\mathrm{K}$. For $T \ll \varepsilon_\mathrm{F}$

$$n^{(\mathrm{F.D.})}(\varepsilon) \approx \begin{cases} 1, & \varepsilon < \varepsilon_\mathrm{F} \\ 0, & \varepsilon > \varepsilon_\mathrm{F} \end{cases} \tag{19}$$

and $\int_0^\infty \mathrm{d}\varepsilon\, n^{(\mathrm{F.D.})}(\varepsilon)\ldots \to \int_0^{\varepsilon_\mathrm{F}} \mathrm{d}\varepsilon\ldots$

The internal energy

$$E(T \ll \varepsilon_\mathrm{F}) = \frac{3}{5}N\varepsilon_\mathrm{F} \tag{20}$$

2.2 Ideal Bose gases

$$n^{(\mathrm{B.E.})} = \frac{1}{\mathrm{e}^{(\varepsilon-\mu)/T}-1} \tag{21}$$

$$-\Omega = PV = \frac{2}{3}E = \frac{Vg_\mathrm{s}T}{\lambda^3}g_{5/2}(z) \tag{22}$$

$$N_\mathrm{e} = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V} = \frac{Vg_\mathrm{s}}{\lambda^3}g_{3/2}(z) \tag{23}$$

$$\frac{PV}{NT} = \frac{g_{5/2}(z)}{g_{3/2}(z)} \tag{24}$$

Thermal wavelength $\lambda = h/\sqrt{2\pi mT} =: \Lambda/\sqrt{T}$, Λ is a constant.

Bose functions

$$g_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \mathrm{d}x\frac{x^{\nu-1}}{z^{-1}\mathrm{e}^x-1} \tag{25}$$

For bosons $\mu \le 0$, meaning that $z = \mathrm{e}^{\mu/T} \le 1$.

$$z\frac{\partial g_\nu(z)}{\partial z} = \frac{\partial g_\nu(z)}{\partial(\ln z)} = g_{\nu-1}(z) \tag{26}$$

At $z = 1$, $g_\nu(z=1) = \zeta(\nu)$.

Critical temperature Critical teperature for ideal Bose gas (3 dim.)

$$T_\mathrm{c} = \frac{2\pi\hbar^2}{m}\left(\frac{N}{V g_\mathrm{s}\,\zeta(3/2)}\right)^{3/2} \tag{27}$$

Number of condensed particles

$$N_0 = N\left[1-\left(\frac{T}{T_\mathrm{c}}\right)^{3/2}\right], \quad T \le T_\mathrm{c}. \tag{28}$$

$$\begin{aligned} \lambda &= \frac{h}{\sqrt{2\pi m T}}, \quad \lambda_\mathrm{c} = \left[v\zeta(3/2)\right]^{1/3} \\ v &= \frac{1}{n} = \frac{V}{N}, \quad\quad v_\mathrm{c} = \frac{\lambda^3}{\zeta(3/2)} \end{aligned} \tag{29}$$

2.2.1 Photons

Photons are bosons with $\mu = 0$.

$$n^{(\text{photons})} = \frac{1}{\mathrm{e}^{\hbar\omega/T}-1}. \tag{30}$$

Number of photons in the interval ω to $\omega + \mathrm{d}\omega$:

$$\mathrm{d}N_\omega = \frac{V}{\pi^2 c^3}\frac{\omega^2\,\mathrm{d}\omega}{\mathrm{e}^{\hbar\omega/T}-1}. \tag{31}$$

Energy in the same interval:

$$\mathrm{d}E_\omega = \hbar\omega\,\mathrm{d}N_\omega = V u_\omega\,\mathrm{d}\omega, \tag{32}$$

where u_ω is the radiation energy density. Total radiation energy

$$E = \int_0^\infty \mathrm{d}E_\omega = \frac{4\sigma}{c}VT^4, \tag{33}$$

where $\sigma = \pi^2/(60\hbar^3 c^2)$

2.2.2 Phonons

Chain of N particles connected with springs, mean distance a. Hamiltonian

$$H = \sum_{s=0}^{N-1}\left[\frac{p_s^2}{2m} + \frac{\kappa}{2}(q_{s+1}-q_s)^2\right] \tag{34}$$

Also periodoc boundary conditions: $q_{s+N} = q_s$ and $p_{s+N} = p_s$.

Fourier transform (momentum space)

$$H = \sum_{k=0}^{N-1}\left[\frac{1}{2m}p_{-k}p_k + \frac{m\omega_k^2}{2}q_{-k}q_k\right]. \tag{35}$$

This is a Hamiltoninan for N harmonic oscillators:

$$H = \sum_{k=0}^{N-1}\hbar\omega_k\left(\hat{n}_k + \frac{1}{2}\right). \tag{36}$$

The frequency

$$\begin{aligned} \omega_k &= \sqrt{\frac{2\kappa}{m}(1-\cos(ka))} \\ &\overset{ka\ll 1}{\approx} |ka|\sqrt{\frac{\kappa}{m}}. \end{aligned} \tag{37}$$

Speed of sound

$$c_\mathrm{s} = \sqrt{\frac{\kappa a^2}{m}}. \tag{38}$$

For $|ka| \ll 1$,

$$C_V = \frac{2\pi^2}{5\hbar^3 {c_\mathrm{s}}^3}VT^3. \tag{39}$$

3 Second quantaization

Main idea: use a Bogoliubov transformation to transform the Hamiltonian to to a hamiltonian of an ideal gas of quasi-particles.

