Analysis on locally convex topological spaces

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Prologue

This is an expository text summarising some of my learnings, mainly from the great text by Conway [Con94], on the functional-analysis on locally convex spaces. This is usually somewhat under-appreciated in the undergraduate syllabus for functional analysis (maybe because we spent all the time doing Hahn-Banach theorems and analysis of operators). The flavour of this document is more about the geometry of semi-norms and the construction of locally convex topology from semi-norms. I have tried to include most of the necessary ideas from analysis on an abstract topological space, where I took some inspiration from Folland's book [Fol13], but it is likely incomplete. It is quite fascinating to see how the geometric form of Hahn-Banach theorem continues to have a reasonable interpretation on more abstract topological spaces. Another interesting part is the metrisability and normability section, where we can see some constructions similar to ones in metric geometry and optimal transport. It is likely that I have introduced some errors either in compilation or some of the proofs (most likely because I couldn't find a reference).

1 Topological convergence and compactness

We first recall some notions of topological convergence, which are useful for the discussion on weak topologies.

Definition 1.1 (Directed set and nets)**.** *A directed set is the pair $(A, \preceq)$ of a non-empty set with a binary relation(in fact a pre-order) $\preceq$ such that:*

1. $\alpha \preceq \alpha \forall \alpha \in A$;
2. $\alpha \preceq \beta, \beta \preceq \lambda \Rightarrow \alpha \preceq \lambda$;
3. $\forall \alpha, \beta \in A, \exists \lambda \in A$ *with* $\alpha, \beta \preceq \lambda$ *(i.e. every pair in A has an upper bound).*

A net in a set X indexed by a directed set $(A, \preceq)$ is a mapping

$$\langle x_\alpha \mid \alpha \in A \rangle \colon A \to X, \quad \alpha \longmapsto x_\alpha$$

Definition 1.2 (Convergence of nets)**.** *Let $\langle x_\alpha \mid \alpha \in A \rangle$ be a net in a topological space X and $E \subset X$ a subset. Then we say:*

(i) $\langle x_\alpha \mid \alpha \in A \rangle \in E$ *e.v. if* $\exists \alpha_0 \in A$ *such that* $x_\alpha \in E \forall \alpha \succeq \alpha_0$;

(ii) $\langle x_\alpha \mid \alpha \in A \rangle \in E$ *i.o. if* $\forall \alpha \in A$, $\exists \beta \succeq \alpha$ *such that* $x_\beta \in E$;

(iii) $x_\alpha \to x$ *if* $\forall U \subset X$ *neighbourhood of* x, $\langle x_\alpha \rangle_{\alpha \in A} \in E$ *e.v.*

A point $x \in X$ is a cluster point for $\langle x_\alpha \mid \alpha \in A \rangle$ if for every neighbourhood $U \subset X$ of x

$$\langle x_\alpha \mid \alpha \in A \rangle \in U \text{ i.o.}$$

The above notions generalise the related concepts for convergence of sequences in a metric space. Indeed, every sequence is a net indexed by the natural numbers $\mathbb{N}$. An important remark is that if $x_\alpha \to x \in X$, then x is a cluster point for the net $\langle x_\alpha \mid \alpha \in A \rangle$. The existence of a cluster point is simply the natural generalisation of the existence of a convergent subsequence in the usual sense. In the following, we formulate compactness in terms of the existence of cluster points, the proof of which relies crucially on the finite intersection property:

Proposition 1.1 (Cluster compactness criterion)**.** *Let X be a Hausdorff topological space and $K \subset X$. Then K is compact if and only if every net $\langle x_\alpha \mid \alpha \in A \rangle$ in K admits a cluster point $x \in K$.*

Proof. If $K \subset X$ compact and $\langle x_\alpha \mid \alpha \in A \rangle$ net in K, then for each $\beta \in A$, set $F_\beta := \overline{\{x_\alpha \mid \alpha \succeq \beta\}}$, which is a closed subset of K. Since A is directed, any finite string $(\alpha_1, \ldots, \alpha_N)$ in A admits an upper bound β via an induction argument. In particular, we have

$$\beta \succeq \alpha_i \text{ for each } i = 1, \ldots, N \Rightarrow F_\beta \subset \bigcap_{i=1}^{N} F_{\alpha_i},$$

thus $\{F_\alpha \mid \alpha \in A\}$ satisfies the finite intersection property. If $\bigcap_{\beta \in A} F_\beta = \emptyset$, then $\{K \setminus F_\beta \mid \beta \in A\}$ forms an open cover for the compact set K. Thus there exists a finite subcover $\{K \setminus F_{\beta_i} \mid i = 1, \ldots N\}$, which has nonempty intersection by above. However, the covering property gives:

