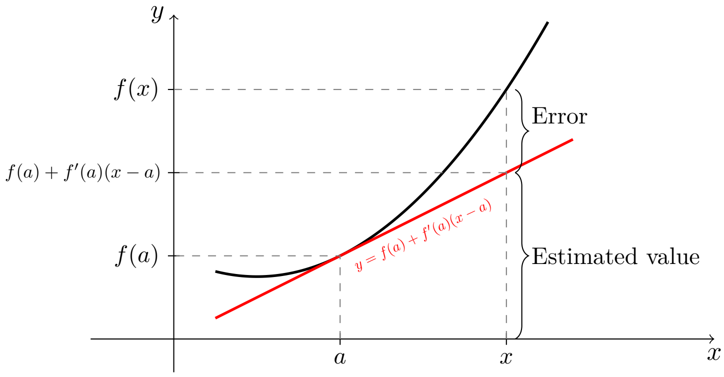
TAYLOR POLYNOMIALS

Theoretical notes The differentiability of a function f at a point $a \in \operatorname{int} \mathcal{D}_f$ means that the function can be *well approximated* by a linear polynomial in the neighborhood of the point. This is visible in the picture, where the graph of the function and the tangent line almost merge in the vicinity of the point a.



Linear approximation can be used to estimate function values. If we know the exact values $f(a)$ and $f'(a)$, we can estimate the value of the function at any point x that lies within a *small radius* around a by the expression

$$f(a) + f'(a)(x-a).$$

For instance, we can estimate the value of $\sqrt{1.1}$ using linear approximation on the function

$$f(x) = \sqrt{1+x} \qquad (x \geq -1)$$

at $a = 0$. For that we first need to calculate the values

$$f(x) = \sqrt{1+x} \implies f(0) = 1 \qquad \text{and} \qquad f'(x) = \frac{1}{2\sqrt{1+x}} \implies f'(0) = \frac{1}{2}.$$

Therefore

$$\sqrt{1+x} \approx f(0) + f'(0)(x-0) = 1 + \frac{1}{2}(x-0) = 1 + \frac{x}{2}$$

where x is in a small neighborhood of $a = 0$. Hence, for $x = 0.1$ we obtain

$$\sqrt{1.1} = \sqrt{1+0.1} = f(0.1) \approx 1 + \frac{0.1}{2} = 1.05.$$

The obtained value 1.05 is a good approximation, since the first ten decimal places of the true value is 1.0488088480.

If the linear approximation is not accurate enough, then we also tried higher-degree polynomials. Polynomials are widely used in approximation because they only contain algebraic operations. We assume that the function f is n-times differentiable at the point a, where $n \in \mathbb{N}$. Then the polynomial

$$\begin{aligned} T_{n,a}f(x) :&= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \qquad (x \in \mathbb{R}) \end{aligned}$$

is called the n-th ***Taylor polynomial*** centered at a of the function f. The following statement gives us an estimation for the approximation using Taylor polynomials.

***Theorem* (Taylor's Formula)** *Let $n \in \mathbb{N}$, $K(a)$ be a neighborhood of a, and f be an $n+1$-times differentiable function on $K(a)$. Then for all $x \in K(a)$, there exists ξ between the points a and x such that*

$$f(x) - T_{n,a}f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

The function on the right-hand side of the above equation is called the **Lagrange remainder term**.

