

Statistical Physics – PHYS 704

Course summary

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1 Some Thermodynamic relations

- Energy: E , $dE = TdS - PdV + \mu dN$. Min. in equilibrium when S and V are const.
- Helmholtz free energy: $F = E - TS$, $dF = -SdT - PdV + \mu dN$. Min. in equil. when T and V are const.
- Enthalpy: $W = E + PV$, $dW = TdS + VdP + \mu dN$. Min. in equil. when S (adiabatic) and P are const.
- Gibbs free energy: $\Phi = E + PV - TS$, $d\Phi = -SdT + VdP + \mu dN$. Min. in equil. when T and P are const.
- Grand potential: $\Omega = -PV$, $d\Omega = -SdT + PdV + Nd\mu$.

Derivative relations

$$\begin{aligned}
 + \left(\frac{\partial T}{\partial V} \right)_S &= - \left(\frac{\partial P}{\partial S} \right)_T = + \frac{\partial^2 E}{\partial S \partial V} \\
 + \left(\frac{\partial T}{\partial P} \right)_S &= + \left(\frac{\partial V}{\partial S} \right)_P = + \frac{\partial^2 W}{\partial S \partial P} \\
 + \left(\frac{\partial S}{\partial V} \right)_T &= + \left(\frac{\partial P}{\partial T} \right)_V = - \frac{\partial^2 F}{\partial T \partial V} \\
 - \left(\frac{\partial S}{\partial P} \right)_T &= + \left(\frac{\partial V}{\partial T} \right)_P = + \frac{\partial^2 \Phi}{\partial T \partial P}
 \end{aligned} \tag{1}$$

Temperature

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_{V,N} \tag{2}$$

Heat-capacity

$$\begin{aligned}
 C_V &= \left(\frac{dE}{dT} \right)_V = T \left(\frac{\partial S}{\partial T} \right)_V = -T \left(\frac{\partial^2 F}{\partial T^2} \right)_V \\
 C_P &= \left(\frac{dW}{dT} \right)_P = \left(\frac{\partial E}{\partial T} \right)_P + P \left(\frac{\partial V}{\partial T} \right)_P
 \end{aligned} \tag{3}$$

Compressibility

$$\kappa_X = - \frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_X \tag{4}$$

1.1 Some def. in stat. mech.

Definition of entropy

$$S := - \sum_{n,(N)} \rho_{n,(N)} \ln(\rho_{n,(N)}), \quad (5)$$

where ρ is the density function or distribution fuction.

Canonical distribution (N constant)

$$\rho_n = \frac{1}{Z} e^{-E_n/T}, \quad Z = \sum_n e^{-E_n/T}. \quad (6)$$

$$F = -T \ln Z \quad (7)$$

Grand canonical distribution

$$\rho_{n,N} = \frac{1}{\mathcal{Z}} e^{-(E_n - \mu N)/T}, \quad \mathcal{Z} = \sum_n e^{-(E_n - \mu N)/T}. \quad (8)$$

$$\Omega = -T \ln \mathcal{Z} \quad (9)$$

2 Theory of ideal gases

Ideal gas means that there is no interaction between particles, $\varepsilon = \hbar^2 k^2 / (2m)$. In 3 dim.

$$\frac{1}{V} \sum_k \rightarrow \int \frac{d^3 k}{(2\pi)^3} = \int d\varepsilon g(\varepsilon), \quad (10)$$

$$\frac{d^3 k}{(2\pi)^3} = d\varepsilon g(\varepsilon), \quad g(\varepsilon) = \frac{m^{3/2}}{\sqrt{2\pi}\hbar^3} \sqrt{\varepsilon}. \quad (11)$$

See assignment 3 for other dim.

2.1 Ideal Fermi gases

$$n^{(F.D.)} = \frac{1}{e^{(\varepsilon - \mu)/T} + 1} \quad (12)$$

$$-\Omega = PV = \frac{2}{3} E = \frac{V g_s T}{\lambda^3} f_{5/2}(z) \quad (13)$$

$$N = - \left(\frac{\partial \Omega}{\partial \mu} \right)_{T,V} = \frac{V g_s}{\lambda^3} f_{3/2}(z) \quad (14)$$

$$\frac{PV}{NT} = \frac{f_{5/2}(z)}{f_{3/2}(z)} \quad (15)$$

Thermal wavelength $\lambda = h / \sqrt{2\pi m T} =: \Lambda / \sqrt{T}$, Λ is a constant.

