Fall 2016

AMATH 732: Assignment 4.

Due Monday, November 7

1. Consider the IVP

$$y'' + (1 - \epsilon x)y = 0; \qquad y(0) = 1, \qquad y'(0) = 0,$$

where $0 < \epsilon \ll 1$, $0 \le x < \infty$. Find an approximate solution by RPT (i.e., as in first attempt for nonlinear pendulum) and discuss it. Is the solution uniformly ordered?

2. Use the method of multiple scales to find a perturbation solution of the underdamped linear damped oscillator

$$\begin{aligned} \ddot{x} + 2\epsilon\dot{x} + x &= 0, \\ x(0) &= 1, \\ \dot{x}(0) &= 0. \end{aligned} \tag{1}$$

where $0 < \epsilon \ll 1$.

(a) Introduce two long time scales $\tau_1 = \epsilon t$ and $\tau_2 = \epsilon^2 t$ and set $x(t) = f(t, \tau_1, \tau_2; \epsilon)$. What are the equations and initial conditions for f?

(b) Expand f via $f = f_0(t, \tau_1, \tau_2) + \epsilon f_1(t, \tau_1, \tau_2) + \epsilon^2 f_2(t, \tau_1, \tau_2) + \cdots$. Show that the leading-order problem gives

$$f_0 = A_0(\tau_1, \tau_2)\cos(t) + B_0(\tau_1, \tau_2)\sin(t). \tag{2}$$

What initial conditions must A_0 and B_0 satisfy?

(c) The solvability condition in the $O(\epsilon)$ problem (i.e., elimination of terms that lead to a disordered solution) gives

$$\begin{aligned} A_0 &= \tilde{A}_0(\tau_2)e^{-\tau_1}, \\ B_0 &= \tilde{B}_0(\tau_2)e^{-\tau_1}. \end{aligned} \tag{3}$$

What initial conditions do $\tilde{A}_0$ and $\tilde{B}_0$ satisfy? Show that f_1 has the form

$$f_1 = A_1(\tau_1, \tau_2)\cos(t) + B_1(\tau_1, \tau_2)\sin(t). \tag{4}$$

What initial conditions are satisfied by A_1 and B_1?

(d) Use solvability conditions at the next order to determine $\tilde{A}_0$ and $\tilde{B}_0$. What is your complete $O(1)$ solution? Compare with the exact solution?

3. Use Pritulo's technique to find a solution to $O(\epsilon)$ of the following IVP for the free Duffing oscillator:

$$x'' + x = \epsilon x^3; \qquad x(0) = 1, \qquad x'(0) = 0.$$

4. Consider the IVP

$$y'' + y = \epsilon y^2; \qquad y(0) = \alpha, \qquad y'(0) = 0. \tag{5}$$

(a) Find an RPT to $O(\epsilon^2)$. Is the solution uniformly ordered on $0 \le x < \infty$?

(b) Use the Poincaré-Linstedt method to find a solution that is uniformly ordered to $O(\epsilon^2)$.

(c) Use the method of Pritulo to find a solution uniformly ordered to $O(\epsilon^2)$. Compare your solution with the previous one.

5. Use Pritulo's technique to find the solution to the nonlinear-pendulum problem

$$\theta'' + \sin\theta = 0; \qquad \theta(0) = a; \qquad \theta'(0) = 0, \tag{6}$$

to $O(a^4)$.

6. Consider the following one-dimensional linear dispersive wave equation (linearized KdV):

$$\eta_t + c_0\eta_x + \beta\eta_{xxx} = 0. \tag{7}$$

Consider the propagation of a localized wave packet initially of the form

$$\eta(x, 0) = \eta_o(x) = A(\epsilon x)\cos(kx), \tag{8}$$

where $\epsilon \ll 1$ and $k = O(1)$. The wave packet has the form of fast oscillations modulated by a slowly varying wave envelope. Use the method of multiple scales (in both space and time) to look for an approximate solution of the wave equation of the form

$$\eta(x, t) = A(\chi, \tau)\cos(kx - \omega t), \tag{9}$$

where $\chi = \epsilon x$ and $\tau = \epsilon t$ are slow space and time scales. Show that to $O(\epsilon)$ the fast oscillations move with phase speed $c = \omega/k = c_0 - \beta k^2$ (i.e., at leading order the frequency and wavenumber are related by $\omega = \omega(k) = c_0 k - \beta k^3$) while the wave envelope moves with the group velocity $\omega_k = c - 3\beta k^2$. The wave energy moves with the envelope since there are only waves in regions where $A \neq 0$. The wave envelope moves with a different speed than wave crests do. This is a general property of dispersive waves. For this reason the group velocity is usually much more important than the phase speed: it tells you where the waves go.