Now we estimate the value of $\sqrt{1.1}$ using the second-degree Taylor polynomial centered at 0, and we also estimate the approximation error. We have already calculated the values of $f(0) = 1$ and $f'(0) = 1/2$, but now we need also the value of $f''(0)$:

$$f''(x) = \left(\frac{1}{2\sqrt{1+x}}\right)' = \frac{1}{2}\left((1+x)^{-1/2}\right)' = -\frac{1}{4}(1+x)^{-3/2} \implies f''(0) = -\frac{1}{4}.$$

Therefore

$$\sqrt{1+x} \approx T_{2,0}f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 = 1 + \frac{1}{2}(x-0) + \frac{-1/4}{2}(x-0)^2 = 1 + \frac{x}{2} - \frac{1}{8}x^2$$

where x is in a small neighborhood of $a = 0$. Hence, for $x = 0.1$ we obtain

$$\sqrt{1.1} = \sqrt{1+0.1} = f(0.1) \approx 1 + \frac{0.1}{2} - \frac{(0.1)^2}{8} = 1.04875.$$

For the estimation of the error, let $a = 0$ and $x = 0.1$. According to the Taylor formula, there exists $0 < \xi < 0.1$ such that

$$f(x) - T_{2,0}f(x) = \frac{f'''(\xi)}{3!}x^3.$$

Since

$$f'''(x) = \frac{3}{8}(1+x)^{-5/2} = \frac{3}{8\sqrt{(1+x)^5}} \qquad (x > -1),$$

we have

$$|f(x) - T_{2,0}f(x)| = \frac{|f'''(\xi)|}{3!}|x|^3 = \frac{1}{6}\cdot\frac{3}{8\sqrt{(1+\xi)^5}}|x|^3$$

where $x = 0.1$, but the value of ξ is unknown, we only know that $0 < \xi < 0.1$. In order to still make the estimate, we must recognize that the above expression increases as the value of ξ decreases. Therefore

$$\frac{1}{6}\cdot\frac{3}{8\sqrt{(1+\xi)^5}}|x|^3 \le \frac{1}{6}\cdot\frac{3}{8\sqrt{(1+0)^5}}\cdot(0.1)^3 = \frac{1}{6}\cdot\frac{3}{8}\cdot\frac{1}{1000} = 0.000375.$$

The second-degree approximation used for estimating $\sqrt{1.1}$ here is more accurate than the linear approximation. The question arises whether increasing the degree of Taylor polynomials leads to a smaller error bound. Unfortunately, this is not true in general.

Exercise 1. *Let $f(x) := \ln(1+x)$ $(x > -1)$.*

a) *Find the second Taylor-polynomial centered at 0, denoted by $T_{2,0}f(x)$.*

b) *Approximate the value of $\ln 2$ by $T_{2,0}f(1)$, and estimate the error.*

c) *Estimate the error of approximation $f(x) \approx T_{2,0}f(x)$ for $x \in [-1/2, 0]$ and $x \in [0, 2]$.*

Solution

a) First we find the coefficient of $T_{2,0}f(x)$:

$$\begin{aligned} f(x) &= \ln(1+x) &&\implies && f(0) = 0,\\ f'(x) &= \frac{1}{1+x} = (1+x)^{-1} &&\implies && f'(0) = 1,\\ f''(x) &= -(1+x)^{-2} &&\implies && f''(0) = -1. \end{aligned}$$

Hence

$$T_{2,0}f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = x - \frac{1}{2}x^2 \qquad (x \in \mathbb{R}).$$

b) Since $\ln(2) = \ln(1+1) = f(1)$, then

$$\ln 2 = f(1) \approx T_{2,0}f(1) = 1 - \frac{1}{2} \cdot 1^2 = \frac{1}{2}.$$

For the estimation of the error, let $a = 0$ and $x = 1$. According to the Taylor formula, there exists $0 < \xi < 1$ such that

$$f(x) - T_{2,0}f(x) = \frac{f'''(\xi)}{3!}x^3.$$

Since

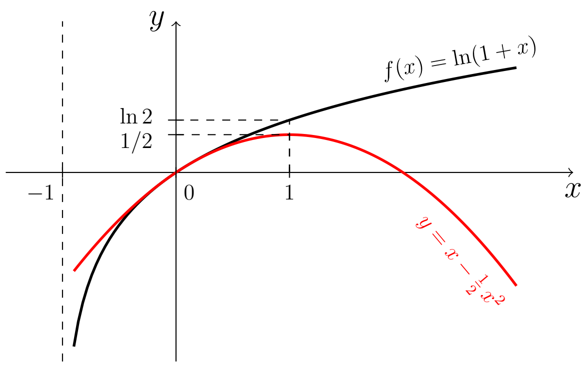
$$f'''(x) = 2(1+x)^{-3} = \frac{2}{(1+x)^3} \qquad (x > -1),$$

