Fall 2016

AMATH 732: Problem Set 5.

1. Consider the following boundary layer problem.

$$\epsilon y'' + (1+x^2)y' - x^3 y = 0, \qquad y(0) = y(1) = 1, \qquad \text{on } 0 \le x \le 1. \tag{1}$$

   (a) Find the outer solution $y_{out}(x)$ and show that $x = 1$ is in the outer region while $x = 0$ is not. Find the region in which the outer solution is valid by comparing the neglected term with one of the retained terms.

   (b) Find the inner solution $y_{in}(x)$. Where is it valid?

   (c) Match the two solutions. Find a *uniform* approximation y_{unif} using the expression

$$y_{unif} = y_{out}(x) + y_{in}(x) - y_{match}(x). \tag{2}$$

   What must $y_{match}(x)$ be. Explain why this gives an approximation that is valid on the whole interval $[0,1]$.

2. Consider the problem

$$(x + \epsilon y)y' - \frac{1}{2}y = 1 + x^2, \qquad y(1) = 1 \tag{3}$$

   for $x \in [0,1]$.

   (a) Find the RPT solution to $O(\epsilon)$ and discuss its uniformity.

   (b) Use renormalization to render the RPT solution uniformly valid to $O(\epsilon)$.

   (c) Use Lighthill's method to find a solution to $O(\epsilon)$.

3. A thin rod of length L and bending stiffness b lies along the x-axis between $x = 0$ and $x = L$. It is clamped at each end and subjected to a compressive force of magnitude F. Let θ be the angle the rod between the rod and the x-axis. When the rod is stationary (not varying in time) its shape is determined by the nondimensional boundary value problem

$$\begin{aligned} \frac{d^2\theta}{ds^2} + \alpha^2 \sin\theta &= 0, \\ \theta(0) &= 0, \\ \theta(1) &= 0, \end{aligned} \tag{4}$$

   where $s = x/L$ and $\alpha^2 = FL^2/b$. α^2 is a bifurcation parameter.

   (a) Approximate solutions are easily obtained in a certain limit. What are the solutions when $\alpha^2 = n^2\pi^2$? In what limit are they valid?

   (b) Suppose α is slightly larger than the critical value π, say $\alpha^2 = \pi^2 + \epsilon^2$, where $\epsilon \ll 1$. Seek a solution for small θ by first scaling θ via $\theta = \epsilon\phi$. Expand ϕ via

$$\phi \sim \phi_0 + \epsilon^2\phi_1 + O(\epsilon^4), \tag{5}$$

and find ϕ_0. The solution involves an undetermined constant A. Next write down the problem for ϕ_1. It should involve a resonant forcing term. Note that we are not interested in solutions for large θ since θ is restricted to lie between 0 and 1 so this is not necessarily a problem. Show, however, that it is by multiplying the differential equation for ϕ_1 by ϕ_0 and integrating from 0 to 1. You should obtain a solvability condition which determines the value of A. What is its value?

(c) Consider the problem again, assuming α is slightly less than the critical value π. Show that the only solution is the zero solution.

(d) Let $M_\theta(\alpha)$ be the amplitude of the solution $\theta(s;\alpha)$, i.e., $M_\theta(\alpha) = \max\big\{|\theta|\big\}$ on $0 \le s \le 1$. Show that for α close to π

$$M_\theta(\alpha) = \begin{cases} 0, & \text{if } \alpha < \pi; \\ \pm\frac{2\sqrt{2}}{\pi}\sqrt{\alpha^2-\pi^2}, & \text{if } \alpha > \pi. \end{cases} \tag{6}$$

This is an example of what is called a pitch-fork bifurcation.

4. **Cable laying problem**

This problem concerns the laying of an undersea cable from a ship. The angle θ between the cable and the horizontal satisfies

$$\begin{aligned} \epsilon\frac{d^2\theta}{ds^2} - F^*\sin\theta + (F_0+s)\cos\theta = 0,& \\ \theta(0) = 0,& \\ \frac{\theta}{ds}(0) = 0& \end{aligned} \tag{7}$$

where s is distance along the cable measured from the point of contact with the sea floor. The boundary conditions are applied at the point where the cable contacts the sea flow. F_0 is an unknown force (the vertical force applied to the end of the cable where it touches the sea floor) and F^* is a known $O(1)$ constant. ϵ is a small dimensionless positive parameter.

(a) What is the outer solution? Can it satisfy the boundary conditions?

(b) Introduce an inner variable via $s = \delta(\epsilon)\xi$. You will also have to scale θ and F_0. Set $\theta = \mu(\epsilon)\phi$ and $F_0 = \alpha(\epsilon)f_0$. $\delta(\epsilon)$, $\mu(\epsilon)$, and $\alpha(\epsilon)$ all go to zero as $\epsilon \to 0$. You can assume that the dominant balance in the inner layer includes the first two terms which determines $\delta(\epsilon)$. What is the approximate equation in the inner layer? Apply the boundary conditions and match to the outer solution. In so doing determine $\mu(\epsilon)$ and $\alpha(\epsilon)$ and f_0.