Statistical Physics – PHYS 704
Course summary

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1 Some Thermodynamic relations

- Energy: E, $\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N$. Min. in equilibrium when S and V are const.
- Helmholtz free energy: $F = E - TS$, $\mathrm{d}F = -S\mathrm{d}T - P\mathrm{d}V + \mu\mathrm{d}N$. Min. in equil. when T and V are const.
- Enthalph: $W = E + PV$, $\mathrm{d}W = T\mathrm{d}S + V\mathrm{d}P + \mu\mathrm{d}N$. Min. in equil. when S (adiabatic) and P are const.
- Gibbs free energy: $\Phi = E + PV - TS$, $\mathrm{d}\Phi = -S\mathrm{d}T + V\mathrm{d}P + \mu\mathrm{d}N$. Min. in equil. when T and P are const.
- Grand potential: $\Omega = -PV$, $\mathrm{d}\Omega = -S\mathrm{d}T + P\mathrm{d}V + N\mathrm{d}\mu$.

Derivative relations

$$\begin{aligned}
+\left(\frac{\partial T}{\partial V}\right)_S &= -\left(\frac{\partial P}{\partial S}\right)_T = +\frac{\partial^2 E}{\partial S\partial V}\\
+\left(\frac{\partial T}{\partial P}\right)_S &= +\left(\frac{\partial V}{\partial S}\right)_P = +\frac{\partial^2 W}{\partial S\partial P}\\
+\left(\frac{\partial S}{\partial V}\right)_T &= +\left(\frac{\partial P}{\partial T}\right)_V = -\frac{\partial^2 F}{\partial T\partial V}\\
-\left(\frac{\partial S}{\partial P}\right)_T &= +\left(\frac{\partial V}{\partial T}\right)_P = +\frac{\partial^2 \Phi}{\partial T\partial P}
\end{aligned} \tag{1}$$

Temperature

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N} \tag{2}$$

Heat-capacity

$$\begin{aligned}
C_V &= \left(\frac{\mathrm{d}E}{\mathrm{d}T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V\\
C_P &= \left(\frac{\mathrm{d}W}{\mathrm{d}T}\right)_P = \left(\frac{\partial E}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P
\end{aligned} \tag{3}$$

Compressibility

$$\kappa_X = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_X \tag{4}$$

1.1 Some def. in stat. mech.

Definition of entropy

$$S := -\sum_{n,(N)} \rho_{n,(N)} \ln\Big(\rho_{n,(N)}\Big), \tag{5}$$

where ρ is the density function or distribution fuction.

Canonical distribution (N constant)

$$\rho_n = \frac{1}{Z}\mathrm{e}^{-E_n/T}, \quad Z = \sum_n \mathrm{e}^{-E_n/T}. \tag{6}$$

$$F = -T\ln Z \tag{7}$$

Grand canonical distribution

$$\rho_{n,N} = \frac{1}{\mathcal{Z}}\mathrm{e}^{-(E_n-\mu N)/T}, \quad \mathcal{Z} = \sum_n \mathrm{e}^{-(E_n-\mu N)/T}. \tag{8}$$

$$\Omega = -T\ln\mathcal{Z} \tag{9}$$

2 Theory of ideal gases

Ideal gas means that there is no interaction between particles, $\varepsilon = \hbar^2k^2/(2m)$. In 3 dim.

$$\frac{1}{V}\sum_k \to \int\frac{\mathrm{d}^3k}{(2\pi)^3} = \int\mathrm{d}\varepsilon\, g(\varepsilon), \tag{10}$$

$$\frac{\mathrm{d}^3k}{(2\pi)^3} = \mathrm{d}\varepsilon\, g(\varepsilon), \quad g(\varepsilon) = \frac{m^{3/2}}{\sqrt{2}\pi\hbar^3}\sqrt{\varepsilon}. \tag{11}$$

See assignment 3 for other dim.

2.1 Ideal Fermi gases

$$n^{(\text{F.D.})} = \frac{1}{\mathrm{e}^{(\varepsilon-\mu)/T}+1} \tag{12}$$

$$-\Omega = PV = \frac{2}{3}E = \frac{V g_\text{s} T}{\lambda^3} f_{5/2}(z) \tag{13}$$

$$N = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V} = \frac{V g_\text{s}}{\lambda^3} f_{3/2}(z) \tag{14}$$

$$\frac{PV}{NT} = \frac{f_{5/2}(z)}{f_{3/2}(z)} \tag{15}$$

Thermal wavelength $\lambda = h/\sqrt{2\pi m T} =: \Lambda/\sqrt{T}$, Λ is a constant.