Fermi functions

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty dx \frac{x^{\nu-1}}{z^{-1} e^x + 1} \quad (16)$$

Fugacity $z = e^{\mu/T}$.

$$z \frac{\partial f_\nu(z)}{\partial z} = \frac{\partial f_\nu(z)}{\partial (\ln z)} = f_{\nu-1}(z) \quad (17)$$

Fermi energy As $T \rightarrow 0$, the ccamical potential will go to

$$\mu(T \rightarrow 0) =: \varepsilon_F = \frac{\hbar^2}{2m} \left(\frac{6}{g_s} \pi^2 n \right)^{2/3}, \quad (18)$$

where g_s is the spin deganareacy, and $n = N/V$.

In a regular metal, $\varepsilon_F \sim 10^4$ K. For $T \ll \varepsilon_F$

$$n^{(F.D.)}(\varepsilon) \approx \begin{cases} 1, & \varepsilon < \varepsilon_F \\ 0, & \varepsilon > \varepsilon_F \end{cases} \quad (19)$$

$$\text{and } \int_0^\infty d\varepsilon n^{(F.D.)}(\varepsilon) \dots \rightarrow \int_0^{\varepsilon_F} d\varepsilon \dots$$

The internal energy

$$E(T \ll \varepsilon_F) = \frac{3}{5} N \varepsilon_F \quad (20)$$

2.2 Ideal Bose gases

$$n^{(B.E.)} = \frac{1}{e^{(\varepsilon - \mu)/T} - 1} \quad (21)$$

$$-\Omega = PV = \frac{2}{3} E = \frac{V g_s T}{\lambda^3} g_{5/2}(z) \quad (22)$$

$$N_e = - \left(\frac{\partial \Omega}{\partial \mu} \right)_{T,V} = \frac{V g_s}{\lambda^3} g_{3/2}(z) \quad (23)$$

$$\frac{PV}{NT} = \frac{g_{5/2}(z)}{g_{3/2}(z)} \quad (24)$$

Thermal wavelength $\lambda = h / \sqrt{2\pi m T} =: \Lambda / \sqrt{T}$, Λ is a constant.

Bose functions

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty dx \frac{x^{\nu-1}}{z^{-1} e^x - 1} \quad (25)$$

For bosons $\mu \leq 0$, meaning that $z = e^{\mu/T} \leq 1$.

$$z \frac{\partial g_\nu(z)}{\partial z} = \frac{\partial g_\nu(z)}{\partial (\ln z)} = g_{\nu-1}(z) \quad (26)$$

At $z = 1$, $g_\nu(z = 1) = \zeta(\nu)$.

Critical temperature Critical teperature for ideal Bose gas (3 dim.)

$$T_c = \frac{2\pi\hbar^2}{m} \left(\frac{N}{V g_s \zeta(3/2)} \right)^{3/2} \quad (27)$$

Number of condensed particles

$$N_0 = N \left[1 - \left(\frac{T}{T_c} \right)^{3/2} \right], \quad T \leq T_c. \quad (28)$$

$$\lambda = \frac{h}{\sqrt{2\pi m T}}, \quad \lambda_c = [v \zeta(3/2)]^{1/3} \quad (29)$$

$$v = \frac{1}{n} = \frac{V}{N}, \quad v_c = \frac{\lambda^3}{\zeta(3/2)}$$

2.2.1 Photons

Photons are bosons with $\mu = 0$.

$$n^{(\text{photons})} = \frac{1}{e^{\hbar\omega/T} - 1}. \quad (30)$$

Number of photons in the interval ω to $\omega + d\omega$:

$$dN_\omega = \frac{V}{\pi^2 c^3} \frac{\omega^2 d\omega}{e^{\hbar\omega/T} - 1}. \quad (31)$$

Energy in the same interval:

$$dE_\omega = \hbar\omega dN_\omega = V u_\omega d\omega, \quad (32)$$

where u_ω is the radiation energy density. Total radiation energy

$$E = \int_0^\infty dE_\omega = \frac{4\sigma}{c} V T^4, \quad (33)$$

where $\sigma = \pi^2/(60\hbar^3 c^2)$

2.2.2 Phonons

Chain of N particles connected with springs, mean distance a . Hamiltonian

$$H = \sum_{s=0}^{N-1} \left[\frac{p_s^2}{2m} + \frac{\kappa}{2} (q_{s+1} - q_s)^2 \right] \quad (34)$$

Also periodoc boundary conditions: $q_{s+N} = q_s$ and $p_{s+N} = p_s$.

