Varifold-theoretic approach to minimal surface theory

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Abstract

This expository note summarises some aspects in the varifold-theoretic methods for problems arising in classical minimal surface theory. The notion of (rectifiable) varifolds, is introduced as the weaker notion of smooth submanifolds, which due to its measure-theoretic nature, is extremely versatile in terms of good compactness properties etc. I have also tried to include some hints towards the use of homothety maps in blow-up analysis of singularities on varifolds. A main focus was to derive the first order variation of the volume functional and discuss the monotonicity statements for varifolds as a generalisation of minimal submanifold theory. The results are then used to present some of the materials from the work in [EWW04], which concerns the embeddedness of classical minimal immersions with a bound on the total curvature of its boundary. It would certainly be nice to have some pictures, illustrating some of the steps in the longer proofs. This might be updated in the future. The note is primarily based on the materials I have presented to Aris Mercier, who has studied the work by Ekholm-White-Wienholtz together and provided me with valuable feedbacks. All mistakes and typos would certainly be my own.

1 Constructions of varifolds as generalised manifolds

Roughly speaking, varifold is a generalised variational notion of smooth manifolds, which allows the existence of singularities. Varifolds were first introduced by Almgren in solving the Plateau's problem for a more general class of surfaces, which has good compactness and convergence properties. The general construction of a varifold involves some results from basic geometric measure theory and a couple of algebraic preliminaries:

Definition 1.1 (Grassmannian algebra)**.** *Let V be a n-dimensional $\mathbb{K}$-vector space. For $k = 1, \ldots, n$, define the k-Grassmannian $\mathbb{G}(k;V)$ of V to be the space of all k-dimensional unoriented vector subspaces of V. $\mathbb{G}(k;V)$ can be equipped with the metric d defined via:*

$$\mathrm{d}\colon \mathbb{G}(k;V)^2 \to \mathbb{R}_+, \quad (W,P) \longmapsto \mathrm{d}(W,P) := \|\pi_W - \pi_P\|_F,$$

where $\pi_W\colon V \to W$ and $\pi_P\colon V \to P$ denote the orthogonal projections onto the subspaces $W, P \subset V$ respectively, and $\|\cdot\|_F$ denotes the Frobenius norm.
Let $\Omega \subset V$. We will sometimes denote $\mathbf{G}_k(\Omega;V) = \Omega \times \mathbb{G}(k;V)$, which can be equipped with the product metric. For simplicity, we write $\mathbb{G}(k;\mathbb{R}^n) = \mathbb{G}(k;n)$ and $\mathbf{G}_k(\Omega;\mathbb{R}^n) = \mathbf{G}_k(\Omega)$. The metric space structure induces a topology on $\mathbf{G}_k(\Omega;V)$, thus the Borel σ-algebra $\mathcal{B}(\Omega \times \mathbb{B}(k,V))$ is well-defined.

Remark 1.1. In fact, $\mathbb{G}(k;V)$ can be given the structure of a smooth manifold, which we will not discuss further here. We will however remark a consequence of this fact, that $\mathbb{G}(k;V)$ is locally compact. In particular, $\mathbf{G}_k(\Omega;V)$ is a locally compact metric space and thus Riesz representation theorem applies to give us the following identification of Radon measures via duality: $\mathcal{M}(\mathbf{G}_k(\Omega;V);\mathbb{R}) = (C_0(\mathbf{G}_k(\Omega;V)))^*$.

Definition 1.2 (Rectifiability of sets and measures)**.** *Let $\Omega \in \mathcal{B}(\mathbb{R}^n)$ and $\mu \in \mathcal{M}(\mathbb{R}^n;\mathbb{R}^m)$ a Radon measure. Then we say:*

(i) Ω is countably $\mathcal{H}^k$-rectifiable if Ω is $\mathcal{H}^k$-measurable and there exists Lipschitz functions $f_i\colon \mathbb{R}^k \to \mathbb{R}^n$ such that $E \subset \bigcup_{i=1}^\infty f_i(\mathbb{R}^k)$ upto a $\mathcal{H}^k$-negligible set;

(ii) μ is countably k-rectifiable if there exist a countably $\mathcal{H}^k$-rectifiable set $S \subset \mathbb{R}^n$ and a Borel function $\vartheta\colon S \to \mathbb{R}^m$ such that $\mu = \vartheta\mathcal{H}^k|_S$.

We will also need the notion of rectifiability in a general (locally compact) metric space, which we state below:

Definition 1.3 (Rectifiable sets in metric spaces, [Fed14])**.** *Let (X,d) be a metric space and $E \subset X$ an arbitrary subset. Given a fixed $k \in \mathbb{N}$, we then say that:*

(i) E is k-rectifiable in X if there exists a bounded subset $K \subset \mathbb{R}^k$ and a Lipschitz map $f\colon K \subset \mathbb{R}^k \to X$ with $f(K) = E$ upto $\mathcal{H}^k|_X$-a.e. equality;

(ii) *E is countably k-rectifiable in X if there exists a countably family of k-rectifiable sets $(E_i \mid i \in \mathbb{N})$ such that $E = \bigcup_{i=1}^{\infty} E_i$ upto $\mathcal{H}^k|_X$-a.e. equality;*

(iii) *E is countably (ν, k)-rectifiable in X if $\nu \in \mathcal{M}^+(X)$ and there exists countably k-rectifiable set $E' \subset X$ such that $E = E'$ ν-a.e.*

Let $\nu \in \mathcal{M}^+(X)$. A countably (ν, k)-rectifiable set E is (ν, k)-rectifiable in X if $\nu E < \infty$.

We introduce the rescaling of a Radon measure $\mu \in \mathcal{M}(\mathbb{R}^n; \mathbb{R}^m)$ around $x \in \mathbb{R}^d$ by a factor of $\rho > 0$, denoted by $\mu_{x,\rho} B := \mu(x + \rho B)$ for any $B \in \mathcal{B}_\mu(\mathbb{R}^n)$. This is commonly used for discussing tangent measures.

Definition 1.4 (Approximate tangent space)**.** *Let $M \subset \mathbb{R}^n$ be $\mathcal{H}^k$-measurable such that $\mathcal{H}^k|_M$ defines a Radon measure. We say M admits an approximate tangent space $T_x^k M \in \mathbb{G}(k; n)$ of multiplicity $\theta(x) \in (0, \infty)$ at the point $x \in M$ if $\rho^{-k}(\mathcal{H}^k|_M)_{x,\rho} \overset{*}{\rightharpoonup} \vartheta(x)\mathcal{H}^k|_{T_x^k M}$ locally in $\mathbb{R}^n$ as $\rho \searrow 0$.*
Similarly, let $\mu \in \mathcal{M}(\mathbb{R}^n; \mathbb{R}^m)$ be a vector-valued Radon measure. A k-dimensional subspace $\pi \in \mathbb{G}(k; n)$ is the approximate tangent space of multiplicity $\vartheta(x) \in (0, \infty)$ to μ at a point $x \in \mathbb{R}^n$, for which we write as $\operatorname{Tan}^k(\mu, x) = \vartheta(x)\mathcal{H}^k|_\pi$, if $\rho^{-k}\mu_{x,\rho} \overset{*}{\rightharpoonup} \vartheta(x)\mathcal{H}^k|_\pi$ as $\rho \searrow 0$.

Remark 1.2. If M is a $\mathcal{H}^k$-measurable set with $\mathcal{H}^k(M \cap K) < \infty$ for any $K \subset \mathbb{R}^n$ compact, then M is countably k-rectifiable if and only if it admits an approximate tangent space $T_x^k M$ for $\mathcal{H}^k$-a.e. $x \in M$ of multiplicity $\vartheta(x) = 1$. The approximate tangent space to a Radon measure is a local property, i.e. if $\mu_i = \vartheta_i \mathcal{H}^k|_{S_i}$ are rectifiable measures with $\operatorname{Tan}^k(\mu, x) = \vartheta_i \mathcal{H}^k|_{\pi_i}$, then $\pi_1(x) = \pi_2(x)$ for $\mathcal{H}^k$-a.e. $x \in S_1 \cap S_2$.

With the above concepts, we can now define the notion of a k-varifold and rectifiability of varifolds:

Definition 1.5 (k-dimensional varifold)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open subset. A k-varifold in Ω is simply a positive Radon measure $V \in \mathcal{M}^+(\Omega \times \mathbb{G}(k; n)) = \mathcal{M}^+(\mathbf{G}_k(\Omega))$. The weight corresponding to the k-varifold V is a μ_V defined on Ω via $\mu_V A := V(\pi^{-1}(A))$ for any $A \subset \Omega$, where $\pi \colon \mathbf{G}_k(\Omega) \to \Omega$ is the projection onto Ω. We define the mass $\mathbb{M}(V)$ of a k-varifold V in Ω to be the weight of Ω, i.e. $\mathbb{M}(V) := \mu_V \Omega = V(\mathbf{G}_k(\Omega))$. For $k \in \mathbb{N}$, we denote the space of integral k-varifolds in the open set $\Omega \subset \mathbb{R}^n$ by $IV_k(\Omega)$.*

When it comes to define the notion of a rectifiable varifold, there are two equivalent formulations.

Definition 1.6 (Rectiable varifolds as equivalent classes)**.** *Let $M \subset \mathbb{R}^n$ be a countably k-rectifiable set and $\vartheta \colon M \to [0, \infty)$ locally $\mathcal{H}^k$-integrable. Consider the equivalence relation $\sim$ on the collection of such pairs (M, ϑ), where we say $(M, \vartheta) \sim (M', \vartheta')$ if $\mathcal{H}^k(M \Delta M') = 0$ and $\vartheta = \vartheta'$ $\mathcal{H}^k$-a.e. on M. Then a rectifiable k-varifold V is the $\sim$ equivalence class $V = \mathbf{v}(M, \vartheta)$, where we say ϑ is the multiplicity of V.*

A generalisation of Remark 1.2 allows us to discuss the tangents to rectifiable varifolds.

Theorem 1.1 (Existence of approximate tangent spaces)**.** *Let $M \subset \mathbb{R}^n$ be a $\mathcal{H}^k$-measurable set and the multiplicity function $\vartheta \colon M \to [0, \infty)$ a $\mathcal{H}^k$-measurable function such that for any $K \subset \mathbb{R}^n$ compact:*

$$\int \chi_{M \cap K}(x)\vartheta(x) d\mathcal{H}^k(x) < \infty.$$

Then M is countably k-rectifiable if and only if it admits an approximate tangent space $T_x^k M \in \mathbb{G}(k, n)$ with multiplicity given by $\vartheta(x)$ at $\mathcal{H}^k$-a.e. $x \in M$. We denote by $M_* = \{x \in M \mid T_x^k M \text{ exists}\}$.

Definition 1.7 (Rectifiable varifolds as measures)**.** *Let $\Omega \subset \mathbb{R}^n$ be open. Then a rectifiable k-varifold in Ω is a Radon measure $V = \vartheta \mathcal{H}^k|_M \otimes \delta_{T_x^k M}$ on $\mathbf{G}_k(\Omega) = \Omega \times \mathbb{G}(k, n)$, i.e. we define via duality:*

$$\int_{\Omega \times \mathbb{G}(k,n)} \varphi(x, T) dV(x, T) = \int_M \varphi(x, T_x^k M)\vartheta(x) d\mathcal{H}^k(x) \quad \text{for all } \varphi \in C_0(\mathbf{G}_k(\Omega); \mathbb{R}),$$

where $M \subset \Omega$ is a k-rectifiable subset and $\vartheta \colon M \to [0, \infty]$ is a non-negative locally $\mathcal{H}^k$-integrable function.
For Borel subsets $B \in \mathcal{B}(\mathbf{G}_k(\Omega)) = \mathcal{B}(\Omega \otimes \mathbb{G}(k, n))$, we can define the measure of B via:

$$V(B) := \int_{\pi_M(TM \cap B)} \vartheta(x) d\mathcal{H}^k(x), \quad \text{where } TM := \{(x, T_x^k M) \mid x \in M_*\}.$$

Remark 1.3. We can also define the notion of k-varifolds in a given Riemannian manifold (M^n, g). In this case, the Nash-Kuiper embedding theorem allows us to embed M^n isometrically into some Euclidean space $\mathbb{R}^N$. Thus the formulation in fact coincides with k-varifolds initially defined on a Euclidean space.

Let $V = \mathbf{v}(M, \vartheta)$ be a rectifiable k-varifold in the sense of Definition 1.6. There exists an associated k-varifold $V \in \mathcal{M}(\mathbf{G}_k(M))$, which is given by the analogous construction in Definition 1.7: for $B \in \mathcal{B}(\mathbf{G}_k(M))$

$$V(B) := \vartheta\mathcal{H}^k|_M(\pi_M(TM \cap B)) = \int_{M\times\mathbb{G}(k;n)} \chi_B(x,T)\vartheta(x)d(\mathcal{H}^k \otimes \delta_{T_x^k M})(x,T).$$

Indeed, since ϑ is locally $\mathcal{H}^k$-integrable on M, we deduce V is a Radon measure on $\mathbf{G}_k(M)$ from the regularity of the Hausdorff measure. Furthermore, by using the Carathéodory-Hahn criterion, we can deduce the second equality above, i.e. $V = \vartheta\mathcal{H}^k|_M \otimes \delta_{T_x^k M}$. In particular $\mathbf{v}(M,\vartheta)$ corresponds to a rectifiable k-varifold V in the sense of Definition 1.7. Conversely, it is easy to see that a rectifiable k-varifold $V = \vartheta\mathcal{H}^k|_M \otimes \delta_{T_x^k M}$ in the view of Definition 1.7 induces a representative pair (M,ϑ) for a rectifiable k-varifold $\mathbf{v}(M,\vartheta)$.
The above allows us to use the two definitions interchangeably, whereas we will primarily use Definition 1.6 for the discussion on the monotonicity formula. The equivalence in fact gives us hint towards the question of the rectifiability of a given k-varifold, which we try to answer in the remainder of this section. We will first establish some differentiation theory for Radon measures and a technical lemma.

Definition 1.8 (Density functions)**.** *Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ and $\nu \in \mathcal{M}(\mathbb{R}^n;\mathbb{R})$. We define the lower and upper density of ν with respect to μ at each $x \in \operatorname{supp}\mu$ via:*

$$D_\mu^- \nu(x) := \liminf_{\rho\downarrow 0} \frac{\nu\mathbb{B}(x,\rho)}{\mu\mathbb{B}(x,\rho)} \quad \textit{and} \quad D_\mu^+ \nu(x) := \limsup_{\rho\downarrow 0} \frac{\nu\mathbb{B}(x,\rho)}{\mu\mathbb{B}(x,\rho)}.$$

If the lower and upper densities at $x \in \operatorname{supp}\mu$ coincide, then we say that blow-up quotient $\frac{d\nu}{d\mu}$ exists at x.

Lemma 1.1 (Density lemma)**.** *Let $\mu,\nu \in \mathcal{M}(\mathbb{R}^n;\mathbb{R})$ be Radon measures and $E \subset \operatorname{supp}\mu$ a Borel subset. Then $\{D_\mu^+\nu = \infty\}$ is μ-negligible and moreover for any $t \in \mathbb{R}$, we have:*

(i) if $D_\mu^-\nu \leqslant t$ on E, then $\nu E \leqslant t\mu E$;

(ii) if $D_\mu^+\nu \geqslant t$ on E, then $\nu E \geqslant t\mu E$.

Proof. (i) Assume without loss of generality that E is bounded. For any open neighbourhood $U \supset E$ and $\varepsilon > 0$, we want to apply the Besicovitch-Vitali theorem on the following collection of closed balls:

$$\mathcal{F}_\varepsilon := \{\overline{\mathbb{B}}(x,\rho) \mid x \in E,\, \mathbb{B}(x,\rho) \subset U,\, \nu\overline{\mathbb{B}}(x,\rho) \leqslant (t+\varepsilon)\mu\overline{\mathbb{B}}(x,\rho)\}.$$

Indeed, suppose $D_\mu^-\nu(x) = \liminf_{\rho\downarrow 0} \frac{\nu\mathbb{B}(x,\rho)}{\mu\mathbb{B}(x,\rho)} \leqslant t$ for all $x \in E$. Then for any $\rho > 0$ sufficiently small, we must have for all $x \in E$, $\nu\overline{\mathbb{B}}(x,\rho) \leqslant (t+\varepsilon)\mu\overline{\mathbb{B}}(x,\rho)$. In particular $\mathcal{F}_\varepsilon$ defines a fine cover for E. Therefore, the Besicovitch-Vitali theorem applies and there is an at most countable, disjoint subfamily $\mathcal{F}'_\varepsilon \subset \mathcal{F}_\varepsilon$ such that:

$$\nu(E \setminus \bigcup \mathcal{F}'_\varepsilon) = 0 \Rightarrow \nu E \leqslant \sum_{B\in\mathcal{F}'_\varepsilon} \nu B \leqslant (t+\varepsilon) \sum_{B\in\mathcal{F}'_\varepsilon} \mu B \leqslant (t+\varepsilon)\mu U,$$

where the last inequality follows from disjointness and $B \subset U$ for all $B \in \mathcal{F}'_\varepsilon \subset \mathcal{F}_\varepsilon$ by definition. Sending $\varepsilon \searrow 0$ allows us to conclude by outer regularity of μ on Borel sets.
(ii) Similarly, suppose now $D_\mu^-\nu(x) = \limsup_{\rho\downarrow 0} \frac{\nu\mathbb{B}(x,\rho)}{\mu\mathbb{B}(x,\rho)} \geqslant t$ for all $x \in E$. Then $\nu\overline{\mathbb{B}}(x,\rho) \geqslant (t-\varepsilon)\mu\overline{\mathbb{B}}(x,\rho)$ for all $\rho > 0$ small. We can again define the fine cover for E for any arbitrary open neighbourhood $U \supset E$:

$$\mathcal{F}^\varepsilon := \{\overline{\mathbb{B}}(x,\rho) \mid x \in E,\, \mathbb{B}(x,\rho) \subset U,\, \nu\overline{\mathbb{B}}(x,\rho) \geqslant (t-\varepsilon)\mu\overline{\mathbb{B}}(x,\rho)\},$$

from which we can find a countable, disjoint sub-collection $\mathcal{F}^\varepsilon_* \subset \mathcal{F}^\varepsilon$ satisfying:

$$\mu(E \setminus \bigcup \mathcal{F}^\varepsilon_*) = 0 \rightarrow (t-\varepsilon)\mu E \leqslant (t-\varepsilon)\sum_{B\in\mathcal{F}^\varepsilon_*} \mu B \leqslant \sum_{B\in\mathcal{F}^\varepsilon_*} \nu B = \nu U.$$

Then we can conclude as in (i). Let $E := \{D_\mu^+\nu = \infty\}$ and notice $E = \bigcap_k E_k$ where $E_k := \{D_\mu^+\nu \geqslant k\}$ for $k \in \mathbb{N}$. In particular, for each fixed k, applying (ii) to $E_k \in \mathcal{B}(\mathbb{R}^n)$ gives us $\nu E_k \geqslant t\mu E_k$ for all $t \in \mathbb{R}$. Now sending $t \to \pm\infty$ forces $\mu E_k = 0$ for all k. Thus $E \subset E_k$ is μ-negligible. □

Theorem 1.2 (Besicovitch differentiation theorem). *Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ and $\nu \in \mathcal{M}(\mathbb{R}^n; V)$ be Radon measures where V is some finite-dimensional normed vector space. Then we have:*

$$\frac{d\nu}{d\mu}(x) = \lim_{\rho \downarrow 0} \frac{\nu \mathbb{B}(x,\rho)}{\mu \mathbb{B}(x,\rho)} \text{ exists in } V \quad \mu\text{-a.e. on } \operatorname{supp} \mu.$$

In particular, we obtain the Lebesgue-Radon-Nikodym decomposition of ν:

$$\nu = \frac{d\nu}{d\mu}\mu + \nu^s, \quad \text{where } \begin{cases} \nu^s = \nu|_E \in \mathcal{M}(\mathbb{R}^n; V); \\ E := (\mathbb{R}^n \setminus \operatorname{supp} \mu) \cup \{x \in \operatorname{supp}\mu \mid \lim_{\rho \downarrow 0} \frac{|\nu|(\mathbb{B}(x,\rho))}{\mu \mathbb{B}(x,\rho)} = \infty\}. \end{cases}$$

Proof. Arguing componentwise, we may reduce to the case where ν is a signed measure. Then by the density lemma, $D_\mu^+\nu$ is finite μ-a.e. We consider the following set functions defined on $\mathcal{B}(\mathbb{R}^n)$:

$$\lambda^+(B) := \int_B D_\mu^+ \nu d\mu, \quad \lambda^-(B) := \int_B D_\mu^- \nu d\mu \quad \text{for all } B \in \mathcal{B}(\mathbb{R}^n).$$

Fix $t > 1$ and choose a Borel set $B \subset \operatorname{supp} \mu$ with $D_\mu^+\nu < \infty$ everywhere on B. Now partition our choice of B as the disjoint union of $B_k := \{x \in B \mid D_\mu^+\nu(x) \in (t^k, t^{k+1}]\} \subset \mathbb{R}^n \setminus E$ for $k \in \mathbb{Z}$. By the density lemma:

$$\lambda^+(B_k) \leq t^{k+1}\mu B_k \leqslant t\nu B_k \Rightarrow \lambda^+(B) = \sum_{k \in \mathbb{Z}} \lambda^+(B_k) \leqslant t \sum_{k \in \mathbb{Z}} \nu B_k = t(\nu|_{\mathbb{R}^\kappa \setminus E})(B),$$

which follows from the disjointness property. Sending $t \searrow 1$ gives us $\lambda^+(B) \leqslant t(\nu|_{\mathbb{R}^n \setminus E})(B)$ for any Borel subset $B \subset \mathbb{R}^n \setminus E$. Using an analogous approach, for any Borel set $B \subset \operatorname{supp} \mu$ with $D_\mu^-\nu \in (0, \infty)$ on B, we define $B_j := \{x \in B \mid D_\mu^-\nu(x) \in (t^k, t^{k+1}]\}$. We deduce again from the density lemma:

$$\nu B_k \leqslant t^{k+1}\mu B_k \Rightarrow \lambda^-(B) = \sum_{k \in \mathbb{Z}} \lambda^-(B_k) \geqslant \frac{1}{t} \sum_{k \in \mathbb{Z}} \nu B_k = \frac{1}{t}(\nu|_{\mathbb{R}^n \setminus E})(B).$$

