Convex Integration: From Nash embedding to Euler equations

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1 Introduction

We construct weak solutions to the homogeneous incompressible Euler equations using the method of convex integration, i.e. We find $(u,p) \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t, \mathbb{R}^n) \times L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ solving:

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0 \text{ and } \nabla \cdot u = 0. \tag{1}$$

The resulting solutions are compactly supported in spacetime. Inspired by the proof of the Nash-Kuiper theorem, these irregular solutions are generated from oscillatory solutions to an inhomogeneous system. More precisely, the desired solution pair (u,p) arise from two sequences $(u_k)_{k\in\mathbb{N}}, (p_k)_{k\in\mathbb{N}} \in \mathcal{C}^\infty_c$, where $\forall k$, we have:

$$\partial_t u_k + \nabla \cdot (u_k \otimes u_k) + \nabla p_k = f_k \text{ and } \nabla \cdot u_k = 0, \tag{2}$$

where $(f_k)_{k\in\mathbb{N}} \in \mathcal{C}^\infty_c$ converges weakly in the Sobolev space $H^1 = W^{1,2}$ i.e. $f_k \to 0$ in H^{-1}; and we have $(u_k, p_k) \to (u,p)$ in L^q for every $q \in [1,\infty)$. We follow the idea from [LS09] in the sequel.
The oscillatory elements originate from the planewave framework by L.Tartar, where we rewrite the Euler equation as a system of differential inclusions. Each element in the sequence is then constructed iteratively by adding planewaves to the preceding element. More precisely, given u_k, we establish a new sequence $(u_{k,j})_{j\in\mathbb{N}_0}$ by perturbing(using planewaves) $u_{k,0} := u_k$. This construction is parallel to the stages established in Nash's proof in [Nas54] for the existence of $\mathcal{C}^1$-isometric immersions. The intuition comes from by perturbing using oscillatory elements, only the nonlinear differential terms are affected while the linear information is preserved. The rough idea is that the sequence of inhomogeneous solutions can be perturbed to converge to a homogeneous solution to the Euler equation.

2 Set-up

2.1 Statement of the main theorem

We state the main theorem here:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n_x \times \mathbb{R}_t$ be an open, bounded domain. Then $\exists$ weak solutions $(u,p) \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ for the incompressible Euler equations (1) s.t.*

$$|u(x,t)| = 1 \text{ a.e. } (x,t) \in \Omega \tag{3}$$

$$u = 0, p = 0 \text{ a.e. in } (\mathbb{R}^n_x \times \mathbb{R}_t) \setminus \Omega \tag{4}$$

Furthermore, $\exists$ sequence $(u_k, p_k, f_k)_{k\in\mathbb{N}} \in \mathcal{C}^\infty_c(\Omega)$ satisfying $f_k \to 0$ in H^{-1} solving 2 s.t.

$$\sup_k (\|u_k\|_\infty + \|p_k\|_\infty) < \infty \tag{5}$$

$$\forall q \in (1,\infty),\ (u_k, p_k) \to (u,p) \text{ in } L^q \tag{6}$$

2.2 Euler equations in the framework of planewave analysis

Consider the PDE problem involving a (system of) conservation law(s) coupled with a pointwise constraint for the state variable:

$$\sum_{i=1}^{N} C_i \partial_{x_i} g = 0 \text{ and } g(x) \in K \subset \mathbb{R}^d \text{ a.e. } x \in \mathbb{R}^N \tag{7}$$

Here, $g : \Omega \subset \mathbb{R}^N \to \mathbb{R}^d$ denotes a solution to the PDE problem 7, which we refer to as the state variable here with dependence on $x \in \mathbb{R}^N$; $(C_i)_{i=1}^N$ are scalar constants. Ideally, we would want to consider planewave solutions taking the form $g(x) = h(\xi \cdot x)c$ for some scalar function $h : \mathbb{R} \to \mathbb{R}$ and constant vector $c \in \mathbb{R}^d$. Note we have:

$$\sum_{i=1}^{N} C_i \partial_{x_i}(ch(\xi \cdot x)) = \sum_{i=1}^{N} C_i \xi_i h'(\xi \cdot x)c = h'(\xi \cdot x) \sum_{i=1}^{N} C_i \xi_i c$$

Thus whether a planewave solves the conservation law is in general independent from the scalar field h. The wave cone(collection of planewave solutions) is then given by

$$\Lambda = \{c \in \mathbb{R}^d \mid \exists \xi \in \mathbb{R}^N \setminus \{0\} \text{ with } \sum_{i=1}^{N} C_i \xi_i c = 0\}$$

