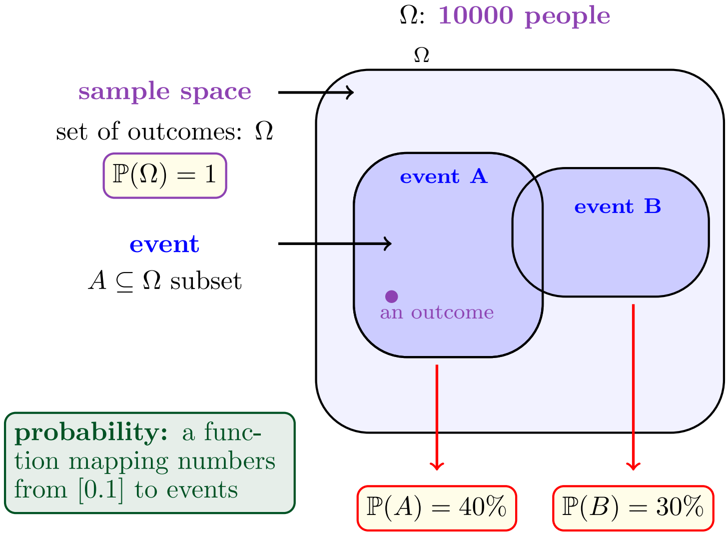


Solutions to the problem set 1, 10 February, 2025

Remark 1 *Problems on probability can be solved by simple elementary counting when we have finitely many possible outcomes and we may assume that they are equiprobable, by the formula*

p=(number of the favourable outcomes)/(number of all outcomes)

(1) In a country the number plates are labelled by five digit numbers from 00000 to 99999. We randomly pick one number plate. What is the probability of the event that

a) there is a six among the digits on the plate

$p = 1 - \frac{9^5}{10^5} = 0.41$, as only those plates are bad that do not contain the digit 6.

b) the digits on the plate are different

$p = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{10^5} = 0.3$.

(2) At a soccer training session 20 players participate, two of them are Simon and Garfunkel. We divide the participants randomly into two groups of 10 persons. What is the probability that Simon and Garfunkel play against each other?

Let us fix the team of Simon. There are 9 spaces left in it, while there are 19 spots left altogether. So for Garfunkel 10 out of the 19 choices result in being in a different team than Simon.

(3) Peter fires at a target with his gun. The first shot has a 60% chance of hitting the target, the second one has 70%, and the last one has 80%. The shots are independent. What is the probability of the following events:

a) Peter not hitting the target at all?

A, B, C: hitting the target for the first, second, third shot. Notice that if certain events are independent, then we get independent events if we replace any of the events with its complement.

$$\mathbb{P}(\overline{A} \cap \overline{B} \cap \overline{C}) = \mathbb{P}(\overline{A}) \cdot \mathbb{P}(\overline{B}) \cdot \mathbb{P}(\overline{C}) = 0.4 \cdot 0.3 \cdot 0.2 = 0.024$$

b) Peter hits the target only at the third shot?

$$\mathbb{P}(\overline{A} \cap \overline{B} \cap C) = \mathbb{P}(\overline{A}) \cdot \mathbb{P}(\overline{B}) \cdot \mathbb{P}(C) = 0.4 \cdot 0.3 \cdot 0.8 = 0.096$$

c) does not hit that target at all, given that the first shot was a miss?

$\frac{0.4\cdot 0.3\cdot 0.2}{0.4} = 0.3 \cdot 0.2 = 0.06$

Independence:

$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$

A_j: the event that the jth shot hits

Definition 2 *The events $A, B \in \mathcal{A}$ are* **independent** *if*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B),$$

that is, **the probability of the intersection is the product of the probabilities***.*

The events $A_1, A_2, \ldots \in \mathcal{A}$ are **independent** *if for every $k \geq 1$ and $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ we have*

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2})\ldots\mathbb{P}(A_{i_k}).$$

$\boxed{\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)},}$

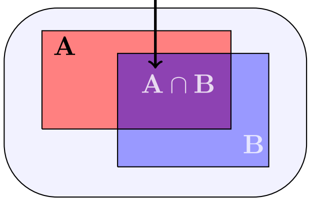
probability of the intersection

both A and B occur

probability of the condition

$\mathbb{P}(B) > 0$, we can divide by it

Conditional probability: given that B occurs, what is the probability that A also occurs



Given B, what is the conditional probability that event A occurs?