In particular, we have $\lambda^+ \leqslant \nu|_{\mathbb{R}^n \setminus E} \leqslant \lambda^-$ on $\mathbb{R}^n \setminus E$, which allows us to conclude:

$$D_\mu^+\nu = D_\mu^-\nu = \frac{d\nu}{d\mu} \quad \mu\text{-a.e. on } \operatorname{supp} \mu \setminus E.$$

Furthermore, the decomposition of measure holds with $d\nu|_{\mathbb{R}^d \setminus E} = \frac{d\nu}{d\mu} d\mu$ and $\nu^s = \nu|_E$. ☐

Lemma 1.2. *Let $V \in \mathcal{M}^+(\mathbf{G}_k(\Omega))$ be a k-varifold on an open set $\Omega \subset \mathbb{R}^n$. Then for μ_V-a.e. $x \in \Omega$, there exists $\eta_V^x \in \mathcal{M}(\mathbb{G}(k;n))$ such that for any $\varphi \in C^0(\mathbb{G}(k;n))$:*

$$\int_{\mathbb{G}(k;n)} \varphi(W) d\eta_V^x(W) = \lim_{\rho \downarrow 0} \frac{1}{\mu_V \mathbb{B}(x,\rho)} \int_{\mathbf{G}_k(\mathbb{B}(x,\rho))} \varphi(W) dV(y, W).$$

Moreover, for any non-negative $\varphi \in C^0(\mathbb{G}(k;n))$, we have for all $B \in \mathcal{B}(\Omega)$:

$$\int_{\mathbf{G}_k(B)} \varphi(W) dV(x, W) = \int_B \int_{\mathbb{G}(k;n)} \varphi(W) d\eta_V^x(W) d\mu_V(x).$$

Proof. Recall that $C_c^0(\mathbb{G}(k;n))$ is separable. For any dense sequence $(\varphi_i) \in C_c^0(\mathbb{G}(k;n))$, set:

$$\nu_i \colon \mathcal{B}(\mathbb{R}^n) \longrightarrow [0, \infty], \quad \nu_i B := \int_{\mathbf{G}_k(B)} \varphi_i(W) dV(y, W),$$

which defines a Radon measure $\nu_i \in \mathcal{M}(\mathbb{R}^n; \mathbb{R})$. Thus by Besicovitch differentiation theorem, for all $i \in \mathbb{N}$, there exists a μ_V-negligible set $Z_i \subset \mathbb{R}^n$ such that:

$$\frac{d\nu_i}{d\mu_V}(x) := \lim_{\rho \downarrow 0} \frac{\nu_i \mathbb{B}(x,\rho)}{\mu_V \mathbb{B}(x,\rho)} \text{ exists for all } x \in \mathbb{R}^n \setminus Z_i \text{ with } \frac{d\nu_i}{d\mu_V} d\mu_V = d\nu_i,$$

by the Radon-Nikodym theorem since $\nu_i \ll \mu_V$ for all i as each φ_i is non-negative. In particular, there exists a set $E = \mathbb{R}^n \setminus (\bigcup_{i=1}^\infty Z_i)$ of full measure such that the Radon-Nikodym density $\frac{d\nu_i}{d\mu_V}$ exists on E for all $i \in \mathbb{N}$. Fix arbitrary $\varepsilon > 0$,

$\varphi \in C_c^0(\mathbb{G}(k;n))$, and $x \in E$. By uniform density, we can choose φ_i such that $\|\varphi_i - \varphi\|_\infty \leqslant \varepsilon$. In particular, we have for all $\rho \in (0,\infty)$:

$$\Big|\frac{\nu_i \mathbb{B}(x,\rho)}{\mu_V \mathbb{B}(x,\rho)} - \frac{\nu_\varphi \mathbb{B}(x,\rho)}{\mu_V \mathbb{B}(x,\rho)}\Big| \leqslant \frac{\varepsilon V(\mathbf{G}_k(\mathbb{B}(x,\rho))}{\mu_V \mathbb{B}(x,\rho)} = \varepsilon, \quad \text{where } \nu_\varphi B := \int_{\mathbf{G}_k(B)} \varphi(W) dV(y,W).$$

By the means of triangle inequality, we have the Radon-Nikodym density of ν_φ with respect to the weight measure μ_V exists at every $x \in E$ for all $\varphi \in C_c^0(\mathbb{G}(k;n))$. In particular, we have for μ_V-a.e. $x \in \mathbb{R}^n$:

$$\eta_V^x \colon C_c^0(\mathbb{G}(k;n);[0,\infty)) \longrightarrow \mathbb{R}, \quad \varphi \longmapsto \lim_{\rho \downarrow 0} \frac{\nu_\varphi \mathbb{B}(x,\rho)}{\mu_V \mathbb{B}(x,\rho)} = \frac{d\nu_\varphi}{d\mu_V}(x),$$

defines a linear functional on $C_c^0(\mathbb{G}(k;n);[0,\infty))$ satisfying the following estimates:

$$|\langle \varphi, \eta_V^x \rangle| \leqslant \|\varphi\|_{C^0(\mathbb{G}(k;n))} \frac{V(\mathbf{G}_k(\mathbb{B}(x,\rho))}{\mu_V \mathbb{B}(x,\rho)} = \|\varphi\|_{C^0(\mathbb{G}(k;n))} \quad \text{for all } \varphi \in C_c^0(\mathbb{G}(k;n)).$$

This thus defines a continuous linear functional on $C_c^0(\mathbb{G}(k;n))$, which is in fact induced by a Radon measure by the Riesz-Markov-Kakutani theorem, denoted by the same symbol $\eta_V^x \in \mathcal{M}(\mathbb{G}(k;n))$, i.e.

$$\lim_{\rho \downarrow 0} \frac{1}{\mu_V \mathbb{B}(x,\rho)} \int_{\mathbf{G}_k(\mathbb{B}(x,\rho))} \varphi(W) dV(y,W) = \frac{d\nu_\varphi}{d\mu_V}(x) = \langle \varphi, \eta_V^x \rangle = \int_{\mathbb{G}(k;n)} \varphi(W) d\eta_V^x(W),$$

for all $\varphi \in C_c^0(\mathbb{G}(k;n))$, where the equality extends to all continuous functions by the means of monotone convergence theorem. The remaining statement is a consequence of the Radon-Nikodym decomposition. ☐

We are now ready to establish the rectifiability criterion for a general k-varifold.

Theorem 1.3 (First varifold rectifiability theorem)**.** *Let $V \in \mathcal{M}(\mathbf{G}_k(U))$ be a k-varifold in the open set $\Omega \subset \mathbb{R}^n$ such that V admits an approximate tangent space with multiplicity $\vartheta(x) \in (0,\infty)$ at μ_V-a.e. $x \in \Omega$, i.e. $\operatorname{Tan}^k(V,x) = \vartheta(x)\mathcal{H}^k|_{\pi(x)}$ for some $\pi(x) \in \mathbb{G}(k;n)$. Then V is a rectifiable k-varifold with:*

$$M := \{(x,W) \in \operatorname{supp} V \mid \operatorname{Tan}^k(V,(x,W)) = \vartheta(x)\mathcal{H}^k|_V \otimes \delta_{\pi(x)}\},$$

being $\mathcal{H}^k$-measurable and countably k-rectifiable; $\vartheta \colon M \to (0,\infty)$ locally $\mathcal{H}^k$-integrable; and $V = \mathbf{v}(M,\vartheta)$.

Proof. The above assumption implies that the approximate tangent space $\operatorname{Tan}^k(\mu_V,x) = \vartheta(x)\mathcal{H}^k|_{\pi(x)}$ exists for μ_V-a.e. $x \in \Omega$. In particular, we have $\mu_V = \vartheta\mathcal{H}^k|_M$, from which we deduce M is countably k-rectifiable and $\vartheta \in L^1_{loc}(\Omega, \mathcal{B}(\Omega), \mathcal{H}^k)$. Now μ_V-a.e. $x \in M$ admits a Radon measure $\eta_V^x \in \mathcal{M}(\mathbb{G}(k;n))$ by the above technical lemma. For any $\psi \in C_c^0(\mathbb{G}(k;n))$, we necessarily have, e.g. via Lebesgues differentiation theorem:

$$\langle \psi, \eta_V^x \rangle = \lim_{\rho \downarrow 0} \frac{1}{\mu_V \mathbb{B}(x,\rho)} \int_{\mathbf{G}_k(\mathbb{B}(x,\rho))} \psi(W) dV(y,W) = \vartheta(x)\psi(T_x^k M).$$

In particular, for aribtrary Borel subset $B \in \mathcal{B}(\Omega)$, we obtain by density:

$$\int_{\mathbf{G}_k(\Omega)} f(x,W) dV(x,W) = \int_M f(x, T_x^k M) d\mu_V(x) \quad \text{for all } f \in C_c^0(\mathbf{G}_k(\Omega)).$$

By duality, we obtain $\mu_V = \vartheta\mathcal{H}^k|_M$, which effectively shows us $V = \mathbf{v}(M,\vartheta)$. ☐

Let $\Sigma^k \hookrightarrow \mathbb{R}^n$ be a smooth, properly embedded submanifold. The associated tangent bundle $T\Sigma = \bigsqcup_{x\in\Sigma} T_x\Sigma$, or more generally in the categorical sense $\pi_\Sigma \colon T\Sigma \to \Sigma$, is a subset of the Grassmannian k-bundle over Σ. There is a natural choice of Radon measure $V_\Sigma \in \mathcal{M}^+(\mathbf{G}_k(\Sigma))$, which in fact gives rise to a k-varifold in $\mathbb{R}^n$:

$$V_\Sigma \colon \mathcal{B}(\mathbf{G}_k(\Sigma)) \to [0,\infty], \quad V_\Sigma B := \operatorname{Vol}_g(\pi_\Sigma(B \cap T\Sigma)) = \mathcal{H}^k(\pi_\Sigma(B \cap T\Sigma)),$$

where Vol_g denotes the intrinsic k-dimensional volume. The weight measure μ_Σ coincides with the k-dimensional Hausdorff measure restricted to Σ with multiplicity $\vartheta \equiv 1$. In particular, the class of k-varifolds includes the collection of embedded k-submanifolds. The advantage of this generalised notion is that we can enjoy the weak compactness properties when viewing a sequence of submanifolds as Radon measures. The price to pay, however, is that the limiting measure may not exhibit geometric features a priori.
We conclude this introductory section by stating some more preliminary results in geometric measure theory where we refer to [Fed14] for a detailed treatment.

Definition 1.9 (Density of measures)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open subset and $k \in \mathbb{N}$. We define the k-dimensional upper and lower densities of $\mu \in \mathcal{M}^+(\Omega)$ at $x \in \Omega$ to be respectively:*

$$\Theta^{*k}(\mu, x) := \limsup_{\rho \downarrow 0} \frac{\mu\mathbb{B}(x,\rho)}{\omega_k \rho^k} \quad \textit{and} \quad \Theta^k_*(\mu, x) := \liminf_{\rho \downarrow 0} \frac{\mu\mathbb{B}(x,\rho)}{\omega_k \rho^k}.$$

If both the upper and lower densities exists at x, we then say the k-dimensional density of μ exists at x and define:

$$\Theta^k(\mu, x) := \lim_{\rho \downarrow 0} \frac{\mu\mathbb{B}(x,\rho)}{\omega_k \rho^k}.$$

Analogously, we can define the k-dimensional density of a Borel set $E \subset \mathbb{R}^n$ as follows:

$$\Theta^{*k}(E) = \limsup_{\rho \downarrow 0} \frac{\mathcal{H}^k(E \cap \mathbb{B}(x,\rho))}{\omega_k \rho^k} \quad \Theta^k_*(E) = \liminf_{\rho \downarrow 0} \frac{\mathcal{H}^k(E \cap \mathbb{B}(x,\rho))}{\omega_k \rho^k} \quad \Theta(E) = \lim_{\rho \downarrow 0} \frac{\mathcal{H}^k(E \cap \mathbb{B}(x,\rho))}{\omega_k \rho^k}.$$

Definition 1.10 (Tangent differential)**.** *Let $\varphi \colon \mathbb{R}^k \to \mathbb{R}^n$ be an injective Lipschitz function, $D \subset \mathbb{R}^k$ be an arbitrary $\mathcal{L}^k$-measurable subset, and denote by $E = \varphi(D) \subset \mathbb{R}^n$. Then for $\mathcal{H}^k$-a.e. $x \in E$, the approximate tangent space to E at x is given in the following parametric form $\operatorname{Tan}^k(E, x) = d\varphi_{\varphi^{-1}(x)}\mathbb{R}^k$.*
We then say a Lipschitz function $f \colon \mathbb{R}^k \to \mathbb{R}^m$ is tangentially differentiable at $x \in E$ if $f|_{x+\operatorname{Tan}^k(E,x)}$ is differentiable at x. The notion of tangential differentiability induces an (equivalence class of) linear map(s):

$$d^E f_x \colon \operatorname{Tan}^k(E, x) \longrightarrow \mathbb{R}^m,$$

which we will refer to as the tangential differential of f at x(not well-defined pointwise).

Suppose that $f \colon E \to \mathbb{R}^m$ is differentiable at $x \in E$ with E countably k-rectifiable. Then the existence of approximate tangent space at $\mathcal{H}^k$-a.e. $x \in E$ allows us to deduce the following version of Rademacher's theorem.

Theorem 1.4 (Rademacher's theorem for tangential differential)**.** *Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function and $E \subset \mathbb{R}^n$ countably k-rectifiable. Then $d^E f_x$ exists for $\mathcal{H}^k$-a.e. $x \in E$ and more precisely:*

$$d^E f_x = df_x|_{\operatorname{Tan}^k(E,x)} \quad \textit{for } \mathcal{H}^k\textit{-a.e. } x \in E.$$

In particular, we can state a more general version of the area and coarea formula:

Theorem 1.5 (Area formulas)**.** *Let $f \colon \mathbb{R}^k \to \mathbb{R}^n$ be Lipschitz. Then for any $\mathcal{L}^k$-measurable subset $\Omega \subset \mathbb{R}^k$, we have $y \mapsto \mathcal{H}^0(\Omega \cap f^{-1}(\{y\}))$ is a $\mathcal{H}^k$-measurable map in $\mathbb{R}^n$ and the following formula holds:*

$$\int_{\mathbb{R}^n} \mathcal{H}^0(\Omega \cap f^{-1}(\{y\})) d\mathcal{H}^k(y) = \int_\Omega \mathbb{J}_k df_x dx.$$

Furthermore, using a monotone class argument, we have for any Borel measurable function $h \colon \Omega \to [0, \infty]$:

$$\int_{\mathbb{R}^n} \int_{\Omega \cap f^{-1}(\{y\})} h(x) d\mathcal{H}^0(x) d\mathcal{H}^k(y) = \int_\Omega h(x) \mathbb{J}_k df_x dx.$$

If $f \colon \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz-continuous and $E \subset \mathbb{R}^m$ countably k-rectifiable, then $y \mapsto \mathcal{H}^0(E \cap f^{-1}(\{y\}))$ is $\mathcal{H}^k$-measurable in $\mathbb{R}^n$ and the following formula holds:

$$\int_{\mathbb{R}^n} \mathcal{H}^0(E \cap f^{-1}(\{y\})) d\mathcal{H}^k(y) = \int_E \mathbb{J}_k d^E f_x d\mathcal{H}^k(x).$$

Theorem 1.6 (Coarea formula)**.** *Let $f \colon \mathbb{R}^m \to \mathbb{R}^k$ be Lipschitz-continuous and $E \subset \mathbb{R}^m$ countably $\mathcal{H}^n$-rectifiable. Then, denoting by $\mathbf{C}_k d^E f_x = \sqrt{\det((d^E f_x) \circ (d^E f_x)^*)}$, the following coarea formula holds:*

$$\int_E \mathbf{C}_k d^E f_x d\mathcal{H}^n(x) = \int_{\mathbb{R}^k} \mathcal{H}^{n-k}(E \cap f^{-1}(\{t\})) d\mathcal{L}^k(t).$$

More generally, similar to the area formula, for any $\mathcal{H}^k$-measurable function $h \colon E \to [0, \infty)$:

$$\int_E h(x) \mathbf{C}_k d^E f_x d\mathcal{H}^n(x) = \int_{\mathbb{R}^k} \int_{E \cap f^{-1}(\{t\})} h(x) d\mathcal{H}^{n-k}(x) d\mathcal{L}(t).$$

2 Derivation of the first variation of mass and the generalised mean curvature

We will first discuss the push-forward of a rectifiable k-varifold, relative to a Lipschitz mapping, which is essential for defining the first variation of rectifiable varifolds.

Definition 2.1 (Image varifold). *Let $V = \mathbf{v}(M, \vartheta)$ be a rectifiable k-varifold in the open set $\Omega \subset \mathbb{R}^n$ and $f\colon \Omega \to \mathbb{R}^n$ a proper, Lipschitz mapping. The image varifold f_*V is defined as follows:*

$$f_*V := \mathbf{v}(f(M), \tilde{\vartheta}), \quad \text{where } \tilde{\vartheta}(y) := \int_{f^{-1}(\{y\})\cap M} \vartheta d\mathcal{H}^0.$$

The definition needs some further verification that it is not initially clear whether the image varifold in fact defines a rectifiable k-varifold. Any Lipschitz image of a countably k-rectifiable set is necessarily k-rectifiable, whereas we can obtain a $\mathcal{H}^k$-a.e. covering by Lipschitz graphs via composition. By the virtue of the generalised area formula, we obtain for any compact subset $K \subset \mathbb{R}^n$:

$$\int_{f(M)\cap K} \tilde{\vartheta}(y) d\mathcal{H}^k(y) = \int_{f(M)\cap K} \sum_{x\in f^{-1}(\{y\})\cap M} \vartheta(x) d\mathcal{H}^k(y) = \int_{M\cap f^{-1}(K)} \vartheta(x)\mathbb{J}_k df_x dx,$$

whence $\tilde{\vartheta}$ is locally $\mathcal{H}^k$-integrable. This shows that f_*V indeed defines a rectifiable k-varifold in $f(\Omega)$.
We will now establish the set-up for computing the first variation of varifolds, which is effectively the Gauteaux derivative of the area functional. This allows us to define a generalised notion of minimal surfaces in the lower regularity setting, corresponding to the stable element under compactly supported deformation. Let $X \in C^2_c(\Omega; \mathbb{R}^n)$ be a compactly supported C^2-vector field in Ω and construct a 1-parameter variation f of Ω satisfying the following conditions:

$$f\colon I \times \Omega \longrightarrow \mathbb{R}^n \quad \text{with} \begin{cases} f_0 \equiv \mathrm{Id}_\Omega \quad \text{and} \quad X = \frac{\partial f}{\partial t}|_{t=0} = \mathrm{d}f \cdot \frac{\partial}{\partial t}|_{t=0}; \\ f(t,x) = x \text{ for all } (t,x) \in I \times \Omega \setminus K \text{ with } K \subset \Omega \text{ compact.} \end{cases}$$

Using the area formula and an mollification argument, we obtain for any compact $K \subset \mathbb{R}^n$:

$$\mathbb{M}((f_t)_*(V|_K)) = \int_{f_t(M\cap K)} \sum_{x\in f_t^{-1}(\{y\})\cap M} \vartheta(x) d\mathcal{H}^k(y) = \int_{M\cap K} \vartheta(x)\mathbb{J}_k d(f_t)_x dx.$$

The above can be extended to the more general setting, where the deformation is in a C^2-Riemannian manifold.

$$\mathbb{J}_k d(f_t) = \sqrt{\det Df_t^T Df_t} = \sqrt{\det(\langle \mathrm{d}f_t \cdot \frac{\partial}{\partial x^i}, \mathrm{d}f_t \cdot \frac{\partial}{\partial x^j}\rangle_{g_{\mathbb{R}^n}})} = \sqrt{\det g(t)}, \quad \text{where } g(t) := f_t^* g_{\mathbb{R}^n}.$$

In particular, we can write the mass of the deformation in terms of their Riemannian volume:

$$\mathbb{M}((f_t)_*(V|_K)) = \int_{M\cap K} \mathbb{J}_k d(f_t)_x d\mu_V(x) = \int_{M\cap K} \vartheta(x) \,\mathrm{dvol}_{g(t)} .$$

We use this chance to define the divergence operator for a general Riemannian manifold.