We rewrite the Euler equation into this form as follows. Denote (u, p) a solution to 1, we introduce the following:

$$q = p + \frac{1}{n}|u|^2 \text{ and } v = u \otimes u - \frac{1}{n}|u|^2 I_n \tag{8}$$

where we have $v \in \mathcal{S}_0^n = \{A \in S^n \mid \operatorname{tr}(A) = 0\}$ and q is a scalar field. The state variable becomes $g = (u, v, q)$, whereas the following is now clear:

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0 \Leftrightarrow \partial_t u + \nabla \cdot v + \nabla q = 0 \tag{9}$$

We want to introduce(as in the planewave framework) a nonlinear condition allowing us to induce a 1-1 correspondence between weak solutions to the equations above. Indeed, for weak solutions (u, v, q) to 10 satisfying $v = u \otimes u - \frac{1}{n}|u|^2 I_n$ a.e., then we obtain a weak solution pair (u, p) with $p := q - \frac{1}{n}|u|^2$ to 1 and vice versa. We state the problem with the reformulated Euler equation and the nonlinear constraint:

$$\partial_t u + \nabla \cdot v + \nabla p = 0 \text{ and } \nabla \cdot u = 0 \tag{10}$$

$$v = u \otimes u - \frac{1}{n}|u|^2 I_n \text{ a.e. in } \mathbb{R}_x^n \times \mathbb{R}_t \tag{11}$$

For a given solution triple (u, v, q) to 10, we define in block form

$$U := \begin{pmatrix} v + qI_n & u \\ u^T & 0 \end{pmatrix} \in S^{n+1}$$

Now 10 is equivalent to $\nabla_{(x,t)} \cdot U = 0$, from which we induce the wave cone for the Euler equation:

$$\Lambda_{Euler} = \{(u, v, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R} \mid \det \begin{pmatrix} v + qI_n & u \\ u^T & 0 \end{pmatrix} = 0\} \tag{12}$$

Linear algebra tells us $\forall (u, v) \in \mathbb{R}^n \times \mathcal{S}_0^n$, $\exists q \in \mathbb{R}$ s.t. $\det \begin{pmatrix} v + qI_n & u \\ u^T & 0 \end{pmatrix} = 0$. Thus, we focus on finding u satisfying the non-linear constraint 11. Note clearly $0 \in \Lambda_{Euler}$.

3 Inspiration from the Nash-Kuiper Theorem

We give a brief discussion of Nash's technique used to construct an isometric embedding of a given n-dimensional Riemannian manifold into $\mathbb{R}^{n+1}$. This is the content of the Nash-Kuiper theorem.
Given a Riemannian n-manifold $(\mathcal{M}, g)$ with a short immersion or embedding $h : \mathcal{M} \to \mathbb{R}^N$ for some $N \geqslant n+1$, we would like to construct an isometric immersion/embedding based on the initial short immersion. Nash's idea is to create a sequence of perturbed short map such that the sequence converges to a limiting map which is an isometry, i.e. length-preserving. The construction of our sequence of short maps relies on adding corrugation to the initial embedding iteratively. Then, for each n, the nth short map compensates 2^{-n} of the missing length from the initial short map. By relaxing the regularity constraint to $\mathcal{C}^1$, the perturbation process controls the first-order derivatives while higher-order derivatives become unbounded, thus producing a $\mathcal{C}^1$ limiting immersion/embdedding. The result is surprising in the context of Weyl's problem: if $\mathbb{S}$ is a 2-dimensional sphere with a $\mathcal{C}^2$-metric of positive Gauss curvature K, then the image of every $\mathcal{C}^2$-isometric embedding $u : \mathbb{S} \to \mathbb{R}^3$ is the boundary of a convex set unique upto rigid motion. Note the curvature is not preserved in the case of a Nash $\mathcal{C}^1$-isometric immersion/embedding due to low regularity. An interesting question would be to discuss the weak notion of curvature in the limiting case.
In the Euler equation case, we would like to mirror the the construction above to obtain non-standard weak solution of the Euler equations. In Section 4, we will construct a compactly supported solution to the reformulated Euler equation 10. This solution will then be perturbed to form a sequence of inhomogeneous weak solution converging to an irregular homogeneous solution with respect to the L^∞-weak-* topology, thus showing non-uniqueness of weak solution to the Euler equations. Similar to the Nash embedding case, the constructed L^∞-solution is discontinuous thus we are in a low-regularity situation. We observe an analogy between weak solutions of Euler equations and isometric immersions/embeddings: solutions(or embeddings) are uniquely determined with appropriate constraints in the case of sufficiently high regularity, whereas relaxing the regularity condition gives us irregular solutions while we lose properties which would be conserved in higher regularity.