Harmonic oscillator interpretation in momentum space Occupation number representation:

$$|\psi\rangle = |n_{k_1}, n_{k_2}, \ldots, n_{k_i}, \ldots\rangle \tag{40}$$

$$\hat{n}_{k_i}\,|n_{k_1}, \ldots, n_{k_i}, \ldots\rangle = n_{k_i}\,|n_{k_1}, \ldots, n_{k_i}, \ldots\rangle \tag{41}$$

For bosons:

$$\left[a_k, a_{k'}^\dagger\right] = \delta_{k,k'}, \tag{42}$$

whereas for fermions

$$\left\{c_k, c_{k'}^\dagger\right\} = \delta_{k,k'}. \tag{43}$$

The Hamiltonian for interacting bosons

$$H = \overbrace{\sum_k \varepsilon_k a_k^\dagger a_k}^{\text{non-interating}} + \frac{1}{2V}\sum_{k,k',q}U(q)a_{k+q}^\dagger a_{k'-q}^\dagger a_{k'}a_k \tag{44}$$

and for fermions

$$H = \overbrace{\sum_{k,\sigma} \varepsilon_k c_{k,\sigma}^\dagger c_{k,\sigma}}^{\text{non-interating}} + \frac{1}{2V}\sum_{k,k',q,\sigma,\sigma'}U(q)c_{k+q,\sigma}^\dagger c_{k'-q,\sigma}^\dagger c_{k',\sigma'}c_{k,\sigma}. \tag{45}$$

The inteaction, $U(q)$, is given by

$$U(\boldsymbol{r}-\boldsymbol{r'}) = \frac{1}{V}\sum_q U(\boldsymbol{q})\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r}')}, \tag{46}$$

or

$$U(\boldsymbol{q}) = \int \mathrm{d}^3 r U(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}. \tag{47}$$

We can also talk about annihilation and creation operators in real space: $\psi^\dagger(\boldsymbol{r})$ creates a particle at position $\boldsymbol{r}$.

$$\int \mathrm{d}^3 r\, \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r}) = \sum_k a_k^\dagger a_k = \hat{N} \tag{48}$$

These operators faoolw the same commuttation relations as the momentum space equivalence.

3.1 Superfluidity in ^{4}He

At low enough temperatures most atoms will be in the ground state. Our goual is to bring the Hamiltonian to a form:

$$H = \sum_k E_k \xi_k^\dagger \xi_k + \text{const.} \tag{49}$$

Mean-field theory:

$$a_0 a_0^\dagger \approx a_0^\dagger a_0 = |a_0| = N_0 \tag{50}$$

meaning that

$$a_0 = \sqrt{N_0}\mathrm{e}^{\mathrm{i}\theta}. \tag{51}$$

Bogoliubov transformation

$$\begin{cases} \xi_k = u_k a_k + v_k a_{-k}^\dagger \\ \xi_k^\dagger = u_k a_k^\dagger + v_k a_{-k} \end{cases} \Leftrightarrow \begin{cases} a_k = u_k \xi_k - v_k \xi_{-k}^\dagger \\ a_k^\dagger = u_k \xi_k^\dagger - v_k \xi_{-k} \end{cases} \tag{52}$$

We still want the commutation relation $\left[\xi_k, \xi_k^\dagger\right] = 1$, which gives

$$u_k = \cosh\theta_k, \quad v_k \sinh\theta_k. \tag{53}$$

And

$$\cosh 2\theta_k = \frac{A_k}{\sqrt{A_k^2 - B_k^2}}, \quad \sinh 2\theta_k = \frac{B_k}{\sqrt{A_k^2 - B_k^2}}, \tag{54}$$

where

$$A_k = \varepsilon_k + nU(k), \quad B_k = nU(k). \tag{55}$$

Energy of the quasi-particles

$$\begin{aligned} E_k =& A_k \cosh 2\theta_k - B_k \sinh 2\theta_K \\ =& \sqrt{A_k^2 - B_k^2} = \sqrt{\varepsilon_k[\varepsilon_k + 2nU(k)]} \end{aligned} \tag{56}$$

Long wavelength limit ($k \approx 0$)

$$E_k \approx \hbar|k|\sqrt{\frac{nU(0)}{m}} \tag{57}$$

same form as phonons, and $c_\mathrm{s} = \sqrt{nU(0)/m}$. The super fluid behaves more like a solid than a liquid.