Fourier transform (momentum space)

$$H = \sum_{k=0}^{N-1} \left[\frac{1}{2m} p_{-k} p_k + \frac{m\omega_k^2}{2} q_{-k} q_k \right]. \quad (35)$$

This is a Hamiltoninan for N harmonic oscillators:

$$H = \sum_{k=0}^{N-1} \hbar\omega_k \left(\hat{n}_k + \frac{1}{2} \right). \quad (36)$$

The frequency

$$\omega_k = \sqrt{\frac{2\kappa}{m} (1 - \cos(ka))} \quad (37)$$

$$\stackrel{ka \ll 1}{\approx} |ka| \sqrt{\frac{\kappa}{m}}.$$

Speed of sound

$$c_s = \sqrt{\frac{\kappa a^2}{m}}. \quad (38)$$

For $|ka| \ll 1$,

$$C_V = \frac{2\pi^2}{5\hbar^3 c_s^3} V T^3. \quad (39)$$

3 Second quantaization

Main idea: use a Bogoliubov transformation to transform the Hamiltonian to to a hamiltonian of an ideal gas of quasi-particles.

Harmonic oscillator interpretation in momentum space Occupation number representation:

$$|\psi\rangle = |n_{k_1}, n_{k_2}, \dots, n_{k_i}, \dots\rangle \quad (40)$$

$$\hat{n}_{k_i} |n_{k_1}, \dots, n_{k_i}, \dots\rangle = n_{k_i} |n_{k_1}, \dots, n_{k_i}, \dots\rangle \quad (41)$$

For bosons:

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'}, \quad (42)$$

whereas for fermions

$$\{c_k, c_{k'}^\dagger\} = \delta_{k,k'}. \quad (43)$$

The Hamiltonian for interacting bosons

$$H = \overbrace{\sum_k \varepsilon_k a_k^\dagger a_k}^{\text{non-interating}} + \frac{1}{2V} \sum_{k,k',q} U(q) a_{k+q}^\dagger a_{k'-q}^\dagger a_{k'} a_k \quad (44)$$

and for fermions

$$H = \overbrace{\sum_{k,\sigma} \varepsilon_k c_{k,\sigma}^\dagger c_{k,\sigma}}^{\text{non-interating}} + \frac{1}{2V} \sum_{k,k',q,\sigma,\sigma'} U(q) c_{k+q,\sigma}^\dagger c_{k'-q,\sigma}^\dagger c_{k',\sigma'} c_{k,\sigma}. \quad (45)$$

The interaction, $U(q)$, is given by

$$U(\mathbf{r} - \mathbf{r}') = \frac{1}{V} \sum_{\mathbf{q}} U(\mathbf{q}) e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')}, \quad (46)$$

or

$$U(\mathbf{q}) = \int d^3r U(\mathbf{r}) e^{-i\mathbf{q} \cdot \mathbf{r}}. \quad (47)$$

We can also talk about annihilation and creation operators in real space: $\psi^\dagger(\mathbf{r})$ creates a particle at position \mathbf{r} .

$$\int d^3r \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) = \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} = \hat{N} \quad (48)$$

These operators follow the same commutation relations as the momentum space equivalence.

3.1 Superfluidity in ^4He

At low enough temperatures most atoms will be in the ground state. Our goal is to bring the Hamiltonian to a form:

$$H = \sum_{\mathbf{k}} E_{\mathbf{k}} \xi_{\mathbf{k}}^\dagger \xi_{\mathbf{k}} + \text{const.} \quad (49)$$

Mean-field theory:

$$a_0 a_0^\dagger \approx a_0^\dagger a_0 = |a_0| = N_0 \quad (50)$$

meaning that

$$a_0 = \sqrt{N_0} e^{i\theta}. \quad (51)$$

Bogoliubov transformation

$$\begin{cases} \xi_{\mathbf{k}} = u_{\mathbf{k}} a_{\mathbf{k}} + v_{\mathbf{k}} a_{-\mathbf{k}}^\dagger \\ \xi_{\mathbf{k}}^\dagger = u_{\mathbf{k}} a_{\mathbf{k}}^\dagger + v_{\mathbf{k}} a_{-\mathbf{k}} \end{cases} \Leftrightarrow \begin{cases} a_{\mathbf{k}} = u_{\mathbf{k}} \xi_{\mathbf{k}} - v_{\mathbf{k}} \xi_{-\mathbf{k}}^\dagger \\ a_{\mathbf{k}}^\dagger = u_{\mathbf{k}} \xi_{\mathbf{k}}^\dagger - v_{\mathbf{k}} \xi_{-\mathbf{k}} \end{cases} \quad (52)$$

We still want the commutation relation $[\xi_{\mathbf{k}}, \xi_{\mathbf{k}}^\dagger] = 1$, which gives

$$u_{\mathbf{k}} = \cosh \theta_{\mathbf{k}}, \quad v_{\mathbf{k}} = \sinh \theta_{\mathbf{k}}. \quad (53)$$