4 Construction of compactly supported (planewave) solutions

Recall we want to construct irregular solution with bounded support. Reflecting the idea from the Nash-Kuiper theorem, heuristically we want to perturb a given initial function with a sequence of "almost planewave" solutions. The limit of the perturbed sequence should then both solve the Euler equation satisfying the nonlinear condition 11 $v = u \otimes u - \frac{1}{n}|u|^2 I_n$ a.e.. The reason why we could not use exact planewaves is that the only compactly supported planewaves are identically zero, we thus relax to allow the perturbations to take value in an ε-neighbourhood of exact planewaves with $\varepsilon > 0$ arbitrarily small.

Proposition 4.1 (Localised solutions)**.** *Let $a = (u_0, v_0, q_0) \in \Lambda_{Euler}$ s.t. $v_0 \neq 0$. Denote $\gamma = [-a, a] \subset \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}$ the line segment connecting $-a, a$. Then $\forall \varepsilon > 0$, $\exists$ smooth solution (u, v, q) to 10 s.t.*

1. *$spt((u, v, q)) \subset \mathbb{B}_1 \subset \mathbb{R}^n_x \times \mathbb{R}_t$*
2. *$(u, v, q)(\mathbb{B}_1)$ lies in an ε-neighbourhood of γ*
3. *$\int |u(x,t)| dxdt \geqslant \alpha |u_0|$ for some $\alpha = \alpha_n = const.$*

In order to reduce the problem to group with symmetric structures, we restate the proposition using 10. Denote $\mathcal{M}_0 = \{A \in S^{n+1} \mid A_{(n+1)(n+1)=0}\}$, we have $\mathcal{M}_0 \cong \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}$ via the following isomorphism(which induces an equivalent form of 4.1)

$$T : \mathcal{M}_0 \to \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R},\ (u, v, q) \mapsto \begin{pmatrix} v + qI_n & u \\ u^T & 0 \end{pmatrix}$$

Proposition 4.2. *Let $U_0 \in \mathcal{M}_0$ with $\det U_0 = 0$, $U_0 e_{n+1} \neq 0$. Denote by $\Gamma := [-U_0, U_0] \subset \mathcal{M}_0$ the line segment connecting $-U_0$ and U_0. Then $\exists \alpha = \alpha_n =$ const. > 0 s.t.$\forall \varepsilon > 0$, $\exists$ smooth divergence-free matrix field $U : \mathbb{R}^{n+1} \to \mathcal{M}_0$ satisfying:*

1. *spt $U \subset \mathbb{B}_1$*

2. $\forall y \in \mathbb{B}_1$ $\operatorname{dist}(U(y),\Gamma) < \varepsilon$*(contained in an ε-neighbourhood)*

3. $\int |U(y)e_{n+1}|dy \geqslant \alpha|U_0 e_{n+1}|$

The construction above provides candidate perturbations, which will then be used to construct a sequence in the form of partial sums. By scaling(1. in Proposition 3.1), we can always choose the support of localised planewaves to be smaller in order to achieve strong convergence of the partial sum. The iteration then gives a limit which is the desired irregular solution satisfying 11. Condition 3, which is analogous in both propositions above, is not negligible to ensure with our perturbation process the limiting solution solves the Euler equation in the whole domain. A similar condition is posed in Nash's proof of his embedding theorem, where the geometric idea is more intuitive. Indeed, in the Nash embedding case, we want the short embedding to converge to an isometry so that the "missing length" is compensated everywhere for the limiting map, without any pointwise compensation at any elements in the sequence. We only outline the main idea of the proof for existence of compactly supported solutions, since this is less relevant to the idea of convex integration.

In order to prove the existence of compactly supported solutions, we introduce the Galilean group to reduce the problem by symmetry to some particular Λ_{Euler}-directions and a potential to achieve cutoff preserving 10. We state these here without proof.