As long as some disturbance is slower than c_s, there can not be any dissipation of energy into the super fluid.

Spontaneously broken symmetry We can choose θ in the mean-field therory description of a_0 freely. And the particles curren will be

$$\boldsymbol{j} = \frac{\hbar n}{m}\boldsymbol{\nabla}\theta \tag{58}$$

But if the geomery has holes, we must have a periodically varying θ, meaning that the vorticity

$$\oint \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{\nabla}\theta(\boldsymbol{r}) = \ell 2\pi \tag{59}$$

is quantized.

3.2 BCS theory of superconductivity

Assume localized interaction $U(\boldsymbol{r}-\boldsymbol{r}') = (U/2)\delta(\boldsymbol{r}-\boldsymbol{r}')$, meaning that $U(q) \equiv U/2$, const. The Hmiltonian becomes

$$H = \sum_{k,\sigma} \varepsilon_k c_{k,\sigma}^\dagger c_{k,\sigma} + \frac{U}{V}\sum_{k,k',q} c_{k+q,\uparrow}^\dagger c_{k'-q,\downarrow}^\dagger c_{k',\downarrow} c_{k,\uparrow}. \tag{60}$$

Bogoliubov transformation Ideal gas of quasi-particles (fermions) with annihilation and creation operators γ_k and $\gamma_k^\dagger$:

$$\left\{\gamma_{k,\sigma}, \gamma_{k',\sigma}^\dagger\right\} = \delta_{k,k'}. \tag{61}$$

$$\begin{cases} \gamma_{k,\uparrow}^\dagger = u_k c_{k,\uparrow}^\dagger - v_k c_{-k,\downarrow} \\ \gamma_{k,\downarrow}^\dagger = u_k c_{k,\downarrow}^\dagger + v_k c_{-k,\uparrow} \end{cases} \Leftrightarrow \begin{cases} c_{k,\uparrow}^\dagger = u_k \gamma_{k,\uparrow}^\dagger + v_k \gamma_{-k,\downarrow} \\ c_{k,\downarrow}^\dagger = u_k \gamma_{k,\downarrow}^\dagger - v_k \gamma_{-k,\uparrow} \end{cases} \tag{62}$$

$$u_k = \cos\theta_k, \quad v_k = \sin\theta_k. \tag{63}$$

And

$$\cos 2\theta_k = \frac{\xi_k}{\sqrt{\xi_k^2 + \Delta^2}}, \quad \sin 2\theta_k = \frac{\Delta}{\sqrt{\xi_k^2 + \Delta^2}}. \tag{64}$$

The new Hamiltonian

$$H = \sum_{k,\sigma} E_k \gamma_{k\sigma}^\dagger \gamma_{k\sigma} + \overbrace{\frac{V\Delta^2}{U} + \sum_k (\xi_k - E_k)}^{\text{const.}}. \tag{65}$$

3.2.1 BCS mean-field theory

$$\begin{aligned}\Delta &= \frac{U}{2V} \sum_k \frac{\Delta}{\sqrt{\xi_k^2 + \Delta^2}} \left[1 - 2n^{(\text{F.D.})}(E_k)\right] \\ &= \frac{U}{2V} \sum_k \frac{\Delta}{\sqrt{\xi_k^2 + \Delta^2}} \tanh\left(\frac{E_k}{2T}\right).\end{aligned} \tag{66}$$

Here $E_k = \sqrt{\xi_k^2 + \Delta^2}$, and $\xi_k = (\varepsilon_k - \varepsilon_\text{F})$ varies from $-\varepsilon_\text{F}$ to ∞.

Sums transfroms according to

$$\frac{1}{V} \sum_k \to g(\varepsilon_\text{F}) \int_{-\hbar\omega_\text{D}}^{\hbar\omega_\text{D}} \mathrm{d}\xi, \tag{67}$$

where ω_D is the Debye frequency, and $\hbar\omega_\text{D} \ll \varepsilon_\text{F}$ (typ. values $\hbar\omega_\text{D} \sim 10^2\,\text{K}$, while $\varepsilon_\text{F} \sim 10^4\,\text{K}$). Only the states around $\varepsilon = \varepsilon_\text{F}$ ($\xi = 0$) affects superconductivity.

Some limit cases for the BCS eqn.

$T \to 0$:
Here, $\tanh\left(\frac{E_k}{2T}\right) \to 1$, and

$$\begin{aligned}1 =& \frac{U}{2} g(\varepsilon_\text{F}) \int_{-\hbar\omega_\text{D}}^{\hbar\omega_\text{D}} \mathrm{d}\xi \frac{1}{\sqrt{\xi^2 + \Delta^2}} \\ =& U g(\varepsilon_\text{F}) \ln\left(\frac{\hbar\omega_\text{D} + \sqrt{\hbar^2\omega_\text{D}{}^2 + \Delta^2}}{\Delta}\right).\end{aligned} \tag{68}$$

Since $\Delta \ll \hbar\omega_\text{D}$, then $1 \approx U g(\varepsilon_\text{F}) \ln\left(\frac{2\hbar\omega_\text{D}}{\Delta}\right)$ and

$$\Delta(T = 0) = 2\hbar\omega_\text{D} \exp\left[-\frac{1}{U g(\varepsilon_\text{F})}\right] \ll \hbar\omega_\text{D} \tag{69}$$

This is the maximum value of Δ.

