Homework 2
Due Wednesday, October 26

1. (Problem 8.4 of PB)

   (a) Show that the isothermal compressibility κ_T and the adiabatic compressibility κ_S of an ideal Fermi gas are given by

   $$\kappa_T = \frac{1}{nT}\frac{f_{1/2}(z)}{f_{3/2}(z)}, \quad \kappa_S = \frac{3}{5nT}\frac{f_{3/2}(z)}{f_{5/2}(z)},$$

   where $n = N/V$ is the particle density, $z = e^{\mu/T}$ is the fugacity, and

   $$f_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}}{z^{-1}e^x + 1}dx.$$

   Show that at low temperatures

   $$\kappa_T \approx \frac{3}{2n\epsilon_F}\left[1 - \frac{\pi^2}{12}\left(\frac{T}{\epsilon_F}\right)^2\right], \quad \kappa_S \approx \frac{3}{2n\epsilon_F}\left[1 - \frac{5\pi^2}{12}\left(\frac{T}{\epsilon_F}\right)^2\right].$$

   (b) Prove a general thermodynamic relation

   $$C_P - C_V = T\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_P = TV\kappa_T\left(\frac{\partial P}{\partial T}\right)_V^2.$$

   and show that

   $$\frac{C_P - C_V}{C_V} = \frac{4C_V}{9N}\frac{f_{1/2}(z)}{f_{3/2}(z)} \approx \frac{\pi^2}{3}\left(\frac{T}{\epsilon_F}\right)^2.$$

2. (Problem 8.9 of PB) Show that the chemical potential of a Fermi gas at low temperatures is given by

   $$\mu \approx \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{T}{\epsilon_F}\right)^2 - \frac{\pi^4}{80}\left(\frac{T}{\epsilon_F}\right)^4 + \dots\right],$$

   and the internal energy per particle is

   $$\frac{E}{N} \approx \frac{3\epsilon_F}{5}\left[1 + \frac{5\pi^2}{12}\left(\frac{T}{\epsilon_F}\right)^2 - \frac{\pi^4}{16}\left(\frac{T}{\epsilon_F}\right)^4 + \dots\right].$$

   Hence determine the T^3 correction to the T-linear result for the specific heat of the electron gas.

3. (Problem 7.8 of PB) The velocity of sound in a fluid is given by

$$v_S = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S},$$

where ρ is the mass density of the fluid. Show that for an ideal Bose gas

$$v_S^2 = \frac{5T}{3m}\frac{g_{5/2}(z)}{g_{3/2}(z)} = \frac{5}{9}\langle u^2 \rangle,$$

where $\langle u^2 \rangle$ is the mean square velocity of the particles in the gas and

$$g_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}}{z^{-1}e^x - 1}dx.$$

4. Consider an ideal gas of spinless bosons with an energy spectrum $\epsilon_{\mathbf{k}} = \hbar^2\mathbf{k}^2/2m$, contained in a box of volume $V = L^d$ in d spatial dimensions.

   (a) Calculate the grand potential Ω and the density $n = N/V$ at a chemical potential μ. Express your answers in terms of d and the function $g_\nu(z)$.

   (b) Calculate the ratio PV/E.

   (c) Find the critical temperature T_c for the onset of Bose condensation and discuss its dependence on d.

   (d) Calculate the specific heat $C_V(T)$ for $T < T_c$.

   (e) Sketch the specific heat as a function of T at all temperatures.

   (f) Find the ratio $C_V^{max}/C_V(T \to \infty)$ of the maximum specific heat to its high-temperature limit and evaluate it in $d = 3$.