Fermi functions

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \mathrm{d}x \frac{x^{\nu-1}}{z^{-1}\mathrm{e}^x+1} \tag{16}$$

Fugacity $z = \mathrm{e}^{\mu/T}$.

$$z\frac{\partial f_\nu(z)}{\partial z} = \frac{\partial f_\nu(z)}{\partial(\ln z)} = f_{\nu-1}(z) \tag{17}$$

Fermi energy As $T \to 0$, the ccamical potential will go to

$$\mu(T\to 0) =: \varepsilon_\text{F} = \frac{\hbar^2}{2m}\left(\frac{6}{g_\text{s}}\pi^2 n\right)^{2/3}, \tag{18}$$

where g_s is the spin deganareacy, and $n = N/V$.

In a regular metal, $\varepsilon_\text{F} \sim 10^4\,\text{K}$. For $T \ll \varepsilon_\text{F}$

$$n^{(\text{F.D.})}(\varepsilon) \approx \begin{cases} 1, & \varepsilon < \varepsilon_\text{F} \\ 0, & \varepsilon > \varepsilon_\text{F} \end{cases} \tag{19}$$

and $\int_0^\infty \mathrm{d}\varepsilon\, n^{(\text{F.D.})}(\varepsilon)\ldots \to \int_0^{\varepsilon_\text{F}} \mathrm{d}\varepsilon \ldots$

The internal energy

$$E(T \ll \varepsilon_\text{F}) = \frac{3}{5}N\varepsilon_\text{F} \tag{20}$$

2.2 Ideal Bose gases

$$n^{(\text{B.E.})} = \frac{1}{\mathrm{e}^{(\varepsilon-\mu)/T}-1} \tag{21}$$

$$-\Omega = PV = \frac{2}{3}E = \frac{V g_\text{s} T}{\lambda^3} g_{5/2}(z) \tag{22}$$

$$N_\text{e} = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V} = \frac{V g_\text{s}}{\lambda^3} g_{3/2}(z) \tag{23}$$

$$\frac{PV}{NT} = \frac{g_{5/2}(z)}{g_{3/2}(z)} \tag{24}$$

Thermal wavelength $\lambda = h/\sqrt{2\pi m T} =: \Lambda/\sqrt{T}$, Λ is a constant.

Bose functions

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \mathrm{d}x \frac{x^{\nu-1}}{z^{-1}\mathrm{e}^x-1} \tag{25}$$

For bosons $\mu \leq 0$, meaning that $z = \mathrm{e}^{\mu/T} \leq 1$.

$$z\frac{\partial g_\nu(z)}{\partial z} = \frac{\partial g_\nu(z)}{\partial(\ln z)} = g_{\nu-1}(z) \tag{26}$$

At $z=1$, $g_\nu(z=1) = \zeta(\nu)$.

Critical temperature Critical teperature for ideal Bose gas (3 dim.)

$$T_\text{c} = \frac{2\pi\hbar^2}{m}\left(\frac{N}{V g_\text{s}\,\zeta(3/2)}\right)^{3/2} \tag{27}$$

Number of condensed particles

$$N_0 = N\left[1-\left(\frac{T}{T_\text{c}}\right)^{3/2}\right], \quad T \leq T_\text{c}. \tag{28}$$

$$\begin{aligned} \lambda &= \frac{h}{\sqrt{2\pi m T}}, & \lambda_\text{c} &= \left[v\zeta(3/2)\right]^{1/3} \\ v &= \frac{1}{n} = \frac{V}{N}, & v_\text{c} &= \frac{\lambda^3}{\zeta(3/2)} \end{aligned} \tag{29}$$

2.2.1 Photons

Photons are bosons with $\mu = 0$.

$$n^{(\text{photons})} = \frac{1}{\mathrm{e}^{\hbar\omega/T} - 1}. \tag{30}$$

Number of photons in the interval ω to $\omega + \mathrm{d}\omega$:

$$\mathrm{d}N_\omega = \frac{V}{\pi^2 c^3} \frac{\omega^2 \,\mathrm{d}\omega}{\mathrm{e}^{\hbar\omega/T} - 1}. \tag{31}$$

Energy in the same interval:

$$\mathrm{d}E_\omega = \hbar\omega \,\mathrm{d}N_\omega = V u_\omega \,\mathrm{d}\omega, \tag{32}$$

where u_ω is the radiation energy density. Total radiation energy

$$E = \int_0^\infty \mathrm{d}E_\omega = \frac{4\sigma}{c} V T^4, \tag{33}$$

where $\sigma = \pi^2/(60\hbar^3 c^2)$

2.2.2 Phonons

Chain of N particles connected with springs, mean distance a. Hamiltonian

$$H = \sum_{s=0}^{N-1} \left[\frac{p_s^2}{2m} + \frac{\kappa}{2}(q_{s+1} - q_s)^2 \right] \tag{34}$$

Also periodoc boundary conditions: $q_{s+N} = q_s$ and $p_{s+N} = p_s$.