$\Delta \to 0$
This will give us T_c.

$$1 = \frac{U}{2} g(\varepsilon_\text{F}) \int_{-\hbar\omega_\text{D}}^{\hbar\omega_\text{D}} \mathrm{d}\xi \frac{\tanh\left(\frac{\xi}{2T_\text{c}}\right)}{\xi}. \tag{70}$$

Numerical integration gives

$$T_\text{c} \approx 1.14\hbar\omega_\text{D} \exp\left[-\frac{1}{U g(\varepsilon_\text{F})}\right] \ll \hbar\omega_\text{D} \tag{71}$$

or

$$\frac{2\Delta(T=0)}{T_\text{c}} \approx 3.51 \tag{72}$$

this value has been confirmed through measurements.

Ground state of a superconductor (no quasi-particles)

$$\gamma_{k,\sigma} \left|\psi_0\right\rangle = 0 \tag{73}$$

$$\left|\psi_0\right\rangle = \prod_k \left[\cos(\theta_k) + \sin(\theta_k) c_{k,\uparrow}^\dagger c_{k,\downarrow}^\dagger\right] \left|0\right\rangle, \tag{74}$$

where $\left|0\right\rangle$ is the vacuum state (no real particles).

$$E = \left\langle\psi_0|H|\psi_0\right\rangle = -V g(\varepsilon_\text{F}) \left[(\hbar\omega_\text{D})^2 + \frac{\Delta^2}{2}\right] \tag{75}$$

or

$$\frac{E - E_\text{normal}}{V} = -\frac{1}{2} g(\varepsilon_\text{F}) \Delta^2. \tag{76}$$

This is the condensation energy.

In a superconductor the density of sates is

$$g_\text{sc}(E) = g(\varepsilon_\text{F}) \frac{E}{\sqrt{E^2 - \Delta^2}}. \tag{77}$$

It takes 2Δ to excite a quasi-paricle.

3.3 Ginsburg-Landau theory of supercond.

The Helmholtz free energy

$$\begin{aligned}F =& -T \ln(Z) \\ =& \sum_k (\xi_k - E_k) + \frac{V}{U}\Delta^2 - 2T \sum_k \ln\left(1 + \mathrm{e}^{-\frac{E_k}{T}}\right).\end{aligned} \tag{78}$$

This expanded in terms of Δ^2, near $T = T_\text{c}$, is

$$F \approx F_0 + a\,\Delta^2 + \frac{1}{2} b\,\Delta^4, \tag{79}$$

where

$$a = V g(\varepsilon_\text{F})\, t, \quad b = 0.107 \frac{V g(\varepsilon_\text{F})}{T_\text{c}{}^2} \tag{80}$$

and

$$t = \frac{T - T_\text{c}}{T_\text{c}}. \tag{81}$$

Minimizing F with resprect to Δ gives

$$\Delta = \begin{cases} \sqrt{-\frac{a}{b}} \approx 3.1\, T_\text{c} \sqrt{\frac{T_\text{c} - T}{T_\text{c}}}, & T < T_\text{c} \\ 0, & T \geq T_\text{c}. \end{cases} \tag{82}$$

The critical exponent in $\Delta \propto t^\beta$, is $\beta = 1/2$.

Non-uniform gap ($\varDelta$) The Hamiltonian

$$H = \int \mathrm{d}^3 r \sum_\sigma \psi_\sigma^\dagger(\boldsymbol{r}) \left(-\frac{\hbar^2}{2m} \nabla^2 \right) \psi_\sigma(\boldsymbol{r}) - U \int \mathrm{d}^3 r \; \psi_\uparrow^\dagger(\boldsymbol{r}) \psi_\downarrow^\dagger(\boldsymbol{r}) \psi_\downarrow(\boldsymbol{r}) \psi_\uparrow(\boldsymbol{r}) \tag{83}$$

is invariant under a transformation of the complex phase $\psi_\sigma(\boldsymbol{r}) \to \psi_\sigma(\boldsymbol{r}) \mathrm{e}^{\mathrm{i}\chi(\boldsymbol{r})}$, for some arbitrary phase function $\chi(\boldsymbol{r})$.