Definition 2.2 (Divergence operator). *Let $M^k \overset{\iota}{\hookrightarrow} (N^n, g_N)$ be an embedded Riemannian k-submanifold, equipped with the induced metric (i.e. first fundamental form) $g := \iota^* g_N$, and $X \in \Gamma_{C^1}(TM^k)$ be a C^1-vector field in M^k. We define the divergence of X on M^k to be:*

$$\mathrm{div}_g\, X(x) := \sum_{i=1}^k \langle \nabla^g_{e_i} X, e_i\rangle_g = g^{ij}\langle \nabla^g_{e_i} X, e_j\rangle_g = \mathrm{tr}\, \nabla X,$$

with respect to the metric g, where $(e_i) \in T_x M^k$ forms a g-orthonormal basis for the tangent space at x and ∇^g denotes the Levi-Civita connection for the metric g on M. A more general notion of the divergence of a C^1-vector field is given with respect to a k-plane $W \in \mathbb{G}(k; n)$:

$$\mathrm{div}_W\, X := \delta^{ij}\langle \nabla_{e_i} X, e_j\rangle_{g_{\mathbb{R}^n}}, \quad \text{where } (e_i) \in W \text{ orthonormal basis for } W.$$

Remark 2.1. Note to define a divergence operator, it suffices to have a notion of orthogonality and a plane of dimension $k \leqslant n$, with respect to which we can take directional derivatives. In our case, $e_i := \mathrm{d}f_t\partial_i$ defines a $g_{\mathbb{R}^n}$-orthonormal frame for the approximate tangent space $T^k_x M$ of $V = \mathbf{v}(M, \vartheta)$. This allows us to define the divergence operator on $T^k_x M$ as follows:

$$\mathrm{div}_g\, X := \mathrm{div}_{T^k_x M}\, X = \delta^{ij}\langle \nabla_{e_i} X, e_j\rangle_{g_{\mathbb{R}^n}} \quad \text{for any } X \in C^1_c(\Omega; \mathbb{R}^n).$$

Taking the time derivative of the mass of the C^2-deformation, since we are integrating over a compact domain:

$$\frac{\partial}{\partial t}\Big|_{t=0}\mathbb{M}((f_t)_*(V|_K)) = \int_{M\cap K}\vartheta(x)\,\frac{\partial}{\partial t}\Big|_{t=0}\sqrt{\det g(t)}dx = \int_{M\cap K}\left(\frac{1}{2\sqrt{\det g(t)}}\cdot\frac{\partial}{\partial t}\det g(t)\right)\Big|_{t=0} d\mu_V(x)$$
$$= \int_{M\cap K}\left(\frac{\mathrm{d}\det|_{g(t)}\cdot\partial_t g(t))}{2\sqrt{\det(g(t))}}\right)\Big|_{t=0} d\mu_V(x),$$

by using the chain rule. Computing the differential of the determinant at $g(t)$ gives us the Jacobi formula:

$$\mathrm{d}\det|_{g(t)}\cdot B = \lim_{\varepsilon\downarrow 0}\frac{\det(g(t)+\varepsilon B)-\det(g(t))}{\varepsilon} = (\det(g(t)))\lim_{\varepsilon\downarrow 0}\frac{\det(I+\varepsilon g(t)^{-1}\cdot B)-\det I}{\varepsilon}$$
$$= \det(g(t))\lim_{\varepsilon\downarrow 0}\frac{1+\varepsilon\,\mathrm{tr}(g(t)^{-1}\cdot B)+\mathcal{O}(t^2)-1}{\varepsilon} = \det(g(t))\cdot\mathrm{tr}(g(t)^{-1}B).$$

On the contrary, since ∇^g is a metric connection, we can compute the derivative of the metric components:

$$\frac{d}{dt}g_{ij}(t) = \langle\nabla^g_{\mathrm{d}f\frac{\partial}{\partial t}}\mathrm{d}f\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\rangle_g + \langle\frac{\partial}{\partial x^i},\nabla_{\mathrm{d}f\frac{\partial}{\partial t}}\mathrm{d}f\frac{\partial}{\partial x^j}\rangle_g = \langle\nabla_X e_i, e_j\rangle_g + \langle e_i,\nabla_X e_j\rangle_g,$$

for all $i,j=1,\dots,k$, where $e_i = \mathrm{d}f_t\partial_i$. In particular, using the Jacobi formula, we can compute the derivative:

$$\frac{\partial}{\partial t}\det g(t) = \det(g(t))\,\mathrm{tr}(g(t)^{-1}\cdot\partial_t g(t)) = 2\det(g(t))\cdot g^{ij}(t)\langle\nabla_X e_i, e_j\rangle_g = 2\det(g(t))g^{ij}(t)\langle\nabla_{e_i}X, e_j\rangle_g,$$

where we have used the fact that ∇^g is torsion-free and thus for any $i,j=1,\dots,k$:

$$\langle\nabla_X e_i, e_j\rangle_g = \langle[X,e_i],e_j\rangle_g + \langle\nabla_{e_i}X,e_j\rangle_g = \langle\mathrm{d}f\cdot\left[\frac{\partial}{\partial t},\frac{\partial}{\partial x^i}\right],e_j\rangle_g + \langle\nabla_{e_i}X,e_j\rangle_g = \langle\nabla_{e_i}X,e_j\rangle_g,$$

since partial derivatives commute. In particular, we obtain the following closed expression for the first variation:

$$\delta V(X) = \frac{\partial}{\partial t}\Big|_{t=0}\mathbb{M}((f_t)_*(V|_K)) = \int_{M\cap K}\sqrt{\det g(t)}\,\mathrm{div}_{g(t)}X(t)\Big|_{t=0} d\mu_V(x) = \int_{M\cap K}\mathrm{div}_g X\,d\mu_V.$$

Definition 2.3 (Stationary varifolds)**.** *A rectifiable k-varifold $V=\mathbf{v}(M,\vartheta)$ in the open set $\Omega\subset\mathbb{R}^n$ is said to be stationary in Ω if $\delta V(X)=0$ for any compactly supported vector field $X\in C^1_c(\Omega;\mathbb{R}^n)$, which is then equivalent to saying:*

$$\int_M \mathrm{div}_g X\,d\mu_V = 0 \quad \textit{for all } X\in C^1_c(\Omega;\mathbb{R}^n).$$

More generally, a k-varifold $V\in\mathcal{M}(\mathbf{G}_k(\Omega))$ in Ω, with $M=\mathrm{supp}\,\mu_V$, is stationary in Ω if:

$$\int_M \mathrm{div}_W X dV(x,W) = 0 \quad \textit{for any } X\in C^1_c(\Omega;\mathbb{R}^n).$$

Remark 2.2. An analogous argument allows us to compute the first variation of the mass functional for a general k-varifold $V\in IV_k(\Omega)$, which reads:

$$\delta V(X) := \int_{\mathbf{G}_k(\Omega)}\mathrm{div}_W X dV(x,W) \quad \text{for all } X\in C^1_c(\Omega;\mathbb{R}^n).$$

Consider a similar notion of rectifiable varifolds: let N^r be an embedded C^2-submanifold of $\mathbb{R}^n$ with $k\leqslant r\leqslant n$.

Definition 2.4 (Rectifiable varifold in submanifolds)**.** *Let $M\subset N$ be a $\mathcal{H}^k$-measurable subset (only in the set-theoretic sense) and $\vartheta\colon N\to(0,\infty]$ locally $\mathcal{H}^k$-integrable on M, i.e. $\vartheta\in L^1_{\mathrm{loc}}(M,\mathcal{B}(M),\mathcal{H}^k)$.*
Then we say that $V=\mathbf{v}(M,\vartheta)$ is a rectifiable k-varifold in N.

By definition, N^r is locally C^2-diffeomorphic to an open subset of $\mathbb{R}^n$. In particular, N is a locally compact topological space and every point $p\in N$ admits a precompact open neighbourhood $\Omega(p)$ with respect to the manifold topology[1]. Hence, setting

[1]Note we identify submanifolds in the categorical sense: $\iota\colon N\to\iota(N)\subset\mathbb{R}^n$.

$\Omega = \bigcup_{p\in N} \Omega(p)$, $V = \mathbf{v}(M,\vartheta)$ defines a rectifiable k-varifold in Ω in the usual sense with $\operatorname{supp}\mu_V \subset N$. This is sometimes easier to work with due to the geometry of manifolds.
Now suppose without loss of generality $M \subset N$ admits an approximate tangent space $T_x^k M \in \mathbb{G}(k;n)$ at every point $x \in M$. We will now consider a C^1-deformation of N generated by a general C^1-vector field on $\mathbb{R}^n$ which is not necessarily tangent to N everywhere. Using the ambient metric, we can obtain an orthogonal decomposition of the generating vector field with respect to M:

$$X = X^T + X^\perp, \quad \text{where } \begin{cases} X^T \in C_c^1(N;\mathbb{R}^n) \text{ with } \operatorname{supp} X^T \subset\subset N \text{ and } X_x^T \in T_x^k M; \\ X_x^\perp \perp_{g_{\mathbb{R}^n}} T_x^k M \text{ for all } x \in M. \end{cases}$$

If M is a C^2-submanifold, the linearity of divergence and the Weingarten equations implies at each $x \in M$:

$$\operatorname{div}_g X = \operatorname{div}_g X^T + g^{ij}\langle \nabla_{e_i} X^\perp, e_j\rangle_{g_{\mathbb{R}^n}} = \operatorname{div}_g X^T - \delta^{ij}\langle X^\perp, \mathrm{I\!I}(e_i,e_j)\rangle_{g_{\mathbb{R}^n}} = \operatorname{div}_g X^T - \langle X^\perp, \mathbf{H}\rangle_{g_{\mathbb{R}^n}},$$

where $(e_i) \in T_x^k M$ is any orthonormal basis and $\mathbf{H} := \delta^{ij}\mathrm{I\!I}(e_i,e_j)$ denotes the mean curvature vector for M. This in fact motivates our discussion in a more general context. We first introduce the notion of mean curvature for rough geometric objects such as rectifiable varifolds.

Definition 2.5 (Generalised mean curvature)**.** *Let $V = \mathbf{v}(M,\vartheta)$ be a rectifiable k-varifold in the open set $\Omega \subset \mathbb{R}^n$. We say V has a generalised mean curvature vector $\mathbf{H} \in L^1_{\mathrm{loc}}(\Omega,\mathcal{B}(\Omega),\mu_V)$ in Ω if:*

$$\int_M \operatorname{div}_g X \, d\mu_V = -\int_M \langle X, \mathbf{H}\rangle_{g_{\mathbb{R}^n}} d\mu_V \quad \text{for all } X \in C_c^1(\Omega;\mathbb{R}^n).$$

In particular, V is a stationary varifold in Ω if and only if it has generalised mean curvature $\mathbf{H} = 0$.

Definition 2.6 (Varifolds of locally bounded first variation)**.** *A k-varifold $V \in IV_k(\Omega)$ is said to have locally bounded first variation if for any compact $K \subset \Omega$, there exists constant $C \in (0,\infty)$ such that:*

$$|\delta V(X)| \leqslant C(K)\cdot \sup_\Omega |X| = C(K)\cdot \|X\|_{C^0(\Omega;\mathbb{R}^n)} \quad \text{for all } X \in C^0(K;\mathbb{R}^n).$$

Notice that the first variation defines a linear functional on $C_c^0(\Omega;\mathbb{R}^n)$, which implies that any varifold of locally bounded variation in fact defines a continuous linear functional $\delta V \in (C_c^0(\Omega;\mathbb{R}^n))$. Using the Riesz-Markov-Kakutani theorem, we can find a Radon measure $\delta V \in \mathcal{M}(\Omega;\mathbb{R}^n)$ characterised by:

$$\delta V(X) = \int_\Omega X\cdot\delta V \quad \text{for all } X \in C_c^0(\Omega;\mathbb{R}^n) \quad \text{and} \quad \|\delta V\|(W) = \sup\{|\delta V(X)| \mid X \in C_c^0(W;\mathbb{B}(0,1))\}.$$

Using the Besicovitch differentiation theorem and the uniqueness statement, we get the following decomposition:

$$\delta V = -\mathbf{H}\cdot\mu_V + d\sigma \quad \text{with } \begin{cases} \mathbf{H} = -D_{\mu_V}|\delta V| \in L^1_{\mathrm{loc}}(\Omega,\mathcal{B}(\Omega),\mu_V); \\ \sigma = (\delta V)^s \perp \mu_V. \end{cases}$$

The above Radon-Nikodym decomposition is another way to define the generalised mean curvature vector for $V = \mathbf{v}(M,\vartheta)$. In this case, motivated by the Stokes' theorem, we call $d\sigma$ the generalised surface measure on the rectifiable k-varifolds in a given open set. The characterisation of stationary varifolds via generalised mean curvature allows us to show that stationarity is preserved under varifold convergence formulated below.

Definition 2.7 (Convergence in varifold sense)**.** *Let $(V_i) \in IV_k(\Omega)$ for some open subset $\Omega \subset \mathbb{R}^n$. We say that V_i converges to $V \in IV_k(\Omega)$ in the varifold sense if $V_i \overset{*}{\rightharpoonup} V$ with respect to the weak-* topology on the space of Radon measures $\mathcal{M}^+(\mathbf{G}_k(\Omega))$, i.e.*

$$\int_{\mathbf{G}_k(\Omega)} \varphi(x,W)dV_i(x,W) \longrightarrow \int_{\mathbf{G}_k(\Omega)} \varphi(x,W)dV(x,W) \quad \text{for all } \varphi \in C_c^0(\mathbf{G}_k(\Omega)) = C_c^0(\Omega\times\mathbb{G}(k;n)).$$

A consequence of convergence in the varifold sense is the convergence of the total variation measure and the first variation distribution, i.e. $V_i \overset{*}{\rightharpoonup} V$ implies

(i) $|V_i| \overset{*}{\rightharpoonup} |V|$ by the definition of the total variation measure;

(ii) $\delta V_i \overset{*}{\rightharpoonup} \delta V$ by the first variation formula for general k-varifolds.

Proposition 2.1 (Stationarity compactness)**.** *Let $(V_i) \in \mathcal{M}(\mathbf{G}_k(\Omega))$ be a sequence of stationary k-varifolds in Ω. Suppose $V_i \overset{*}{\rightharpoonup} V$ in the sense of Radon measures. Then V is a stationary k-varifold.*

Proof. The divergence of any compactly supported C^1-vector field with respect to any k-plane is a compactly supported continuous function on $\mathbf{G}_k(\Omega)$. Now the result follows from the assumed weak-* convergence and the Riesz-Markov-Kakutani theorem. □

3 Monotonicity formula with L^p-bounded generalised mean curvature

Let $V = \mathbf{v}(M, \vartheta)$ be a rectifiable k-varifold in the open set $\Omega \subset \mathbb{R}^n$. Suppose for the rest of this subsection that V admits a generalised mean curvature vector $\mathbf{H} \in L^1_{\mathrm{loc}}(\Omega, \mathcal{B}(\Omega), \mu_V)$ such that $|\mathbf{H}|_{g_{\mathbb{R}^n}} \in L^p(\mu_V)$. Fix an arbitrary reference point $y \in M$. Consider a C^1-deformation of the following form: let $h \in C^1(\Omega; \mathbb{R})$ and $\eta \in C^1([0, \infty))$ such that $\operatorname{supp} h(\cdot)\eta(|\cdot - y|) \subset\subset \Omega$, the deformation $f\colon M \times I \to \Omega$ is generated by:

$$X|_x = \mathrm{d}f \cdot \frac{\partial}{\partial t} := h(x)\eta(|x - y|) \cdot (x - y).$$

Notice that the covariant derivative of variation field arises as the Euclidean derivative projected onto the approximate tangent space $T_x^k M$. In particular, we can reformulate the divergence operator $\mathcal{H}^k$-a.e. as below:

$$\operatorname{div}_g X = \nabla_i^g X^i = g^{ij} \langle \nabla_{e_i}^g X, e_j \rangle_g = g^{ij} \langle \pi_{T_x^k M}(DX^i), e_j \rangle_g.$$

Computing the Euclidean derivative of each component gives:

$$DX^i(x) = (x^i - y^i)\eta(|x - y|)Dh(x) + h(x)\eta(|x - y|)E_i + (x^i - y^i)h(x)\eta'(|x - y|)\frac{x - y}{|x - y|} \quad \text{for all } i = 1, \ldots, n,$$

where E_i denotes the standard coordinate vector in $\mathbb{R}^n$. Projecting the total derivative onto $T_x^k M$ gives:

$$\pi_{T_x^k M}(DX^i(x)) = (x^i - y^i)\eta(|x - y|)\nabla^g h|_x + h(x)\eta(|x - y|)e_i + h(x)|x - y|\eta'(|x - y|)\pi_{T_x^k M}\left(\frac{(x^i - y^i)(x - y)}{|x - y|^2}\right).$$

In particular, the divergence of our choice of generating vector field is pointwise given by:

$$\operatorname{div}_g X(x) = \langle \eta(r(x))(x - y), \nabla^g h|_x \rangle_g + k \cdot h(x)\eta(r(x)) + r(x)\eta'(r(x))h(x)|\nabla^g r|^2,$$

where $r\colon x \mapsto |x - y|$ denotes the distance function from y. Now since V admits a generalised mean curvature:

$$\int_M h(x)|\nabla^g r|^2(k\eta(r(x)) + r\eta'(r(x)))d\mu_V(x) = -\int_M \eta(r(x))\langle x - y, h(x)\mathbf{H} + \nabla^g h|_x \rangle_{g_{\mathbb{R}^n}} d\mu_V(x).$$

Let $\varepsilon \in (0, 1)$ be arbitrary and pick a decreasing function $\varphi = \varphi_\varepsilon \in C^1(\mathbb{R}; [0, 1])$ such that $\varphi \equiv 1$ on $(-\infty, 1]$ and $\varphi \equiv 0$ on $[1 + \varepsilon, \infty)$. Set $\eta(r) := \varphi(\rho^{-1}r)$ for some $\rho > 0$ with $(1 + \varepsilon)\rho < R$, where $R > 0$ is chosen such that $\overline{\mathbb{B}}(y, R) \subset \Omega$. Using the chain rule, we can deduce $r\eta'(r) = r\rho^{-1}\varphi'(\rho^{-1}r) = -\rho\partial_\rho(\varphi(\rho^{-1}r))$ and thus:

$$k \int_M h(x)\varphi(\frac{r(x)}{\rho})d\mu_V(x) - \rho\frac{\partial}{\partial\rho}\left(\int_M h(x)\varphi(\frac{r(x)}{\rho})|\nabla^g r|^2 \, d\mu_V(x)\right) = I(h, \rho),$$

where we define via scaling the 1-parameter functional $I(\cdot, \rho) = I_\rho\colon C^1(\Omega; \mathbb{R}) \to \mathbb{R}$ by the following formula:

$$I(h, \rho) := \int_M \langle h(x)\mathbf{H} + \nabla^g h|_x, \varphi(\frac{r(x)}{\rho})(x - y) \rangle_{g_{\mathbb{R}^n}} d\mu_V(x).$$

Assume for the sequel that $h \equiv 1$ and $p = \infty$, i.e. $R \cdot \sup_{\mathbb{B}(y,R)} |\mathbf{H}| \leqslant \Lambda < \infty$. We notice either via a direct computation $|Dr| = \left|\frac{x-y}{|x-y|}\right| = 1$ or from the fact that r defines a distance function. In particular, the Pythagoras theorem gives us $|\nabla^g r|^2 = 1 - |\nabla^\perp r|^2$ since $Dr = \pi_{T_x^k M}(Dr) + \pi_{T_x^k M}^\perp(Dr) = \nabla^g r + \nabla^\perp r$. By chain rule:

$$\frac{\partial}{\partial\rho}(\rho^{-k}F(\rho)) = -k\rho^{-(k+1)}\int_M \varphi(\frac{r(x)}{\rho})d\mu_V(x) + \rho^{-k}\int_M \partial_\rho\left(\varphi(\frac{r(x)}{\rho})\right)d\mu_V(x),$$

where $\mathcal{F}(\rho) := \int_M \varphi(\frac{r(x)}{\rho})d\mu_V(x)$. Using the g-orthogonal decomposition above, we obtain:

$$\frac{k}{\rho^{k+1}}\int_M \varphi(\frac{r(x)}{\rho})d\mu_V(x) - \rho^{-k}\int_M \partial_\rho\left(\varphi(\frac{r(x)}{\rho})\right)(1 - |\nabla^\perp r|^2)d\mu_V(x) = I(h \equiv 1, \rho),$$

whereas we can now apply the chain rule as above to deduce:

$$-\frac{\partial}{\partial\rho}(\rho^{-k}\mathcal{F}(\rho)) + \int_M \partial_\rho\left(\varphi(\frac{r(x)}{\rho})\right)|\nabla^\perp r|^2 d\mu_V(x) = -\rho^{-(k+1)}\int_M \langle \mathbf{H}, \varphi(\frac{r(x)}{\rho})(x - y) \rangle_{g_{\mathbb{R}^n}} d\mu_V(x).$$

Rearranging the above equation gives us the radial derivative:

$$\frac{\partial}{\partial\rho}\left(\rho^{-k}\mathcal{F}(\rho)\right)=\rho^{-k}\frac{\partial}{\partial\rho}\left(\int_M|\nabla^\perp r|^2\varphi_\varepsilon\left(\frac{r(x)}{\rho}\right)d\mu_V(x)\right)+\rho^{-(k+1)}\int_M\langle\mathbf{H},\varphi_\varepsilon(\frac{r(x)}{\rho})(x-y)\rangle_{g_{\mathbb{R}^n}}d\mu_V(x).$$

Now sending $\varepsilon\searrow 0$ gives us $\varphi_\varepsilon\searrow\chi_{(-\infty,1]}$ in $L^1_{\mathrm{loc}}(\mu_V)$, which in turn implies $\varphi_\varepsilon(\frac{r}{\rho})\searrow\chi_{\overline{\mathbb{B}}(y,\rho)}$ in $L^1_{\mathrm{loc}}(\mu_V)$. In particular, by the monotone convergence theorem, it yields in the sense of distribution and for $\mathcal{L}^1$-a.e. $\rho\in(0,R)$:

$$\frac{\partial}{\partial\rho}\left(\frac{\mu_V\overline{\mathbb{B}}(y,\rho)}{\rho^k}\right)=\rho^{-k}\frac{\partial}{\partial\rho}\left(\int_M\chi_{\overline{\mathbb{B}}(y,\rho)}|\nabla^\perp r|^2d\mu_V(x)\right)+\rho^{-(k+1)}\int_{\overline{\mathbb{B}}(y,\rho)}\langle\mathbf{H},x-y\rangle_{g_{\mathbb{R}^n}}d\mu_V(x).$$

Using the coarea formula with the radial distance function f from y, we deduce the following equality:

$$\begin{aligned}\rho^{-k}\int_{\overline{\mathbb{B}}(y,\rho)}|\nabla^\perp r(x)|^2|\nabla f(x)|\vartheta(x)d\mathcal{H}^k(x)&=\rho^{-k}\int_{\mathbb{R}}\int_{f^{-1}(t)\cap M}\chi_{\mathbb{B}(y,\rho)}|\nabla^\perp r(x)|^2\vartheta(x)d\mathcal{H}^{k-1}(x)dt\\&=\rho^{-k}\int_0^\rho\int_{\partial\mathbb{B}(y,t)}|\nabla^\perp r|^2\vartheta d\mathcal{H}^{k-1}dt.\end{aligned}$$

Similarly, we have:

$$\int_{\overline{\mathbb{B}}(y,\rho)}\frac{|\nabla^\perp r(x)|^2}{r(x)^k}|\nabla f(x)|\vartheta(x)d\mathcal{H}^k(x)=\int_0^\rho\int_{\partial\mathbb{B}(y,t)}\frac{|\nabla^\perp r(x)|^2}{r(x)^k}\vartheta(x)d\mathcal{H}^{k-1}(x)dt.$$

