Statistical Physics – PHYS 704
Course summary

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1 Some Thermodynamic relations

- Energy: E, $\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N$. Min. in equilibrium when S and V are const.
- Helmholtz free energy: $F = E - TS$, $\mathrm{d}F = -S\mathrm{d}T - P\mathrm{d}V + \mu\mathrm{d}N$. Min. in equil. when T and V are const.
- Enthalph: $W = E + PV$, $\mathrm{d}W = T\mathrm{d}S + V\mathrm{d}P + \mu\mathrm{d}N$. Min. in equil. when S (adiabatic) and P are const.
- Gibbs free energy: $\Phi = E + PV - TS$, $\mathrm{d}\Phi = -S\mathrm{d}T + V\mathrm{d}P + \mu\mathrm{d}N$. Min. in equil. when T and P are const.
- Grand potential: $\Omega = -PV$, $\mathrm{d}\Omega = -S\mathrm{d}T + P\mathrm{d}V + N\mathrm{d}\mu$.

Derivative relations

$$\begin{aligned}
+\left(\frac{\partial T}{\partial V}\right)_S &= -\left(\frac{\partial P}{\partial S}\right)_T = +\frac{\partial^2 E}{\partial S\partial V}\\
+\left(\frac{\partial T}{\partial P}\right)_S &= +\left(\frac{\partial V}{\partial S}\right)_P = +\frac{\partial^2 W}{\partial S\partial P}\\
+\left(\frac{\partial S}{\partial V}\right)_T &= +\left(\frac{\partial P}{\partial T}\right)_V = -\frac{\partial^2 F}{\partial T\partial V}\\
-\left(\frac{\partial S}{\partial P}\right)_T &= +\left(\frac{\partial V}{\partial T}\right)_P = +\frac{\partial^2 \Phi}{\partial T\partial P}
\end{aligned} \tag{1}$$

Temperature

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N} \tag{2}$$

Heat-capacity

$$\begin{aligned}
C_V &= \left(\frac{\mathrm{d}E}{\mathrm{d}T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V\\
C_P &= \left(\frac{\mathrm{d}W}{\mathrm{d}T}\right)_P = \left(\frac{\partial E}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P
\end{aligned} \tag{3}$$

Compressibility

$$\kappa_X = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_X \tag{4}$$

1.1 Some def. in stat. mech.

Definition of entropy

$$S := -\sum_{n,(N)} \rho_{n,(N)} \ln\Big(\rho_{n,(N)}\Big), \tag{5}$$

where ρ is the density function or distribution fuction.

Canonical distribution (N constant)

$$\rho_n = \frac{1}{Z}\mathrm{e}^{-E_n/T}, \quad Z = \sum_n \mathrm{e}^{-E_n/T}. \tag{6}$$

$$F = -T\ln Z \tag{7}$$

Grand canonical distribution

$$\rho_{n,N} = \frac{1}{\mathcal{Z}}\mathrm{e}^{-(E_n-\mu N)/T}, \quad \mathcal{Z} = \sum_n \mathrm{e}^{-(E_n-\mu N)/T}. \tag{8}$$

$$\Omega = -T\ln\mathcal{Z} \tag{9}$$

2 Theory of ideal gases

Ideal gas means that there is no interaction between particles, $\varepsilon = \hbar^2k^2/(2m)$. In 3 dim.

$$\frac{1}{V}\sum_k \to \int\frac{\mathrm{d}^3k}{(2\pi)^3} = \int\mathrm{d}\varepsilon\, g(\varepsilon), \tag{10}$$

$$\frac{\mathrm{d}^3k}{(2\pi)^3} = \mathrm{d}\varepsilon\, g(\varepsilon), \quad g(\varepsilon) = \frac{m^{3/2}}{\sqrt{2}\pi\hbar^3}\sqrt{\varepsilon}. \tag{11}$$

See assignment 3 for other dim.

2.1 Ideal Fermi gases

$$n^{(\mathrm{F.D.})} = \frac{1}{\mathrm{e}^{(\varepsilon-\mu)/T} + 1} \tag{12}$$

$$-\Omega = PV = \frac{2}{3}E = \frac{V g_{\mathrm{s}} T}{\lambda^3} f_{5/2}(z) \tag{13}$$

$$N = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{T,V} = \frac{V g_{\mathrm{s}}}{\lambda^3} f_{3/2}(z) \tag{14}$$

$$\frac{PV}{NT} = \frac{f_{5/2}(z)}{f_{3/2}(z)} \tag{15}$$

Thermal wavelength $\lambda = h/\sqrt{2\pi m T} =: \Lambda/\sqrt{T}$, Λ is a constant.

Fermi functions

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \mathrm{d}x \frac{x^{\nu-1}}{z^{-1}\mathrm{e}^x + 1} \tag{16}$$

Fugacity $z = \mathrm{e}^{\mu/T}$.

$$z\frac{\partial f_\nu(z)}{\partial z} = \frac{\partial f_\nu(z)}{\partial(\ln z)} = f_{\nu-1}(z) \tag{17}$$

Fermi energy As $T \to 0$, the ccamical potential will go to

$$\mu(T \to 0) =: \varepsilon_{\mathrm{F}} = \frac{\hbar^2}{2m}\left(\frac{6}{g_{\mathrm{s}}}\pi^2 n\right)^{2/3}, \tag{18}$$

where g_{s} is the spin deganareacy, and $n = N/V$.