When applying a magnetic field

$$\boldsymbol{p} \to \boldsymbol{p} - \frac{q}{c} \boldsymbol{A} \tag{84}$$

and the Hamiltonian changes accordingly. The free energy will get a form

$$F = \int \mathrm{d}^3 r \left[a |\varDelta(\boldsymbol{r})|^2 + \frac{1}{2} b |\varDelta(\boldsymbol{r})|^4 + \kappa \left| \left(-\mathrm{i}\hbar \boldsymbol{\nabla} + \frac{2e}{c} \boldsymbol{A} \right) \varDelta(\boldsymbol{r}) \right|^2 \right] \tag{85}$$

Rescale

$$\phi := \frac{\sqrt{2m\kappa}}{\hbar} \varDelta(\boldsymbol{r}) \tag{86}$$

$$\frac{\hbar^2}{2m\kappa} a \to a, \quad \frac{\hbar^2}{2m\kappa} b \to b \tag{87}$$

which gives

$$F = \int \mathrm{d}^3 r \left[\frac{1}{2m} \left| \left(-\mathrm{i}\hbar \boldsymbol{\nabla} + \frac{2e}{c} \boldsymbol{A} \right) \phi(\boldsymbol{r}) \right|^2 + a |\phi(\boldsymbol{r})|^2 + b |\phi(\boldsymbol{r})|^4 + \frac{\boldsymbol{B}^2}{8\pi} \right] \tag{88}$$

We minimize F with respect to ϕ and ϕ^*, and we get the **first Ginsburg-Landau equation**

$$a\phi + b|\phi|^2 \phi + \frac{1}{2m} \left(-\mathrm{i}\hbar \boldsymbol{\nabla} + \frac{2e}{m} \boldsymbol{A} \right)^2 \phi = 0 \tag{89}$$

and the **second G-L eqn.**

$$\boldsymbol{j} = \frac{\mathrm{i}\hbar e}{m} (\phi^* \boldsymbol{\nabla} \phi - \phi \boldsymbol{\nabla} \phi^*) - \frac{4e^2}{mc} |\phi| \boldsymbol{A}. \tag{90}$$

Define $n_\mathrm{s} := |\phi|^2 = -a/b$ (in absense of EM-fields), which is almost the pair density. Then

$$\phi(\boldsymbol{r}) = \sqrt{n_\mathrm{s}(\boldsymbol{r})} \, \mathrm{e}^{\mathrm{i}\theta(\boldsymbol{r})} \tag{91}$$

for some phase $\theta(\boldsymbol{r})$. If n_s is uniform (but θ is not), then

$$\boldsymbol{j} = -\frac{2\hbar e n_\mathrm{s}}{m} \left(\boldsymbol{\nabla}\theta + \frac{2e}{\hbar c} \boldsymbol{A} \right). \tag{92}$$

As for superfluids, the vorticity

$$\oint \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{\nabla}\theta(\boldsymbol{r}) = \ell 2\pi \tag{93}$$

is quantized. But now $\boldsymbol{j}$ is electrical current, so the magnetic flux trough a hole in the superconductor

$$\varPhi = \int \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{B} = \oint \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{A} = \ell \, \frac{hc}{2e} =: \ell \Phi_0 \tag{94}$$

is also quantized in terms of $\Phi_0 = hc/(2e)$.

3.3.1 The Meissner effect

This effect forces magnetic fields out of a superconductor. Say a superconductor begins at $x = 0$ and $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ outside the super sonductor, then

$$\boldsymbol{B}(x) = \boldsymbol{B}(0) \mathrm{e}^{-x/\lambda}, \quad \text{for } x > 0, \tag{95}$$

where

$$\lambda = \sqrt{\frac{mc^2}{16\pi e^2 n_\mathrm{s}}} \tag{96}$$

is the penetration depth ($\lambda \sim 100\,\text{Å}$).

The Meissner effect is what separates a true super conductor from a "regular" conductor with arbitrarily low resistivity.

There is a critical (magnetic) field strength

$$H_\mathrm{c} = \sqrt{\frac{4\pi a^2}{b}}, \tag{97}$$

above which the super conductor brakes down.

3.3.2 Type I or II

With no magnetic field ($\boldsymbol{A} = 0$) the first G-L eqn. in 1 dim. reads

$$-\frac{\hbar^2}{2m} \frac{\mathrm{d}\phi}{\mathrm{d}x} + a\phi + b|\phi|^2 \phi = 0 \tag{98}$$

and has the solution

$$\phi(x) = \frac{1}{2} \sqrt{-\frac{a}{b}} \left[\tanh\left(\frac{x}{\sqrt{2}\xi} \right) + 1 \right] \tag{99}$$

where

$$\xi = \sqrt{-\frac{\hbar^2}{2ma}} \tag{100}$$

is the coherence length.

It turns out that the ratio

$$\kappa = \frac{\lambda}{\xi} = \frac{mc}{2\hbar e}\sqrt{\frac{b}{2\pi}} \tag{101}$$

affects the type of super conductor.

$$\begin{cases} \kappa < \frac{1}{\sqrt{2}} & \Longrightarrow \quad \text{Type I,} \\ \kappa > \frac{1}{\sqrt{2}} & \Longrightarrow \quad \text{Type II.} \end{cases} \tag{102}$$

The Meissner effect applies to type I. For type II, it turns out that it's possible to trap small vorticies, which contains a magnetic field, inside the superconductor.