$$\emptyset = X \setminus (\bigcup_{i=1}^{N} X \setminus F_{\beta_i}) = \bigcap_{i=1}^{N} X \setminus (X \setminus F_{\beta_i}) = \bigcap_{i=1}^{N} F_{\beta_i} \neq \emptyset,$$

which is a contradiction. Now $\exists x \in \bigcap_{\beta \in A} F_\beta$ which is necessarily a cluster point for our net. Indeed, this follows from the fact that $x \in F_\beta = \overline{\{x_\alpha \mid \alpha \succeq \beta\}}$ for any $\beta \in A$.
On the contrary, if any net in K admits a cluster point in K, we will show that $\{K_\alpha \mid \alpha \in A\}$, the collection of closed subsets in K with the finite intersection property, necessarily has non-empty intersection, which is equivalent to compactness of K. Set with directed order by inclusion: $\mathfrak{A} := \{I \subset A \mid |I| < \infty\}$. By assumption, $\exists x_I \in \bigcap_{\alpha \in I} K_\alpha$ for any $I \in \mathfrak{A}$ and the net $\langle x_I \mid I \in \mathfrak{A} \rangle$ admits a cluster point $x_\infty \in K$. Now let U be any open neighbourhood of x_∞ and $\alpha \in A$ arbitrary, by definition of a cluster point, $\exists I \in \mathfrak{A}$ with $\alpha \in I$(exists since $\{\alpha\} \in \mathfrak{A}$) such that $x_I \in U$. In particular, $U \cap K_\alpha \neq \emptyset$. Thus: $x_\infty \in \bigcap_{\alpha \in A} K_\alpha \neq \emptyset$. □

The notion of cluster points generalises the more familiar equivalence between compactness and sequential compactness in metric spaces, as shown in the previous Proposition. Another useful compactness result, again using the cluster points, is stated and proved below:

Lemma 1.1 (Cluster convergence criterion). *If X compact Hausdorff topological space, $\langle x_\alpha \rangle$ net in X with a unique cluster point $x_\infty \in X$, then we necessarily have $x_\alpha \to x_\infty$.*

Proof. For any open neighbourhood $U \subset X$ of the cluster point x_∞, set $\mathfrak{A}_U := \{\alpha \in A \mid x_\alpha \notin U\}$. If we assume the contrary, for some $U(x_\infty) \subset X$, there is $\alpha(\beta) \in \mathfrak{A}_{U(x_\infty)}$ for each $\beta \in A$. Note now $\mathfrak{A}_{U(x_\infty)}$ also defines a directed set. In particular, $\langle x_{\alpha(\beta)} \mid \beta \in \mathfrak{A}_{U(x_\infty)} \rangle$ defines a net in the compact space X, thus admits a cluster point y_0 by Proposition 1.1. Since $\mathfrak{A}_{U(x_\infty)} \subset A$, y_0 also defines a cluster point for the whole net $\langle x_\alpha \rangle$. By uniqueness $y_0 = x_0$, which contradicts the construction of $\mathfrak{A}_{U(x_\infty)}$. □

The following lemmas concerns the characterisation of continuous maps via convergence of nets, which again is a generalisation of the sequential continuity in the more familiar setting.

Lemma 1.2 (Continuity via nets). *Let $f \colon X \to Y$ be map between Hausdorff topological spaces. Then f continuous at $x \in X$ if and only if $f(x_\alpha) \to f(x)$ whenever $x_\alpha \to x$.*

Proof. If f continuous at x and $x_\alpha \to x$, then consider open neighbourhood $V \subset Y$ around $f(x)$, where $U := f^{-1}(V)$ is thus an open neighbourhood around x. By convergence of net, $x_\alpha \in U$ e.v. $\Leftrightarrow f(x_\alpha) \in V$ e.v. In particular, $f(x_\alpha) \to f(x)$ in Y.
For the forward implication, suppose f not continuous at x. Denote by $\mathcal{U}$ the collection of open sets containing x. Discontinuity gives the existence of $V \in \mathcal{T}_Y$ such that $f(x) \in V$ with $f^{-1}(V) \notin \mathcal{T}_X$. In particular, for any $U \in \mathcal{U}$, $f(U) \setminus V \neq \emptyset$(otherwise contradicts discontinuity). Order $\mathcal{U}$ by containment gives rise to a directed set, thus we obtain a net $\langle x_U \mid U \in \mathcal{U} \rangle$ with $f(x_U) \in f(U) \setminus V$. By construction $x_U \to x$ while we clearly cannot have $f(x_U) \to f(x)$ by the reverse implication. □

Lemma 1.3 (Homeomorphism onto compact space). *Let X, Y be Hausdorff topological spaces and X compact. If $f \colon X \to Y$ bijective continuous map, then f is necessarily a homeomorphism.*

Proof. Follows since f necessarily a closed map. □

2 Topology induced by family of semi-norms

We start by examining the vital connection between the linear and topological structures on a given topological vector space.