The distributional derivative of the above functionals of ρ must agree for $\mathcal{L}^1$-a.e. $\rho\in(0,R)$ by the fundamental lemma of calculus of variations, which gives us:

$$\rho^{-k}\frac{\partial}{\partial\rho}\left(\int_M\chi_{\overline{\mathbb{B}}(y,\rho)}|\nabla^\perp r|^2d\mu_V(x)\right)=\rho^{-k}\int_{\partial\mathbb{B}(y,\rho)}|\nabla^\perp r|^2\vartheta d\mathcal{H}^{k-1}=\frac{\partial}{\partial\rho}\left(\int_{\overline{\mathbb{B}}(y,\rho)}\frac{|\nabla^\perp r(x)|^2}{r(x)^k}d\mu_V(x)\right).$$

A more straightforward argument is based on the monotonicity of φ_ε. Notice that $\varphi_\varepsilon\equiv 1$ on $(-\infty,1]$ and $\varphi_\varepsilon\equiv 0$ on $[1+\varepsilon,\infty)$. Since φ_ε is everywhere decreasing, we thus have:

$$\varphi_\varepsilon'\left(\frac{r}{\rho}\right)\geqslant 0\text{ for }\rho\leqslant r\leqslant(1+\varepsilon)\rho;\quad\varphi_\varepsilon'\left(\frac{r}{\rho}\right)=0\text{ otherwise.}$$

Writing $\mathcal{J}_\varepsilon(\rho)=\int_M\varphi_\varepsilon(\frac{r}{\rho})|\nabla^\perp r|^2d\mu_V$ for simplicity, we can derive the following integral estimate:

$$((1+\varepsilon)\rho)^{-k}\mathcal{J}_\varepsilon'(\rho)\leqslant\int_M\varphi_\varepsilon\left(\frac{r}{\rho}\right)\frac{|\nabla^\perp r|^2}{r^k}d\mu_V\leqslant\rho^{-k}\mathcal{J}_\varepsilon'(\rho).$$

In particular, sending $\varepsilon\searrow 0$, we obtain the following equality in the sense of distribution:

$$\rho^{-k}\frac{\partial}{\partial\rho}\left(\int_M\chi_{\overline{\mathbb{B}}(y,\rho)}|\nabla^\perp r|^2d\mu_V\right)=\frac{\partial}{\partial\rho}\left(\int_{\overline{\mathbb{B}}(y,\rho)}\frac{|\nabla^\perp r|^2}{r^k}d\mu_V\right).$$

We can again deduce via the monotonicity of both sides that the equality in fact holds for $\mathcal{L}^1$-a.e. $\rho\in(0,R)$. Recall that we have assumed a priori $R\|\mathbf{H}\|_{L^\infty(\overline{\mathbb{B}}(y,R))}\leqslant\Lambda<\infty$. Now the integral involving the generalised mean curvature vector can be estimated using Cauchy-Schwarz inequality and the identity $(1+\varepsilon)\rho\in(0,R]$:

$$\left|\rho^{-(k+1)}\int_M\langle\mathbf{H},\varphi_\varepsilon\left(\frac{r}{\rho}\right)(x-y)\rangle_{g_{\mathbb{R}^n}}d\mu_V\right|\leqslant\rho^{-k}\int_M\varphi_\varepsilon\left(\frac{r}{\rho}\right)r|\mathbf{H}|d\mu_V\leqslant\frac{1+\varepsilon}{R}\Lambda\cdot\rho^{-k}\mathcal{F}_\varepsilon(\rho).$$

In particular, we obtain the following representation for the above integral:

$$\rho^{-(k+1)}\int_M\langle\mathbf{H},\varphi_\varepsilon\left(\frac{r}{\rho}\right)(x-y)\rangle_{g_{\mathbb{R}^n}}d\mu_V=C(\rho)\cdot\rho^{-k}\mathcal{F}(\rho),\qquad\text{where }C(\rho)\in[-(1+\varepsilon)R^{-1}\Lambda,(1+\varepsilon)R^{-1}\Lambda].$$

Using an argument involving integrating factors, we can deduce the monotonicity formula for rectifiable varifolds with bounded generalised mean curvature in the following form:

Theorem 3.1 (L^∞-monotonicity formula)**.** *Let $\Omega \subset \mathbb{R}^n$ be open with $\mathbb{B}(y,R) \subset\subset \Omega$ and $V = \mathbf{v}(M,\vartheta)$ a rectifiable k-varifold in Ω with $R\|\mathbf{H}\|_{L^\infty(\overline{\mathbb{B}}(y,R))} \leqslant \Lambda$ for some $\Lambda \in (0,\infty)$. Then for any $0 < \sigma \leqslant \tau < R$:*

$$e^{C_1(\tau)}\frac{\mu_V\overline{\mathbb{B}}(y,\tau)}{\omega_k\tau^k} - e^{C_1(\sigma)}\frac{\mu_V\overline{\mathbb{B}}(y,\sigma)}{\omega_k\sigma^k} = \frac{C_2(\sigma,\tau)}{\omega_k}\int_{\mathbb{A}_{\sigma,\tau}(y)}\frac{|\nabla^\perp r|^2}{r^k}d\mu_V,$$

where $\mathbb{A}_{\sigma,\tau}(y) := \overline{\mathbb{B}}(y,\tau)\setminus\overline{\mathbb{B}}(y,\sigma)$, $|C_1(\tau)| \leqslant \Lambda\tau R^{-1}$, and $C_2(\sigma,\tau) \in [e^{-\Lambda}, e^{\Lambda}]$ for all $0 < \sigma \leqslant \tau < R$.

Proof. Consider an integrating factor $e^{-C_1(\rho)}$ where $C_1^\varepsilon(\rho) := \int_0^\rho C^\varepsilon(s)ds$. An simple estimate is given by:

$$|C_1^\varepsilon(\rho)| \leqslant \int_0^\rho |C^\varepsilon(s)|ds \leqslant (1+\varepsilon)R^{-1}\Lambda \quad \text{ for all } (1+\varepsilon)\rho \in (0,R].$$

Multiply the equation involving the radial derivatives by the integrating factor gives us the following:

$$\begin{aligned}\frac{\partial}{\partial\rho}\left(e^{-C_1^\varepsilon(\rho)}\rho^{-k}\mathcal{F}_\varepsilon(\rho)\right) &= e^{-C_1^\varepsilon(\rho)}\cdot\left(\rho^{-k}\mathcal{J}_\varepsilon'(\rho) + C^\varepsilon(\rho)\rho^{-k}\mathcal{F}_\varepsilon(\rho)\right) - C^\varepsilon(\rho)e^{-C_1^\varepsilon(\rho)}\rho^{-k}\mathcal{F}_\varepsilon(\rho)\\ &= e^{-C_1^\varepsilon(\rho)}\frac{\partial}{\partial\rho}\left(\int_M \varphi_\varepsilon\left(\frac{r}{\rho}\right)\frac{|\nabla^\perp r|^2}{r^k}d\mu_V.\right)\end{aligned}$$

Now sending $\varepsilon \searrow 0$ allows us to deduce the following in the weak sense:

$$\frac{\partial}{\partial\rho}(e^{-C_1(\rho)}\rho^{-k}\mu_V\overline{\mathbb{B}}(y,\rho)) = e^{C_1(\rho)}\frac{\partial}{\partial\rho}\left(\int_{\overline{\mathbb{B}}(y,\rho)}\frac{|\nabla^\perp r|^2}{r^k}d\mu_V\right),$$

which we can integrate from σ to τ with respect to ρ to conclude:

$$\begin{aligned}e^{-C_1(\tau)}\frac{\mu_V\overline{\mathbb{B}}(y,\tau)}{\omega_k\tau^k} - e^{-C_1(\sigma)}\frac{\mu_V\overline{\mathbb{B}}(y,\sigma)}{\omega_k\sigma^k} &= \int_{\overline{\mathbb{B}}(y,\tau)}\frac{e^{-C_1(\tau)}}{\omega_k}\frac{|\nabla^\perp r|^2}{r^k}d\mu_V - \int_{\overline{\mathbb{B}}(y,\sigma)}\frac{e^{-C_1(\sigma)}}{\omega_k}\frac{|\nabla^\perp|^2}{r^k}d\mu_V\\ &\quad + \int_\sigma^\tau C(\rho)e^{-C_1(\rho)}\int_{\overline{\mathbb{B}}(y,\rho)}\frac{|\nabla^\perp r|^2}{r^k}d\mu_V\,d\rho.\end{aligned}$$

Rearranging and relabeling the constants gives us:

$$e^{-C_1(\tau)}\frac{\mu_V\overline{\mathbb{B}}(y,\tau)}{\omega_k\tau^k} - e^{-C_1(\sigma)}\frac{\mu_V\overline{\mathbb{B}}(y,\sigma)}{\omega_k\sigma^k} = \frac{C_2(\sigma,\tau)}{\omega_k}\int_{\mathbb{A}_{\sigma,\tau}(y)}\frac{|\nabla^\perp r|^2}{r^k}d\mu_V,$$

which is the desired monotonicity result for the density ratio. □

More generally, when it comes to studying minimisation problems for general integral functionals depending on the mean curvature, a monotonicity formula assuming L^p-bounded generalised mean curvature becomes useful.

Theorem 3.2 (L^p-monotonicity formula, [Mon12])**.** *Let $V = \mathbf{v}(M,\vartheta)$ be a rectifiable k-varifold in the open subset $\Omega \subset \mathbb{R}^n$, admitting a generalised mean curvature $\mathbf{H} \in L^p(\mu_V)$ for some $p \in (k,\infty)$. Then for any fixed $\mathbb{B}(y,R) \subset\subset \Omega$ and $0 < \sigma < \tau < R$, we obtain the following fundamental inequality:*

$$\left(\frac{\mu_V\overline{\mathbb{B}}(y,\tau)}{\omega_k\tau^k}\right)^{\frac{1}{p}} - \left(\frac{\mu_V\overline{\mathbb{B}}(y,\sigma)}{\omega_k\sigma^k}\right)^{\frac{1}{p}} \geqslant \frac{1}{p-k}\left(\frac{\sigma^{1-\frac{k}{p}}}{\omega_k^{\frac{1}{p}}}\left(\int_{\overline{\mathbb{B}}(y,\sigma)}|\mathbf{H}|^p d\mu_V\right)^{\frac{1}{p}} - \frac{\tau^{1-\frac{k}{p}}}{\omega_k^{\frac{1}{p}}}\left(\int_{\overline{\mathbb{B}}(y,\tau)}|\mathbf{H}|^p d\mu_V\right)^{\frac{1}{p}}\right).$$

Proof. For the remainder of the proof, we adapt the notations introduced above for deriving the L^∞-monotonicity formula. Denoting by $\mathcal{E}_\varepsilon(\rho) = \int_M \langle\mathbf{H}, \varphi_\varepsilon(\frac{r}{\rho})(x-y)\rangle_{g_{\mathbb{R}^n}}d\mu_V$, we can write the radial derivative as:

$$\frac{\partial}{\partial\rho}(\rho^{-k}\mathcal{F}_\varepsilon(\rho)) = \rho^{-k}\mathcal{J}_\varepsilon'(\rho) + \rho^{-(k+1)}\mathcal{E}_\varepsilon(\rho) \geqslant \rho^{-(k+1)}\mathcal{E}_\varepsilon(\rho),$$

where the inequality follows since $\partial_\rho(\varphi_\varepsilon(\frac{r}{\rho}) = -\frac{r}{\rho^2}\varphi_\varepsilon'(\frac{r}{\rho}) \geqslant 0$ everywhere. By the Cauchy-Schwarz inequality:

$$\frac{\partial}{\partial\rho}(\rho^{-k}\mathcal{F}_\varepsilon(\rho)) \geqslant \rho^{-(k+1)}\mathcal{E}_\varepsilon(\rho) \geqslant -\rho^{-(k+1)}\int_M(\varphi_\varepsilon\left(\frac{r}{\rho}\right)^{\frac{1}{p}}|\mathbf{H}|)(r\varphi_\varepsilon\left(\frac{r}{\rho}\right)^{1-\frac{1}{p}})\,d\mu_V.$$

Recall $\varphi_\varepsilon(\frac{r}{\rho}) = 0$ whenever $r \geqslant (1+\varepsilon)\rho$ and φ_ε everywhere non-negative. In particular, we obtain the radial estimate $r \leqslant (1+\varepsilon)\rho$ and thus it yields together with Hölder's inequality:

$$\frac{\partial}{\partial\rho}(\rho^{-k}\mathcal{F}_\varepsilon(\rho)) \geqslant -\frac{1+\varepsilon}{\rho^k}\int_M \varphi_\varepsilon\left(\frac{r}{\rho}\right)^{\frac{1}{p}} |\mathbf{H}| \varphi_\varepsilon\left(\frac{r}{\rho}\right)^{1-\frac{1}{p}} d\mu_V \geqslant -\frac{1+\varepsilon}{\rho^k}\mathcal{F}_\varepsilon(\rho)\left(\int_M \varphi_\varepsilon\left(\frac{r}{\rho}\right)|\mathbf{H}|^p d\mu_V\right)^{\frac{1}{p}}.$$

Multiply the above inequality by $p^{-1}\rho^{k(1-\frac{1}{p})}\mathcal{F}_\varepsilon(\rho)^{-(1-\frac{1}{\rho})}$ gives us the L^p-version of radial derivatives:

$$\frac{\partial}{\partial\rho}((\rho^{-k}\mathcal{F}_\varepsilon(\rho))^{\frac{1}{\rho}}) = \frac{1}{p}\rho^{k(1-\frac{1}{p})}\mathcal{F}_\varepsilon(\rho)^{-(1-\frac{1}{p})}\frac{\partial}{\partial\rho}(\rho^{-k}\mathcal{F}_\varepsilon(\rho)) \geqslant -\frac{1+\varepsilon}{p}\rho^{-\frac{k}{p}}\left(\int_M \varphi_\varepsilon\left(\frac{r}{\rho}\right)|\mathbf{H}|^p d\mu_V\right)^{\frac{1}{p}}.$$

Now integrating the above inequality from σ to τ and we obtain via integration by parts:

$$\begin{aligned}\frac{\mathcal{F}_\varepsilon(\tau)^{\frac{1}{p}}}{\tau^{\frac{k}{p}}} - \frac{\mathcal{F}_\varepsilon(\sigma)^{\frac{1}{p}}}{\sigma^{\frac{k}{p}}} \geqslant -\frac{1+\varepsilon}{k+p}&\left(\tau^{1-\frac{k}{p}}\left(\int_M \varphi_\varepsilon\left(\frac{r}{\rho}\right)|\mathbf{H}|^p d\mu_V\right)^{\frac{1}{p}} - \sigma^{1-\frac{k}{p}}\left(\int_M \varphi_\varepsilon\left(\frac{r}{\sigma}\right)|\mathbf{H}|^p d\mu_V\right)^{\frac{1}{p}}\right)\\ &+\frac{1+\varepsilon}{k+p}\int_\sigma^\tau \rho^{1-\frac{k}{p}}\frac{\partial}{\partial\rho}\left(\int_M \varphi_\varepsilon\left(\frac{r}{\rho}\right)|\mathbf{H}|^p d\mu_V\right)^{\frac{1}{p}} d\rho.\end{aligned}$$

Similar to before, we can deduce the following from the monotonicity of φ_ε:

$$\frac{\partial}{\partial\rho}\left(\int_M \varphi_\varepsilon\left(\frac{r}{\rho}\right)|\mathbf{H}|^p d\mu_V\right) = \int_M -\frac{r}{\rho^2}\varphi_\varepsilon'\left(\frac{r}{\rho}\right)|\mathbf{H}|^p d\mu_V \geqslant 0,$$

from which it yields together with the non-negativity of φ_ε:

$$\frac{\partial}{\partial\rho}\left(\left(\int_M \varphi_\varepsilon\left(\frac{r}{\rho}\right)|\mathbf{H}|^p d\mu_V\right)^{\frac{1}{p}}\right) = \frac{1}{p}\left(\int_M \varphi_\varepsilon\left(\frac{r}{\rho}\right)|\mathbf{H}|^p d\mu_V\right)^{\frac{1}{p}-1}\frac{\partial}{\partial\rho}\left(\int_M \varphi_\varepsilon\left(\frac{r}{\rho}\right)|\mathbf{H}|^p d\mu_V\right) \geqslant 0.$$

In particular, since we restrict to the case where $p > k$, we obtain the following inequality:

$$\frac{\mathcal{F}_\varepsilon(\tau)^{\frac{1}{p}}}{\tau^{\frac{k}{p}}} - \frac{\mathcal{F}_\varepsilon(\sigma)^{\frac{1}{p}}}{\sigma^{\frac{k}{p}}} \geqslant -\frac{1+\varepsilon}{k+p}\left(\tau^{1-\frac{k}{p}}\left(\int_M \varphi_\varepsilon\left(\frac{r}{\rho}\right)|\mathbf{H}|^p d\mu_V\right)^{\frac{1}{p}} - \sigma^{1-\frac{k}{p}}\left(\int_M \varphi_\varepsilon\left(\frac{r}{\sigma}\right)|\mathbf{H}|^p d\mu_V\right)^{\frac{1}{p}}\right).$$

Now sending $\varepsilon \searrow 0$ in the above inequality gives us via the dominated convergence theorem:

$$\frac{\mu_V\overline{\mathbb{B}}(y,\tau)^{\frac{1}{p}}}{\tau^{\frac{k}{p}}} - \frac{\mu_V\overline{\mathbb{B}}(y,\sigma)^{\frac{1}{p}}}{\sigma^{\frac{k}{p}}} \geqslant \frac{1}{p-k}\left(\sigma^{1-\frac{k}{p}}\left(\int_{\overline{\mathbb{B}}(y,\sigma)}|\mathbf{H}|^p d\mu_V\right)^{\frac{1}{p}} - \tau^{1-\frac{k}{p}}\left(\int_{\overline{\mathbb{B}}(y,\tau)}|\mathbf{H}|^p d\mu_V\right)^{\frac{1}{p}}\right),$$

which is equivalent to the stated monotonicity statement for the density ratio. □

We remark that a monotonicity statement for general k-varifolds of locally bounded first variation, whose proof follows the same computation as the derivation above with L^∞-bounded generalised mean curvature.

Theorem 3.3 (Monotonicity formula for general varifolds, [Sim84]). *Let $V \in IV_k(\Omega)$ for some open subset $\Omega \subset \mathbb{R}^n$ with locally bounded first variation in Ω. Let $\mathbb{B}(y,R) \subset\subset \Omega$ with $\Lambda \in (0,\infty)$ such that:*

$$|\delta V|(\overline{\mathbb{B}}(y,\rho)) \leqslant \Lambda\cdot\mu_V\overline{\mathbb{B}}(x,\rho) \quad \text{for any } \rho \in (0,R).$$

Then for any $0 < \sigma \leqslant \tau < R$, we have:

$$e^{\Lambda\tau}\frac{\mu_V\overline{\mathbb{B}}(y,\tau)}{\omega_k\tau^k} - e^{\Lambda\sigma}\frac{\mu_V\overline{\mathbb{B}}(y,\sigma)}{\omega_k\sigma^k} \geqslant \frac{1}{\omega_k}\int_{\mathbf{G}_k(\mathbb{A}_{\sigma,\tau}(y))}\frac{|\pi_{W^\perp}(x-y)|^2}{r^{k+2}}dV(x,W).$$

Furthermore, for any $h \in C^1(\Omega)$, we obtain a generalised version of the above monotonicity statement:

$$\begin{aligned}\frac{\partial}{\partial\rho}\left(\rho^{-k}\int\varphi\left(\frac{r(x)}{\rho}\right)h(x)d\mu_V(x)\right) &= \rho^{-k}\frac{\partial}{\partial\rho}\left(\int h(x)\varphi\left(\frac{r(x)}{\rho}\right)\frac{|\pi_{W^\perp}(x-y)|^2}{r(x)^2}dV(x,W)\right)\\ &+\rho^{-(k+1)}\left(\langle h\varphi\left(\frac{r}{\rho}\right)(\cdot-y),\delta V\rangle + \int\langle\varphi\left(\frac{r(x)}{\rho}\right)(x-y), D_W h(x)dV(x,W)\right).\end{aligned}$$

Two useful consequences of the above monotonicity formula, which will be used to establish a rectifiability theorem for general k-varifolds of locally bounded first variation, are stated and proved below.

