Clarification with definition of event space and some basic set theory

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Recall a sample space Ω is simply a set, where each element $\omega \in \Omega$ is an outcome to some random process, or as I phrased it today, an outcome of an (random) experiment. We would like to define a so-called event space(or more precisely, a σ-algebra on the space Ω) $\mathcal{F}$ which should contain all the "events" we would be interested in. The purpose of this exposition is to give some examples of σ-algebras(this name is also in your course notes and is the one you should always use!) and hopefully motivate you to measure theory(which as the name suggests is the study of how to measure objects in the mathematical sense).

We first want to recall what it means for a set A to be a subset of another set Ω(for the interest of this exposition, it would be our sample space), written as $A \subset \Omega$. Some of you have confused the difference between set inclusion $\subset$ and element inclusion $\in$. A hopefully really simple example is as follows:

Example 1. *Consider the set of all natural numbers* $\mathbb{N} = \{1, 2, 3, ...\}$*. A subset would be some finite or countable collection of natural numbers. For instance, define* $A_1 := \{1, 2, 3\}, A_2 := \{56, 900, 21\}$*. Clearly, each element of* A_1 *and* A_2 *is a natural number, thus we can write* $A_1 \subset \mathbb{N}$ *and* $A_2 \subset \mathbb{N}$*. We also want to be able to talk about an element of a set. Continuing with the example of natural numbers,* 1 *is a natural number, so is* 196883*. Thus they are both elements of the set of natural numbers, which we write as* $1 \in \mathbb{N}$ *and* $196883 \in \mathbb{N}$*. Note the seemingly identical statement* $\{1\} \subset \mathbb{N}$ *is actually slightly different: the curly brackets around* $\{1\}$ *indicates that it is a set of one element(i.e. a singleton), which is different to an element in the ambient space* $\mathbb{N}$*. Thus the statement means in words: the set* $\{1\}$*, containing of one element 1, is a subset of the natural numbers* $\mathbb{N}$*. For completeness we also talk about the empty set* $\emptyset$ *as a subset of any set* Ω*, written as* $\emptyset \subset \Omega$*. Hopefully this makes sense after the next section.*

In the context of probability, we are interested in "events" from a random experiment. Usually we are interested in the chance of an event occurring, in other words, the probability of an event. So we would want our event to consist of more than one outcome. But we might want to describe the collection of all outcomes first: this can be done by defining an ambient sample space. After the next example it should become clear why we want the σ-algebra(collection of events) to be some collection of subsets of the sample space, or in other words, a subset of the power set of our sample space.

Example 2. *Consider a game of rolling dice against an octopus: whomever rolls a greater number wins. We are interested in describing the event* $A := \{\text{The octopus wins}\}$*. An outcome of this game should be a pair of two numbers where each number is 1, 2, 3, 4, 5, or 6. In set-theoretic language: the sample space is given by* $\Omega := \{(a_1, a_2) | a_i = 1, 2, 3, 4, 5, 6 \text{ for } i = 1, 2\} = \{1, 2, 3, 4, 5, 6\}^2$*, where* a_1 *is the number you roll and* a_2 *is the number the octopus rolls. Now what would be an outcome where the octopus wins look like? If the octopus rolls 3 and you roll 1, the octopus clearly wins, in which case the outcome can be written as the pair* $(1, 3)$*. But in the same way if you roll 2 instead of 1 and the octopus still rolls 3, we would still have the octopus winning. However, this outcome would be the pair* $(2, 3)$*, where it should be hopefully clear* $(2, 3) \neq (1, 3)$*. We then have* $(2, 3) \in A$ *and* $(1, 3) \in A$*, which shows that an event consisting of more than one outcome is natural.*

Thus considering the collections of singletons only is not enough for our σ-algebra to be useful. Recall we can identify the set operations intersection $\cap$ and union $\cup$ with the logical symbols "and" $\wedge$ and "or" $\vee$. So it is natural to include unions and intersections of sets in our σ-algebra to translate a wordy description of an event into a purely set-theoretic formulation using the events we know. It is also natural to consider countable unions and intersections of sets for some events which can be hopefully demonstrated in the following example.

