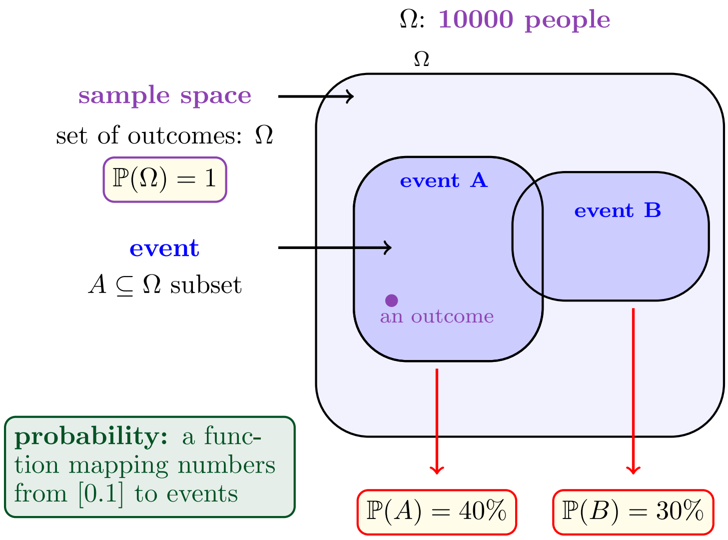


Problem set 1, 10 February, 2025

(1) In a country the number plates are labeled by five digit numbers from 00000 to 99999. We randomly pick one number plate. What is the probability of the event that

a) there is a six among the digits on the plate

b) the digits on the plate are different

Remark 1 *Problems on probability can be solved by simple elementary counting when we have finitely many possible outcomes and we may assume that they are equiprobable, by the formula*

p=(number of the favorable outcomes)/(number of all outcomes)

(2) At a soccer training session 20 players participate, two of them are Simon and Garfunkel. We divide the participants randomly into two groups of 10 persons. What is the probability that Simon and Garfunkel play against each other?

(3) Peter fires at a target with his gun. The first shot has a 60% chance of hitting the target, the second one has 70%, and the last one has 80%. The shots are independent. What is the probability of the following events:

a) Peter not hitting the target at all?

b) Peter hits the target only at the third shot?

c) does not hit that target at all, given that the first shot was a miss?

Definition 2 *The events $A, B \in \mathcal{A}$ are* **independent** *if*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B),$$

that is, **the probability of the intersection is the product of the probabilities***.*
The events $A_1, A_2, \ldots \in \mathcal{A}$ are **independent** *if for every $k \geq 1$ and $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ we have*

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2}) \ldots \mathbb{P}(A_{i_k}).$$

Given B, what is the conditional probability that event A occurs?

Definition 3 (Conditional probability) *Let $A, B \in \mathcal{A}$ be two events, and suppose that $\mathbb{P}(B) > 0$. The conditional probability of A with respect to B is defined as follows:*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Notice that if A and B are independent, then we have:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$$

$$\boxed{\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)},}$$

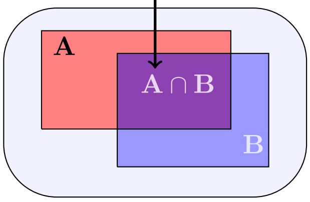
probability of the intersection

both A and B occur

probability of the condition

$\mathbb{P}(B) > 0$, we can divide by it

Conditional probability: given that B occurs, what is the probability that A also occurs



(4) What is the probability that maximum of the numbers that we get is equal to 5 if we throw 2 (generally n) regular dice? (Regular dice: $1, 2, 3, 4, 5, 6$ with equal probabilities.).

Definition 4 *The triple $(\Omega, \mathcal{A}, \mathbb{P})$ is a* **probability field**, *if*

- *the sample set Ω is a non-empty set;*
- *$\mathcal{A} \subseteq \mathcal{P}(\Omega)$ (where $\mathcal{P}(\Omega)$ is the set of all subsets of Ω). is the* **set of events** *(or σ-algebra of events), that is, for all $A \in \mathcal{A}$ we have $A \subseteq \Omega$ such that*
  - *(i) $\Omega \in \mathcal{A}$;*
  - *(ii) if $A_1, A_2, \ldots \in \mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ (that is, the union of countably many sets in $\mathcal{A}$ is also in $\mathcal{A}$);*
  - *(iii) if $A \in \mathcal{A}$, then $\Omega \setminus A \in \mathcal{A}$ (that is, the complement of sets in $\mathcal{A}$ is also in $\mathcal{A}$.*
- **probability** *$\mathbb{P} : \mathcal{A} \to [0.1]$ is a function such that*
  - *(i) $\mathbb{P}(\Omega) = 1$. that is. the probability of the whole sample set is 1;*
  - *(ii) if $A_1, A_2, \ldots \in \mathcal{A}$ and for all $1 \le i < j$ we have $A_i \cap A_j = \emptyset$, then*

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

That is, the probability of the union of countably many pairwise disjoint sets is the sum of the probabilities.

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(5) Odysseus arrives at a junction while wandering on the road. One leads to Athens, one to Sparta and one to Mycenae. The athenians tell the truth every third time when asked, the mycanaeans every second time, while the spartans never lie. Odysseus does not know which way leads to which city so he chooses randomly. Arriving in the city he asks a person the question: How much is 2×2. The answer is 4. What is the probability that he is in Athens?

(6) There is a disease which affects 2% of the population. There is a blood test which is positive with probability 95% for people who have the disease, and it is positive with probability 1% for healthy people.

(a) What is the probability that Peter's test will be positive? (Peter is a randomly chosen person.)