Corollary 3.1. *Suppose $V \in IV_k(\Omega)$ has locally bounded first variation in Ω. Then for μ_V-a.e. $x \in \Omega$, the density $\Theta^k(\mu_V, x) \in \mathbb{R}$ exists. Furthermore, $x \mapsto \Theta^k(\mu_V, x)$ is μ_V-measurable and thus approximately continuous with respect to μ_V, i.e. $\mu_V \overline{\mathbb{B}}(x, \rho) > 0$ for all $\rho > 0$ and:*

$$\lim_{\rho \downarrow 0} \frac{\mu_V\{y \in \overline{\mathbb{B}}(x, \rho) \mid |\Theta^k(\mu_V, y) - \Theta^k(\mu_V, x)| > \varepsilon\}}{\mu_V \overline{\mathbb{B}}(x, \rho)} = 0 \quad \textit{for all } \varepsilon > 0.$$

Proof. The existence of density follows from reformulating the monotonicity formula above:

$$0 \leqslant \Theta^k(\mu_V, x) \leqslant e^{\Lambda\sigma} \frac{\mu_V \overline{\mathbb{B}}(x, \sigma)}{\omega_k \sigma^k} \leqslant e^{\Lambda\tau} \frac{\mu_V \overline{\mathbb{B}}(x, \tau)}{\omega_k \tau^k} - \frac{1}{\omega_k} \int_{\mathbf{G}_k(\mathbb{A}_{\sigma,\tau}(y))} \frac{|\pi_{W^\perp}(x-y)|^2}{r^{k+2}} dV(x, W),$$

for any $0 < \sigma \leqslant \tau < \infty$. Notice that it suffices to show $x \mapsto \Theta^k(\mu_V, x)$ is Borel measurable which would then imply μ_V-measurability. Let $x \in \Omega$ and consider a sequence $(x_i) \in \Omega$ such that $x_i \to x$ as $i \to \infty$. Fix an arbitrary radius $\rho > 0$. Consider any infinitesimal sequence (ε_i) and for any $i \in \mathbb{N}$, pick $k(i)$ such that $|x_{k(i)} - x| \leqslant \varepsilon_i$. Thus we obtain for any $y \in \overline{\mathbb{B}}(x_{k(i)}, \rho)$:

$$|x - y| \leqslant |x_{k(i)} - y| + |x_{k(i)} - x| \leqslant \rho + \varepsilon_i \Rightarrow \limsup_{i \to \infty} \chi_{\overline{\mathbb{B}}(x_i, \rho)} \leqslant \chi_{\overline{\mathbb{B}}(x, \rho)}.$$

Since μ_V is a Radon measure, we can apply Fatou's lemma to obtain:

$$\limsup_{i \to \infty} \mu_V \overline{\mathbb{B}}(x_i, \rho) = \int_\Omega \chi_{\overline{\mathbb{B}}(x_i, \rho)} d\mu_V \leqslant \int_\Omega \chi_{\overline{\mathbb{B}}(x, \rho)} d\mu_V = \mu_V \overline{\mathbb{B}}(x_i, \rho) < \infty.$$

In particular, the density ratio $x \mapsto \Theta^k(\mu_V, x, \rho) = \frac{\mu_V \overline{\mathbb{B}}(x_i, \rho)}{\omega_k \rho^k}$ is upper-semicontinuous and thus Borel measurable. We can then deduce that $x \mapsto \Theta^k(\mu_V, x) := \lim_{\rho \searrow 0} \Theta^k(\mu_V, x, \rho)$ is a Borel function.
The approximate continuity in fact holds in greater generality. Indeed, the following proof shows that any measurable function is necessarily approximately continuous a.e. with respect to a Radon measure. By above:

$$f \colon \mathbb{R}^n \longrightarrow \mathbb{R}, \quad x \longmapsto \Theta^k(\mu_V, x),$$

defines a μ_V-measurable function. Now choose a compact exhaustion (E_i) of $\mathbb{R}^n$ (e.g. $E_i = \overline{\mathbb{B}}(0, i)$), whereas since $\mu_V \in \mathcal{M}^+(\mathbb{R}^n)$ defines a Radon measure, we have $\mu_V E_i < \infty$ for any $i \in \mathbb{N}$. By Lusin's theorem, we can find a compact subset $K_1 \subset E_1$ such that $\mu_V(E_1 \setminus K_1) \leqslant 1$ and $f|_{K_1}$ is continuous. Now construct a disjoint sequence of compact subsets (K_i) inductively such that for any $i \in \mathbb{N}$, the restriction $f|_{K_i}$ is continuous with:

$$K_i \subset\subset E_i \setminus \left(\bigcup_{j=1}^{i-1} K_j \right) \quad \text{and} \quad \mu_V \left(E_i \setminus \left(\bigcup_{j=1}^{i} K_j \right) \right) \leqslant \frac{1}{i}.$$

By the Lebesgue's density theorem, for any fixed $i \in \mathbb{N}$, μ_V-a.e. $x \in K_i$ is a density point, i.e.

$$\lim_{\rho \downarrow 0} \frac{\mu_V(K_i \cap \overline{\mathbb{B}}(x, \rho))}{\mu_V \overline{\mathbb{B}}(x, \rho)} = 1 \quad \text{for } \mu_V\text{-a.e. } x \in K_i.$$

In particular, the set of μ_V-density points in $\bigcup_{i=1}^\infty K_i$ is of full measure in $\mathbb{R}^n$. Fix such a density point $x \in K_i$ for some i. The continuity of $f|_{K_i}$ implies that for any $\varepsilon > 0$, we can find $\delta > 0$ such that:

$$|\Theta^k(\mu_V, y) - \Theta^k(\mu_V, x)| \leqslant \varepsilon \quad \text{for all } y \in K_i \text{ with } |y - x| \leqslant \delta.$$

Then for any $r \in (0, \delta)$, we have:

$$\frac{\mu_V\{y \in \overline{\mathbb{B}}(x, r) \mid |\Theta^k(\mu_V, y) - \Theta^k(\mu_V, x)| \geqslant \varepsilon\}}{\mu_V \overline{\mathbb{B}}(x, r)} \leqslant \frac{\mu_V(\overline{\mathbb{B}}(x, r) \setminus K_i)}{\mu_V \overline{\mathbb{B}}(x, r)} \to 0 \quad \text{as } r \downarrow 0,$$

which gives us the desired μ_V-approximate continuity at μ_V-a.e. $x \in \mathbb{R}^n$. $\square$

Corollary 3.2. *Let $(V_i) \in IV_k(\Omega)$ of locally bounded first variation in Ω such that $V_i \overset{*}{\rightharpoonup} V \in IV_k(\Omega)$ as Radon measure on $\mathbf{G}_k(\Omega)$. Suppose $\Theta^k(\mu_{V_i}, y) \geqslant 1$ for all $y \in \Omega \setminus E_i$ where $\mu_V(E_i \cap K) \to 0$ as $i \to \infty$ for any $K \subset\subset \Omega$. Then the following holds:*

(i) $|\delta V|(K) \leqslant \liminf_{i\to\infty} |\delta V_i|(K)$ *for all* $K \subset\subset \Omega$*;*

(ii) $\Theta^k(\mu_V, y) \geqslant 1$ *for* μ_V*-a.e.* $y \in \Omega$*.*

Proof. (i) follows by definition of the total variation measure and the lower-semicontinuity under weak-* convergence of Radon measures.
It thus remains to prove (ii). Fix some $K \subset\subset \Omega$ and $R \in (0, \mathrm{dist}(K, \partial\Omega))$. Consider for any $i, j \in \mathbb{N}$:

$$E_{ij} := \{y \in K \setminus E_i \mid |\delta V_i|(\overline{\mathbb{B}}(y,r)) \leqslant j\mu_{V_i}\overline{\mathbb{B}}(y,r) \text{ for all } r \in (0,R)\} \quad \text{and} \quad F_{ij} := K \setminus E_{ij}.$$

Thus, $x \in F_{ij}$ implies either $x \in E_i \cap K$ or there exists $s \in (0,R)$ with $\mu_{V_i}\overline{\mathbb{B}}(x,s) \leqslant j^{-1}|\delta V_i|(\overline{\mathbb{B}}(x,s))$. Set:

$$\mathcal{B} := \{\overline{\mathbb{B}}(x,\sigma) \mid x \in F_{ij},\ \sigma \in (0,R),\ \mu_{V_i}\overline{\mathbb{B}}(x,\sigma) \leqslant j^{-1}|\delta V_i|(\overline{\mathbb{B}}(x,\sigma))\}.$$

The Besicovitch covering theorem gives rise to disjoint, at most countable subfamilies $\mathcal{B}_1, \dots, \mathcal{B}_{\beta(n)} \subset \mathcal{B}$ for some dimensional constant $\beta(n) \in \mathbb{N}$ such that $F_{ij} \setminus E_i \subset \bigcup_{i=1}^{\beta(n)} (\bigcup \mathcal{B}_i)$. In particular, by disjointness:

$$\mu_{V_i}F_{ij} - \mu_{V_i}(E_i \cap K) = \mu_{V_i}(F_{ij} \setminus E_i) \leqslant \sum_{k=1}^{\beta(n)} \sum_{B \in \mathcal{B}_k} \frac{1}{j}|\delta V_i|(B) \Rightarrow \mu_{V_i}F_{ij}.$$

Writing $K' := \{x \in \Omega \mid \mathrm{dist}(x,K) < R\}$, we obtain by rearranging the above inequality:

$$\mu_{V_i}F_{ij} \leqslant \mu_{V_i}(E_i \cap K) + \sum_{k=1}^{\beta(n)} \sum_{B \in \mathcal{B}_k} \frac{1}{j}|\delta V_i|(B) \leqslant \mu_{V_i}(E_i \cap K) + \frac{\beta(n)}{j}|\delta V_i|(K') \leqslant \frac{C}{j} + \mu_{V_i}(E_i \cap K).$$

Thus we have for any $j, l \in \mathbb{N}$, since $\mu_{V_i}(E_i \cap K) \to 0$ as $i \to \infty$:

$$\mu_V(\mathrm{int}(\bigcap_{k=j}^{\infty} F_{kl})) \leqslant \liminf_{i\to\infty} \mu_{V_i}(\mathrm{int}(\bigcap_{k=j}^{\infty} F_{kl})) \leqslant \frac{C}{l}.$$

If there exists $x \in K \setminus \mathrm{int}(\bigcap_{j=i}^{\infty} F_{jk})$ for some fixed $k \in \mathbb{N}$ and for any $i \in \mathbb{N}$, then we can find $(x_i) \in W$ and $(k_i) \in \mathbb{N}$ such that $x_i \in K \setminus \bigcap_{j=i}^{\infty} F_{jk}$ and $x_i \to x$. Moreover, we can require $x_i \notin F_{k(i),k}$ for all $i \in \mathbb{N}$. Then $x_i \in E_{k(i),k}$ for all i and each V_i has bounded first variation around the point x_i: i.e.

$$|\delta V_{k(i)}|(\overline{\mathbb{B}}(x_i,r)) \leqslant k\mu_{V_{k(i)}}\overline{\mathbb{B}}(x_i,r) \quad \text{ for all } r \in (0,R).$$

In particular, the monotonicity formula for V_i applies at x_i:

$$1 \leqslant \Theta^k(\mu_{V_{k(i)}}, x_i) \leqslant e^{kr}\frac{\mu_{V_{k(i)}}\overline{\mathbb{B}}(x_i,r)}{\omega_k r^k} \quad \text{ for all } r \in (0,R),$$

which gives us the density lower bound $\Theta^k(\mu_V, x) \geqslant 1$ for any $x \in K \setminus (\bigcap_{k=1}^{\infty}\bigcup_{i=1}^{\infty} \mathrm{int}(\bigcap_{j=i}^{\infty} F_{jk}))$. However:

$$\mu_V(\bigcap_{k=1}^{\infty}\bigcup_{i=1}^{\infty}\mathrm{int}(\bigcap_{j=i}^{\infty} F_{jk})) \leqslant \mu_V(\bigcup_{i=1}^{\infty}\mathrm{int}(\bigcap_{j=i}^{\infty} F_{jk})) = \lim_{i\to\infty} \mu_V(\mathrm{int}(\bigcap_{j=i}^{\infty} F_{jk})) \leqslant \frac{C}{k} \to 0,$$

as $k \to \infty$. Thus, the density lower bound holds on a set of full μ_V-measure in K. Using a compact exhaustion argument and the fact that countable any union of μ_V-negligible sets remains μ_V-negligible, the density lower bound $\Theta^k(\mu_V, x) \geqslant 1$ holds for μ_V-a.e. $x \in \Omega$. $\square$

4 Compactness theorems for varifolds and tangent cones

The measure-theoretic formulation of the class of varifolds results in weaker regularity than the class of smooth submanifolds, whereas the loss of regularity is compensated by good compactness properties and the possibility to discuss regularity near the singular points on a minimal submanifold. We will be working primarily in the setting of a k-varifold with locally bounded first variation on some open subset $\Omega \subset \mathbb{R}^n$, which automatically implies rectifiability. We will discuss this in the sequel for deriving a more intuitive formulation of the extended monotonicity theorem.

It is necessary to first clarify the meaning of mappings of a general k-varifold. Let $\Omega \subset \mathbb{R}^n$ and $f \in C^1(\Omega; \mathbb{R}^n)$ such that $f|_{\operatorname{supp}(\mu_V)\cap\Omega}$ defines a proper map. We can thus interpret the image varifold f_*V on $f(\Omega)$ analogously to the rectifiable case as follows: for all Borel subset $E \subset \mathbf{G}_k(f(\Omega))$:

$$f_*V(E) := \int_{F^{-1}(E)} \mathbb{J}_W df_x dV(x,W) = \int_{F^{-1}(E)} \sqrt{\det((df_x|_W)^* \circ (df_x|_W))} dV(x,W),$$

where we define the mapping of Grassmannian bundles $F\colon \mathbf{G}_k(\Omega) \to \mathbf{G}_k(f(\Omega)),\ (x,W) \mapsto (f(x), df_x(W))$. For any $W \in \mathbb{G}(k;n)$, $\mathbb{J}_W df_x$ is the analogue of the k-dimensional Jacobian restricted to the k-plane W. Thus the above interpretation generalises the notion of image of rectifiable varifolds.

Now recall the rescaling function $\eta_{x,\lambda}$ defined by setting $\eta_{x,\lambda}(y) := \lambda^{-1}(y-x)$. Using the above interpretation of image varifolds and the compactness of Radon measures, we can consider a sequence of rescaled minimal submanifold $(\eta_{x,\lambda}V \mid \lambda > 0)$ at a given singularity and discuss the regularity properties near the singularity in the subsequential varifold limit. A blow-up analysis can thus be conducted using the asymptotic structure of the limiting varifolds, which are known as the varifold tangents.

Definition 4.1 (Varifold tangent cone). *Let $V \in IV_k(\Omega)$ for some open subset $\Omega \subset \mathbb{R}^n$ and $x \in \Omega$ with*

$$\Theta^k(\mu_V, x) \in (0,\infty) \quad \textit{and} \quad \lim_{\rho \downarrow 0} \frac{|\delta V|(\mathbb{B}(x,\rho))}{\omega_k \rho^{k-1}} = 0.$$

We define a tangent cone of V at x to be any weak- subsequential limit of $((\eta_{x,\lambda})_*V \mid \lambda > 0)$ in the varifold sense as $\lambda \searrow 0$. The collections of all tangent cones of V at x is referred to as the varifold tangents $\operatorname{VarTan}(V,x)$ of V. Note clearly, if $V = \mathbf{v}(M,\vartheta)$ defines a rectifiable k-varifold, then denoting by $T_x^k M$ the approximate tangent space, $\operatorname{VarTan}(V,x)$ consists of the unique tangent cone $\vartheta(x)\mathcal{H}^k \llcorner T_x^k M$.*

Note that $d\eta_{x,\lambda} = \lambda^{-1}\operatorname{Id}$ and any $W \in \mathbb{G}(k,n)$ is invariant under dilation. In particular, the divergence is preserved in the following sense:

$$\operatorname{div} \circ (\eta_{x,\lambda}, d\eta_{x,\lambda})(X,W) = \operatorname{div}_{d\eta_{x,\lambda}W}(X \circ \eta_{x,\lambda}^{-1}) = \lambda(\operatorname{div}_W X) \circ \eta_{x,\lambda}^{-1}$$

Notice that $\operatorname{supp}(X \circ \eta_{x,\lambda}) = \lambda(\operatorname{supp} X) + \{x\}$, which collapses to μ_V-negligible set (i.e. a singleton). In the view of the Riesz-Markov-Kakutani theorem, after passing to any weak-* subsequential limit $(\eta_{x,\lambda_i})_*V \xrightarrow{*} V_\infty$ as $\lambda_i \searrow 0$, we have:

$$\begin{aligned}\int_{\mathbf{G}_k(\eta_{x,\lambda}(\Omega))} \operatorname{div}_W X d(\eta_{x,\lambda})_*V(y,W) &= \int_{\mathbf{G}_k(\Omega)} \operatorname{div}_{d\eta_{x,\lambda}W} X \mathbb{J}_W d\eta_{x,\lambda}|_y dV(y,W) \\ &= \int_{\mathbf{G}_k(\Omega)} (\operatorname{div}_W X) \circ \eta_{x,\lambda}^{-1} dV(y,W) \to 0 = \langle \operatorname{div}_W X, V_\infty \rangle = \delta V_\infty(X),\end{aligned}$$

as $i \to \infty$, which in particular implies that any weak-* subsequential limit is in fact a stationary varifold. Furthermore, we notice that by construction, weak-* convergence of varifolds implies the convergence of their respective weight measures, i.e. $V_i = (\eta_{x,\lambda_i})_*V \xrightarrow{*} V_\infty$ implies $\mu_{V_i} \xrightarrow{*} \mu_{V_\infty}$, where (λ_i) is infinitesimal. In particular:

$$\begin{aligned}\frac{\mu_{V_\infty}\overline{\mathbb{B}}(0,\rho)}{\omega_k \rho^k} &= \lim_{i\to\infty} \frac{\mu_{V_i}\overline{\mathbb{B}}(0,\rho)}{\omega_k \rho^k} = \lim_{i\to\infty} \frac{1}{\omega_k \rho^k} \int_{F_i^{-1}(\mathbf{G}_k(\overline{\mathbb{B}}(0,\rho)))} \mathbb{J}_W d\eta_{x,\lambda}\, dV(y,W) \\ &= \lim_{i\to\infty} \frac{1}{\omega_k \rho^k} \int_{\mathbf{G}_k(\overline{\mathbb{B}}(x,\lambda\rho))} \mathbb{J}_W d\eta_{x,\lambda}\, dV(y,W) = \lim_{i\to\infty} \frac{\mu_{V_i}\overline{\mathbb{B}}(x,\lambda\rho)}{\omega_k \lambda \rho^k} = \Theta^k(\mu_V, x),\end{aligned}$$

for any $\rho > 0$, where the convergence follows upto passing to a further subsequence. We will first prove a result which guarantees any stationary varifolds to have constant multiplicity on regions of sufficiently high regularity.

Theorem 4.1 (Constancy theorem). *Let $M^k \hookrightarrow \mathbb{R}^n$ be an embedded C^2-submanifold and $\Omega \subset \mathbb{R}^n$ some open subset. Suppose $V \in IV_k(\Omega)$ is stationary in Ω and $\operatorname{supp}(\mu_V) \cap \Omega \subset M$. Then $V = \mathbf{v}(M,\vartheta)$ with $\vartheta \equiv$ const.*

Proof. Let $f \in C^2(\Omega)$ any function such that we have $M \subset \{f = 0\}$. Then by Leibniz's rule:

$$\operatorname{div}_W(fDf)(x) = \langle D_W f(x), Df(x)\rangle_{g_{\mathbb{R}^n}} + f(x)\operatorname{div}_g(Df)(x) = |D_W f|^2 \quad \text{for all } (x, W) \in \operatorname{supp} V$$

since $f \equiv 0$ on M by construction. Thus the stationarity condition tells us:

$$0 = \delta V(fDf) = \int_M \operatorname{div}_W(fDf)(x)dV(x, W) = \int_M |D_W f(x)|^2 dV(x, W),$$

from which we deduce $\pi_W(Df)(x) = D_W f(x) = 0$ whenever $(x, W) \in \operatorname{supp} V$. Now since $M^k \hookrightarrow \mathbb{R}^n$, using a version of the pre-image theorem, we can choose slice coordinates $(x^1, \ldots, x^n) \in C^2(\Omega)$. Then upto reordering, M is locally given by the vanishing set of $x^{k+1}, \ldots, x^n$ and $(T_x M)^\perp = \operatorname{span}_{\mathbb{R}}(\{\partial_i \mid i = k+1, \ldots, n\})$. Thus using the above we can deduce $\pi_W((T_x M)^\perp) = 0$ for all $(x, W) \in \operatorname{supp} V$. By orthogonality, we can see that $W = T_x M$ for all $(x, W) \in \operatorname{supp} V$ and then by duality:

$$\int_M \varphi(x, W)dV(x, W) = \int_M \varphi(x, T_x M)d\mu_V(x) \quad \text{for all } \varphi \in C_c^0(\mathbf{G}_k(\Omega)).$$

Using the monotonicity formula and C^2-regularity of M, writing $\vartheta := \Theta^k(\mu_V, \cdot)$ gives us:

$$\vartheta(x) = \lim_{\rho \downarrow 0} \frac{\mu_V \overline{\mathbb{B}}(x, \rho)}{\omega_k \rho^k} = \lim_{\rho \downarrow 0} \frac{\mu_V \overline{\mathbb{B}}(x, \rho)}{\mathcal{H}^k(\overline{\mathbb{B}}(x, \rho))} = D_{\mu_V}\mathcal{H}^k(x) \quad \text{for } \mu_V\text{-a.e. } x \in M.$$

Thus by the Lebesgue-Besicovitch differentiation theorem, we have:

$$\int_M \varphi(x, W)dV(x, W) = \int_M \varphi(x, T_x M)\vartheta(x)d\mathcal{H}^k(x) \quad \text{for all } \varphi \in C_c^0(\mathbf{G}_k(\Omega)).$$

In particular, $V = \mathbf{v}(M, \vartheta)$ admits a multiplicity function $\vartheta = \Theta^k(\mu_V, \cdot)$. Now taking $X \in C_c^1(\Omega; \mathbb{R}^n)$ such that $X(x) \in T_x M$ for all $x \in M$, then on local neighbourhoods, the stationarity condition reduces to:

$$\delta V(X) = \int_M \operatorname{div}_W(X)(x)dV(x, W) = \int_M \operatorname{div}_g X(x)\vartheta(x)d\mathcal{H}^k(x) = \int D_i X^i \vartheta d\mathcal{H}^k.$$

Now choosing for each fixed $i = 1, \ldots, n$, $X = \varphi e_i$ for an arbitrary $\varphi \in C_c^\infty(\Omega)$, the above tells us:

$$\int_\Omega D_i\varphi\vartheta d\mathcal{H}^k = \delta V(X) = 0,$$

which tells us ϑ is weakly differentiable on Ω with $D\vartheta = 0$ pointwise $\mathcal{H}^k$-a.e. By connectedness of M, ϑ is constant upto some $\mathcal{H}^k$-negligible set. ☐

Theorem 4.2 (First variation rectifiability theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $V \in IV_k(\Omega)$. Suppose V has locally bounded first variation in Ω. Then V is a rectifiable k-varifold.*

Proof. Suppose that $C \in \operatorname{VarTan}(V, x)$. Then C is a stationary varifold in $\mathbb{R}^n$ with constant density ratio at the origin. By the lower-semicontinuity of norm and the weak-* convergence in the assumption, we obtain:

$$|\delta C|(X) \leqslant \liminf_{i \to \infty} |\delta V_i|(X) \leqslant C(V, E)\sup_E |X| \quad \text{for all } X \in C_c^0(E; \mathbb{R}^n),\ E \subset\subset \mathbb{R}^n.$$