Definition 2.1 (Topological vector spaces). *A topological vector space(TVS) is the pair $(X, \mathcal{T}_X)$ consisting of a linear vector space X and a topology $\mathcal{T}_X$ defined on X such that:*

(i) $X \times X \to X, \quad (x, y) \longmapsto x + y$*;*

(ii) $X \times \mathbb{K} \to X, \quad (x, \lambda) \longmapsto \lambda x$*,*

both define continuous operators with respect to this topology.

Definition 2.2 (Semi-norms and norms)**.** *Let X be a $\mathbb{K}$-vector space. A semi-norm is a function $p \colon X \to [0, \infty)$ satisfying the following properties:*

- $p(x + y) \leqslant p(x) + p(y)$ *for any $x, y \in X$;*
- $p(\lambda x) = |\lambda| \cdot p(x)$ *for any $\lambda \in \mathbb{K}$ and $x \in X$.*

A norm $\|\|_X : X \to [0, \infty)$ is a semi-norm such that $\|x\|_X = 0$ if and only if $x = \mathbf{0} \in X$.

Lemma 2.1. *Let $p, q \colon X \to [0, \infty)$ be semi-norms on the $\mathbb{K}$-vector space X. Then the following are equivalent:*

(i) $p \leqslant q$*, i.e.* $p(x) \leqslant q(x)$ *for all* $x \in X$*;*

(ii) $\{x \in X \mid q(x) < 1\} \subset \{x \in X \mid p(x) < 1\}$*, i.e.* $q(x) < 1 \Rightarrow p(x) < 1$*;*

(iii) $\{x \in X \mid q(x) \leqslant 1\} \subset \{x \in X \mid p(x) \leqslant 1\}$*, i.e.* $q(x) \leqslant 1 \Rightarrow p(x) \leqslant 1$*;*

(iv) $\{x \in X \mid q(x) < 1\} \subset \{x \in X \mid p(x) \leqslant 1\}$*, i.e.* $q(x) < 1 \Rightarrow p(x) \leqslant 1$.

Proof. (i) clearly implies all the remaining properties and both (ii), (iii) imply (iv). Thus it suffices to show $(iv) \Rightarrow (i)$. For any arbitrary $\varepsilon > 0$, $x \in X$ with $q(x) = \lambda$, we have $q((\lambda + \varepsilon)^{-1} x) = \frac{\lambda}{\lambda + \varepsilon} < 1$. If (iv) holds, then we must have $(\lambda + \varepsilon)^{-1} x$ is contained in the closed p-unit ball. In particular:

$$q((\lambda + \varepsilon)^{-1} x) < 1 \Rightarrow p((\lambda + \varepsilon)^{-1} x) = \frac{p(x)}{\lambda + \varepsilon} \leqslant 1 \overset{\varepsilon \downarrow 0}{\Rightarrow} p(x) \leqslant \liminf_{\varepsilon \downarrow 0} \lambda + \varepsilon = \lambda = q(x).$$

□

Let X be a vector space and $\mathcal{P}$ a family of semi-norms defined on X. We can construct a topology $\mathcal{T}_{\mathcal{P}}$ on X by declaring as a sub-basis for the desired topology, the collection $\mathcal{B}$ of ρ-balls $\{x \in X \mid \rho(x - x_0) < \varepsilon\}$, where $x_0 \in X$ and $\varepsilon > 0$. In particular, a subset $U \in \mathcal{T}_{\mathcal{P}}$ if and only if for any $x \in U$, we can find $(\rho_k)_{k=1}^N \in \mathcal{P}$ and $(\varepsilon_k)_{k=1}^N \in (0, \infty)$ such that $x \in \bigcap_{k=1}^N \{y \in X \mid \rho_k(y - x_k) < \varepsilon_k\}$.