we have

$$|f(x) - T_{2,0}f(x)| = \frac{|f'''(\xi)|}{3!}|x|^3 = \frac{1}{6} \cdot \frac{2}{(1+\xi)^3}|x|^3$$

where $x = 1$, but the value of ξ is unknown, we only know that $0 < \xi < x = 1$. In order to still make the estimate, we must recognize that the above expression increases as the value of ξ decreases. Therefore

$$\frac{1}{6} \cdot \frac{2}{(1+\xi)^3}|x|^3 \leq \frac{1}{6} \cdot \frac{2}{(1+0)^3} \cdot 1^3 = \frac{1}{6} \cdot 2 \cdot 1 = \frac{1}{3}.$$



c) We use the estimation that appears in part b)

$$|f(x) - T_{2,0}f(x)| = \frac{|f'''(\xi)|}{3!}|x|^3 = \frac{1}{6} \cdot \frac{2}{(1+\xi)^3}|x|^3$$

where the value of x is between 0 and x.

- In the first case $-\frac{1}{2} \leq x < \xi < 0$, so

$$|f(x) - T_{2,0}f(x)| \leq \frac{1}{6} \cdot \frac{2}{(1-\frac{1}{2})^3} \cdot \left|-\frac{1}{2}\right|^3 = \frac{1}{3}.$$

- In the second case $0 < \xi < x < 2$, so

$$|f(x) - T_{2,0}f(x)| \leq \frac{1}{6} \cdot \frac{2}{(1+0)^3} \cdot |2|^3 . = \frac{8}{3}$$

Exercise 2. *Find the second Taylor polynomial of the following function centered at 0, and provide an estimate of the approximation error on the interval* $\left[0, \frac{1}{8}\right]$.

$$f(x) := \frac{1}{\sqrt{1+2x}} \qquad \left(x > -\frac{1}{2}\right).$$

Solution The function is differentiable as many times as needed, and for every $x > -\frac{1}{3}$, we have:

$$\begin{aligned}
f(x) &= (1+2x)^{-1/2} && \Longrightarrow && f(0) = 1,\\
f'(x) &= -\frac{1}{2}(1+2x)^{-3/2} \cdot 2 = -(1+2x)^{-3/2} && \Longrightarrow && f'(0) = -1,\\
f''(x) &= \frac{3}{2}(1+2x)^{-5/2} \cdot 2 = 3(1+2x)^{-5/2} && \Longrightarrow && f''(0) = 3.
\end{aligned}$$

Hence, the required Taylor polynomial is

$$T_{2,0}f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1 - x + \frac{3}{2}x^2 \qquad (x \in \mathbb{R}).$$

To estimate the error, we apply Taylor's formula with the Lagrange remainder term. For every $0 < x \leq \frac{1}{8}$, there exists $0 < \xi < x$ such that

$$f(x) - T_{2,0}f(x) = \frac{f'''(\xi)}{3!}x^3.$$

Moreover,

$$f'''(x) = -\frac{15}{2}(1+2x)^{-7/2} \cdot 2 = \frac{-15}{\sqrt{(1+2x)^7}}.$$

Thus,

$$\left|f'''(\xi)\right| = \frac{15}{\sqrt{(1+2\xi)^7}} \leq \frac{15}{\sqrt{(1+2\cdot 0)^7}} = 15.$$

Therefore,

$$|f(x) - T_{2,0}f(x)| = \frac{|f'''(\xi)|}{6}|x|^3 \leq \frac{15}{6} \cdot \left|\frac{1}{8}\right|^3 = \frac{5}{1024} \approx 0.004883.$$