In particular, C has locally bounded first variation and thus the monotonicity formula in Theorem 3.3 applies:

$$\frac{\mu_C \overline{\mathbb{B}}(y, r)}{\omega_k r^k} \leqslant \frac{\mu_C \overline{\mathbb{B}}(y, R)}{\omega_k R^k} \quad \text{for all } 0 < r \leqslant R < \infty,\ y \in \mathbb{R}^n.$$

Reformulating the above and using the constancy of density ratio at the origin gives:

$$\frac{\mu_C \overline{\mathbb{B}}(y, r)}{\omega_k r^k} \leqslant \frac{\mu_C \overline{\mathbb{B}}(y, R)}{\omega_k R^k} \leqslant \frac{\mu_C \overline{\mathbb{B}}(0, R + |y|)}{\omega_k (R + |y|)^k}\left(1 + \frac{|y|}{R}\right)^k = \Theta^k(\mu_V, x)\left(1 + \frac{|y|}{R}\right)^k \to \Theta^k(\mu_V, x),$$

as $R \to \infty$ for any $r > 0$, since $\overline{\mathbb{B}}(y, R) \subset \overline{\mathbb{B}}(0, R + |y|)$. Thus again using the monotonicity formula for general k-varifolds with locally bounded first variation, we obtain the following density bound:

$$\Theta^k(\mu_C, y) = \lim_{r \downarrow 0} \frac{\mu_C \overline{\mathbb{B}}(y, r)}{\omega_k r^k} \leqslant \Theta^k(\mu_V, x) \quad \text{for all } y \in \mathbb{R}^n.$$

Recall that by Corollary 3.1, the density function is approximately continuous with respect to the weight measure at μ_V-a.e. $x \in \Omega$. Without loss of generality, we assume μ_V-approximate continuity at x which gives:

$$|\Theta^k(\mu_V, y) - \Theta^k(\mu_V, x)| \leqslant \varepsilon \Rightarrow \Theta(\mu_V, y) \geqslant \Theta^k(\mu_V, x) - \varepsilon \quad \text{for all } y \in K \cap \overline{\mathbb{B}}(x, \rho),$$

where $\rho > 0$ is sufficiently small and $K \subset \Omega$ with $\mu_V(\overline{\mathbb{B}}(x, \rho) \setminus K) \leqslant \varepsilon\rho^k$. We can now pick an infinitesimal sequence (ε_i) and substitute $\rho = \lambda_i$ in the above inequalities, where $V_i = (\eta_{x,\lambda_i})_* V \overset{*}{\rightharpoonup} C$ as $i \to \infty$. Then:

$$\Theta^k(\mu_{V_i}, y) \geqslant \Theta^k(\mu_V, x) - \varepsilon_i \text{ for all } y \in K_i \cap \overline{\mathbb{B}}(0, 1), \text{ where } \mu_{V_i}(\overline{\mathbb{B}}(0, 1) \setminus K_i) \leqslant \varepsilon_i.$$

Thus by Corollary 3.2, we can pass the above to the weak-* limit such that:

$$\Theta^k(\mu_V, x) \geqslant \Theta^k(\mu_C, y) = \lim_{i\to\infty} \Theta^k(\mu_{V_i}, y) \geqslant \lim_{i\to\infty} \Theta^k(\mu_V, x) - \varepsilon_i = \Theta^k(\mu_V, x)$$

In particular, we have for any $y \in \operatorname{supp} \mu_C$ and $r \in (0, \infty)$:

$$\Theta^k(\mu_C, y) = \Theta^k(\mu_V, x) = \frac{\mu_C \overline{\mathbb{B}}(y, r)}{\omega_k r^k},$$

whence using the monotonicity formula again, we have for all $y \in \operatorname{supp} \mu_C$:

$$\pi_{W^\perp}(x - y) = 0 \Rightarrow x - y \in W \quad \text{for all } (x, W) \in \operatorname{supp} C \subset \mathbf{G}_k(\Omega).$$

Now since $0 \in \operatorname{supp} \mu_C = \pi_\Omega(\operatorname{supp} C)$, we can find $T \in \mathbb{G}(k; n)$ such that $(0, T) \in \operatorname{supp} C$ and in particular, $\operatorname{supp} \mu_C \subset T$. The constancy theorem then implies that $C = c\mathcal{H}^k|_M \otimes \delta_T$ for some constant $c \in (0, \infty)$. The collection of tangent cones of V is thus given by:

$$\operatorname{VarTan}(V, x) \subset \{c\mathcal{H}^k|_M \otimes \delta_T \mid T \in \mathbb{G}(k; n),\ c \in (0, \infty)\}.$$

Recall by Lemma 1.2, for μ_V-a.e. $x \in \Omega$, there exists $\eta^x_V \in \mathcal{M}^+(\mathbb{G}(k; n))$ such that for all $\varphi \in C^0_c(\mathbb{G}(k; n))$:

$$\int_{\mathbb{G}(k;n)} \varphi(W) d\eta^x_V(W) = \lim_{\rho\downarrow 0} \frac{1}{\mu_V \overline{\mathbb{B}}(x, \rho)} \int_{\mathbf{G}_k(\overline{\mathbb{B}}(x,\rho))} \varphi(W) dV(y, W).$$

In particular, since for $C = c\mathcal{H}^k|_M \otimes \delta_T \in \operatorname{VarTan}(V, x)$, there exists an infinitesimal sequence (λ_i) such that $(\eta_{x,\lambda_i})_* V \overset{*}{\rightharpoonup} C$, we have for all $i \in \mathbb{N}$:

$$\frac{1}{\mu_{V_i} \overline{\mathbb{B}}(0, 1)} \int_{\mathbf{G}_k(\overline{\mathbb{B}}(0,1))} \varphi(W) dV_i(y, W) = \frac{1}{\mu_V \overline{\mathbb{B}}(x, \lambda_i)} \int_{\mathbf{G}_k(\overline{\mathbb{B}}(x,\lambda_i))} \varphi(W) dV(y, W),$$

for all $\varphi \in C^0_c(\mathbb{G}(k; n))$. In particular, sending $i \to \infty$, we obtain the following by duality:

$$\varphi(T) = \frac{1}{\mu_C \overline{\mathbb{B}}(0, 1)} \int_{\overline{\mathbb{B}}(0,1)} \varphi(W) d(\mu_C \otimes \delta_T) = \int_{\mathbb{G}(k;n)} \varphi(W) d\eta^x_V(W).$$

Thus $C = c\mathcal{H}^k|_M \otimes \delta_T$ is the unique tangent cone in $\operatorname{VarTan}(V, x)$ and thus defines the approximate tangent space of M at x. We can now use the first rectifiability theorem to conclude. □

Remark 4.1. It is possible to construct rectifiable varifolds with locally unbounded first variation. A simple example is to explicitly define a rectifiable varifold $V = \mathbf{v}(M, \vartheta)$ where the multiplicity function is not of bounded variation, i.e. $\vartheta \notin BV(\Omega)$. In particular, the distributional derivative of ϑ does not induce a Radon measure. An example can then be constructed where we have interior blow-up.

Using the lower-semicontinuity of the mass functional, we can prove a compactness result for varifolds of locally bounded first variation, which asserts the preservation of integer multiplicity under varifold convergence. We will first establish a technical lemma for an upper bound on the density function.

Lemma 4.1. *For any $\delta \in (0, 1)$ and $\Lambda \in [1, \infty)$, there exists some $\varepsilon(\delta, \Lambda, k) \in (0, \delta^2)$ such that if V is a rectifiable k-varifold in $\mathbb{B}(0, 3)$ with integer-valued multiplicity and locally bounded first variation such that:*

$$\mu_V \mathbb{B}(0, 3) \leqslant \Lambda; \quad |\delta V|(\mathbb{B}(0, 3)) \leqslant \varepsilon^2; \quad \int_{\mathbb{B}(0,3)} \|\pi_W - \pi_{\mathbb{R}^k\times\{0\}}\| dV(y, W) \leqslant \varepsilon^2,$$

then there exists $E \subset \mathbb{B}^k(0, 1)$ with $\mathcal{H}^k(E) \leqslant \delta$ and:

$$\sum_{y\in\pi^{-1}_{\mathbb{R}^k}(x)\cap\operatorname{supp}(\mu_V)\cap\{|\pi_{\mathbb{R}^{n-1}}|<\varepsilon\}} \Theta^k(\mu_V, y) \leqslant (1 + \delta)\frac{\mu_V \mathbb{B}(0, 2)}{\omega_k 2^k} + \delta \quad \textit{for all } x \in \mathbb{B}^k(0, 1) \cap E.$$

Proof. Fix an arbitrarily small $\varepsilon > 0$ to be determined later. We will consider the following two collections:

$$E_{1,\varepsilon} := \bigcup_{\rho\in(0,1)} \left\{y \in \mathbb{B}(0,2) \cap \operatorname{supp}(\mu_V) \mid |\delta V|(\overline{\mathbb{B}}(y,\rho)) \geqslant \varepsilon \mu_V \overline{\mathbb{B}}(y,\rho)\right\};$$

$$E_{2,\varepsilon} := \bigcup_{\rho\in(0,1)} \left\{y \in \mathbb{B}(0,2) \cap \operatorname{supp}(\mu_V) \mid \int_{\mathbb{B}(y,\rho)} \|\pi_W - \pi_{\mathbb{R}^k}\| dV(\zeta,W) \geqslant \varepsilon\rho^k\right\}.$$

Based on the choice of $\varepsilon > 0$, our varifold $V \in IV_k(\mathbb{B}(0,3))$ is assumed to satisfy:

$$\mu_V \mathbb{B}(0,3); \quad |\delta V|(\mathbb{B}(0,3)) \leqslant \varepsilon^2; \quad \int_{\mathbb{B}(0,3)} \|\pi_W - \pi_{\mathbb{R}^k}\| dV(\zeta,W)) \leqslant \varepsilon^2.$$

Notice if $y \in (\mathbb{B}(0,2) \cap \operatorname{supp}(\mu_V)) \setminus E_1$, we obtain by using the monotonicity formula from Theorem 3.3:

$$\frac{\mu_V \overline{\mathbb{B}}(y,\rho)}{\omega_k \rho^k} \leqslant e^{\varepsilon(1-\rho)} \frac{\mu_V \overline{\mathbb{B}}(y,1)}{\omega_k} \leqslant e^{\varepsilon} \frac{\mu_V \overline{\mathbb{B}}(y,1)}{\omega_k} \leqslant \frac{e^{\varepsilon}}{\omega_k} \mu_V \mathbb{B}(0,3) \leqslant \frac{\Lambda e^{\varepsilon}}{\omega_k} \quad \text{for all } \rho \in (0,1),$$

since $|\delta V|(\overline{\mathbb{B}}(y,\rho)) \leqslant \varepsilon \mu_V \overline{\mathbb{B}}(y,\rho)$ for all $\rho \in (0,1)$ and $\overline{\mathbb{B}}(y,1) \subset \mathbb{B}(0,3)$ by triangle inequality. Now notice for any $y \in E_{2,\varepsilon} \setminus E_{1,\varepsilon}$, there exists some $\rho(y) \in (0,1)$ such that:

$$\int_{\mathbb{B}(y,\rho(y))} \|\pi_W - \pi_{\mathbb{R}^n}\| dV(\zeta,W) \geqslant \varepsilon\rho(y)^k \geqslant \frac{\varepsilon}{e^{\varepsilon}} \mu_V \overline{\mathbb{B}}(y,\rho(y)).$$

On the other hand, if $y \in E_{1,\varepsilon}$, we can find some $\rho(y) \in (0,1)$ with $\varepsilon\mu_V \overline{\mathbb{B}}(y,\rho(y)) \leqslant |\delta V|(\overline{\mathbb{B}}(y,\rho(y)))$. In particular, we obtain a covering $\mathcal{B} = \{\overline{\mathbb{B}}(y,\rho(y)) \mid y \in E_{1,\varepsilon} \cup E_{2,\varepsilon}\}$ by closed balls for the set $E_{1,\varepsilon} \cup E_{2,\varepsilon}$. Then by Besicovitch's covering theorem, we obtain finitely many disjoint sub-families $\mathcal{B}_1, \ldots, \mathcal{B}_{\beta(n)} \subset \mathcal{B}$ such that $E_{1,\varepsilon} \cup E_{2,\varepsilon} \subset \bigcup_{i=1}^{\beta(n)} \bigcup \mathcal{B}_i$. Suppose upto reordering that for some $k \leqslant \beta(n)$, the centres of balls in $\mathcal{B}_i$ lie in $E_{1,\varepsilon}$ for all $i = 1, \ldots, k$. Then we have by disjointness of the sub-families and our assumptions on V:

$$\begin{aligned}
\mu_V(E_{1,\varepsilon} \cup E_{2,\varepsilon}) &\leqslant \sum_{i=1}^{k} \sum_{B\in\mathcal{B}_i} \mu_V B + \sum_{i=k+1}^{\beta(n)} \sum_{B\in\mathcal{B}_i} \mu_V B \\
&\leqslant \sum_{i=1}^{k} \sum_{B\in\mathcal{B}_i} \frac{1}{\varepsilon} |\delta V|(B) + \sum_{i=k+1}^{\beta(n)} \sum_{B\in\mathcal{B}_i} \int_B \frac{e^{\varepsilon}}{\varepsilon} \|\pi_W - \pi_{\mathbb{R}^k}\| dV(\zeta,W) \\
&\leqslant \frac{\beta(n)\max\{1,e^{\varepsilon}\}}{\varepsilon} \left(|\delta V|(\mathbb{B}(0,3)) + \int_{\mathbb{B}(0,3)} \|\pi_W - \pi_{\mathbb{R}^k}\| dV(\zeta,W)\right) \leqslant C(k,\Lambda)\varepsilon.
\end{aligned}$$

We remark that it suffices to show the following a priori estimate for all $x \in \mathbb{B}^k(0,1) \setminus \pi_{\mathbb{R}^k}(E_{1,\varepsilon} \cup E_{2,\varepsilon})$:

$$\sum_{i=1}^{N} \Theta^k(\mu_V, y_i) \leqslant (1+\delta) \frac{\mu_V \overline{\mathbb{B}}(x,2)}{\omega_k 2^k} + \delta \quad \text{for all } (y_i)_{i=1}^N \in \pi_{\mathbb{R}^k}^{-1}(\{x\}) \cap \operatorname{supp}(\mu_V) \cap \{|\pi_{\mathbb{R}^{n-k}}| \leqslant \varepsilon\},$$

where the required statement then follows by picking a suitable choice of $\delta \in (\sqrt{\varepsilon}, 1)$. We will introduce some more notations first: for any $y \in \mathbb{B}^k(0,1) \setminus \pi_{\mathbb{R}^n}(E_{1,\varepsilon} \cup E_{2,\varepsilon})$ and $\tau, \sigma > 0$, define the set:

$$U_\sigma^\tau(y) := \overline{\mathbb{B}}(y,\sigma) \cap \{\zeta \in \mathbb{R}^n \mid |\pi_{\{0\}\times\mathbb{R}^{n-k}}(\zeta - y)| \leqslant \tau\}.$$

Now note for all $W \in \mathbb{G}(k;n)$, the W-tangential derivative $D_W(f|\pi_{\mathbb{R}^{n-k}}(\zeta - y)|)$ satisfies the following:

$$|D_W(f(|\pi_{\mathbb{R}^{n-k}}(\zeta - y)|))| = |f'(|\pi_{\mathbb{R}^{n-k}}(\zeta - y)|) D_W(|\pi_{\mathbb{R}^{n-k}}(\zeta - y)|)| \leqslant f'(|\pi_{\mathbb{R}^{n-k}}(\zeta - y)|)\|\pi_W - \pi_{\mathbb{R}^k}\|,$$

for any $\zeta \in \mathbb{R}^n$. Now by the first monotonicity formula in Theorem 3.3, we obtain for $\tau \geqslant \sigma$:

$$\Theta^k(\mu_V, y) \leqslant e^{\varepsilon\sigma} \frac{\mu_V \overline{\mathbb{B}}(y,\sigma)}{\omega_k \sigma^k} = e^{\varepsilon\sigma} \frac{\mu_V U_\sigma^{2\tau}(y)}{\omega_k \sigma^k} \leqslant e^{\varepsilon\sigma} \frac{\mu_V U_\sigma^{2\tau}(y)}{\omega_k \sigma^k} + C\varepsilon\frac{\sigma}{\tau},$$

where we notice $U_\sigma^{2\tau}(y) = \overline{\mathbb{B}}(y,\sigma)$ since $|\pi_{\mathbb{R}^{n-k}}(\zeta - y)| \leqslant |\zeta - y| \leqslant \sigma < 2\tau$ for any $\zeta \in \overline{\mathbb{B}}(y,\sigma)$. On the other hand, assume $\tau < \sigma$, then we can consider the second monotonicity formula by using a cut-off function $h(\zeta) = f(|\pi_{\mathbb{R}^{n-k}}(\zeta - y)|)$ satisfying $f \equiv 1$ on $(-\infty, \tau)$ and $f \equiv 0$ on $(2\tau, \infty)$. Thus we obtain:

$$\Theta^k(\mu_V, y) \leqslant e^{\varepsilon\tau}\frac{\mu_V\overline{\mathbb{B}}(y,\tau)}{\omega_k\tau^k} \leqslant e^{\varepsilon\sigma}\frac{\mu_V U_\sigma^{2\tau}(y)}{\omega_k\sigma^k} + C\varepsilon\frac{\sigma}{\tau},$$

where we have integrated the second monotonicity formula along with an approximation argument. Similarly, we can integrate the second monotonicity formula and use the a priori bound on the density ratio (which holds at μ_V-a.e. $y \in \operatorname{supp}(\mu_V) \cap \mathbb{B}(0,2)$) to deduce:

$$\frac{\mu_V U_\sigma^{\tau}(y)}{\omega_k\sigma^k} \leqslant e^{\varepsilon\sigma}\frac{\mu_V U_\sigma^{2\tau}(y)}{\omega_k\sigma^k} + C\varepsilon\frac{\sigma}{\tau} \quad \text{for all } 0 < \sigma < \rho < 1.$$

The remainder of the proof follows from an induction argument. ☐

Theorem 4.3 (Allard's compactness theorem)**.** *Let $(V_i) \in IV_k(\Omega)$ with locally bounded first variation in the open subset $\Omega \subset \mathbb{R}^n$ and thus define rectifiable k-varifolds $V_i = \mathbf{v}(M_i, \vartheta_i)$. Suppose that $\vartheta_i \geqslant 1$ μ_{V_i}-a.e. and*

$$\sup_{i\in\mathbb{N}} \mu_{V_i}K + |\delta V_i|(K) \leqslant C(K) < \infty \quad \textit{for all } K \subset\subset \Omega.$$

Then there exists a subsequence (V_{k_i}) and $V_\infty \in IV_k(\Omega)$ of locally bounded first variation in Ω such that $V_{k_i} \overset{*}{\rightharpoonup} V_\infty$ as $i \to \infty$. Furthermore, $V_\infty = \mathbf{v}(M_\infty, \vartheta_\infty)$ with $\vartheta_\infty \geqslant 1$ μ_{V_∞}-a.e. and

$$|\delta V_\infty|(K) \leqslant \liminf_{i\to\infty} |\delta V_{k_i}|(K) \quad \textit{for all } K \subset\subset \Omega.$$

Moreover, if $V_i = \mathbf{v}(M_i, \vartheta_i)$ is a rectfiable k-varifold of integer multiplicity, i.e. $\vartheta_i(x) \in \mathbb{N}$ for $\mathcal{H}^k$-a.e. $x \in \Omega$, then V_∞ also has integer multiplicity.