Fourier transform (momentum space)

$$H = \sum_{k=0}^{N-1} \left[\frac{1}{2m} p_{-k} p_k + \frac{m\omega_k^2}{2} q_{-k} q_k \right]. \tag{35}$$

This is a Hamiltoninan for N harmonic oscillators:

$$H = \sum_{k=0}^{N-1} \hbar\omega_k \left(\hat{n}_k + \frac{1}{2} \right). \tag{36}$$

The frequency

$$\begin{aligned} \omega_k &= \sqrt{\frac{2\kappa}{m}(1 - \cos(ka))} \\ &\overset{ka \ll 1}{\approx} |ka| \sqrt{\frac{\kappa}{m}}. \end{aligned} \tag{37}$$

Speed of sound

$$c_\mathrm{s} = \sqrt{\frac{\kappa a^2}{m}}. \tag{38}$$

For $|ka| \ll 1$,

$$C_V = \frac{2\pi^2}{5\hbar^3 {c_\mathrm{s}}^3} V T^3. \tag{39}$$

3 Second quantaization

Main idea: use a Bogoliubov transformation to transform the Hamiltonian to to a hamiltonian of an ideal gas of quasi-particles.

Harmonic oscillator interpretation in momentum space Occupation number representation:

$$|\psi\rangle = |n_{k_1}, n_{k_2}, \ldots, n_{k_i}, \ldots\rangle \tag{40}$$

$$\hat{n}_{k_i} |n_{k_1}, \ldots, n_{k_i}, \ldots\rangle = n_{k_i} |n_{k_1}, \ldots, n_{k_i}, \ldots\rangle \tag{41}$$

For bosons:

$$\left[a_k, a_{k'}^\dagger \right] = \delta_{k,k'}, \tag{42}$$

whereas for fermions

$$\left\{ c_k, c_{k'}^\dagger \right\} = \delta_{k,k'}. \tag{43}$$

The Hamiltonian for interacting bosons

$$H = \overbrace{\sum_k \varepsilon_k a_k^\dagger a_k}^{\text{non-interating}} + \frac{1}{2V} \sum_{k,k',q} U(q) a_{k+q}^\dagger a_{k'-q}^\dagger a_{k'} a_k \tag{44}$$

and for fermions

$$H = \overbrace{\sum_{k,\sigma} \varepsilon_k c_{k,\sigma}^\dagger c_{k,\sigma}}^{\text{non-interating}} + \frac{1}{2V} \sum_{k,k',q,\sigma,\sigma'} U(q) c_{k+q,\sigma}^\dagger c_{k'-q,\sigma}^\dagger c_{k',\sigma'} c_{k,\sigma}. \tag{45}$$

The inteaction, $U(q)$, is given by

$$U(\boldsymbol{r} - \boldsymbol{r'}) = \frac{1}{V} \sum_q U(\boldsymbol{q}) \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r'})}, \tag{46}$$

or

$$U(\boldsymbol{q}) = \int \mathrm{d}^3 r U(\boldsymbol{r}) \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}. \tag{47}$$

We can also talk about annihilation and creation operators in real space: $\psi^\dagger(\boldsymbol{r})$ creates a particle at position $\boldsymbol{r}$.

$$\int \mathrm{d}^3 r \, \varphi^\dagger(\boldsymbol{r}) \varphi(\boldsymbol{r}) = \sum_k a_k^\dagger a_k = \hat{N} \tag{48}$$

These operators faoolw the same commuttation relations as the momentum space equivalence.

3.1 Superfluidity in ^{4}He

At low enough temperatures most atoms will be in the ground state. Our goual is to bring the Hamiltonian to a form:

$$H = \sum_k E_k \xi_k^\dagger \xi_k + \text{const.} \tag{49}$$

Mean-field theory:

$$a_0 a_0^\dagger \approx a_0^\dagger a_0 = |a_0| = N_0 \tag{50}$$

meaning that

$$a_0 = \sqrt{N_0}\mathrm{e}^{\mathrm{i}\varphi}. \tag{51}$$

Bogoliubov transformation

$$\begin{cases} \xi_k = u_k a_k + v_k a_{-k}^\dagger \\ \xi_k^\dagger = u_k a_k^\dagger + v_k a_{-k} \end{cases} \tag{52}$$