Exercise 3. *Using the fourth Taylor polynomial centered at 0 of the function* $f(x) = e^x$, *provide an estimate for the value of* $\sqrt{e}$ *and determine an error bound. When calculating the estimates, only the four basic arithmetic operations may be used.*

Solution Since the nth derivative of the function $f(x) = e^x$ is also $f^(n)(x) = e^x$, it is easy to obtain

$$T_{4,0}f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24},$$

whose value at the point $x = 1/2$ is 1.6484375. Thus,

$$\sqrt{e} = e^{1/2} \approx T_{4,0}f(1/2) = 1.6484375.$$

To estimate the error, we apply Taylor's formula. Let $x = 1/2$. Then, there exists some ξ such that $0 < \xi < x = 1/2$, for which

$$f(x) - T_{4,0}f(x) = \frac{f^{(5)}(\xi)}{5!} \cdot x^5 = \frac{e^\xi}{5!} \cdot x^5.$$

Note that the value of the formula above increases when the value of ξ also increases. Thus,

$$\Big|f(x) - T_{4,0}f(x)\Big| = \frac{e^\xi}{5!} \cdot |x|^5 < \frac{2}{120} \cdot \left(\frac{1}{2}\right)^5 = 0.00052084,$$

since $e^\xi < e^{1/2} < 4^{1/2} = \sqrt{4} = 2$. Therefore, the error bound is 0.00052084.

Exercise 4.

a) *Show that for any polynomial P of degree at most n and any center point $a \in \mathbb{R}$, the following holds for all $x \in \mathbb{R}$:*

$$P(x) = \sum_{k=0}^{n} \frac{P^{(k)}(a)}{k!}(x-a)^k.$$

b) *Rewrite the polynomial x^5 as a sum of powers of $(x-1)$ and use it to compute the exact value of* 1.1^5.

Solution

a) Consider an arbitrary polynomial P of degree at most n, and a given point $a \in \mathbb{R}$. By Taylor's formula, for all $x \in \mathbb{R}$, there exists a ξ between a and x such that

$$P(x) - \sum_{k=0}^{n} \frac{P^{(k)}(a)}{k!}(x-a)^k = \frac{P^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} = 0,$$

since $P^{(n+1)}(\xi) = 0$ (because the mth derivative of a polynomial of degree at most n is is the constant zero function if $m > n$ ($P^{(m)} \equiv 0$ for all $m > n$). Thus, the statement is proven.

b) Let $f(x) := x^5$ $(x \in \mathbb{R})$. For all $x \in \mathbb{R}$:

$$f'(x) = 5x^4, \quad f''(x) = 20x^3, \quad f'''(x) = 60x^2, \quad f^{(4)}(x) = 120x, \quad f^{(5)}(x) = 120.$$

Since $f(1) = 1$, $f'(1) = 5$, $f''(1) = 20$, $f'''(1) = 60$, $f^{(4)}(1) = 120$, and $f^{(5)}(1) = 120$, the fifth Taylor polynomial at $a = 1$ is:

$$T_{5,1}(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$$
$$+\frac{f^{(5)}(1)}{5!}(x-1)^5 = 1 + 5(x-1) + 10(x-1)^2 + 10(x-1)^3 + 5(x-1)^4 + (x-1)^5.$$

According to the previous part, f and $T_{5,1}$ are identical at every point, so:

$$x^5 = 1 + 5(x-1) + 10(x-1)^2 + 10(x-1)^3 + 5(x-1)^4 + (x-1)^5 \quad (x \in \mathbb{R}).$$

If $x = 1.1$, we obtain the desired exact value:

$$\begin{aligned} 1.1^5 &= 1 \\ &+0.5 \\ &+0.1 \\ &+0.01 \\ &+0.0005 \\ &+0.00001 \\ \hline &= 1.61051 \end{aligned}$$