There is a second critical field strength

$$H_{c_2} = -\frac{amc}{\hbar e} = \sqrt{2}\kappa H_c. \tag{103}$$

If this is larger than H_c, we can have vortecies

4 Landau theory phase transitions

Characterized by an order parameter which varies continously.

4.1 Ising model

A model for ferromagnetism. It consists of a lattice of "bits" which can have the value -1 or $+1$.

Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \, \sigma_j, \tag{104}$$

where $\sigma_n = \pm 1$, and the sum is over all nearest neighbours.

Define the magnitazation

$$M = \langle \sigma_i \rangle = \langle \sigma_j \rangle \,. \tag{105}$$

Mean-field approach

$$\sigma_i \sigma_j \approx M^2 + M\left(\sigma_i + \sigma_j\right) \tag{106}$$

$$H \approx -JzM \sum_i \sigma_i + \frac{JM^2Nz}{2}, \tag{107}$$

where z is the connectivity (number of nearest neighbours).

The free energy

$$\frac{\partial F}{\partial M} = JzN\left[M - \tanh\left(\frac{JzM}{T}\right)\right]. \tag{108}$$

Minimizing F means that

$$M = \tanh\left(\frac{JzM}{T}\right). \tag{109}$$

This has non-zero solutions only if $Jz/T < 1$. The critical temperature is

$$T_c = Jz. \tag{110}$$

Taylor expansion

$$f = \frac{F}{N} \approx \frac{T_c}{2}\left(1 - \frac{T_c}{T}\right)^2 + \frac{T_c}{12}M^4. \tag{111}$$

At $T \leq T_c$

$$M = \pm\sqrt{3\left(1 - \frac{T}{T_c}\right)} \propto t^{1/2}. \tag{112}$$

Critical exponent $\beta = 1/2$.

4.2 Heissenberg model

$$H = -J \sum_{\langle i,j \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j, \tag{113}$$

$$\boldsymbol{M} = \langle \boldsymbol{S}_i \rangle \tag{114}$$

$$F = a(T)\boldsymbol{M}\cdot\boldsymbol{M} + \frac{1}{2}b(\boldsymbol{M}\cdot\boldsymbol{M})^2 \tag{115}$$

Minimized ($T < T_c$) when

$$|\boldsymbol{M}| = M_0 = \sqrt{-\frac{a}{b}}. \tag{116}$$

This gives the (Landau) free energy if condensation

$$\Delta f = -\frac{a_1^2 t^2}{2b}, \tag{117}$$

where $a(T) = a_1 t$. And the entropy of condensation is

$$\Delta s = -\left(\frac{\partial f}{\partial T}\right)\Bigg|_{M=M_0} = \frac{a_1^2}{b{T_c}^2}(T - T_c). \tag{118}$$

And the heat capacity of condensation

$$\Delta c_V = T\frac{\partial S}{\partial T} = \frac{a_1 T}{b{T_c}^2} \tag{119}$$

4.2.1 External magnetic field

If we apply an external magnetic field, we get

$$f = a(T)\boldsymbol{M}\cdot\boldsymbol{M} + \frac{1}{2}b(\boldsymbol{M}\cdot\boldsymbol{M})^2 - \boldsymbol{B}\cdot\boldsymbol{M}. \tag{120}$$

The symmetry is now broken; use $M := \boldsymbol{M}\cdot\boldsymbol{B}/|\boldsymbol{B}|$. The derivative

$$\frac{\partial f}{\partial M} = 2aM + 2bM^3 - B = 0. \tag{121}$$

Differentiate w.r.t. B:

$$2a\frac{\partial M}{\partial B} + 6bM^2\frac{\partial M}{\partial B} = 1. \tag{122}$$

The sucepebility

$$\chi = \left.\frac{\partial M}{\partial B}\right|_{B=0} = \frac{1}{2a+6bM^2} \tag{123}$$

At $T > T_\mathrm{c}$, $M = \chi B$ for weak fields.

In absens of B

$$\chi = \frac{T_\mathrm{c}}{2a_1(T-T_\mathrm{c})} \times \begin{cases} 1, & T > T_\mathrm{c} \\ -\frac{1}{2}, & T < T_\mathrm{c}, \end{cases} \tag{124}$$

which is divergen at T_c from both directions. Critical exponent

$$\chi \propto |t|^\gamma, \quad \gamma = 1. \tag{125}$$

Scaling relations (mean-field)

$$M(t,B) = |t|^{1/2}\,\phi\left(\frac{B}{|t|^{1/2}}\right) \tag{126}$$

$$f = |t|^2\,F\left(\frac{B}{|t|^{3/2}}\right) \tag{127}$$

for some function F.

With mean-field, we only get these exponents.