Lemma 1 (Galilean Group)**.** *Let $\mathcal{G} := \{A \in GL(n,\mathbb{R}) \mid Ae_{n+1} = e_{n+1}\}$. Then $\forall$ divergence-free $U : \mathbb{R}^{n+1} \to \mathcal{M}_0$, $A \in \mathcal{G}$, $V : \mathbb{R}^{n+1} \to \mathcal{M}$, $V(y) := A^T \cdot U((A^{-1})^T y) \cdot A$ is divergence-free.*

Lemma 2 (Generalised Potential)**.** *Let $(E_{ij}^{kl})_{i,j,k,l\leqslant n+1} \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$ s.t. tensor $E = (E_{ij}^{kl})$ skew-symmetric in ij and kl. Then: $U_{ij} = \mathcal{L}(E) := \frac{1}{2}\sum_{k,l}\partial_{kl}^2(E_{kj}^{il} + E_{jl}^{ki})$ is symmetric, divergence-free. If in addition $E_{(n+1)j}^{(n+1)i} = 0$ $\forall i,j$, then U takes values in $\mathcal{M}_0$.*

Proof of Proposition 3.2. (i) Suppose $U_0 \in \mathcal{M}_0$ s.t. $U_0 e_1 = 0, U_0 e_{n+1} \neq 0$. Set tensor $E = (E_{ij}^{kl})$:

$$E_{j1}^{i1}(y) = -E_{j1}^{1i}(y) = E_{1j}^{1i}(y) = -E_{j1}^{1i}(y) = (U_0)_{ij}\frac{\sin(Ny_1)}{N^2}$$

$$E_{ij}^{kl} = 0, \text{ otherwise}$$

Thus the tensor(well-defined by assumption on U_0) satisfies constraints in lemma 2. We fix a smooth cutoff function φ satisfying

- $|\varphi| \leqslant 1$
- $\varphi = 1$ on $\mathbb{B}_{\frac{1}{2}}$
- *spt*$(\varphi) \subset \mathbb{B}_1$

Consider smooth map $U = \mathcal{L}(\varphi E)$, which is $\mathcal{M}_0$-valued divergence-free, and supported in $\mathbb{B}_1$ by lemma 2. We have, for N sufficiently large:

$$\forall y \in \mathbb{B}_{\frac{1}{2}},\, U(y) = U_0 \sin(Ny_1) \Rightarrow \int |U(y)e_{n+1}|dy \geqslant 2\alpha_n|U_0 e_{n+1}|$$

Note if we define $U_1 := \mathcal{L}(E)$, then $U - \varphi U_1 = \mathcal{L}(\varphi E) - \varphi\mathcal{L}(E)$ is controlled by derivatives of φ and components in E upto second order. We thus obtain the following estimate:

$$\|U - \varphi U_1\|_{L^\infty} \leqslant C\|\varphi\|_{\mathcal{C}^2}\|E\|_{\mathcal{C}^1} \leqslant \frac{C'}{N}\|\varphi\|_{\mathcal{C}^2} < \varepsilon$$

for N sufficiently large. Thus we have $\varphi U_1(\mathbb{B}_1) \subset \Gamma$, giving $U(\mathbb{B}_1)$ lies in ε-neighbourhood of Γ.
(ii)(Sketch) For a general $U_0 \in \mathcal{M}_0$, we choose some $f \in \mathbb{R}^{n+1}\setminus\{0\}$ s.t. $\{f, e_{n+1}\}$ linearly independent. We complete this to a basis for $\mathbb{R}^{n+1}$ with $f_1 = f$, $f_{n+1} = e_{n+1}$ and set $A \in \mathcal{G}$ s.t. $Ae_i = f_i$. Now reduce to (i) by setting $V_0 := A^T U_0 A \in \mathcal{M}_0$. We can thus obtain matrix field V by using (i) on V_0, whereas we can invert this to the desired matrix field U which satisfies the constraints by lemma 1. The L^1 lower bound for Ue_{n+1} can be obtained using a standard covering(by finite disjoint collection of balls contained in $\mathbb{B}_1$) argument. ∎

5 Geometric set-up and main iteration

Consider the set of points satisfying the pointwise constraint 11 and unit-norm condition:

$$K := \{(u,v) \in \mathbb{R}^n \times \mathcal{S}_0^n \mid v = u \otimes u - \frac{1}{n}|u|^2 I_n, |u| = 1\} \tag{13}$$

We define $\mathcal{U} := int(conv(K) \times [-1,1])$ which acts as our ambient space for us to select perturbations from. We will develop a sequence to approximate a solution to 10 taking values in the extreme set of $\overline{\mathcal{U}}$, which is equivalent to satifying the pointwise constraint 11, thus a desired solution. We state two lemmas here without proof which make the construction possible.