Definition 3 (Conditional probability) *Let $A, B \in \mathcal{A}$ be two events, and suppose that $\mathbb{P}(B) > 0$. The conditional probability of A with respect to B is defined as follows:*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}.$$

Notice that if A and B are independent, then we have:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$$

(4) What is the probability that maximum of the numbers that we get is equal to 5 if we throw 2 (generally n) regular dice? (Regular dice: $1, 2, 3, 4, 5, 6$ with equal probabilities.)

Set of all possible outcomes:

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

11	12	13	14	**15**	16
21	22	23	24	**25**	26
31	32	33	34	**35**	36
41	42	43	44	**45**	46
51	**52**	**53**	**54**	**55**	56
61	62	63	64	65	66

$$\mathbb{P}(\text{the maximum is 5}) = \frac{9}{36} = \frac{1}{4}.$$

It is important here that due to symmetry, all possible outcomes have the same probability.

For n dice:

$$\mathbb{P}(\text{the maximum is at most 5}) = \frac{5^n}{6^n} = \left(\frac{5}{6}\right)^n.$$

as the rolls are independent. In general:

$$\mathbb{P}(\text{the maximum is 5}) = \mathbb{P}(\text{the maximum is at most 5}) - \mathbb{P}(\text{the maximum is at most 4}) =$$
$$= \left(\frac{5}{6}\right)^n - \left(\frac{4}{6}\right)^n.$$

Definition 4 *The triple* $(\Omega, \mathcal{A}, \mathbb{P})$ *is a* **probability field**, *if*

- *the sample set* Ω *is a non-empty set;*
- $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ *(where* $\mathcal{P}(\Omega)$ *is the set of all subsets of* Ω*). is the* **set of events** *(or* σ*-algebra of events), that is, for all* $A \in \mathcal{A}$ *we have* $A \subseteq \Omega$ *such that*
  (i) $\Omega \in \mathcal{A}$*;*
  (ii) if $A_1, A_2, \ldots \in \mathcal{A}$*, then* $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ *(that is, the union of countably many sets in* $\mathcal{A}$ *is also in* $\mathcal{A}$*);*
  (iii) if $A \in \mathcal{A}$*, then* $\Omega \setminus A \in \mathcal{A}$ *(that is, the complement of sets in* $\mathcal{A}$ *is also in* $\mathcal{A}$*.*
- **probability** $\mathbb{P} : \mathcal{A} \to [0.1]$ *is a function such that*
  (i) $\mathbb{P}(\Omega) = 1$*. that is. the probability of the whole sample set is* 1*;*
  (ii) if $A_1, A_2, \ldots \in \mathcal{A}$ *and for all* $1 \leq i < j$ *we have* $A_i \cap A_j = \emptyset$*, then*

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

That is, the probability of the union of countably many pairwise disjoint sets is the sum of the probabilities.

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(5) Odesseus arrives at a junction while wandering on the road. One leads to Athens, one to Sparta and one to Mycenae. The athenians tell the truth every third time when asked, the mycanaeans every second time, while the spartans never lie. Odesseus does not know which way leads to which city so he chooses randomly. Arriving in the city he asks a person the question: How much is 2×2. The answer is 4. What is the probability that he is in Athens?

T: the answer is true; A: Athens; M: Mycenae. S: Sparta

By using Bayes's theorem for the partition $A.M.S$ and the event T we obtain that

$$\mathbb{P}(A|T) = \frac{\mathbb{P}(A \cap T)}{\mathbb{P}(T)} = \frac{\mathbb{P}(A)\mathbb{P}(T|A)}{\mathbb{P}(T)} = \frac{\frac{1}{3} \cdot \frac{1}{3}}{\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1} = 0.182 < \mathbb{P}(A) = \frac{1}{3}.$$

We can also see that A and T are not independent.

(6) There is a disease which affects 2% of the population. There is a blood test which is positive with probability 95% for people who have the disease, and it is positive with probability 1% for healthy people.

(a) What is the probability that Peter's test will be positive? (Peter is a randomly chosen person.)

(b) Given that the test of Peter is positive, what is the conditional probability that he has the disease?