(b) Given that the test of Peter is positive, what is the conditional probability that he has the disease?

(c) Now suppose that the test was repeated k times independently for a randomly chosen person, and all results were positive. What is the conditional probability that this person has the disease? Calculate this probability for $k = 2$ and $k = 3$

Theorem 5 (Law of total probability) *Let $A \in \mathcal{A}$ be an event and $B_1, B_2, \ldots, B_n$ a partition of the sample space. Then we have*

$$\mathbb{P}(A) = \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) + \mathbb{P}(A|B_3)\mathbb{P}(B_3) + \ldots + \mathbb{P}(A|B_n)\mathbb{P}(B_n) = \sum_{j=1}^{n} \mathbb{P}(A|B_j)\mathbb{P}(B_j).$$

Theorem 6 (Bayes' theorem) *Let $A \in \mathcal{A}$ be an event such that $\mathbb{P}(A) > 0$, $B_1, B_2, \ldots, B_n$ a partition of the sample space. Then for all $k = 1, 2, \ldots .n$ we have*

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) + \mathbb{P}(A|B_3)\mathbb{P}(B_3) + \ldots + \mathbb{P}(A|B_n)\mathbb{P}(B_n)} =$$
$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\sum_{j=1}^{n} \mathbb{P}(A|B_j)\mathbb{P}(B_j)}.$$

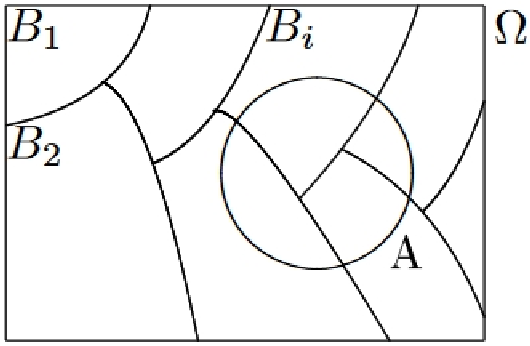

Figure 1: A partition of the probability space and an event A

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(7) (*Monty Hall problem*) In a television show there are 3 doors, such that there is a present behind one of them, and there is nothing behind the other two. The participant of the game chooses one of the doors. Then, the presenter opens one of the other doors, and shows that there is nothing behind it. Then he asks the participant whether he would like to stay with his original choice, or he would like to choose the other door which has not been opened yet. Is it worth changing?

`https://www.randomservices.org/random/apps/MontyHallGame.html`

(8) Suppose that everyone has his or her birthday on a uniformly randomly chosen day among the 365 days of the year (we omit 29 February for sake of simplicity), independently. What is the probability that, among n people, there are at least two who have their birthday on the same day? Calculate this probability for $n = 5, 25, 100$.

`https://www.randomservices.org/random/apps/BirthdayExperiment.html`

(9) Mary is collecting Kinder surprise figurines. There are 10 types of them, and, in each Kinder surprise egg, independently of the others, all of the 10 types are hidden with the same probability. What is the probability that the 20th Kinder surprise egg conatins the last type of figurine?

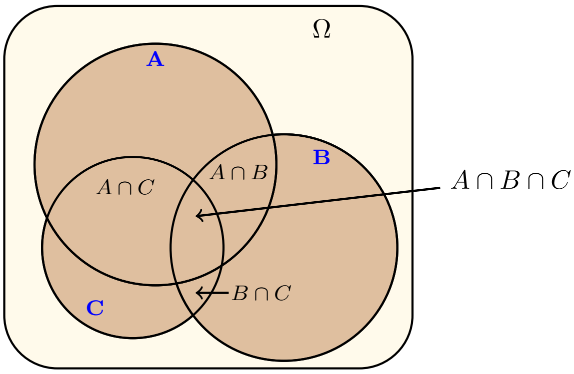

Figure 2: Inclusion-exclusion formula for three events

Proposition 7 *(a)* **Inclusion-exclusion formula for two events.** *The probability that at least one of A and B occurs:*

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

(b) **Inclusion-exclusion formula for three events.** *The probability that at least one of A and B occurs:*

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \\ - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)\end{aligned}$$

(c) **Inclusion-exclusion formula in general***: The probability that at least one of $A_1, A_2, \ldots, A_n$ occurs:*

$$\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots + \mathbb{P}(A_n) - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) -$$
$$- \ldots - \mathbb{P}(A_2 \cap A_3) - \mathbb{P}(A_2 \cap A_4) - \ldots - \mathbb{P}(A_{n-1} \cap A_n) + \mathbb{P}(A_1 \cap A_2 \cap A_3) + \ldots$$
$$= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}).$$

Remark 8 *We shall use the statistical programming language* `R` *during the semester. You may find a tutorial on the webpage:* ***https://zempleni.elte.hu/Stat_R_Prohle_Zempleni***

Homework until 16 February, Sunday, 23:59 A company produces computers at two locations. A computer from factory A breaks down within one year with probability 1.5%, while a computer from factory B breaks down within one year with probability 3%. We also know that 40% of the computers are made in factory A. Suppose that Anne brought a computer from this company, and her computer did not broke down in the first year. Conditionally to this event, what is the probability that Anne's computer was from factory A?

The solution has to be sent in the Canvas group of the course.

There will be a mid-term and a final exam paper, for 50 points each. The minimum is 15 points, if someone has less than 15 points, then he or she has to write the re-take of that exam. On each problem set (10 times), there will be a homework for 2 points.

Planned grades after the two mid-terms and the homeworks:

0-35: 1; 36-53: 2; 54-66: 3; 67-83: 4; 84-: 5.