Lemma 3. *Set* $K := \{(u,v) \in \mathbb{R}^n \times \mathcal{S}_0^n \mid v = u \otimes u - \frac{1}{n}|u|^2 I_n, |u| = 1\}$ *Then* $0 \in int(conv(K)) \subset \mathcal{U}$

Lemma 4. $\exists C = C_n = const.$ *s.t.* $\forall (u,v,q) \in \mathcal{U}, \exists (\overline{u},\overline{v}) \in \mathbb{R}^n \times \mathcal{S}_0^n$ *s.t.* $(\overline{u},\overline{v},0) \in \Lambda_{Euler}$*, the line segment* $\gamma = [(u,v,q) - (\overline{u},\overline{v},0), (u,v,q) + (\overline{u},\overline{v},0)] \subset \mathcal{U}$ *and*

1. $|\overline{u}| \geqslant C(1 - |u|)$
2. $\inf_{p\in\gamma} \operatorname{dist}(p, \partial\mathcal{U}) \geqslant \frac{1}{2}\operatorname{dist}((u,v,q), \partial\mathcal{U})$

Lemma 3 establishes $0 \in \mathcal{U}$, thus we apply lemma 4 with $0 = (u,v,q) \in \mathcal{U}$ to find $\mathcal{U}$-valued planewaves. Heuristically, each gives us a direction to perturb the current solution triple (u,v,q). The lower bound on the distance between γ and $\partial\mathcal{U}$ ensures that none of the elements in our sequence reaches the desired solution before the solution triple converges to the limiting functions. This reflects the idea in the Nash-Kuiper theorem that the short map at any step of any stage is short everywhere and the missing length is only compensated uniformly(i.e. when the sequence of short maps converges to an isometry) on the whole of our Riemannian manifold $\mathcal{M}$. We make this more precise now by introducing the ambient space which should contain our sequence of perturbed solutions. This also gives a precise notion of convergence (hopefully) clarifying why the construction is possible.

Let X_0 be the set of $\mathcal{C}^\infty$-functions (u,v,q) s.t. the following hold:

1. $spt((u,v,q)) \subset \Omega$
2. (u,v,q) solves $\partial_t u + \nabla \cdot v + \nabla p = 0$ and $\nabla \cdot u = 0$ in $\mathbb{R}^n_x \times \mathbb{R}_t$
3. $(u,v,q) \in \mathcal{U}$ everywhere in $\mathbb{R}^n_x \times \mathbb{R}_t$

Note $X_0 \subset L^\infty(\Omega)$, thus we equip X_0 with the L^∞ weak-* topology. Define X to be the closure of X_0 wrt. the topology of weak-* convergence of triple (u,v,q). This gives us a closed and bounded subset of $L^\infty(\Omega; \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R})$. Thus by Banach-Alaoglu theorem, we have X equipped with the L^∞ weak-* topology forms a compact space. With the induced metric, we have X is a complete metric space. We notice $X \subset \overline{\mathcal{U}}$ by the third constraint for X_0. Then for any $(u,v,q) \in X$, we have $\operatorname{supp}((u,v,q)) \subset \Omega$ and (u,v,q) is $\overline{\mathcal{U}}$-valued since $\overline{\mathcal{U}}$ is compact convex. If $|u(x,t)| = 1$ a.e $(x,t) \in \Omega$, then we must have $(u,v) \in K$ a.e. . Therefore, this gives us a solution to the problem described by 10 and 11.

5.1 Main iterative argument

We construct the converging sequence of solutions here. Note we have $0 \in \mathcal{U}$, thus we can start with $(u_0,v_0,q_0) = 0$ and proceed to define the next solution triple inductively. More precisely, we would like to produce, for a given triple (u_k,v_k,q_k), a sequence $(u_{k,j},v_{k,j},q_{k,j})_{j\in\mathbb{N}}$ converging weak-*ly to (u_k,v_k,q_k). Then we can select $(u_{k+1},v_{k+1},q_{k+1})$ to be a term in the sequence sufficiently close to the limit. This differs from the more direct approach to construct converging sequence of short map for proving the Nash-Kuiper theorem.