4.2.2 Improvements to mean-field theory

We can generalize the scaling relations to

$$f = |t|^{2-\alpha}\,F\left(\frac{B}{|t|^{\Delta}}\right) \tag{128}$$

All other critical exponents can be derived from any two, e.g. α and Δ.

$$M = -\frac{\partial f}{\partial B} \propto |t|^{2-\alpha-\Delta}, \quad \beta = 2-\alpha-\Delta \tag{129}$$

$$\chi = \left.\frac{\partial M}{\partial B}\right|_{B=0} \propto |t|^{2-\alpha-2\Delta}, \quad \gamma = 2\Delta+\alpha-2 \tag{130}$$

Calculating the critical exponents We need to introduce some non-uniformity, $\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{r})$. The full free energy becomes

$$F[\boldsymbol{M}] = \int \mathrm{d}^3 r\Big[a(\boldsymbol{M}\cdot\boldsymbol{M}) + \frac{b}{2}(\boldsymbol{M}\cdot\boldsymbol{M})^2 + \frac{1}{2}\rho(\boldsymbol{\nabla}\boldsymbol{M}\cdot\boldsymbol{\nabla}\boldsymbol{M})\Big] \tag{131}$$

where

$$\boldsymbol{\nabla}\boldsymbol{M}\cdot\boldsymbol{\nabla}\boldsymbol{M} = \sum_{\alpha\in\{x,y,z\}} \boldsymbol{\nabla}M_\alpha\cdot\boldsymbol{\nabla}M_\alpha. \tag{132}$$

This last term penalizes non-uniformity.

Regard F as an effective Hamiltonian

$$\mathcal{Z} = \mathrm{tr}\Big(\mathrm{e}^{-H/T}\Big) \to \int \mathrm{D}\boldsymbol{M}\,\exp\Big(-\frac{F[\boldsymbol{M}]}{T}\Big). \tag{133}$$

This is a functional integral, over all functions $\boldsymbol{M}(\boldsymbol{r})$.

Switch to momentum space

$$\boldsymbol{M}(\boldsymbol{k}) = \boldsymbol{M}^*(-\boldsymbol{k}) \tag{134}$$

$$F[\boldsymbol{M}] = \sum_{\boldsymbol{k}}\left(a + \frac{\rho k^2}{2}\right)|\boldsymbol{M}(\boldsymbol{k})| + \frac{b}{2V}\sum_{\boldsymbol{k}_{1,2,3,4}} M_\alpha(\boldsymbol{k}_1)M_\alpha(\boldsymbol{k}_2)M_\beta(\boldsymbol{k}_3)M_\beta(\boldsymbol{k}_4) \tag{135}$$

Sum over both $\alpha,\beta\in\{x,y,z\}$.

The partition function

$$\mathcal{Z}\int\prod_{\boldsymbol{k},\alpha}\mathrm{d}M_\alpha(\boldsymbol{k})\exp\Big(-\frac{F[\boldsymbol{M}]}{T}\Big). \tag{136}$$

This is just a countably infinit-dimensional integral. Mean-filed theory is equivalent to evaluating this integral using the saddle point method (Laplace's method).

Correlation function

$$\begin{aligned} G(\boldsymbol{r}-\boldsymbol{r}') &= \Big\langle \boldsymbol{M}(\boldsymbol{r})\cdot\boldsymbol{M}(\boldsymbol{r}')\Big\rangle \\ &= \frac{1}{V}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}\Big\langle \boldsymbol{M}(\boldsymbol{k}_1)\cdot\boldsymbol{M}(\boldsymbol{k}_2)\Big\rangle \mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}'} \end{aligned} \tag{137}$$

$$\Big\langle \boldsymbol{M}(\boldsymbol{k}_1)\cdot\boldsymbol{M}(\boldsymbol{k}_2)\Big\rangle = \int \ldots = \Big\langle |\boldsymbol{M}(\boldsymbol{k})|^2\Big\rangle \tag{138}$$

For $T > T_\mathrm{c}$

$$\Big\langle |\boldsymbol{M}(\boldsymbol{k})|^2\Big\rangle = \frac{3T}{\rho}\frac{1}{k^2+\xi^{-2}} \tag{139}$$

where

$$\xi = \sqrt{\frac{\rho}{2a}} \propto |t|^{-\nu}, \quad \nu = 1/2 \tag{140}$$

is the correlation length. It diverges at $T \to T_{\mathrm{c}}$. Back to the real correlation function

$$G(\boldsymbol{r}-\boldsymbol{r}') = \frac{3T}{4\pi\rho} \frac{\exp\left(\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{\xi}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}, \quad T > T_{\mathrm{c}}. \tag{141}$$

For $T < T_{\mathrm{c}}$, we have some magetization

$$\langle \boldsymbol{M} \rangle = \boldsymbol{M}_0 \overset{\text{choose}}{=} M_z \hat{\boldsymbol{z}} \tag{142}$$

and

$$G(\boldsymbol{r}-\boldsymbol{r}') = \Big\langle \delta\boldsymbol{M}(\boldsymbol{r}) \cdot \delta\boldsymbol{M}(\boldsymbol{r}') \Big\rangle \tag{143}$$

is with respect to the deviations in $\boldsymbol{M}$ from $\boldsymbol{M}_0$.