We still want the commutation relation $\left[\xi_k,\xi_k^\dagger\right] = 1$, which gives

$$u_k = \cosh\theta_k, \quad v_k \sinh\theta_k. \tag{53}$$

And

$$\cosh 2\theta_k = \frac{A_k}{\sqrt{A_k^2 - B_k^2}}, \quad \sinh 2\theta_k = \frac{B_k}{\sqrt{A_k^2 - B_k^2}}, \tag{54}$$

where

$$A_k = \varepsilon_k + nU(k), \quad B_k = nU(k). \tag{55}$$

Energy of the quasi-particles

$$\begin{aligned} E_k =& A_k \cosh 2\theta_k - B_k \sinh 2\theta_K \\ =& \sqrt{A_k^2 - B_k^2} = \sqrt{\varepsilon_k[\varepsilon_k + 2nU(k)]} \end{aligned} \tag{56}$$

Long wavelength limit ($k \approx 0$)

$$E_k \approx \hbar|k|\sqrt{\frac{nU(0)}{m}} \tag{57}$$

same form as phonons, and $c_\mathrm{s} = \sqrt{nU(0)/m}$. The super fluid behaves more like a solid than a liquid.

As long as some disturbance is slower than c_s, there can not be any dissipation of energy into the super fluid.

Spontaneously broken symmetry We can choose φ in the mean-field theory description of a_0 freely. And the particles curren will be

$$\boldsymbol{j} = \frac{\hbar n}{m}\boldsymbol{\nabla}\varphi \tag{58}$$

But if the geomery has holes, we must have a periodically varying φ, meaning that $\boldsymbol{\nabla}\varphi = \ell 2\pi$ becomes quantized.

3.2 BCS theory of superconductivity

The BCS (mean-field) equation:

$$\begin{aligned} \varDelta &= \frac{U}{2V}\sum_k \frac{\varDelta}{\sqrt{\xi_k^2 + \varDelta^2}}\left[1 - 2n^{(\mathrm{F.D.})}(E_k)\right] \\ &= \frac{U}{2V}\sum_k \frac{\varDelta}{\sqrt{\xi_k^2 + \varDelta^2}}\tanh\left(\frac{E_k}{2T}\right). \end{aligned} \tag{59}$$

Here $E_k = \sqrt{\xi_k^2 + \varDelta^2}$, and $\xi_k = (\varepsilon_k - \varepsilon_\mathrm{F})$ varies from $-\varepsilon_\mathrm{F}$ to ∞.

Sums transfroms according to

$$\frac{1}{V}\sum_k \to g(\varepsilon_\mathrm{F})\int_{-\hbar\omega_\mathrm{D}}^{\hbar\omega_\mathrm{D}} \mathrm{d}\xi, \tag{60}$$

where ω_D is the Debye frequency, and $\hbar\omega_\mathrm{D} \ll \varepsilon_\mathrm{F}$ (typ. values $\hbar\omega_\mathrm{D} \sim 10^2\,\mathrm{K}$, while $\varepsilon_\mathrm{F} \sim 10^4\,\mathrm{K}$). Only the states around $\varepsilon = \varepsilon_\mathrm{F}$ ($\xi = 0$) affects superconductivity.

3.3 Ginsburg-Landau theory of superconductivity

4 Second order phase transitions

4.1 Ising model

4.2 Landau theory of cont. phase transitions

A Special functions

A.1 Gamma function

$$\Gamma(\nu) = \int_0^\infty x^{\nu-1} \mathrm{e}^{-x} \, \mathrm{d}x \tag{61}$$

$$\Gamma(n) = (n-1)!, \quad n \in \mathbb{Z}^+ \tag{62}$$

x	$^1/_2$	$^3/_2$	$^5/_2$	$^7/_2$	$^9/_2$
$\Gamma(x)$	$\sqrt{\pi}$	$\sqrt{\pi}/2$	$3\sqrt{\pi}/4$	$15\sqrt{\pi}/8$	$105\sqrt{\pi}/16$

A.2 Zeta function

The Riemann zeta function

$$\zeta(\nu) = \sum_{n=1}^{\infty} \frac{1}{n^\nu} = \int_0^\infty \frac{x^{\nu-1}}{\mathrm{e}^x - 1} \, \mathrm{d}x \tag{63}$$

x	2	4	6
$\zeta(x)$	$\pi^2/6$	$\pi^4/90$	$\pi^6/945$
x	$3/2$	$5/2$	$7/2$
$\zeta(x)$	2.61238	1.34149	1.12673
x	3	5	7
$\zeta(x)$	1.20206	1.03693	1.00835

B The harmonic oscillator