Lemma 5 (Main Iteration)**.** *Let* $(u_0,v_0,q_0) \in X_0$*. Then* $\exists$ *sequence* $((u_k,v_k,q_k))_{k\in\mathbb{N}} \in X_0$ *satisfying for some* $\beta = \beta_n = const.$*:*

$$\|u_k\|^2_{L^2(\Omega)} \geqslant \|u_0\|^2_{L^2(\Omega)} + \beta(|\Omega| - \|u_0\|^2_{L^2(\Omega)})^2 \ \ \& \ \ (u_k,v_k,q_k) \overset{*}{\rightharpoonup} (u_0,v_0,q_0) \text{ in } L^\infty(\Omega) \tag{14}$$

Proof. *(i)* Let $(u_0, v_0, q_0) \in X_0$. By lemma 4, $\forall (x_0, t_0) \in \Omega$, $\exists$ direction $(\overline{u}(x_0,t_0), \overline{v}(x_0,t_0)) \in \mathbb{R}^n \times \mathcal{S}^n_0$ s.t.

$$\gamma_0 := \begin{bmatrix} (u_0(x_0,t_0), v_0(x_0,t_0), q_0(x_0,t_0)) - (\overline{u}(x_0,t_0), \overline{v}(x_0,t_0), 0) \\ (u_0(x_0,t_0), v_0(x_0,t_0), q_0(x_0,t_0)) + (\overline{u}(x_0,t_0), \overline{v}(x_0,t_0), 0) \end{bmatrix} \subset \mathcal{U},$$

and $|\overline{u}(x_0,t_0)| \geqslant C(1 - |u_0(x_0,t_0)|^2)$. Uniform continuity gives us a choice of $\varepsilon > 0$ s.t. an ε-neighbourhood of line segment of form $\gamma_0 = [(u_0(x,t), v_0(x,t), q_0(x,t)) - (\overline{u}(x,t), \overline{v}(x,t), 0), (u_0(x,t), v_0(x,t), q_0(x,t)) + (\overline{u}(x,t), \overline{v}(x,t), 0)]$ is contained in $\mathcal{U}$ whenever $|x - x_0| + |t - t_0| < \varepsilon$.

By Proposition 3.1(with $a = (\overline{u}(x_0,t_0), \overline{v}(x_0,t_0), 0) \in \Lambda_{Euler}$), we obtain C^∞-solution (u, v, q) satisfying stated properties(i.e. compactly supported in $\mathbb{B}_1$). Set $\forall r < \varepsilon$, $(u_r, v_r, q_r)(x,t) := (u,v,q)(\frac{x-x_0}{r}, \frac{t-t_0}{r})$. Then (u_r, v_r, q_r) gives a smooth solution to 10 satisfying:

1. $\mathrm{supp}((u_r, v_r, q_r)) \subseteq \mathbb{B}_r((x_0,t_0)) \subset \mathbb{R}^n_x \times \mathbb{R}_t$
2. $(u_r, v_r, q_r)(\mathbb{B}_r((x_0,t_0))) \subset \gamma + \mathbb{B}_\varepsilon$, where $\gamma := [-(\overline{u}(x,t), \overline{v}(x,t), 0), (\overline{u}(x,t), \overline{v}(x,t), 0)]$
3. $\int |u_r(x,t)| dxdt \geqslant \alpha |\overline{v}(x_0,t_0)||\mathbb{B}_r|$

Thus $(u_0, v_0, q_0) + (u_r, v_r, q_r) \in X_0$ $\forall r < \varepsilon$.

(ii) Note since u_0 is compactly supported smooth function, thus uniformly continuous. We can therefore find $\delta > 0$ s.t. $\forall r < \delta$, we find by using 3 iteratively a finite disjoint collection $\mathbb{B}_{r_j}(x_j)_{j \in \mathcal{J}} \in \mathcal{P}(\Omega)$ with $|\mathcal{J}| < \infty$ and $r_j < r$ $\forall j \in \mathcal{J}$ satisfying:

$$\int_\Omega (1 - |u_0(x,t)|^2) dxdt \leqslant 2 \sum_j (1 - |u_0(x_j,t_j)|^2) |\mathbb{B}_{r_j}| \tag{15}$$

Now fix $k \in \mathbb{N}$ s.t. $\frac{1}{k} < min\{\delta, \varepsilon\}$, we find a finite collection $\{\mathbb{B}_{r_{j,k}}(x_{j,k})\}_{j \in \mathcal{J}}$ of disjoint balls as above s.t. $\sup_j r_{j,k} < \frac{1}{k}$. Perform the construction as in *(i)* within $\mathbb{B}_{r_{j,k}}(x_{j,k})$ for each j to obtain sequence $((u_{k,j}, v_{k,j}, q_{k,j}))_j$ s.t. $(u_k, v_k, q_k) := (u_0, v_0, q_0) + \sum_j (u_{k,j}, v_{k,j}, q_{k,j}) \in X_0$. From 3 we have:

$$\int |u_k(x,t) - u_0(x,t)| dxdt = \sum_j \int |u_{k,j}(x,t)| dxdt \geqslant \alpha \sum_j |\overline{u}(x_{k,j}, t_{k,j})||\mathbb{B}_{r_{k,j}}| \tag{16}$$