And

$$\cosh 2\theta_{\mathbf{k}} = \frac{A_{\mathbf{k}}}{\sqrt{A_{\mathbf{k}}^2 - B_{\mathbf{k}}^2}}, \quad \sinh 2\theta_{\mathbf{k}} = \frac{B_{\mathbf{k}}}{\sqrt{A_{\mathbf{k}}^2 - B_{\mathbf{k}}^2}}, \quad (54)$$

where

$$A_{\mathbf{k}} = \varepsilon_{\mathbf{k}} + nU(\mathbf{k}), \quad B_{\mathbf{k}} = nU(\mathbf{k}). \quad (55)$$

Energy of the quasi-particles

$$\begin{aligned} E_{\mathbf{k}} &= A_{\mathbf{k}} \cosh 2\theta_{\mathbf{k}} - B_{\mathbf{k}} \sinh 2\theta_{\mathbf{k}} \\ &= \sqrt{A_{\mathbf{k}}^2 - B_{\mathbf{k}}^2} = \sqrt{\varepsilon_{\mathbf{k}} [\varepsilon_{\mathbf{k}} + 2nU(\mathbf{k})]} \end{aligned} \quad (56)$$

Long wavelength limit ($k \approx 0$)

$$E_{\mathbf{k}} \approx \hbar |k| \sqrt{\frac{nU(0)}{m}} \quad (57)$$

same form as phonons, and $c_s = \sqrt{nU(0)/m}$. The super fluid behaves more like a solid than a liquid.

As long as some disturbance is slower than c_s , there can not be any dissipation of energy into the super fluid.

Spontaneously broken symmetry We can choose θ in the mean-field theory description of a_0 freely. And the particles current will be

$$\mathbf{j} = \frac{\hbar n}{m} \nabla \theta \quad (58)$$

But if the geometry has holes, we must have a periodically varying θ , meaning that the vorticity

$$\oint d\mathbf{r} \cdot \nabla \theta(\mathbf{r}) = \ell 2\pi \quad (59)$$

is quantized.

3.2 BCS theory of superconductivity

Assume localized interaction $U(\mathbf{r} - \mathbf{r}') = (U/2)\delta(\mathbf{r} - \mathbf{r}')$, meaning that $U(q) \equiv U/2$, const. The Hamiltonian becomes

$$H = \sum_{\mathbf{k}, \sigma} \varepsilon_{\mathbf{k}} c_{\mathbf{k}, \sigma}^\dagger c_{\mathbf{k}, \sigma} + \frac{U}{V} \sum_{\mathbf{k}, \mathbf{k}', q} c_{\mathbf{k}+q, \uparrow}^\dagger c_{\mathbf{k}'-q, \downarrow}^\dagger c_{\mathbf{k}', \downarrow} c_{\mathbf{k}, \uparrow}. \quad (60)$$

Bogoliubov transformation Ideal gas of quasi-particles (fermions) with annihilation and creation operators $\gamma_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}^\dagger$:

$$\{\gamma_{\mathbf{k}, \sigma}, \gamma_{\mathbf{k}', \sigma}^\dagger\} = \delta_{\mathbf{k}, \mathbf{k}'}. \quad (61)$$

$$\begin{cases} \gamma_{\mathbf{k}, \uparrow}^\dagger = u_{\mathbf{k}} c_{\mathbf{k}, \uparrow}^\dagger - v_{\mathbf{k}} c_{-\mathbf{k}, \downarrow} \\ \gamma_{\mathbf{k}, \downarrow}^\dagger = u_{\mathbf{k}} c_{\mathbf{k}, \downarrow}^\dagger + v_{\mathbf{k}} c_{-\mathbf{k}, \uparrow} \end{cases} \Leftrightarrow \begin{cases} c_{\mathbf{k}, \uparrow}^\dagger = u_{\mathbf{k}} \gamma_{\mathbf{k}, \uparrow}^\dagger + v_{\mathbf{k}} \gamma_{-\mathbf{k}, \downarrow} \\ c_{\mathbf{k}, \downarrow}^\dagger = u_{\mathbf{k}} \gamma_{\mathbf{k}, \downarrow}^\dagger - v_{\mathbf{k}} \gamma_{-\mathbf{k}, \uparrow} \end{cases} \quad (62)$$

$$u_{\mathbf{k}} = \cos \theta_{\mathbf{k}}, \quad v_{\mathbf{k}} = \sin \theta_{\mathbf{k}}. \quad (63)$$