In a regular metal, $\varepsilon_{\mathrm{F}} \sim 10^4\,\mathrm{K}$. For $T \ll \varepsilon_{\mathrm{F}}$

$$n^{(\mathrm{F.D.})}(\varepsilon) \approx \begin{cases} 1, & \varepsilon < \varepsilon_{\mathrm{F}} \\ 0, & \varepsilon > \varepsilon_{\mathrm{F}} \end{cases} \tag{19}$$

and $\int_0^\infty \mathrm{d}\varepsilon\, n^{(\mathrm{F.D.})}(\varepsilon) \ldots \to \int_0^{\varepsilon_{\mathrm{F}}} \mathrm{d}\varepsilon \ldots$

The internal energy

$$E(T \ll \varepsilon_{\mathrm{F}}) = \frac{3}{5} N \varepsilon_{\mathrm{F}} \tag{20}$$

2.2 Ideal Bose gases

$$n^{(\mathrm{B.E.})} = \frac{1}{\mathrm{e}^{(\varepsilon-\mu)/T} - 1} \tag{21}$$

$$-\Omega = PV = \frac{2}{3}E = \frac{V g_{\mathrm{s}} T}{\lambda^3} g_{5/2}(z) \tag{22}$$

$$N_{\mathrm{e}} = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{T,V} = \frac{V g_{\mathrm{s}}}{\lambda^3} g_{3/2}(z) \tag{23}$$

$$\frac{PV}{NT} = \frac{g_{5/2}(z)}{g_{3/2}(z)} \tag{24}$$

Thermal wavelength $\lambda = h/\sqrt{2\pi m T} =: \Lambda/\sqrt{T}$, Λ is a constant.

Bose functions

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \mathrm{d}x \frac{x^{\nu-1}}{z^{-1}\mathrm{e}^x - 1} \tag{25}$$

For bosons $\mu \leq 0$, meaning that $z = \mathrm{e}^{\mu/T} \leq 1$.

$$z\frac{\partial g_\nu(z)}{\partial z} = \frac{\partial g_\nu(z)}{\partial(\ln z)} = g_{\nu-1}(z) \tag{26}$$

At $z = 1$, $g_\nu(z = 1) = \zeta(\nu)$.

Critical temperature Critical teperature for ideal Bose gas (3 dim.)

$$T_{\mathrm{c}} = \frac{2\pi\hbar^2}{m}\left(\frac{N}{V g_{\mathrm{s}}\, \zeta(3/2)}\right)^{3/2} \tag{27}$$

Number of condensed particles

$$N_0 = N\left[1 - \left(\frac{T}{T_{\mathrm{c}}}\right)^{3/2}\right], \quad T \leq T_{\mathrm{c}}. \tag{28}$$

$$\begin{aligned} \lambda &= \frac{h}{\sqrt{2\pi m T}}, & \lambda_{\mathrm{c}} &= \left[v\zeta(3/2)\right]^{1/3} \\ v &= \frac{1}{n} = \frac{V}{N}, & v_{\mathrm{c}} &= \frac{\lambda^3}{\zeta(3/2)} \end{aligned} \tag{29}$$

2.2.1 Photons

2.2.2 Phonons

3 Second quantaization

3.1 Superfluidity

3.2 BCS theory of superconductivity

The BCS (mean-field) equation:

$$
\begin{aligned}
\Delta &= \frac{U}{2V}\sum_k \frac{\Delta}{\sqrt{\xi_k^2+\Delta^2}}\Big[1-2n^{(\mathrm{F.D.})}(E_k)\Big] \\
&= \frac{U}{2V}\sum_k \frac{\Delta}{\sqrt{\xi_k^2+\Delta^2}}\tanh\!\left(\frac{E_k}{2T}\right).
\end{aligned}
\tag{30}
$$

Here $E_k = \sqrt{\xi_k^2+\Delta^2}$, and $\xi_k = (\varepsilon_k - \varepsilon_{\mathrm{F}})$ varies from $-\varepsilon_{\mathrm{F}}$ to ∞.

Sums transfroms according to

$$
\frac{1}{V}\sum_k \to g(\varepsilon_{\mathrm{F}}) \int\limits_{-\hbar\omega_{\mathrm{D}}}^{\hbar\omega_{\mathrm{D}}} \mathrm{d}\xi, \tag{31}
$$

where ω_{D} is the Debye frequency, and $\hbar\omega_{\mathrm{D}} \ll \varepsilon_{\mathrm{F}}$ (typ. values $\hbar\omega_{\mathrm{D}} \sim 10^2\,\mathrm{K}$, while $\varepsilon_{\mathrm{F}} \sim 10^4\,\mathrm{K}$). Only the states around $\varepsilon = \varepsilon_{\mathrm{F}}$ ($\xi = 0$) affects superconductivity.

3.3 Ginsburg-Landau theory of superconductivity

4 Second order phase transitions

4.1 Ising model

4.2 Landau theory of cont. phase transitions

A Special functions

A.1 Gamma function

$$\Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x} \, dx \tag{32}$$

$$\Gamma(n) = (n-1)!, \quad n \in \mathbb{Z}^+ \tag{33}$$

x	$1/2$	$3/2$	$5/2$	$7/2$	$9/2$
$\Gamma(x)$	$\sqrt{\pi}$	$\sqrt{\pi}/2$	$3\sqrt{\pi}/4$	$15\sqrt{\pi}/8$	$105\sqrt{\pi}/16$

A.2 Zeta function

The Riemann zeta function

$$\zeta(\nu) = \sum_{n=1}^{\infty} \frac{1}{n^\nu} = \int_0^\infty \frac{x^{\nu-1}}{e^x - 1} \, dx \tag{34}$$

x	2	4	6
$\zeta(x)$	$\pi^2/6$	$\pi^4/90$	$\pi^6/945$
x	$3/2$	$5/2$	$7/2$
$\zeta(x)$	2.61238	1.34149	1.12673
x	3	5	7
$\zeta(x)$	1.20206	1.03693	1.00835