(c) Now suppose that the test was repeated k times independently for a randomly chosen person, and all results were positive. What is the conditional probability that this person has the disease? Calculate this probability for $k = 2$ and $k = 3$

Let A be the event that the test is positive. and B the event that Peter has the disease. Then B and its complement $\overline{B}$ is a partition of the sample space, such that both probabilities are positive. By the law of total probability we obtain

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap \overline{B}) = \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|\overline{B})\mathbb{P}(\overline{B}) = 0.95 \cdot 0.02 + 0.01 \cdot 0.98 = 0.028.$$

Theorem 5 (Law of total probability) *Let* $A \in \mathcal{A}$ *be an event and* $B_1, B_2, \ldots, B_n$ *a partition of the sample space. Then we have*

$$\mathbb{P}(A) = \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) + \mathbb{P}(A|B_3)\mathbb{P}(B_3) + \ldots + \mathbb{P}(A|B_n)\mathbb{P}(B_n) = \sum_{j=1}^{n} \mathbb{P}(A|B_j)\mathbb{P}(B_j).$$

Theorem 6 (Bayes' theorem) *Let* $A \in \mathcal{A}$ *be an event such that* $\mathbb{P}(A) > 0$*,* $B_1, B_2, \ldots, B_n$ *a partition of the sample space. Then for all* $k = 1, 2, \ldots .n$ *we have*

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) + \mathbb{P}(A|B_3)\mathbb{P}(B_3) + \ldots + \mathbb{P}(A|B_n)\mathbb{P}(B_n)} =$$
$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\sum_{j=1}^{n} \mathbb{P}(A|B_j)\mathbb{P}(B_j)}.$$

Remark. If we repeat the test k times:

$$\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|\overline{B})\mathbb{P}(\overline{B}) = 0.95^k \cdot 0.02 + 0.01^k \cdot 0.98.$$

The values: 2.8% for $k = 1$, 1.8% for $k = 2$ and 1.7% for $k = 3$.

(b) We apply Bayes' theorem with the above notation. The probability of A is also positive. Hence

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|\overline{B})\mathbb{P}(\overline{B})} = \frac{0.95 \cdot 0.02}{0.95 \cdot 0.02 + 0.01 \cdot 0.98} = 66\%.$$

(c) Similarly, by using independence of the tests:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|\overline{B})\mathbb{P}(\overline{B})} = \frac{0.95^k \cdot 0.02}{0.95^k \cdot 0.02 + 0.01^k \cdot 0.98}.$$

For $k = 1$, we have 66%, for $k = 2$ we have 99.5%, for $k = 3$ we have 99.99%.

(7) (*Monty Hall problem*) In a television show there are 3 doors, such that there is a present behind one of them, and there is nothing behind the other two. The participant of the game chooses one of the doors. Then, the presenter opens one of the other doors, and shows that there is nothing behind it. Then he asks the participant whether he would like to stay with his original choice, or he would like to chose the other door which has not been opened yet. Is it worth changing?

`https://randomservices.org/random/apps/MontyHall.html`

It is worth changing.

First strategy: the player does not change his initial choice. He wins if and only if he has chosen the door with the present in the beginning. which has probability 1/3.

Second strategy: the player changes his initial choice. Notice that if he had chosen an empty door in the beginning, then he would win for sure: among the two other doors, the empty was chosen. and he has chosen the other one. In the other case, if he had chosen the door with the present in the beginning, then he will find an empty door after changing.

Let A be the event that the player wins. B the event that he had chosen the door with the present in the beginning. Then $\{B.\overline{B}\}$ is a partition of the sample space, with both events having positive probability. We apply the law of total probability:

$$\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|\overline{B})\mathbb{P}(\overline{B}) = 0 \cdot 1/3 + 1 \cdot 2/3 = 2/3 > 1/3.$$

(8) Suppose that everyone has his or her birthday on a uniformly randomly chosen day among the 365 days of the year (we omit 29 February for sake of simplicity), independently. What is the probability that, among n people, there are at least two who have their birthday on the same day? Calculate this probability for $n = 5, 25, 100$.

`https://randomservices.org/random/apps/Birthday.html`

Let us first fix $n = 5$. The number of possible outcomes (a possible outcome: the first person was born on 23 November, the second person was born on 31 March, etc.) is 365^5 (every day is possible for all five people, independently of each other), and, due to the conditions. all outcomes have the same probability.

The number of "bad" outcomes, that is, when all five people were born on different days, is the following:

$$365 \cdot 364 \cdot 363 \cdot 362 \cdot 361.$$

The first person can be born on any of the days. Given this date, the number of possibilities is one less for the second person, and it is 364 for all dates chosen for the first person, so we can multiply the two values. This argument can be continued.