Example 3. *Continuing with the setting in the previous example: the dice-rolling game against the octopus, but repeated for infinitely(countably) many times. Then an outcome becomes an infinite sequence which is a bit tedious to write down. We are now interested in the event* $A := \{\text{the octopus wins at least once}\}$*. From the description in words, if we set* $\forall n \in \mathbb{N}$, $A_n := \{\text{the octopus wins exactly } n \text{ times}\}$*, we can write* $A = \bigcup_{n \in \mathbb{N}} A_n$*.*

Another type event we might be interested in is the negation of some event we know. This can be done using the operation of taking complements. Note that by using De Morgan's law, we can think of taking unions and intersections as almost the same thing:

$$A = \bigcup_{n \in \mathbb{N}} A_n = ((\bigcup_{n \in \mathbb{N}} A_n)^c)^c = (\bigcap_{n \in \mathbb{N}} A_n^c) \tag{1}$$

A good exercise would be to use cook up some examples of sets we might be interested in by using De Morgan's law and complements from the same setting as in Example 3(if you are not comfortable with a countable sequence of games, do it in the finite setting first).

We are now in a good position to define the σ-algebra of a given sample space:

Definition 1. *Given sample space* Ω*, a* σ*-algebra of* Ω *is a collection* $\mathcal{F}$ *of subsets of* Ω *such that:*

1. $\emptyset \in \mathcal{F}$ *and* $\Omega \in \mathcal{F}$*;*

2. $A \in \mathcal{F} \Rightarrow A^c = \Omega \setminus A \in \mathcal{F}$*;*

3. if $\{(A_n\}_{n \in \mathbb{N}}$ *is a countable collection such that* $A_n \in \mathcal{F}$ *then* $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}$*.*

Note in this definition, the statement $A \in \mathcal{F}$ means that A, despite being a subset of the sample space Ω, is an element of the σ-algebra $\mathcal{F}$. In particular, we have $\mathcal{F} \subset \mathcal{P}(\Omega)$, where $\mathcal{P}(\Omega) = \{A | A \subset \Omega\}$ is the power set of Ω. I will state some equivalent definitions below whose proof of equivalence is left as an exercise

Exercise 1. *Show that we can replace condition 3 from the definition of a* σ*-algebra with countable intersection. More precisely, if* $\mathcal{F}$ *is a* σ*-algebra(as defined in definition 1) and* $A_n \in \mathcal{F} \forall n \in \mathbb{N}$*, then* $\bigcap_{n \in \mathbb{N}} A_n \in \mathcal{F}$*.*

Exercise 2. *Show that we can equivalently define a* σ*-algebra as a collection of subsets which is closed under taking complements and countable union, where the collection contains the empty set* $\emptyset$*(this is motto for a* σ*-algebra).*

We will give some rather concrete examples below(it's good practice to check these actually define σ-algebras):

1. Borel σ-algebra $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$ containing all the half-open intervals(we have talked about this in today's tutorial);
2. power set $\mathcal{P}(\Omega)$ always defines a σ-algebra;
3. consider a subset $A \subset \Omega$, then $\{\emptyset, A, A^c, \Omega\}$ defines a σ-algebra on Ω;
4. $\{\emptyset, \Omega\}$ defines the trivial σ-algebra on Ω;

We introduce some non-examples in the form of exercises:

Exercise 3. *Show that the collection of all intervals* $\{I \subset \mathbb{R} | I = (a, b),$ *or* $I = [x, y]$ *or* $I = (a, y]$ *or* $I = [x, b), x, y \in \mathbb{R}, a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R} \cup \{+\infty\}\}$ *does not define a* σ*-algebra.* ***Hint:*** *Show that this collection is not closed under unions.*

Exercise 4. *Show that the collection* $\{(a, \infty) | a \in \mathbb{R} \cup \{-\infty\}\}$ *does not define a* σ*-algebra.* ***Hint:*** *Show that this collection is not closed under countable unions.*

Exercise 5. *Show that any arbitrary intersection(finite, countable or even uncountable) of* σ*-algebras defines another* σ*-algebra.*

Another good exercise is contained in your 2nd assignment to show that in general unions of σ-algebras are not σ-algebra. A hint to show this would be to consider 2 σ-algebras $\mathcal{F}_1 := \{\emptyset, A, A^c, \Omega\}$ and $\mathcal{F}_2 := \{\emptyset, B, B^c, \Omega\}$ for some $A, B \subset \Omega$. Then show that some union or intersection of sets among the 8 sets above(they are all subsets of the same sample space Ω, so set operations make sense) is not included in the union of σ-algebras $\mathcal{F}_1 \cup \mathcal{F}_2$. Details are left as an exercise(you will need to use this kind of argument for the assignment).