Homework 5
Due Friday, December 2

1. Consider a system with the following form of the Landau free energy:

$$f(t, m) = -hm + tm^2 + sm^4 + um^6,$$

where u is a fixed positive constant, while s can be either positive or negative (it was strictly positive in the theory of second order phase transitions we discussed in class). m is the magnetization and h is the external magnetic field. First, set $h = 0$ and minimize f with respect to m. Show the following:

(a) For $t > 0$ and $s > -(3ut)^{1/2}$, $m_0 = 0$ is the only real solution.

(b) For $t > 0$ and $-(4ut)^{1/2} < s \leq -(3ut)^{1/2}$, $m_0 = 0$ or $m_0 = \pm m_1$, where $m_1^2 = (\sqrt{s^2 - 3ut} - s)/3u$. However, the minimum of f at $m_0 = 0$ is lower than the two minima at $m_0 = \pm m_1$ and so the ultimate equilibrium value of the magnetization is $m_0 = 0$.

(c) For $t > 0$ and $s = -(4ut)^{1/2}$, $m_0 = 0$ or $m_0 = \pm(t/u)^{1/4}$. Now the minimum f at $m_0 = 0$ is of the same height as the ones at $m_0 = \pm(t/u)^{1/4}$, so a nonzero magnetization is as likely to occur as the zero one.

(d) For $t > 0$ and $s < -(4ut)^{1/2}$, the minima of the free energy are $m_0 = \pm m_1$. This means that a *first order transition* occurs along the line $s = -(4ut)^{1/2}$, as the magnetization jumps between $m_0 = 0$ and $m_0 = \pm m_1$. This demonstrates that Landau theory can be used to describe first order transitions as well as continuous (second order) transitions.

(e) For $t = 0$ and $s < 0$, $m_0 = \pm(2|s|/3u)^{1/2}$.

(f) For $t < 0$, $m_0 = \pm m_1$ for all s. As $t \to 0$, $m_1 \to 0$ if $s > 0$.

(g) For $t = 0$ and $s > 0$, $m_0 = 0$ is the only solution. Combining this results with the previous one, we see that the line $t = 0$ with s positive is the line of second order phase transitions. The lines of second and first order transitions meet at the point $(t = 0, s = 0)$, which is called the *tricritical point*.