Problem set 3, 24 February, 2025

Definition 1 *The* **cumulative distribution function** *of a random variable X is a function $F : \mathbb{R} \to [0, 1]$, where*

$$F(t) = \mathbb{P}(X \leq t)$$

holds for every real number t.

A function $f : \mathbb{R} \to \mathbb{R}$ is **density function** *of a random variable X if*

$$\mathbb{P}(a < X \leq b) = \int_a^b f(x)\, dx$$

holds for every real numbers $a < b$.

Then (only if the density function exists) we have

$$F(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^{t} f(x)\, dx \qquad \text{and} \qquad f(t) = F'(t).$$

where the latter equation is for "almost all" real numbers t.

For every density function f we have $\int_{-\infty}^{\infty} f(x)\, dx = 1$ and $f(x) \geq 0$ for "almost all" real numbers t.

1. We have ordered an item online, and we know that the delivery is between $8:00$ and $10:00$, at a time which is uniformly distributed on the interval $[8, 10]$: the probability that it is between a and b is proportional to $b - a$, if $8 \leq a \leq b \leq 10$. Let X be the time of the delivery (it is a random element of $[8, 10]$, hence this is a random variable).

   (a) Calculate $\mathbb{P}(X \leq 9)$, that is, the probability that the delivery is before 9.

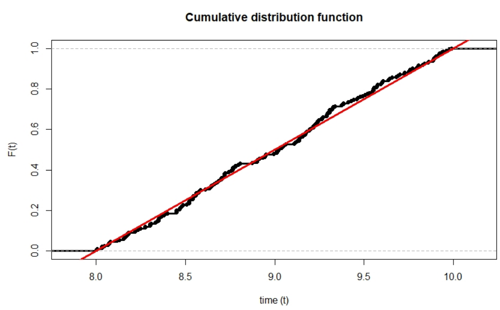
   (b) Calculate the probability $\mathbb{P}(8.5 < X < 9)$, that is, the probability is between $8:30$ and $9:00$.

   (c) Calculate the probability $\mathbb{P}(X > 9.75)$.

   (d) Draw the curve $\mathbb{P}(X \leq t)$ (the cumulative distribution function of X) as a function of t, where $t \in \mathbb{R}$ is a real number.

   Program in R: mean and empirical cumulative distribution function (the proportion of observations not larger than t) for a sample of size $n = 200$ and the theoretical cumulative distribution function of the uniform distribution:

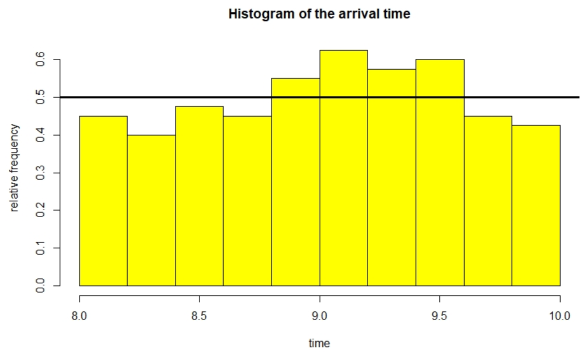
```
sample200<-runif(200, 8, 10)
mean(sample200)
[1] 8.997175
plot(ecdf(sample200), lwd="3", xlab="time (t)", ylab="F(t)", main="Cumulative distribution
function")
abline(-4, 0.5, lwd="3", col="red")
```



   (e) Does there exist a function $f : \mathbb{R} \to \mathbb{R}_+$, such that $\mathbb{P}(a < X < b) = \int_a^b f(x)\, dx$ is satisfied for all real numbers $a < b$?

   Histogram:

```
> sample200<-runif(200, 8, 10)
> hist(sample200, col="yellow", freq=F, main="Histogram of the arrival time", xlab="time",
ylab="relative frequency")
> abline(0.5, 0, lwd="3")
```



2. The density function of the random variable X has the following form:

$$f(x) = \begin{cases} 0, & x \le 0; \\ c \cdot x, & 0 \le x \le 1; \\ 0, & x > 1. \end{cases}$$

Determine the value of c. What is the probability that X is between $1/4$ and $1/2$? What is the probability that it is between $1/2$ and $3/4$? Determine the cumulative distribution function of X, and also its expectation and variance.

3. Let the cumulative distribution function of X be given by:

$$F(t) = \mathbb{P}(X \le t) = \begin{cases} 0, & \text{if } t \le 0; \\ 25t^2, & \text{if } 0 \le t \le 1/5; \\ 1, & \text{if } t \ge 1/5. \end{cases}$$

   (a) Determine the density function of X.
   (b) What is the probability that X is between 0.1 and 0.15?
   (c) Determine the expectation of X.
   (d) Determine the standard deviation of X.