And

$$\cos 2\theta_k = \frac{\xi_k}{\sqrt{\xi_k^2 + \Delta^2}}, \quad \sin 2\theta_k = \frac{\Delta}{\sqrt{\xi_k^2 + \Delta^2}}. \quad (64)$$

The new Hamiltonian

$$H = \sum_{k,\sigma} E_k \gamma_{k\sigma}^\dagger \gamma_{k\sigma} + \overbrace{\frac{V\Delta^2}{U}}^{\text{const.}} + \sum_k (\xi_k - E_k). \quad (65)$$

3.2.1 BCS mean-field theory

$$\begin{aligned} \Delta &= \frac{U}{2V} \sum_k \frac{\Delta}{\sqrt{\xi_k^2 + \Delta^2}} \left[1 - 2n^{(\text{F.D.})}(E_k) \right] \\ &= \frac{U}{2V} \sum_k \frac{\Delta}{\sqrt{\xi_k^2 + \Delta^2}} \tanh\left(\frac{E_k}{2T}\right). \end{aligned} \quad (66)$$

Here $E_k = \sqrt{\xi_k^2 + \Delta^2}$, and $\xi_k = (\varepsilon_k - \varepsilon_F)$ varies from $-\varepsilon_F$ to ∞ .

Sums transforms according to

$$\frac{1}{V} \sum_k \rightarrow g(\varepsilon_F) \int_{-\hbar\omega_D}^{\hbar\omega_D} d\xi, \quad (67)$$

where ω_D is the Debye frequency, and $\hbar\omega_D \ll \varepsilon_F$ (typ. values $\hbar\omega_D \sim 10^2$ K, while $\varepsilon_F \sim 10^4$ K). Only the states around $\varepsilon = \varepsilon_F$ ($\xi = 0$) affects superconductivity.

Some limit cases for the BCS eqn.

$T \rightarrow 0$:

Here, $\tanh\left(\frac{E_k}{2T}\right) \rightarrow 1$, and

$$\begin{aligned} 1 &= \frac{U}{2} g(\varepsilon_F) \int_{-\hbar\omega_D}^{\hbar\omega_D} d\xi \frac{1}{\sqrt{\xi^2 + \Delta^2}} \\ &= U g(\varepsilon_F) \ln\left(\frac{\hbar\omega_D + \sqrt{\hbar^2\omega_D^2 + \Delta^2}}{\Delta}\right). \end{aligned}$$

Since $\Delta \ll \hbar\omega_D$, then $1 \approx U g(\varepsilon_F) \ln\left(\frac{2\hbar\omega_D}{\Delta}\right)$ and

$$\Delta(T=0) = 2\hbar\omega_D \exp\left[-\frac{1}{U g(\varepsilon_F)}\right] \ll \hbar\omega_D \quad (69)$$

This is the maximum value of Δ .

$\Delta \rightarrow 0$

This will give us T_c .

$$1 = \frac{U}{2} g(\varepsilon_F) \int_{-\hbar\omega_D}^{\hbar\omega_D} d\xi \frac{\tanh\left(\frac{\xi}{2T_c}\right)}{\xi}. \quad (70)$$

Numerical integration gives

$$T_c \approx 1.14 \hbar\omega_D \exp\left[-\frac{1}{U g(\varepsilon_F)}\right] \ll \hbar\omega_D \quad (71)$$

or

$$\frac{2\Delta(T=0)}{T_c} \approx 3.51 \quad (72)$$

this value has been confirmed through measurements.

Ground state of a superconductor (no quasi-particles)

$$\gamma_{k,\sigma} |\psi_0\rangle = 0 \quad (73)$$

$$|\psi_0\rangle = \prod_k \left[\cos(\theta_k) + \sin(\theta_k) c_{k,\uparrow}^\dagger c_{k,\downarrow}^\dagger \right] |0\rangle, \quad (74)$$

where $|0\rangle$ is the vacuum state (no real particles).

$$E = \langle \psi_0 | H | \psi_0 \rangle = -V g(\varepsilon_F) \left[(\hbar\omega_D)^2 + \frac{\Delta^2}{2} \right] \quad (75)$$

or

$$\frac{E - E_{\text{normal}}}{V} = -\frac{1}{2} g(\varepsilon_F) \Delta^2. \quad (76)$$

This is the condensation energy.

In a superconductor the density of states is

$$g_{\text{sc}}(E) = g(\varepsilon_F) \frac{E}{\sqrt{E^2 - \Delta^2}}. \quad (77)$$

It takes 2Δ to excite a quasi-paricle.