Proof. This is more or less a direct consequence of what we have established. Indeed, (V_i) admits a weak-* subsequential limit by compactness of Radon measures, which is a priori only a Radon measure $V_\infty \in \mathcal{M}(\mathbf{G}_k(\Omega))$, i.e. a general k-varifold. Now by the lower-semicontinuity property under weak-* convergence, we see that:

$$|\delta V_\infty|(K) \leqslant \liminf_{i\to\infty} |\delta V_{k_i}|(K) \leqslant \sup_{i\in\mathbb{N}} |\delta V_i|(K) < \infty \quad \text{for all } K \subset\subset \Omega,$$

which implies that V_∞ has locally bounded first variation. Thus by the rectifiability theorem, we deduce that V_∞ is in fact a rectifiable k-varifold. Using Corollary 3.2 again, we have the required density lower bound $\Theta^k(\mu_{V_\infty}, \cdot) \geqslant 1$ μ_∞-a.e. on Ω. The more important consequence is when $k \in \mathbb{N}$, the limiting k-varifold necessarily has integer multiplicity if the sequence $(V_i) \in IV_k(\Omega)$ is of integer multiplicity.
We will denote by $V = V_\infty$ the limiting varifold from above. Let $K \subset\subset \Omega$ be arbitrary and consider the subset $E \subset K$ such that for all $N \in \mathbb{N}$ and $x \in E$, there exists $\rho(x) \in (0,\infty)$ and $k(x) \in \mathbb{N}$ satisfying the following condition:

$$N\mu_V\overline{\mathbb{B}}(x,\rho(x)) \leqslant |\delta V_i|(\overline{\mathbb{B}}(x,\rho(x))) \quad \text{for all } i \geqslant k(x).$$

Now $\mathcal{B} = \{\overline{\mathbb{B}}(x,\rho(x)) \mid x \in E\}$ forms a family of closed balls whose centres lie in the bounded subset E. Thus the Besicovitch covering theorem gives us subfamilies of disjoint balls $\mathcal{B}_1, \ldots, \mathcal{B}_{\beta(n)} \subset \mathcal{B}$ such that $E \subset \bigcup_{i=1}^{\beta(n)} \bigcup \mathcal{B}_i$. In particular, writing $E_j = \{x \in E \mid k(x) \leqslant j\} \subset E$ for each $k \in \mathbb{N}$, by disjointness:

$$\mu_V E_j \leqslant \sum_{i=1}^{\beta(n)} \sum_{B\in\mathcal{B}_i} \mu_V B \leqslant \sum_{i=1}^{\beta(n)} \sum_{B\in\mathcal{B}_i} N^{-1}|\delta V_l|(B) = \sum_{i=1}^{\beta(n)} N^{-1}|\delta V_l|(\bigcup \mathcal{B}_i) \leqslant \frac{\beta(n)}{N}|\delta V_l|(K),$$

for all $l \geqslant j \geqslant \sup_{x\in E_j} k(x)$. Thus by noting $\chi_{E_j} \nearrow \chi_E$ as $j \to \infty$, we can deduce:

$$\mu_V E = \lim_{i\to\infty} \mu_V E_i \leqslant \frac{\beta(n)}{N} \limsup_{l\to\infty} |\delta V_l|(K) \leqslant C(K)\frac{\beta(n)}{N} \quad \text{for all } N \in \mathbb{N}.$$

Now sending $N \to \infty$ tells us $\mu_V E = 0$ and thus for μ_V-a.e. $x \in K$, we can find $C(x) \in (0,\infty)$ such that:

$$\liminf_{i\to\infty} |\delta V_i|(\overline{\mathbb{B}}(x,\rho)) \leqslant C(x)\mu_V\overline{\mathbb{B}}(x,\rho) \leqslant C(x)\rho^k \quad \text{where } \rho = \min\{1, \operatorname{dist}(K,\partial\Omega)\}.$$

where the second inequality follows from the monotonicity formula and $\Theta^k(\mu_V, \cdot) \geqslant 1$ μ_V-a.e. on Ω by Corollary 3.2. Now since $V = \mathbf{v}(M, \vartheta)$ is rectifiable, V admits a unique tangent cone since the approximate tangent space $T^k_x M$ exists at μ_V-a.e. $x \in M$. In particular, V admits a unique tangent cone at μ_V-a.e. $x \in M$ with $(\eta_{x,\lambda})_* V \xrightharpoonup{*} \vartheta(x)\mathcal{H}^k|_{T^k_x M} \otimes \delta_{T^k_x M}$. We can thus select an infinitesimal sequence (λ_i) where:

$$(\eta_{x,\lambda_i})_* V_i \to \vartheta(x)\mathcal{H}^k|_{T^k_x M} \otimes \delta_{T^k_x M} \quad \text{and} \quad |\delta((\eta_{x,\lambda_i})_* V_i)|(\mathbb{B}(0,R)) \to 0,$$

for all $R > 0$. Write for simplicity $W_i = (\eta_{x,\lambda_i})_* V_i$ for all $i \in \mathbb{N}$. Note first for all fixed $\varepsilon > 0$:

$$(\pi_{\mathbb{R}^k})_*(W_i \llcorner \mathbf{G}_k(\{|\pi_{\mathbb{R}^{n-k}}| \leqslant \varepsilon\})) \xrightharpoonup{*} \vartheta(x)\mathcal{H}^k \llcorner \mathbb{R}^k \quad \text{as } \lambda \searrow 0,$$

by the weak-* continuity of the push-forward map. Using the mapping formula for varifolds, we also have:

$$\psi_i \mathcal{H}^k \llcorner \mathbb{R}^k \xrightharpoonup{*} \vartheta(x)\mathcal{H}^k \llcorner \mathbb{R}^k \quad \text{where } \psi_i(\zeta) := \sum_{y \in \pi^{-1}_{\mathbb{R}^k}(\{\zeta\}) \cap \{|\pi_{\mathbb{R}^{n-k}}| \leqslant \varepsilon\}} \vartheta_i(y).$$

Using Riesz's theorem, we can reformulate the above convergence by duality:

$$\int_{\mathbb{R}^k} f\psi_i d\mathcal{H}^k \overset{i\to\infty}{\longrightarrow} \vartheta(x) \int_{\mathbb{R}^k} f d\mathcal{H}^k \qquad \text{for all } f \in C^0_c(\mathbb{R}^k).$$

Using a diagonal argument and Lemma 4.1, we can find an infinitesimal sequence (δ_i) such that:

$$\psi_i(\zeta) \leqslant (1+\delta_i)\frac{\mu_{W_i}\mathbb{B}(0,2)}{\omega_k 2^k} + \delta_i = (1+\delta_i)\frac{\mu_{V_i}\mathbb{B}(x,2\lambda_i)}{\omega_k 2^k \lambda^k} + \delta_i \leqslant \Theta^k(\mu_{V_i}, x) + C\delta_i \leqslant \vartheta(x) + C\delta_i,$$

for all $\zeta \in \mathbb{B}(0,1) \backslash E_i$ for some sequence (E_i) with $\mathcal{H}^k(E_i) \to 0$. Thus for all $N \geqslant \vartheta(x)$, we can deduce $\max\{\psi_i, N\} \to \vartheta(x)$ in $L^1(\mathbb{B}(0,1))$ and thus $\mathcal{H}^k$-a.e. This implies that $\vartheta(x) \in \mathbb{Z}$ since $\max\{\psi_i, N\}$ is always integer-valued. This shows that V necessarily has integer-multiplicity. □

The proof method for Allard's integral compactness theorem suggests a method of singularity analysis on stationary varifolds using the homothety map $\eta_{x,\rho}$. We clarify the meaning of singular and regular points below.

Definition 4.2 (Regular point)**.** *Let $V \in IV_k(\Omega)$ be a stationary k-varifold. We say $x \in \operatorname{supp}(\mu_V)$ is a regular point of V if there exists some $\rho > 0$ such that $\operatorname{supp}(\mu_V) \cap \mathbb{B}(x,\rho)$ is in fact a C^1-submanifold. A point x is an (interior) singular point for V if $x \in (\operatorname{supp}(\mu_V) \setminus \operatorname{reg}(V)) \cap \Omega$. The collection of all regular points and singular points of V are denoted by $\operatorname{reg}(V)$ and $\operatorname{sing}(V)$ respectively.*

The constancy theorem guarantees that the density function is locally constant near any regular point and thus constant on connected components of $\operatorname{reg}(V)$. We remark that from the regularity theory for solutions to the minimal surface equation, any C^1-solution is automatically real-analytic. This allows us to equivalently require smooth neighbourhood around a regular point.

5 Extended monotonicity theorem over the exterior cone from the boundary

A generalisation of the monotonicity formula for manifolds with boundary is derived below. The main reference for this subsection is [EWW04], despite that we present a slightly different version of the result.

Definition 5.1 (Strong stationarity)**.** *Let $V = \mathbf{v}(M, \vartheta)$ be a rectifiable k-varifold in $\mathbb{R}^n$ and Γ a closed rectifiable set. We say V is strongly stationary with respect to Γ if for any smooth deformation $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ with $f(0,\cdot) \equiv \operatorname{Id}_{\mathbb{R}^n}$, we have:*

$$\left.\frac{d}{dt}\right|_{t=0} \big(\mathbb{M}((f_t)_* V) + \mathcal{H}^k(f([0,t] \times \Gamma))\big) \geqslant 0 \quad \textit{or equivalently} \quad \delta V(X) = \int_M \operatorname{div}_g X d\mu_V \leqslant \int_\Gamma |X^\perp| d\mathcal{H}^{k-1},$$

where $X = \mathrm{d}f \cdot \partial_t$ is any smooth vector field on $\mathbb{R}^n$.

The equivalence in the above definition comes from a straight-forward computation using the area and coarea formulas. In fact, we will also need another equivalent formulation of strong stationarity, which we state below:

Lemma 5.1. *A rectifiable k-varifold $V = \mathbf{v}(M, \vartheta)$ is strongly stationary with respect to a closed $(k-1)$-rectifiable set Γ if and only if there exists a $\mathcal{H}^{k-1}$-measurable normal vector field ν_Γ on Γ such that:*

$$\delta V(X) = \int_M \operatorname{div}_g X d\mu_V = \int_\Gamma \langle X, \nu_\Gamma \rangle_{g_{\mathbb{R}^n}} d\mathcal{H}^{k-1} \quad \textit{for all } X \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n),$$

and we additionally require $\|\nu_\Gamma\|_{L^\infty(\Gamma, \mathcal{B}(\Gamma), \mathcal{H}^{k-1})} \leqslant 1$. In this case, we have $\Gamma = \operatorname{ess\,supp} \nu_\Gamma$ $\mathcal{H}^{k-1}$-a.e.

Proof. This follows from the Riesz-Markov-Kakutani theorem and the Besicovitch differentiation theorem. ☐

Heuristically, the extended monotonicity formula aims to improve the classical density statement to the boundary of a submanifold. Given a point $x_0 \in \mathbb{R}^n$, where we are interested in the density at this point, the idea relies on constructing the convex cone over the boundary of our surface with vertex x_0. Then using the same argument and an approximation argument, we can effectively show the density ratio is also monotone increasing over the exterior cone. To this end, we first clarify the interpretation of an exterior cone in the varifold case.

Let $\Gamma \subset \mathbb{R}^n$ be a $(k-1)$-rectifiable set in the metric space sense (as in Definition 1.3). Choose $K \subset\subset \mathbb{R}^{k-1}$ and a Lipschitz map $f \colon \mathbb{R}^{k-1} \to \mathbb{R}^n$ with $f(K) = \Gamma$ $\mathcal{H}^k$-a.e. For a given point $p \in \mathbb{R}^n$, the convex cone over Γ with vertex p is given as follows by convexity:

$$C_p(\Gamma) := \{(1-\lambda)p + \lambda x \mid (\lambda, x) \in [0, \infty) \times \Gamma\}.$$

This construction allows us to define a Lipschitz parametrisation of the exterior cone as follows:

$$\widetilde{f_p} \colon [1, \infty) \times K \longrightarrow \mathbb{R}^n, \quad (\lambda, s) \longmapsto (1-\lambda)p + \lambda f(s).$$

It is easy to see that $\Gamma \times [1, \infty)$ defines a countably k-rectifiable subset since any interval is countably 1-rectifiable. Thus, we can see that $\Gamma \times [1, \infty)$ induces a rectifiable k-varifold of constant multiplicity $\vartheta \equiv 1$, which will be denoted by $V_\Gamma = \mathbf{v}(\Gamma \times [1, \infty), 1)$. Now we introduce the adjusted Lipschitz map f_p to define the exterior cone:

$$f_p \colon [1, \infty) \times \Gamma \longrightarrow \mathbb{R}^n, \quad (\lambda, x) \longmapsto (1-\lambda)p + \lambda x$$

Definition 5.2 (Exterior cone over rectifiable set)**.** *The exterior cone $EC_p(\Gamma)$ over $\Gamma \subset \mathbb{R}^n$ with vertex $p \in \mathbb{R}^n$ is defined to be the image varifold of $V_\Gamma = \mathbf{v}(K \times [1, \infty), \vartheta \equiv 1)$ under the Lipschitz map f_p, i.e.*

$$EC_p(\Gamma) \simeq (f_p)_* V_\Gamma := \mathbf{v}(f_p(\Gamma \times [1, \infty)), 1),$$

whereas we will abuse the notation by writing $EC_p(\Gamma) = f_p(\Gamma \times [1, \infty))$ in the sequel.

We are now ready to state the extended monotonicity formula for strongly stationary varifolds:

Theorem 5.1 (Extended monotonicity formula for varifolds)**.** *Let $V \in IV_k(\mathbb{R}^n)$ with locally bounded first variation in $\mathbb{R}^n$ with $\Theta^{*k}(\mu_V, x) > 0$ for μ_V-a.e. $x \in \mathbb{R}^n$ (in particular, $V = \mathbf{v}(M, \vartheta)$ is rectifiable k-varifold) and $\Gamma \subset \mathbb{R}^n$ be a closed countably $(k-1)$-rectifiable set. Suppose $V = \mathbf{v}(M, \vartheta)$ is strongly stationary with respect to Γ. Then for any $p \in \mathbb{R}^n \setminus \Gamma$, we have that the density ratio:*

$$\Theta(M \cup EC_p(\Gamma), p, r) := \frac{1}{\omega_k r^k} \left(\mu_V(\mathbb{B}(p, r) \cap M) + \mathcal{H}^k(\mathbb{B}(p, r) \cap EC_p(\Gamma)) \right) = \frac{\mathbb{M}(V|_{\mathbb{B}(p,r)} + EC_p(\Gamma)|_{\mathbb{B}(p,r)})}{\omega_k r^k},$$

is monotone increasing in r. Furthermore, the density ratio Θ is constant if and only if $M \cup EC_p(\Gamma)$ is conical.

Recall the generalised divergence theorem for manifolds with boundary, using which we can analogously derive a first variation formula and a monotonicity formula with an extra boundary term in either cases.

Lemma 5.2 (Generalised divergence theorem)**.** *Let (M^n, g) be a Riemannian manifold with boundary ∂M and $X \in \mathfrak{T}^0_1(M)$. Then, denoting the Riemannian unit normal vector of ∂M by $\nu_{\partial M} \in \Gamma(N(\partial M))$, we have:*

$$\int_M \operatorname{div}_g X \operatorname{dvol}_g = \int_{\partial M} \langle X^T, \nu_{\partial M} \rangle_g \operatorname{dvol}_{g_{\partial M}} - \int_M \langle X^\perp, \mathbf{H} \rangle_g \operatorname{dvol}_g$$

Proof. First take a g-orthogonal decomposition of $X = X^T + X^\perp$, then this follows from using the Weingarten equation and the usual divergence theorem. ☐

The extension to the varifold case essentially follows from the Riesz representation theorem and the Besicovitch differentiation theorem. Suppose $V \in IV_k(\mathbb{R}^n)$ has locally bounded first variation in $\mathbb{R}^n$. Since we can identify δV as a continuous linear functional on $C_c^0(\mathbb{R}^n;\mathbb{R}^n)$, the Riesz-Markov-Kakutani theorem tells us $\delta V \in \mathcal{M}(\mathbb{R}^n;\mathbb{R}^n)$ induces a vector-valued Radon measure. In particular, since $|\delta V| << \delta V$, we obtain:

$$\delta V(X) = \int_M \operatorname{div}_g X d\mu_V = \langle X, \delta V\rangle = \int_{\mathbb{R}^n} \langle X, \nu\rangle_{Euc} |\delta V| \quad \text{for all } X \in C_c^0(\mathbb{R}^n;\mathbb{R}^n),$$

where the polar $\nu = \frac{d(\delta V)}{d(|\delta V|)}$ is $|\delta V|$-measurable and $|\nu| = 1$ $|\delta V|$-a.e. on $\mathbb{R}^n$. Recall the measure-theoretic decomposition of the first variation distribution δV using the Besicovitch differentiation theorem:

$$\delta V = -\mathbf{H}\, d\mu_V + d\sigma, \quad \text{ where } \begin{cases} E := (\mathbb{R}^n \setminus M) \cup \{x \in M \mid D_{\mu_V}|\delta V|(x) = \infty\}; \\ \sigma = |\delta V| \llcorner E; \end{cases}$$

where we also recall that $\mu_V E = 0$. From the above we get via Riesz's theorem and the first variation formula:

$$\int_M \operatorname{div}_g X d\mu_V = \delta V(X) = -\int_M \langle X, \mathbf{H}\rangle_{Euc} d\mu_V + \int_E \langle X, \nu\rangle_{Euc} |\delta V|.$$

The above identity is the generalisation of Lemma 5.2. By analogy, we refer to E as the generalised boundary, $|\delta V| \llcorner E$ the generalised boundary measure, and ν the generalised co-normal for V with respect to E.
Suppose now that $V \in IV_k(\mathbb{R}^n)$ has locally bounded first variation and strongly stationary with respect to some $(k-1)$-rectifiable set $\Gamma \subset \mathbb{R}^n$. Notice that, as pointed out in [EWW04, Section 7], strong stationarity is simply the usual stationarity in $\mathbb{R}^n \setminus \Gamma$ along with a boundary condition on Γ. Indeed, for any C^1-deformation fixing Γ, the corresponding variation field would be zero in a neighbourhood of Γ. In particular, V would have zero generalised mean curvature μ_V-a.e. on $\mathbb{R}^n$ since E is $\mathcal{H}^k$-negligible. Thus, the above identity gives us:

$$\int_{\mathbb{R}^n} \langle X, \nu_\Gamma d\mathcal{H}^{k-1}\rangle = \delta V(X) = \int_{\mathbb{R}^n} \langle X, (\nu \cdot |\delta V| \llcorner E)\rangle \quad \text{for all } X \in C_c^0(\mathbb{R}^n;\mathbb{R}^n).$$

The uniqueness statement in the Riesz-Markov-Kakutani theorem allows us to deduce that Γ lies in the generalised boundary of V and $\nu \cdot |\delta V| \llcorner E = \nu_\Gamma d\mathcal{H}^{k-1}$ using the following characterisation: for all open $W \subset\subset \mathbb{R}^n$

$$(\nu|\delta V| \llcorner E)(W) = \sup_{X \in C_c^0(W;\mathbb{S}^{n-1})} \langle X, (\nu \cdot |\delta V| \llcorner E)\rangle = \sup_{X \in C_c^0(W;\mathbb{S}^{n-1})} \langle X, \nu_\Gamma d\mathcal{H}^{k-1}\rangle = (\nu_\Gamma \mathcal{H}^{k-1})(W).$$

Proof of the extended monotonicity formula. Consider the variation field $X(x) := \varphi_\varepsilon(\frac{r}{s})(x-p)$ as described in the derivation of the monotonicity formula for an arbitrary $s > 0$, which gives us as before for $\mathcal{H}^k$-a.e. $x \in M$:

$$\operatorname{div}_g X(x) = \varphi_\varepsilon\left(\frac{r(x)}{s}\right) \operatorname{div}_g((x-p)) + r(x)\varphi_\varepsilon'\left(\frac{r(x)}{s}\right)|\nabla^g r|^2 = k \cdot \varphi_\varepsilon\left(\frac{r}{s}\right) - s\frac{\partial}{\partial s}\left(\varphi_\varepsilon\left(\frac{r}{s}\right)\right) \cdot |\nabla^g r|^2,$$

The decomposition of the first variation distribution and strong stationarity then gives us:

$$k\int_M \varphi_\varepsilon\left(\frac{r}{s}\right) d\mu_V - \rho\frac{\partial}{\partial s}\left(\int_M \varphi_\varepsilon\left(\frac{r}{s}\right)|\nabla^g r|^2\, d\mu_V\right) = \int_\Gamma \langle \varphi_\varepsilon\left(\frac{r}{s}\right)(x-p), \nu_\Gamma\rangle_{Euc}\, d\mathcal{H}^{k-1}.$$

Then using integration by parts and proceed as in the proof for the usual monotonicity formula, we get:

$$\frac{\partial}{\partial s}\left(s^{-k}\mathcal{F}_\varepsilon(s)\right) = s^{-k}\frac{\partial}{\partial s}\left(\int_M \varphi_\varepsilon\left(\frac{r}{s}\right)|\nabla^\perp r|^2 d\mu_V\right) - s^{-(k+1)}\int_\Gamma \langle x-p, \nu_\Gamma\rangle_{Euc} d\mathcal{H}^{k-1},$$

where as usual, we denote by $\mathcal{F}_\varepsilon(s) := \int_M \varphi_\varepsilon(\frac{r}{s}) d\mu_V$. For simplicity, we write for the remainder of the proof $E := EC_p(\Gamma)$. Notice that the weight measure for the exterior cone $EC_p(\Gamma)$ over Γ with vertex p is given by:

$$\mu_E := \mu_{EC_p(\Gamma)} = \mathcal{H}^k \llcorner f_p(\Gamma \times [1,\infty)).$$

The k-rectifiability of Γ and regularity of f_p imply by the means of the first variation formula:

$$\begin{aligned}\frac{\partial}{\partial s}\left(s^{-k}\int_E \varphi_\varepsilon\left(\frac{r}{s}\right) d\mathcal{H}^k\right) &= s^{-k}\frac{\partial}{\partial s}\left(\int_E \varphi_\varepsilon\left(\frac{r}{s}\right)|\nabla^\perp r|^2 d\mathcal{H}^k\right) \\ &\quad - s^{-(k+1)}\left(\int_E \langle \varphi_\varepsilon\left(\frac{r}{s}\right)(x-p), \mathbf{H}\rangle_{g_{\mathbb{R}^n}} d\mathcal{H}^k + \int_{\partial E} \langle \varphi_\varepsilon\left(\frac{r}{s}\right)(x-p), \nu_E \cdot |\delta V_E|\rangle\right),\end{aligned}$$

where we denote by $\partial E = \partial EC_p(\Gamma)$ the generalised boundary and $\nu_E|\delta V_E| \llcorner \partial EC_p(\Gamma)$ the generalised boundary measure for the exterior cone $EC_p(\Gamma)$. Notice by construction, the generalised mean curvature vector is orthogonal to the exterior cone while $X = \varphi_\varepsilon(\frac{r}{s})(x-p)$ is tangent. Thus the integral involving the mean curvature vanishes. Using the same argument, the normal derivative of the distance function from the vertex p also vanishes over the cone. Thus we obtain the following:

$$\frac{\partial}{\partial s}\left(s^{-k}\int_E \varphi_\varepsilon\left(\frac{r}{s}\right)d\mathcal{H}^k\right) = -s^{-(k+1)}\int_{\partial E}\langle\varphi_\varepsilon\left(\frac{r}{s}\right)(x-p), \nu_E\cdot|\delta V_E|\rangle.$$

At this stage, a geometric approximation argument is needed. Suppose for now that Γ is a smooth $(k-1)$-submanifold in $\mathbb{R}^n$. In this case, the exterior cone E is a k-dimensional manifold with boundary $\partial E = \Gamma$, whence by the classical divergence theorem and following the above procedure:

$$\frac{\partial}{\partial s}\left(s^{-k}\int_E \varphi_\varepsilon\left(\frac{r}{s}\right)d\mathcal{H}^k\right) = -s^{-(k+1)}\int_{\Gamma}\langle\varphi_\varepsilon\left(\frac{r}{s}\right)(x-p), \nu_E\rangle_{g_{\mathbb{R}^n}}d\mathcal{H}^{k-1},$$

where ν_E denotes the Riemannian unit normal on Γ. For a general closed k-rectifiable set Γ, we can choose a compact exhaustion of $f_p(\Gamma\times[1.\infty))$ by smooth domains (E_i) with boundary $\partial E_i = \Gamma_i$ such that Γ_i converges to Γ as Radon measures. In particular, the associated sequence of multiplicity 1 varifolds converges to the exterior cone $EC_p(\Gamma)$ in the weak-* sense. Denoting the weight measures by $\mu_{E_i} = \mu_{V_{E_i}}$, this gives us:

$$\Theta(E_i, p, s) = \frac{\mu_{E_i}\overline{\mathbb{B}}(p,s)}{\omega_k s^k} \overset{i\to\infty}{\longrightarrow} \Theta(EC_p(\Gamma), p, s) \quad \text{for } \mathcal{L}^1\text{-a.e. } s\in(0,\infty).$$