Definition 2.3 (Locally convex space)**.** *A topological vector space X is locally convex if $\mathcal{T}_X$ is generated by a family $\mathcal{P}$ of semi-norms satisfying the property:*

$$\bigcap_{\rho \in \mathcal{P}} \{x \in X \mid \rho(x) = 0\} = \{0\}.$$

The property in Definition 2.3 is stated to guarantee that the topological space is Hausdorff. In fact, for $x \neq y \in X$, there exists $\rho \in \mathcal{P}$ with $\rho(x - y) =: \delta > 0$. Then via triangle inequality, we can deduce:

$$\rho(y - z) \geqslant \rho(x - y) - \rho(x - z) > \frac{2\delta}{3} \forall z \in \mathbb{B}_\rho(x, \frac{\delta}{3}) \Rightarrow \mathbb{B}_\rho(x, \frac{\delta}{3}) \cap \mathbb{B}_\rho(y, \frac{\delta}{3}) = \emptyset,$$

where for notational convenience, we have denoted $\mathbb{B}_\rho(x, r) = \{z \in X \mid \rho(x - z) < r\}$ for any $r > 0$.
Recall in a topological vector space, translations and dilations are homeomorphisms of the space. A geometric interpretation is that the topology around each point looks the same. In particular, it is natural to simply consider any map near the zero element of the vector space, which makes the following results more intuitive.

Lemma 2.2 (Continuity of semi-norm)**.** *Let X be a topological vector space and $\rho \colon X \to [0, \infty)$ semi-norm on X. Then the following are equivalent:*

(i) ρ *continuous semi-norm;*

(ii) $\{x \in X \mid \rho(x) < 1\} \in \mathcal{T}_X$*;*

(iii) $0 \in \operatorname{int}(\{x \in X \mid \rho(x) < 1\}) \subset \operatorname{int}(\{x \in X \mid \rho(x) \leqslant 1\})$*;*

(iv) ρ *continuous at* $0 \in X$*;*

(v) $\exists$ *continuous semi-norm* $q\colon X \to [0,\infty)$ *with* $\rho \leqslant q$.

Proof. Note $\{x \in X \mid \rho(x) < 1\} = \rho^{-1}([0,1)) \subset \{x \in X \mid \rho(x) \leqslant 1\}$ where $[0,1) \subset [0,\infty)$ relatively open. Thus $(i) \Rightarrow (ii) \Rightarrow (iii)$. By 1-homogeneity of semi-norm, (iii) implies $0 \in \operatorname{int}(\{x \mid \rho(x) \leqslant \varepsilon\})$ for any $\varepsilon > 0$. If $\langle x_\alpha \rangle$ net in X with $x_\alpha \to 0$, then for any $\varepsilon > 0$, $\langle x_\alpha \rangle \in \operatorname{int}(\{x \mid \rho(x) < \varepsilon\})$ e.v. Thus sending $\varepsilon \downarrow 0$ gives $\rho(x_\alpha) \to 0$, thus giving continuity. (iv) clearly implies (i) via reverse triangle inequality. Now since (i) implies (v), it suffices to show the reverse implication, which effectively follows from:

$$x_\alpha \to x \to |\rho(x_\alpha) - \rho(x)| \leqslant |q(x_\alpha) - q(x)| \leqslant q(x_\alpha - x) \to 0.$$

□

If $\mathcal{P}$ is a family of semi-norms on X defining a locally convex topology, we can extend $\mathcal{P}$ such that the resultant family is closed under finite sums and taking supremum of uniformly bounded subfamilies. It is sometimes convenient to assume $\mathcal{P}$ consists of all continuous semi-norms on X, where the topology is invariant under this extension of the defining family, justified by the following result.

Proposition 2.1 (Algebraic structure of continuous semi-norms)**.** *Let X be a topological vector space and $(\rho_i)_{i=1}^N$ continuous semi-norms on X. Then:*

$$\rho_1 + \cdots + \rho_N, \max_{i=1,\ldots,N} \rho_i \colon X \to [0,\infty),$$

define continuous semi-norms. For a (possibly uncountable) family $\{\rho_i \mid i \in \mathcal{I}\}$ of continuous semi-norms with uniform upper bound q, i.e. $\exists$ continuous semi-norm $q\colon X \to [0,\infty)$ with $\sup_{i\in\mathcal{I}} \rho_i \leqslant q$, $x \mapsto \sup_{i\in\mathcal{I}} \rho_i(x)$ defines continuous semi-norm.