Therefore the probability that there are two people with the same birthday is given by

$$1 - \frac{365 \cdot 364 \cdot 363 \cdot 362 \cdot 361}{365^5} = 2.7\%.$$

With a very similar argument, for n people we have

$$1 - \frac{365 \cdot 364 \cdot 363 \cdot \ldots \cdot (366 - n)}{365^n}.$$

For $n = 25$, this probability is equal to 56.9%. For $n = 100$, this is equal to 99.99997%

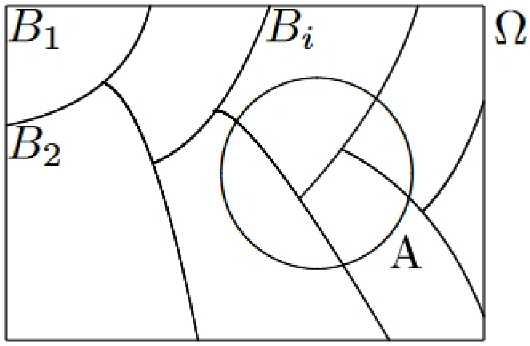

Figure 1: A partition of the probability space and an event A

(9) Mary is collecting Kinder surprise figurines. There are 10 types of them, and, in each Kinder surprise egg, independently of the others, all of the 10 types are hidden with the same probability. What is the probability that the 20th Kinder surprise egg contains the last type of figurine?

Let B be the event that the first 20 Kinder eggs contain all types. Let A_j be the event that type j is missing from the first 20 eggs. Then, by the inclusion-exclusion formula we have

$$\begin{aligned}\mathbb{P}(B) = 1 - \mathbb{P}\Big(\cup_{j=1}^{10} A_j\Big) &= 1 - (\mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots + \mathbb{P}(A_{10}) - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) \\ &- \ldots - \mathbb{P}(A_2 \cap A_3) - \mathbb{P}(A_2 \cap A_4) - \ldots - \mathbb{P}(A_{n-1} \cap A_n) + \mathbb{P}(A_1 \cap A_2 \cap A_3) + \ldots) \\ &= 1 - 10 \cdot \left(\frac{9}{10}\right)^{20} + \binom{10}{2} \cdot \left(\frac{8}{10}\right)^{20} - \binom{10}{3} \cdot \left(\frac{7}{10}\right)^{20} + \ldots.\end{aligned}$$

Similarly, let C be the event that the first 19 Kinder eggs contain all types. The probability of C can be calculated very similarly to the probability of B. In addition, we are interested in the probability of $B \setminus C$. Therefore the answer is

$$\begin{aligned}\mathbb{P}(B) - \mathbb{P}(C) = 1 - 10 \cdot \left(\frac{9}{10}\right)^{20} + \binom{10}{2} \cdot \left(\frac{8}{10}\right)^{20} - \binom{10}{3} \cdot \left(\frac{7}{10}\right)^{20} + \ldots \\ - \left(1 - 10 \cdot \left(\frac{9}{10}\right)^{19} + \binom{10}{2} \cdot \left(\frac{8}{10}\right)^{19} - \binom{10}{3} \cdot \left(\frac{7}{10}\right)^{19} + \ldots\right)\end{aligned}$$

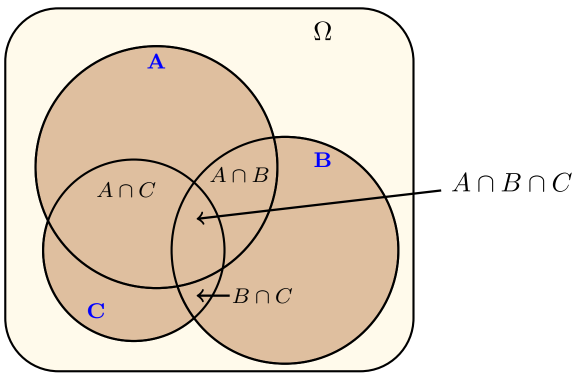

Figure 2: Inclusion-exclusion formula for three events

Proposition 7 *(a)* **Inclusion-exclusion formula for two events.** *The probability that at least one of A and B occurs:*

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

(b) **Inclusion-exclusion formula for three events.** *The probability that at least one of A and B occurs:*

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \\ - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)\end{aligned}$$

(c) **Inclusion-exclusion formula in general***: The probability that at least one of $A_1, A_2, \dots, A_n$ occurs:*

$$\begin{aligned}\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) &= \mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots + \mathbb{P}(A_n) - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - \\ &- \ldots - \mathbb{P}(A_2 \cap A_3) - \mathbb{P}(A_2 \cap A_4) - \ldots - \mathbb{P}(A_{n-1} \cap A_n) + \mathbb{P}(A_1 \cap A_2 \cap A_3) + \ldots \\ &= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}).\end{aligned}$$