Using the same argument as in the usual monotonicity formula, we get by sending $\varepsilon \searrow 0$ in the weak sense:

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\mu_V\overline{\mathbb{B}}(p,s)}{s^k} + \frac{\mathcal{H}^k(E_i\cap\overline{\mathbb{B}}(p,s))}{s^k}\right) &= \frac{\partial}{\partial s}\left(\int_{\overline{\mathbb{B}}(p,s)}\frac{|\nabla^\perp r|^2}{r^k}d\mu_V\right)\\ &\quad - s^{-(k+1)}\left(\int_{\Gamma_i\cap\overline{\mathbb{B}}(p,s)}\langle x-p, \nu_{E_i}d\mathcal{H}^{k-1}\rangle + \int_{\Gamma\cap\overline{\mathbb{B}}(p,s)}\langle x-p,\nu_\Gamma d\mathcal{H}^{k-1}\rangle\right),\end{aligned}$$

whereas the boundary terms can be rearranged as follows since (Γ_i) approximates Γ from below:

$$s^{-(k+1)}\int_{\Gamma_i\cap\overline{\mathbb{B}}(p,s)}\langle x-p, \nu_{E_i}+\nu_\Gamma\rangle_{g_{\mathbb{R}^n}}d\mathcal{H}^{k-1} + s^{-(k+1)}\int_{(\Gamma\setminus\Gamma_i)\cap\overline{\mathbb{B}}(p,s)}\langle x-p,\nu_\Gamma d\mathcal{H}^{k-1}\rangle.$$

The second integral above tends to zero as $i\to\infty$ since Γ_i converges to Γ as Radon measures. Now consider the following linear functional defined on the normal space at an arbitrary point $x\in\Gamma$:

$$N_x\Gamma = (T_x\Gamma)^{\perp_{g_{\mathbb{R}^n}}} \longrightarrow \mathbb{R}, \quad \nu \longmapsto \langle x-p,\nu\rangle_{g_{\mathbb{R}^n}}.$$

Restricting the above linear functional to the unit normal space $N_x^1\Gamma := \{\nu\in N_x\Gamma \mid |\nu|_{g_{\mathbb{R}^n}} = 1\}$, we claim that its maximum is attained at $\nu = \nu_{E_i}$. Indeed, for any $\nu\in N_x^1\Gamma$, consider the $g_{\mathbb{R}^n}$-orthogonal decomposition:

$$T := T_x\Gamma_i \oplus (\mathrm{span}_{\mathbb{R}}(\{\nu\})), \quad \text{where } \nu\perp_{g_{\mathbb{R}^n}} T_x\Gamma_i,$$

via a dimension argument. Thus, via $g_{\mathbb{R}^n}$-orthonormality and the Pythagoras theorem, we get for any $X\in T_xE_i$:

$$|X|^2_{g_{\mathbb{R}^n}} \geqslant |\pi_T(X)|^2_{g_{\mathbb{R}^n}} = |\pi_{T_x\Gamma}(X)|^2_{g_{E_i}} + \langle X,\nu\rangle^2_{g_{\mathbb{R}^n}},$$

where equality is attained at $\nu = \pm\nu_{E_i}$. Since $x-p$ is tangent to T_xE_i, we have in particular for all $x\in\Gamma_i$:

$$\langle x-p,\nu_{E_i}+\nu_\Gamma\rangle_{g_{\mathbb{R}^n}} \leqslant \langle x-p, \nu_{E_i}+(-\nu_{E_i})\rangle_{g_{\mathbb{R}^n}} = 0.$$

Thus for any $i\in\mathbb{N}$, we have the following fundamental inequality for $\mathcal{L}^1$-a.e. $s>0$:

$$\frac{\partial}{\partial s}\left(\frac{\mu_V\overline{\mathbb{B}}(p,s)}{s^k} + \frac{\mathcal{H}^k(E_i\cap\overline{\mathbb{B}}(p,s))}{s^k}\right) \geqslant \frac{\partial}{\partial s}\left(\int_{\overline{\mathbb{B}}(p,s)}\frac{|\nabla^\perp r|^2}{r^k}d\mu_V\right),$$

which we can integrate from r to R with respect to s to get by the means of the fundamental theorem of calculus:

$$\frac{\mu_V\overline{\mathbb{B}}(p,R)}{R^k} + \frac{\mathcal{H}^k(E_i\cap\overline{\mathbb{B}}(p,R))}{R^k} - \left(\frac{\mu_V\overline{\mathbb{B}}(p,r)}{r^k} + \frac{\mathcal{H}^k(E_i\cap\overline{\mathbb{B}}(p,r))}{r^k}\right) \geqslant \int_{\mathbb{A}_{r,R}(p)}\frac{|\nabla^\perp r|^2}{r^k}d\mu_V.$$

Passing the above inequality to the weak-* limit and the convergence of the density ratio gives us:

$$\Theta(M \cup EC_p(\Gamma), p, R) - \Theta(M \cup EC_p(\Gamma), p, r) \geqslant \int_{\mathbb{A}_{r,R}(p)} \frac{|\nabla^\perp r|^2}{\omega_k r^k} d\mu_V,$$

which is the desired monotonicity result. $\square$

Remark 5.1. The original result in [EWW04] is a stronger result, merely requiring rectifiability of the varifold. A detailed proof of the results was however not given for the varifold case. The assumption on locally bounded first variation allows us to use the same proof technique used for minimal surfaces. I am not sure whether there is a cleaner way to present the approximation argument. The main idea is to understand the simpler proof for the smooth case (presented below), then the generalisation to the varifold case should simply follow via an approximation argument along with the convergence in the varifold sense.

Proof of the extended montonicity formula in the smooth case. Suppose $M^k \overset{\iota}{\hookrightarrow} \mathbb{R}^n$ is a smooth k-dimensional minimal submanifold with boundary $\partial M = \Gamma$. Then consider the vector field on $\mathbb{R}^n$ given by $X := x - p$ for some given point $p \in \mathbb{R}^n \setminus \Gamma$. Denoting $M_s := M \cap \mathbb{B}(p, s)$, by the classical divergence theorem:

$$k \operatorname{Vol}_g(M_s) = \int_{M_s} \operatorname{div}_g X \operatorname{dvol}_g = \int_{\partial M_s} \langle x - p, \nu_{\partial M_s} \rangle_g \operatorname{dvol}_{g_{\partial M}},$$

since the mean curvature vanishes by minimality of M. Consider the exterior cone $E = E_p\Gamma$ over $\Gamma = \partial M$ with vertex p. We have $X|_x \in T_x E$ and $\mathbf{H} \perp_{g_{\mathbb{R}^n}} TE$ by construction. Denoting by $E_s = E \cap \mathbb{B}(p, s)$, the generalised divergence theorem again gives us:

$$k \operatorname{Vol}_g(E_s) = \int_{\partial E_s} \langle x - p, \nu_{\partial E_s} \rangle_g \operatorname{dvol}_{g_{\partial E}}.$$

Now notice that we can decompose the boundary components as follows:

$$\partial M_s = (\partial M \cap \mathbb{B}(p, s)) \cup (M \cap \partial\mathbb{B}(p, s)) = \Gamma_s \cup M_s \quad \text{and} \quad \partial E_s = (\partial E \cap \mathbb{B}(p, s)) \cup (E \cap \partial\mathbb{B}(p, s)) = \Gamma_s \cup E_s.$$

In particular, combining the volume integrals gives us:

$$k \operatorname{Vol}_g((M \cup E) \cap \mathbb{B}(p, s)) \leqslant \int_{(\partial(M \cup E))_s} \langle x - p, \nu_{\partial(M \cup E)} \rangle_g \operatorname{dvol}_{g_{\partial(M \cup E)}} + \int_{\Gamma_s} \langle x - p, \nu_{\partial M_s} + \nu_{\partial E_s} \rangle_{g_{\mathbb{R}^n}} \operatorname{dvol}_{g_{\partial M}},$$

for which we conclude the non-positivity of the integral over Γ_s as in the proof above:

$$k \operatorname{Vol}_g((M \cup E) \cap \mathbb{B}(p, s)) \leqslant \int_{(\partial(M \cup E))_s} \langle x - p, \nu_{\partial(M \cup E)} \rangle_g \operatorname{dvol}_{g_{\partial(M \cup E)}} \leqslant s \operatorname{Vol}_{g_{\partial(M \cup E)}}(\partial(M \cup E)_s),$$

where the last inequality follows from the Cauchy-Schwarz inequality. On the other hand, via the coarea formula:

$$\frac{\partial}{\partial s} \operatorname{Vol}_g((M \cup E)_s) = \frac{\partial}{\partial s} \left(\int_0^s \int \chi_{\partial(M \cup E)_r} d\mathcal{H}^{k-1} dr \right) = \mathcal{H}^{k-1}(\partial(M \cup E)_s) \geqslant \mathcal{H}^{k-1}((\partial M \cup E)_s).$$

Now combining the above inequalities gives us:

$$\frac{\partial}{\partial s} \left(\mathcal{H}^k((M \cup E)_s) \right) - \frac{k}{s} \mathcal{H}^k((M \cup E)_s) \geqslant \frac{\partial}{\partial s} \left(\mathcal{H}^k((M \cup E)_s) \right) - \mathcal{H}^{k-1}((\partial(M \cup E))_s) \geqslant 0.$$

Using a technique with an integrating factor gives us the monotonicity statement:

$$\frac{\partial}{\partial s} \left(s^{-k} \mathcal{H}^k((M \cup E) \cap \mathbb{B}(p, s)) \right) = r^{-k} \frac{\partial}{\partial s} \left(\mathcal{H}^k((M \cup E)_s) \right) - k s^{-(k+1)} \mathcal{H}^k((M \cup E)_s) \geqslant 0,$$

from which we deduce the density ratio $s \mapsto \Theta^k(M \cup E, p, s)$ is monotone increasing for fixed $p \in \mathbb{R}^n \setminus \partial M$. $\square$

6 Embeddedness of minimal surfaces

Definition 6.1 (Total curvature for continuous paths)**.** *Let $\Gamma \subset \mathbb{R}^n$ be a polygonal path with vertices $(p_i \mid i = 1, \ldots, N)$. Denote by $\theta(p, \Gamma)$ the exterior angle at $p \in \Gamma$ with respect to the path Γ. Then the total curvature of Γ is given by:*

$$k(\Gamma) := \sum_{i=1}^{N} \theta(p_i, \Gamma).$$

For a general closed continuous path Γ, we define its total curvature by:

$$k(\Gamma) = \sup\{k(\Gamma') \mid \Gamma' \textit{ polygonal path inscribed by } \Gamma\}.$$

Remark 6.1. The above coincides with the classical definition when our path is C^2-regular, given by:

$$k(\Gamma) = \int_0^L k_\gamma(s) ds,$$

where $\gamma \colon [0, L] \to \mathbb{R}^n$ is the arclength parametrisation for Γ and $k_\gamma(s) = |\gamma''(s)|$ is the curvature of Γ at the point $\gamma(s)$. We can convince ourselves of the equivalence of the two definitions via a monotone class argument (note this is not a proof, for a proof, refer to the work by Milnor in [Mil50]).

One of the consequences of extended monotonicity theorem is the μ_V-a.e. existence of the density $\Theta^k(\mu_V, p)$ upto and including the exterior cone over Γ. The monotonicity statement was used to prove a long-open conjecture in the study of classical minimal surfaces in [EWW04], which we will discuss below. By a minimal surface M, we refer to a minimal immersion, i.e. a smooth map $f \colon M \to \mathbb{R}^n$ such that the differential $df_x \colon T_x M \to T_{f(x)}\mathbb{R}^n$ is injective for all $x \in M$ and defines a critical point to the mass functional.

Proposition 6.1. *Suppose $\Gamma \subset \mathbb{R}^n$ is a closed path and $p \in \mathbb{R}^n \setminus \Gamma$. Denoting the radial projection onto the unit sphere centred at p by:*

$$\pi_p \colon \mathbb{R}^n \setminus \mathbb{R}^n \setminus \Gamma \longrightarrow \partial\mathbb{B}(p, 1), \quad \pi_p(x) := p + \frac{x - p}{|x - p|}.$$

Then the length of the radial projection $\pi_p\Gamma$ is bounded above by its total curvature $k(\Gamma)$:

$$\Theta^1(C_p\Gamma, p) = \lim_{r \downarrow 0} \frac{\mathcal{H}^1(\pi_p\Gamma \cap \mathbb{B}(p, r))}{2\pi r} \leqslant \frac{1}{2\pi} k(\Gamma).$$

Proof. Note any polgonal path can be approximated by smooth curves. Using the definition of the total curvature for an arbitrary continuous path, it suffices to consider the case where Γ is a smooth curve. Upto dilations about p, we can assume $\Gamma \cap \mathbb{B}(p, 1) = \emptyset$. Consider the set $A = A_{p,\Gamma} = C_p\Gamma \cap (\mathbb{R}^n \setminus \mathbb{B}(p, 1))$, which describes the annular region between Γ and its projection $\pi_p\Gamma$. Then by the global Gauss-Bonnet theorem:

$$0 = 2\pi\chi(A) = \int_{\partial A} \langle \mathbf{k}_{\partial A}, \mathbf{n} \rangle ds + \int_A K_A d\,\mathrm{vol}_{g_A} = \int_{\pi_p\Gamma} \langle \mathbf{k}_{\pi_p\Gamma}, \mathbf{n} \rangle ds + \int_\Gamma \langle \mathbf{k}_\Gamma, \mathbf{n} \rangle ds,$$

where we have used that A has zero Gaussian curvature and Euler characteristic zero. Thus we have:

$$\mathcal{H}^1(\pi_p\Gamma) = \int_{\pi_p\Gamma} \langle \mathbf{k}_{\pi_p\Gamma}, \mathbf{n} \rangle ds \leqslant \int_\Gamma k_\Gamma ds = k(\Gamma),$$

which is the required result. Note if we canonically embed $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$, then the annulus $A_{p,\Gamma}$ associated with $p \in \mathbb{R}^{n+1}$ will be smooth. In particular, the above holds for all $p \in \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$. ∎

Theorem 6.1 (Density upper bound)**.** *Let $V = \mathbf{v}(M, \vartheta)$ be a rectifiable k-varifold with locally bounded first variation and $\Gamma \subset \mathbb{R}^n$ a $(k-1)$-rectifiable set. Suppose that $\mathcal{H}^{k-1}(\Gamma) < \infty$ and V is strongly stationary with respect to Γ, then:*

$$\Theta^k(\mu_V, p) \leqslant \Theta^k(C_p(\Gamma), p) \quad \textit{for all } p \in \mathbb{R}^n,$$

with equality only when M is conical.

Proof. Assume without loss of generality $\Theta^k(C_p\Gamma, p) < \infty$. Then using the extended monotonicity theorem:

$$\Theta^k(M \cup EC_p\Gamma, p, r) \leqslant \Theta^k(M \cup EC_p\Gamma, p, R) \quad \text{for all } 0 < r < R < \infty.$$

Now simply sending $r \to 0$ and $R \to \infty$ gives us the statement for all $p \in \mathbb{R}^n \setminus \Gamma$. If $p \in \Gamma$ is a boundary point, then the same proof follows through as long as we can show that $\Theta^k(M \cup EC_p\Gamma, p, r) \to \Theta^k(\mu_V, p)$ as $r \searrow 0$. We can first notice that the density ratio can be splitted as follows by strong stationarity:

$$\Theta^k(M \cup EC_p\Gamma, p, r) = \frac{\mu_V \mathbb{B}(p,r)}{\omega_k r^k} + \frac{\mathcal{H}^k(EC_p\Gamma \cap \mathbb{B}(p,r))}{\omega_k r^k} = \Theta^k(\mu_V, p, r) + \Theta^k(EC_p\Gamma, p, r).$$

An intuition is that if Γ is piecewise smooth, the density ratio of the exterior cone at $p \in \Gamma$ clearly decreases to zero as $r \searrow 0$. For a general rectifiable set, we notice for all $r \in (0, 1]$, we have:

$$EC_p\Gamma \cap \mathbb{B}(p,r) \subset C_p\pi_p(\Gamma \cap \mathbb{B}(p,r)),$$

where we denote by π_p the radial projection onto the unit sphere centred at p. Thus:

$$\Theta^k(EC_p\Gamma, p, r) \leqslant \omega_k^{-1}\mathcal{H}^{k-1}(\pi_p(\Gamma \cap \mathbb{B}(p,r))) \quad \text{for all } r \in (0,1].$$

Since Γ is $(k-1)$-rectifiable, we can find countably many Lipschitz maps $\varphi_i \colon \Omega_i \subset \mathbb{R}^{k-1} \to \mathbb{R}^n$ such that $\Gamma \subset \bigcup_{i=1}^\infty \varphi_i(\Omega_i)$ upto some $\mathcal{H}^{k-1}$-negligible subset. Now notice that since $\mathcal{H}^{k-1}(\pi_p\Gamma) < \infty$, the restriction $\mathcal{H}^k \llcorner \pi_p\Gamma \in \mathcal{M}^+(\Omega_i)$ in fact defines a Borel measure on the domain of each Lipschitz map φ_i. By the regularity of Borel measures, $(\varphi_i^{-1}(\mathbb{R}^n \setminus \mathbb{B}(p,r)) \mid r > 0)$ forms an exhaustion for Ω_i and therefore we have:

$$\mathcal{H}^{k-1}(\pi_p(\Gamma \cap \mathbb{B}(p,r))) \leqslant [\varphi_i]_{C^{0,1}(\Omega_i)}\mathcal{H}^{k-1}(\varphi_i^{-1}(\Gamma \cap \mathbb{B}(p,r))) \xrightarrow{i \to \infty} 0.$$

Since the above holds for all φ_i, we can conclude that the density ratio of the exterior cone decreases to zero. $\square$

The above theorems can be applied to the classical theory of minimal surfaces to prove the non-existence of branch points.

Theorem 6.2 (Interior regularity)**.** *Let $\Gamma \subset \mathbb{R}^n$ be a simple closed continuous path with $k(\Gamma) \leqslant 4\pi$. Suppose M is a classical minimal surface with boundary $\partial M = \Gamma$. Then the interior of M is necessarily embedded and admits no interior branch point.*

Proof. This relies on the well-known fact that the density of a minimal surface at any interior branch point or self-intersection point must be at least 2 (see for instance Colding-Minicozzi). Any continuous path with finite total curvature is necessarily rectifiable and in particular of finite length. We can first rule out some easier cases. If M is contained in a cone, then both the mean curvature and the scalar curvature need to vanish. Then M lies in some 2-plane, which makes this case trivial. In general, we can use the above density upper bound and Proposition 6.1 to deduce:

$$\Theta^2(M, p) \leqslant \Theta^2(C_p\Gamma, p) \leqslant \frac{1}{2\pi}k(\Gamma) \leqslant 2,$$

where the equality holds only when M lies in some cone. On the other hand, the strictly inequality shows that p cannot be a branch point nor a self-intersection point. Thus M is necessarily embedded. $\square$

We will now arrive at the main theorem of this section.

Theorem 6.3 (Ekholm-White-Wienholtz)**.** *Let $\Gamma \subset \mathbb{R}^n$ be a simple closed curve with total curvature bounded above by 4π. Suppose M is a minimal surface with $\partial M = \Gamma$ in $\mathbb{R}^n$. Then M is necessarily embedded upto and including the boundary without any interior branch points.*
Furthermore, for any $p \in \Gamma$ with exterior angle $\theta_(p, \Gamma)$, we have the following result on the density of M:

$$\Theta^2(M, p) \in \left\{ -\frac{\theta}{2\pi} \frac{\theta}{2\pi} \right\}.$$

If $p \in \Gamma$ is a cusp, i.e. $\theta(p, \Gamma) = \pi$, then $\Theta^2(M, p) = 0$ unless the boundary Γ is contained in a 2-plane.

Proof in a simpler case. We have already seen the embeddedness of the interior and excluded the existence of interior branch points as discussed above. It now remains to prove the smooth embeddedness of the boundary. Suppose for simplicity that $\Gamma = \partial M$ is in fact smooth curve in $\mathbb{R}^n$. Now for $r > 0$ sufficiently small, set a_r, b_r to be the intersection points of Γ with the r-sphere $\partial\mathbb{B}(p, r)$. Denote by $\theta(r, p)$ the exterior angle of the triangle Δ_{apb} at p and choose some $q \in \mathbb{R}^n$ sufficiently close to

p with $p \in \Delta_{aqb}$ while $p \notin [a, q] \cup [q, b]$.
Now consider the path $\Gamma_{r,q}$ which replaces $\Gamma \cap \mathbb{B}(p, r)$ with the segments $[a, q]$ and $[q, b]$. Then by density upper bound:

$$\mathcal{H}^1(\pi_p(\Gamma \setminus \mathbb{B}(p, r))) + \pi + \theta(r, q) = \mathcal{H}^1(\pi_p \Gamma_{r,q}) \leqslant k(\Gamma_{r,q}),$$

by noticing that $[a, q] \cup [q, b]$ is a geodesic of length $\pi + \theta(r, q)$. Now letting q approach p gives us:

$$\mathcal{H}^1(\pi_p(\Gamma \setminus \mathbb{B}(p, r))) + \pi + \theta(r, p) \leqslant k(\Gamma_{r,p}) \leqslant k(\Gamma),$$

for any $r > 0$ sufficiently small. Now sending $r \searrow 0$ gives us the following upper bound:

$$\mathcal{H}^1(\pi_p \Gamma) = \lim_{r \downarrow 0} \mathcal{H}^1(\pi_p(\Gamma \setminus \mathbb{B}(p, r))) \leqslant k(\Gamma) - \pi - \theta(p, \Gamma).$$

In particular, assuming as usual that M is not conical. Then we have:

$$\Theta^2(M, p) \leqslant \Theta^2(C_p \Gamma, p) \leqslant \frac{1}{2\pi} k(\Gamma) - \frac{1}{2} \leqslant 2 - \frac{1}{2} = \frac{3}{2}.$$

The above density upper bound gives us the non-existence of boundary branch point or self-intersection point since any such points would have density at least $\frac{3}{2}$. $\square$

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