3.3 Ginsburg-Landau theory of supercond.

The Helmholtz free energy

$$\begin{aligned} F &= -T \ln(Z) \\ &= \sum_k (\xi_k - E_k) + \frac{V}{U} \Delta^2 - 2T \sum_k \ln\left(1 + e^{-\frac{E_k}{T}}\right). \end{aligned} \quad (78)$$

(68) This expanded in terms of Δ^2 , near $T = T_c$, is

$$F \approx F_0 + a \Delta^2 + \frac{1}{2} b \Delta^4, \quad (79)$$

where

$$a = V g(\varepsilon_F) t, \quad b = 0.107 \frac{V g(\varepsilon_F)}{T_c^2} \quad (80)$$

and

$$t = \frac{T - T_c}{T_c}. \quad (81)$$

Minimizing F with respect to Δ gives

$$\Delta = \begin{cases} \sqrt{-\frac{a}{b}} \approx 3.1 T_c \sqrt{\frac{T_c - T}{T_c}}, & T < T_c \\ 0, & T \geq T_c. \end{cases} \quad (82)$$

The critical exponent in $\Delta \propto t^\beta$, is $\beta = 1/2$.

Non-uniform gap (Δ) The Hamiltonian

$$H = \int d^3r \sum_{\sigma} \psi_{\sigma}^{\dagger}(\mathbf{r}) \left(-\frac{\hbar^2}{2m} \nabla^2 \right) \psi_{\sigma}(\mathbf{r}) - U \int d^3r \psi_{\uparrow}^{\dagger}(\mathbf{r}) \psi_{\downarrow}^{\dagger}(\mathbf{r}) \psi_{\downarrow}(\mathbf{r}) \psi_{\uparrow}(\mathbf{r}) \quad (83)$$

is invariant under a transformation of the complex phase $\psi_{\sigma}(\mathbf{r}) \rightarrow \psi_{\sigma}(\mathbf{r}) e^{i\chi(\mathbf{r})}$, for some arbitrary phase function $\chi(\mathbf{r})$.

When applying a magnetic field

$$\mathbf{p} \rightarrow \mathbf{p} - \frac{q}{c} \mathbf{A} \quad (84)$$

and the Hamiltonian changes accordingly. The free energy will get a form

$$F = \int d^3r \left[a |\Delta(\mathbf{r})|^2 + \frac{1}{2} b |\Delta(\mathbf{r})|^4 + \kappa \left| \left(-i\hbar \nabla + \frac{2e}{c} \mathbf{A} \right) \Delta(\mathbf{r}) \right|^2 \right] \quad (85)$$

Rescale

$$\phi := \frac{\sqrt{2m\kappa}}{\hbar} \Delta(\mathbf{r}) \quad (86)$$

$$\frac{\hbar^2}{2m\kappa} a \rightarrow a, \quad \frac{\hbar^2}{2m\kappa} b \rightarrow b \quad (87)$$

which gives

$$F = \int d^3r \left[\frac{1}{2m} \left| \left(-i\hbar \nabla + \frac{2e}{c} \mathbf{A} \right) \phi(\mathbf{r}) \right|^2 + a |\phi(\mathbf{r})|^2 + b |\phi(\mathbf{r})|^4 + \frac{B^2}{8\pi} \right] \quad (88)$$

We minimize F with respect to ϕ and ϕ^* , and we get the **first Ginsburg-Landau equation**

$$a\phi + b|\phi|^2\phi + \frac{1}{2m} \left(-i\hbar \nabla + \frac{2e}{m} \mathbf{A} \right)^2 \phi = 0 \quad (89)$$

and the **second G-L eqn.**

$$\mathbf{j} = \frac{i\hbar e}{m} (\phi^* \nabla \phi - \phi \nabla \phi^*) - \frac{4e^2}{mc} |\phi| \mathbf{A}. \quad (90)$$

Define $n_s := |\phi|^2 = -a/b$ (in absense of EM-fields), which is almost the pair density. Then

$$\phi(\mathbf{r}) = \sqrt{n_s(\mathbf{r})} e^{i\theta(\mathbf{r})} \quad (91)$$

for some phase $\theta(\mathbf{r})$. If n_s is uniform (but θ is not), then

$$\mathbf{j} = -\frac{2\hbar e n_s}{m} \left(\nabla \theta + \frac{2e}{\hbar c} \mathbf{A} \right). \quad (92)$$

As for superfluids, the vorticity

$$\oint d\mathbf{r} \cdot \nabla \theta(\mathbf{r}) = \ell 2\pi \quad (93)$$

is quantized. But now \mathbf{j} is electrical current, so the magnetic flux trough a hole in the superconductor

$$\Phi = \int d\mathbf{S} \cdot \mathbf{B} = \oint d\mathbf{r} \cdot \mathbf{A} = \ell \frac{\hbar c}{2e} =: \ell \Phi_0 \quad (94)$$

is also quantized in terms of $\Phi_0 = \hbar c / (2e)$.