Using the lower bound estimate from lemma 4(as in *(i)*), we have $\forall j$, $|\overline{u(x_{k,j}, t_{k,j})}| \geqslant C(1 - |u_0(x_{k,j}, t_{k,j})|)$. Combine this with 15 to see:

$$\alpha \sum_j |\overline{u}(x_{k,j}, t_{k,j})||\mathbb{B}_{r_{k,j}}| \geqslant \alpha \sum_j C(1 - |u_0(x_{k,j}, t_{k,j})|^2)|\mathbb{B}_{r_{k,j}}| \geqslant \frac{C\alpha}{2} \int_\Omega (1 - |u_0(x,t)|^2) dxdt \tag{17}$$

$$\Rightarrow \int |u_k(x,t) - u_0(x,t)| dxdt \geqslant \frac{C\alpha}{2} \int_\Omega (1 - |u_0(x,t)|^2) dxdt \tag{18}$$

We thus obtain a sequence $((u_k, v_k, q_k))_{k \in \mathbb{N}} \in X_0$ s.t. $(u_k, v_k, q_k) \overset{*}{\rightharpoonup} (u_0, v_0, q_0)$ since $\int (u_{k,j}, v_{k,j}, q_{k,j}) = 0$(the sequence is obtained from oscillatory perturbation(i.e. includes trigonometric terms; cf. section 3)). We deduce the following from weak-* convergence:

$$\liminf_{k\to\infty} \|u_k\|^2_{L^2(\Omega)} = \|u_0\| + \liminf_{k\to\infty} 2\langle u_0, u_k - u_0 \rangle_{L^2} + \|u_k - u_0\|^2_{L^2(\Omega)} = \|u_0\| + \liminf_{k\to\infty} + \|u_k - u_0\|^2_{L^2(\Omega)} \tag{19}$$

$$\liminf_{k\to\infty} \|u_k\|^2_{L^2(\Omega)} \geqslant \|u_0\| + \frac{1}{|\Omega|} \liminf_{k\to\infty} (\int |u_k(x,t) - u_0(x,t)| dxdt)^2 \tag{20}$$

by Jensen's inequality(since Ω is bounded, thus has finite volume). We now use 18 to deduce the desired bound:

$$\liminf_{k\to\infty} \|u_k\|^2_{L^2(\Omega)} \geqslant \|u_0\| + \frac{1}{|\Omega|} (\frac{C\alpha}{2} \int_\Omega (1 - |u_0(x,t)|^2) dxdt)^2 = \frac{C^2\alpha^2}{4|\Omega|} (|\Omega| - \|u_0\|^2_{L^2(\Omega)})^2 \tag{21}$$

□

The preceding lemma provides the foundation to inductively construct the desired approximating sequence, which we will make explicit now:

Proof of the main theorem. Let $\eta_\delta(x) := \delta^{-(n+1)}\eta(\frac{x}{\delta})$ where η is the standard mollification kernel in $\mathbb{R}^{n+1}$. Set $(u_1, v_1, q_1) \equiv 0$ in $\mathbb{R}^n_x \times \mathbb{R}_t$. Alongside the approximating sequence of solutions, we will also construct a strictly positive sequence $\delta = (\delta_k) \in \mathfrak{c}_0(\mathbb{R})$ which we use to show strong convergence.
Suppose inductively we have obtained $(z_i)_{i=1}^k := (u_i, v_i, q_i)_{i=1}^k$ with $(\delta_i)_{i=1}^{k-1} \in \mathbb{R}_{>0}$. Choose $\delta_k \in (0, 2^{-k})$ s.t. $\|u_k - u_k * \eta_{\delta_k}\| < 2^{-k}$ and similarly for v_k and q_k. By lemma 5, we can find a sequence $((u_{k,j}, v_{k,j}, q_{k,j}))_{j\in\mathbb{N}} \in X_0$ s.t. $\|u_{k,j}\|^2_{L^2(\Omega)} \geqslant \|u_k\|^2_{L^2(\Omega)} + \beta(|\Omega| - \|u_k\|^2_{L^2(\Omega)})^2$ and $z_{k,j} \overset{*}{\rightharpoonup} z_k$. We can thus choose $z_{k+1} := z_{j_0,k}$ for some j_0 s.t.