4. Suppose that X has uniform distribution on the interval $[0, 1]$, that is, its density function is 1 within $[0, 1]$ and 0 otherwise.

   a) Find the cumulative distribution function and density function of X^2.

   b) Find the cumulative distribution function and density function of X^5.

   c) Determine the expectation and variance of X.

Definition 2 *Let X be a random variable with density function f. Then the* **expectation** *of X is given by*

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx.$$

The **variance** *of X is given by*

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2.$$

The standard deviation of X is given by

$$s.d.(X) = \sqrt{\mathbb{E}(X^2) - \mathbb{E}(X)^2}.$$

Here

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) dx.$$

Definition 3 (Exponential distribution) *A random variable X has exponential distribution with parameter λ is its density function is given by*

$$f(x) = \begin{cases} 0, & x \le 0; \\ \lambda e^{-\lambda x}, & x \ge 0. \end{cases}$$

If X has exponential distribution with parameter λ. then

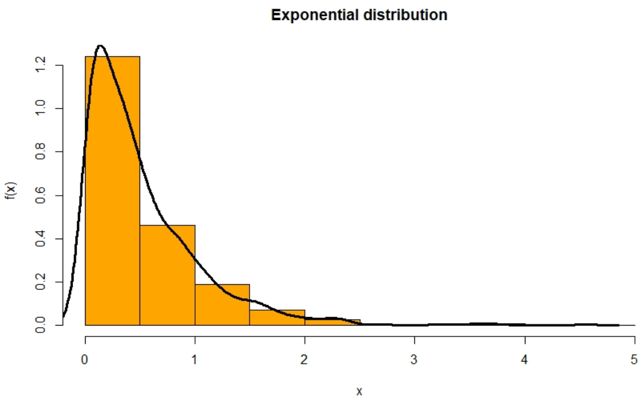
$$\mathbb{E}(X) = D(X) = \frac{1}{\lambda}.$$

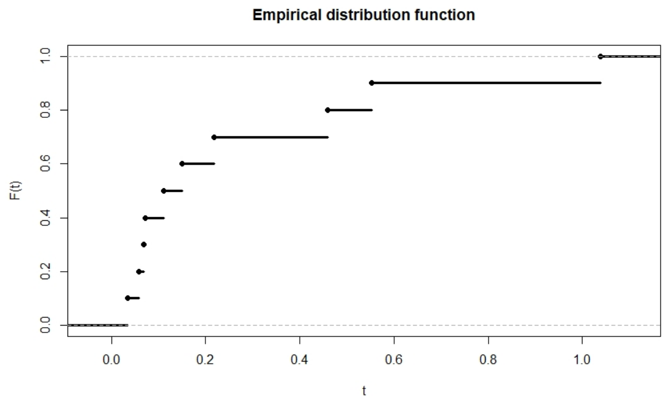
---

5. Suppose that the response time of a server (in seconds) has exponential distribution with parameter $\lambda = 2$.

   (a) Determine the cumulative distribution function and the density function of the response time.
   An illustration: exponentially distributed sample:

```
expsample=rexp(1000, rate=2)
hist(expsample, col="orange", main="Exponential distribution", xlab="x", ylab="f(x)",
freq=FALSE)
lines(density(expsample), lwd="3")
plot(ecdf(expsample[1:10]), lwd="3", xlab="t", ylab="F(t)", main="Empirical distribution
function")
```





   (b) What is the probability that the response time is more than 0.5 seconds?
   (c) the response time is at least 1 second?
   (d) given that the response time is at least 0.5 second, what is probability that it is at least 2.5 seconds?
   (e) What is the probability that the response time is between 1 and 2 seconds?
   (f) For which t is it true that the probability that the response time is at most t is equal to 1/2?
   (g) Find the expectation and standard deviation of X.
   (h) Generate a sample of 1000 random variables from the exponential distribution with parameter $\lambda = 2$. Make a histogram, and find the mean, standard deviation and median of the sample. Compare the results with the values calculated above.

Homework until 2 March, Sunday, 23:59, 2 points

Suppose that the number of bugs in a program code has Poisson distribution with expectation 5.

a) What is the probability that the number of bugs in this code is at most 3?

b) We have already found 2 bugs. What is the conditional probability of the event that the total number of bugs in the code is at most 3, given this information, that is, given that the number of bugs is at least 2? Compare the result to the result of question a).