3.3.1 The Meissner effect

This effect forces magnetic fields out of a superconductor. Say a superconductor begins at $x = 0$ and $\mathbf{B} = B\hat{\mathbf{z}}$ outside the super conductor, then

$$\mathbf{B}(x) = \mathbf{B}(0) e^{-x/\lambda}, \quad \text{for } x > 0, \quad (95)$$

where

$$\lambda = \sqrt{\frac{mc^2}{16\pi e^2 n_s}} \quad (96)$$

is the penetration depth ($\lambda \sim 100 \text{ \AA}$).

The Meissner effect is what separates a true superconductor from a “regular” conductor with arbitrarily low resistivity.

There is a critical (magnetic) field strength

$$H_c = \sqrt{\frac{4\pi a^2}{b}}, \quad (97)$$

above which the super conductor brakes down.

3.3.2 Type I or II

With no magnetic field ($\mathbf{A} = 0$) the first G-L eqn. in 1 dim. reads

$$-\frac{\hbar^2}{2m} \frac{d\phi}{dx} + a\phi + b|\phi|^2\phi = 0 \quad (98)$$

and has the solution

$$\phi(x) = \frac{1}{2} \sqrt{-\frac{a}{b}} \left[\tanh\left(\frac{x}{\sqrt{2}\xi}\right) + 1 \right] \quad (99)$$

where

$$\xi = \sqrt{-\frac{\hbar^2}{2ma}} \quad (100)$$

is the coherence length.

It turns out that the ratio

$$\kappa = \frac{\lambda}{\xi} = \frac{mc}{2\hbar e} \sqrt{\frac{b}{2\pi}} \quad (101)$$

affects the type of super conductor.

$$\begin{cases} \kappa < \frac{1}{\sqrt{2}} & \implies \text{Type I,} \\ \kappa > \frac{1}{\sqrt{2}} & \implies \text{Type II.} \end{cases} \quad (102)$$

The Meissner effect applies to type I. For type II, it turns out that it's possible to trap small vortices, which contains a magnetic field, inside the superconductor.

There is a second critical field strength

$$H_{c2} = -\frac{amc}{\hbar e} = \sqrt{2}\kappa H_c. \quad (103)$$

If this is larger than H_c , we can have vortices

4 Landau theory phase transitions

Characterized by an order parameter which varies continuously.

4.1 Ising model

A model for ferromagnetism. It consists of a lattice of "bits" which can have the value -1 or $+1$.

Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j, \quad (104)$$

where $\sigma_n = \pm 1$, and the sum is over all nearest neighbours.

Define the magnetization

$$M = \langle \sigma_i \rangle = \langle \sigma_j \rangle. \quad (105)$$

Mean-field approach

$$\sigma_i \sigma_j \approx M^2 + M(\sigma_i + \sigma_j) \quad (106)$$

$$H \approx -JzM \sum_i \sigma_i + \frac{JM^2 Nz}{2}, \quad (107)$$

where z is the connectivity (number of nearest neighbours).

The free energy

$$\frac{\partial F}{\partial M} = JzN \left[M - \tanh\left(\frac{JzM}{T}\right) \right]. \quad (108)$$

Minimizing F means that

$$M = \tanh\left(\frac{JzM}{T}\right). \quad (109)$$

This has non-zero solutions only if $Jz/T < 1$. The critical temperature is

$$T_c = Jz. \quad (110)$$

Taylor expansion

$$f = \frac{F}{N} \approx \frac{T_c}{2} \left(1 - \frac{T_c}{T}\right)^2 + \frac{T_c}{12} M^4. \quad (111)$$

At $T \leq T_c$

$$M = \pm \sqrt{3\left(1 - \frac{T}{T_c}\right)} \propto t^{1/2}. \quad (112)$$

Critical exponent $\beta = 1/2$.