$$G(\boldsymbol{r}-\boldsymbol{r}') = \frac{1}{V} \sum_{\boldsymbol{k}} \Big\langle |\delta M_z|^2 + |\delta \boldsymbol{M}_\perp|^2 \Big\rangle \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} \tag{144}$$

$$\begin{aligned} \Big\langle |\delta M_\perp|^2 \Big\rangle &= \frac{2T}{\rho} \frac{1}{k^2 + \xi_{\text{transverse}}^{-2}} \\ \Big\langle |\delta M_z|^2 \Big\rangle &= \frac{T}{\rho} \frac{1}{k^2 + \xi_{\text{along}}^{-2}} \end{aligned} \tag{145}$$

$$\xi_{\text{transverse}} = \infty, \quad \xi_{\text{along}} = \sqrt{-\frac{\rho}{4a}} \tag{146}$$

For $T < T_{\mathrm{c}}$.

A Special functions and integrals

A.1 Gamma function

$$\Gamma(\nu) = \int_0^\infty x^{\nu-1} \mathrm{e}^{-x} \, \mathrm{d}x \tag{147}$$

$$\Gamma(n) = (n-1)!, \quad n \in \mathbb{Z}^+ \tag{148}$$

x	$1/2$	$3/2$	$5/2$	$7/2$	$9/2$
$\Gamma(x)$	$\sqrt{\pi}$	$\sqrt{\pi}/2$	$3\sqrt{\pi}/4$	$15\sqrt{\pi}/8$	$105\sqrt{\pi}/16$

A.2 Zeta function

The Riemann zeta function

$$\zeta(\nu) = \sum_{n=1}^{\infty} \frac{1}{n^\nu} = \int_0^\infty \frac{x^{\nu-1}}{\mathrm{e}^x - 1} \, \mathrm{d}x \tag{149}$$

x	2	4	6
$\zeta(x)$	$\pi^2/6$	$\pi^4/90$	$\pi^6/945$
x	3/2	5/2	7/2
$\zeta(x)$	2.61238	1.34149	1.12673
x	3	5	7
$\zeta(x)$	1.20206	1.03693	1.00835

A.3 Some integrals

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}, \quad \mathrm{Re}[\alpha] \geq 0 \tag{150}$$

For large K:

$$\int_{-\infty}^{\infty} \mathrm{d}x \, f(x) \mathrm{e}^{-\phi(x)K} \approx f(c) \mathrm{e}^{-\phi(c)K} \sqrt{\frac{2\pi}{\phi''(c)}} \tag{151}$$

where c is a min. ponit of ϕ.

$$\int_0^\infty \mathrm{d}x \, \frac{x^3}{\mathrm{e}^x - 1} = \frac{\pi^4}{15} \tag{152}$$

B The harmonic oscillator

$$H = \sum_k \left[\frac{1}{2m} p_{-k} p_k + \frac{m\omega_k^2}{2} q_{-k} q_k \right]. \tag{153}$$

$$\begin{aligned} a_k^\dagger =& \frac{1}{\sqrt{2\hbar}} \left[\sqrt{m\omega_k} q_{-k} - \frac{\mathrm{i}}{\sqrt{m\omega_k}} p_{+k} \right] \\ a_k =& \frac{1}{\sqrt{2\hbar}} \left[\sqrt{m\omega_k} q_{+k} - \frac{\mathrm{i}}{\sqrt{m\omega_k}} p_{-k} \right] \end{aligned} \tag{154}$$

$$\left[a_k, a_{k'}^\dagger \right] = \delta_{k,k'} \tag{155}$$

$$\hat{n}_k = a_k^\dagger a_k \tag{156}$$

$$\hat{n}_k \left| n_k \right\rangle = n_k \left| n_k \right\rangle \tag{157}$$

$$H = \sum_k \hbar\omega_k \left(\hat{n}_k + \frac{1}{2} \right) \tag{158}$$

C General mean-field therory

We assume that two quantities, X and Y, varies very little from their mean values. We can write

$$\begin{aligned} X =& X - \langle X \rangle + \langle X \rangle \\ Y =& Y - \langle Y \rangle + \langle Y \rangle \end{aligned} \tag{159}$$

and

$$\begin{aligned} XY =& \Big[(X - \langle X \rangle) + \langle X \rangle \Big] \Big[(Y - \langle Y \rangle) + \langle Y \rangle \Big] \\ \approx& X \langle Y \rangle + \langle X \rangle Y - \langle X \rangle \langle Y \rangle . \end{aligned} \tag{160}$$

Here, we neglected the term $(X - \langle X \rangle)(Y - \langle Y \rangle) \approx 0$.