$$\|(u_{k+1} - u_k) * \eta_{\delta_i}\|_{L^2(\Omega)} = \|(u_{j_0,k} - u_k) * \eta_{\delta_i}\|_{L^2(\Omega)} < 2^{-k} \; \forall i \leqslant k \tag{22}$$

z_{k+1} also inherits properties from the sequence $(z_{k,j})_{j\in\mathbb{N}}$, i.e.

$$\|u_{k+1}\|^2_{L^2(\Omega)} = \|u_{k,j_0}\|^2_{L^2(\Omega)} \geqslant \|u_k\|^2_{L^2(\Omega)} + \beta(|\Omega| - \|u_k\|^2_{L^2(\Omega)})^2 \tag{23}$$

This gives us a bounded sequence $(z_k)_{k\in\mathbb{N}} \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ and a vanishing sequence $(\delta_k)_{k\in\mathbb{N}}$ satisfying $\delta_k \in (0, 2^{-k}) \; \forall k \in \mathbb{N}$ and conditions 22, 23. Alaoglu's theorem guarantees the existence of a subsequence $(k_i)_{i\in\mathbb{N}}$ s.t. $z_{k_j} \overset{*}{\rightharpoonup} z = (u, v, q) \in X$. We use a telescoping sum argument to deduce:

$$\|(u_k - u) * \eta_{\delta_k}\|_{L^2(\Omega)} \leqslant \sum_{l\in\mathbb{N}_0} \|(u_{k+l} - u_{k+l+1}) * \eta_{\delta_k}\|_{L^2(\Omega)} \leqslant \sum_{l\in\mathbb{N}_0} 2^{-(k+l)} = 2^{-k} \tag{24}$$

Note (z_k) and z are locally integrable(since they are compactly supported by construction). Then $\forall \varepsilon > 0$, by Lebesgue's differentiation theorem, since $\delta_k \downarrow 0$, we can choose k_0 sufficiently large s.t.

$$\|u_k - u_k * \eta_{\delta_k}\|_{L^2(\Omega)} < \frac{\varepsilon}{3} \text{ and } \|u - u * \eta_{\delta_k}\|_{L^2(\Omega)} < \frac{\varepsilon}{3} \; \forall k \geqslant k_0 \tag{25}$$

Let $k_1 \in \mathbb{N}$ s.t. $2^{-k} < \frac{\varepsilon}{3} \; \forall k \geqslant k_1$ and set $N_\varepsilon := \max k_0, k_1$. Combining 24 and 25 gives $u_k \to u$ in $L^2(\Omega)$ since $\forall k \geqslant N_\varepsilon$:

$$\|u_k - u\|_{L^2(\Omega)} \leqslant \|u_k - u_k * \eta_{\delta_k}\|_{L^2(\Omega)} + \|(u_k - u) * \eta_{\delta_k}\|_{L^2(\Omega)} + \|u - u * \eta_{\delta_k}\|_{L^2(\Omega)} < \varepsilon$$

Now we send $k \to \infty$ in 23 to conclude:

$$\|u\|^2_{L^2(\Omega)} = \lim_{k\to\infty} \|u_{k+1}\|^2_{L^2(\Omega)} \geqslant \lim_{k\to\infty} \|u_k\|^2_{L^2(\Omega)} + \beta(|\Omega| - \|u_k\|^2_{L^2(\Omega)})^2 = \|u\|^2_{L^2(\Omega)} + \beta(|\Omega| - \|u\|^2_{L^2(\Omega)})^2$$

Hence, we have $\|u\|^2_{L^2(\Omega)} = |\Omega|$. By assumption on X, we have $u \equiv 0$ in $(\mathbb{R}^n_x \times \mathbb{R}_t) \setminus \Omega$ and $|u| \leqslant 1$ in Ω, which tells us $|u| = \chi_\Omega$. $(u, v, q) \in X$ also satisfies the pointwise constraint: i.e. by definition of X, $(u, v) \in conv(K)$ a.e. in Ω which is, given $|u| = 1$ a.e. in Ω, equivalent to $(u, v)(x, t) \in K$ a.e. $(x, t) \in \Omega$. Thus (u, v, q) is the desired solution triple by discussion in Section 2.2, from which it yields an irregular solution (u, p) to the Euler equation in the original form. □

References

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