4.2 Heisenberg model

$$H = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j, \quad (113)$$

$$\mathbf{M} = \langle \mathbf{S}_i \rangle \quad (114)$$

$$F = a(T) \mathbf{M} \cdot \mathbf{M} + \frac{1}{2} b (\mathbf{M} \cdot \mathbf{M})^2 \quad (115)$$

Minimized ($T < T_c$) when

$$|\mathbf{M}| = M_0 = \sqrt{-\frac{a}{b}}. \quad (116)$$

This gives the (Landau) free energy if condensation

$$\Delta f = -\frac{a_1^2 t^2}{2b}, \quad (117)$$

where $a(T) = a_1 t$. And the entropy of condensation is

$$\Delta s = -\left(\frac{\partial f}{\partial T}\right)\bigg|_{M=M_0} = \frac{a_1^2}{bT_c^2} (T - T_c). \quad (118)$$

And the heat capacity of condensation

$$\Delta c_V = T \frac{\partial S}{\partial T} = \frac{a_1 T}{bT_c^2} \quad (119)$$

4.2.1 External magnetic field

If we apply an external magnetic field, we get

$$f = a(T)\mathbf{M} \cdot \mathbf{M} + \frac{1}{2}b(\mathbf{M} \cdot \mathbf{M})^2 - \mathbf{B} \cdot \mathbf{M}. \quad (120)$$

The symmetry is now broken; use $M := \mathbf{M} \cdot \mathbf{B}/|\mathbf{B}|$.
The derivative

$$\frac{\partial f}{\partial M} = 2aM + 2bM^3 - B = 0. \quad (121)$$

Differentiate w.r.t. B :

$$2a\frac{\partial M}{\partial B} + 6bM^2\frac{\partial M}{\partial B} = 1. \quad (122)$$

The susceptibility

$$\chi = \left. \frac{\partial M}{\partial B} \right|_{B=0} = \frac{1}{2a + 6bM^2} \quad (123)$$

At $T > T_c$, $M = \chi B$ for weak fields.

In absence of B

$$\chi = \frac{T_c}{2a_1(T - T_c)} \times \begin{cases} 1, & T > T_c \\ -\frac{1}{2}, & T < T_c, \end{cases} \quad (124)$$

which is divergent at T_c from both directions.

4.2.2 Improvements to mean-field theory

A Special functions

A.1 Gamma function

$$\Gamma(\nu) = \int_0^{\infty} x^{\nu-1} e^{-x} dx \quad (125)$$

$$\Gamma(n) = (n-1)!, \quad n \in \mathbb{Z}^+ \quad (126) \quad \text{and}$$

x	$1/2$	$3/2$	$5/2$	$7/2$	$9/2$
$\Gamma(x)$	$\sqrt{\pi}$	$\sqrt{\pi}/2$	$3\sqrt{\pi}/4$	$15\sqrt{\pi}/8$	$105\sqrt{\pi}/16$

A.2 Zeta function

The Riemann zeta function

$$\zeta(\nu) = \sum_{n=1}^{\infty} \frac{1}{n^{\nu}} = \int_0^{\infty} \frac{x^{\nu-1}}{e^x - 1} dx \quad (127)$$

x	2	4	6
$\zeta(x)$	$\pi^2/6$	$\pi^4/90$	$\pi^6/945$
x	3/2	5/2	7/2
$\zeta(x)$	2.61238	1.34149	1.12673
x	3	5	7
$\zeta(x)$	1.20206	1.03693	1.00835

B The harmonic oscillator

$$H = \sum_k \left[\frac{1}{2m} p_{-k} p_k + \frac{m\omega_k^2}{2} q_{-k} q_k \right]. \quad (128)$$

$$\begin{aligned} a_k^\dagger &= \frac{1}{\sqrt{2\hbar}} \left[\sqrt{m\omega_k} q_{-k} - \frac{i}{\sqrt{m\omega_k}} p_{+k} \right] \\ a_k &= \frac{1}{\sqrt{2\hbar}} \left[\sqrt{m\omega_k} q_{+k} - \frac{i}{\sqrt{m\omega_k}} p_{-k} \right] \end{aligned} \quad (129)$$

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'} \quad (130)$$

$$\hat{n}_k = a_k^\dagger a_k \quad (131)$$

$$\hat{n}_k |n_k\rangle = n_k |n_k\rangle \quad (132)$$

$$H = \sum_k \hbar\omega_k \left(\hat{n}_k + \frac{1}{2} \right) \quad (133)$$

C General mean-field theory

We assume that two quantities, X and Y , varies very little from their mean values. We can write

$$\begin{aligned} X &= X - \langle X \rangle + \langle X \rangle \\ Y &= Y - \langle Y \rangle + \langle Y \rangle \end{aligned} \quad (134)$$

$$\begin{aligned} XY &= [(X - \langle X \rangle) + \langle X \rangle] [(Y - \langle Y \rangle) + \langle Y \rangle] \\ &\approx X \langle Y \rangle + \langle X \rangle Y - \langle X \rangle \langle Y \rangle. \end{aligned} \quad (135)$$

Here, we neglected the term $(X - \langle X \rangle)(Y - \langle Y \rangle) \approx 0$.