Proof. The sum of continuous semi-norms clearly remains a continuous semi-norm by linear structure of continuous functions. Similarly, let $\rho := \max_{i=1,\ldots,N} \rho_i$, which remains positive-semidefinite and 1-homogeneous. Triangle inequality follows since for any choice of $x, y, z \in X$:

$$\rho_i(x-y) \leqslant \rho_i(x-z) + \rho_i(y-z) \leqslant \rho(x-z) + \rho(y-z) \text{ for all } i = 1,\ldots,N \Rightarrow \rho(x-y) \leqslant \rho(x-z) + \rho(y-z).$$

The resultant semi-norm is continuous at $0 \in X$ using lemma 1.2, thus defines a continuous map. The statement for $\rho := \sup_{i\in\mathcal{I}} \rho_i$ is analogous except we need to observe the supremum is well-defined from the uniform bound. Here, we can simply deduce continuity of ρ from the uniform bound by q using (iv) in lemma 2.2. □

3 Minkowski functional and convexity

Recall the geometric meaning of convexity: a set C is convex if and only if for any points $x, y \in C$, the line segment $[x,y] := \{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\}$ is contained in C. Observe that the intersection of a family of convex sets remains convex. An alternative characterisation of convexity is the following:

Lemma 3.1. *A set C is convex if and only if for any finite strings $x_1, \ldots, x_N \in C$ and $t_1, \ldots t_N \in [0,1]$ with $\sum_{i=1}^N t_i = 1$, we have $\sum_{i=1}^N t_i x_i \in C$.*

Proof. The forward implication follows from an induction argument and definition of convexity. The reverse implication is simply the case $N = 2$ for the stated condition. □

Definition 3.1 (Convex hull)**.** *Let X be a locally convex Hausdorff topological vector space and $U \subset X$ a subset. The convex hull $conv(U)$ of U is the intersection of all convex sets containing U or equivalently the smallest convex set containing U. Alternatively, the following is another equivalent definition:*

$$conv(U) = \{\sum_{i=1}^N \lambda_i u_i \mid n \in \mathbb{N}; u_i \in U \forall i, \lambda_i \in [0,1], \sum_{i=1}^N \lambda_i = 1\}.$$

Similarly, the closed convex hull $\overline{conv}(U)$ of U is the smallest closed convex set containing U.

Remark 3.1. A simple, but often neglected observation is that any linear subspace is a convex set. If X is a normed space, the open and closed unit balls are convex sets in X. For a $\mathbb{R}$-linear map $T\colon X \to Y$, the pre-image of any convex set under T is convex.

Lemma 3.2. *The convex hull $conv(U)$ of an open set $U \in \mathcal{T}_X$ in a TVS is open.*

Proof. Let $x \in conv(U)$. Then we can find constants $(\lambda_i)_{i=1}^N \in [0,1]$ and $(u_i)_{i=1}^N \in U$ such that $x = \sum_{i=1}^N \lambda_i u_i$. In particular, there exists i such that $\lambda_i \neq 0$. Consider the continuous map:

$$f\colon X \to X, \quad y \longmapsto \frac{1}{\lambda_i}(y - \sum_{j\neq i} \lambda_j u_j).$$

Then $f^{-1}(U) = \lambda_i U + \sum_{j\neq i} \lambda_j u_j \in \mathcal{T}_X$ since translations and dilations are homeomorphisms of a TVS. By convexity, $f^{-1}(U) \subset conv(U)$ with $x \in f^{-1}(U)$, hence showing $conv(U) \in \mathcal{T}_X$. ☐

Recall that, in a topological vector space., the topology remains the same pointwise. In particular, since translations define homeomorphisms, an element $x \in S \subset X$ lies in the interior of S if and only if we can find $U \in \mathcal{T}_X$ with $0 \in U$ and $x + U \subset S$. This is the main observation for showing the following property.

Proposition 3.1. *Let X be a Hausdorff topological vector space and $C \subset X$ convex subset. Then:*

(i) $\bar{A}$ *remains convex;*

(ii) $[x, y) := \{(1-\lambda)x + \lambda y \mid \lambda \in [0,1)\} \subset \operatorname{int}(C)$ *for any* $x \in \operatorname{int}(C)$ *and* $y \in \bar{C}$.

Proof. The convexity of $\bar{C}$ can be deduced using net convergence in C. For any $x \in \operatorname{int}(C)$, by translating open neighbourhood around x in C, we can find open neighbourhood V of 0 such that $x + V \subset C$. Now consider the path $\gamma(t) = (1-t)x + ty$ for $t \in (0,1)$. Then for any $t \in (0,1)$ and $z \in C$:

$$C \overset{\text{convexity}}{\supset} tz + (1-t)(x+V) = t(z-y) + ty + (1-t)(x+V) = (t(z-y) + (1-t)V) + \gamma(t).$$

Note $-t^{-1}(1-t)V$ is an open neighbourhood of 0 since dilations are homeomorphisms of X. Then since $y \in \bar{C}$, we can find $z \in C$ with $0 \in t(z-y) + (1-t)V$, where $t(z-y) + (1-t)V$ remains open in X. Thus, $\gamma(t) \in \operatorname{int}(C)$ by the above remark ☐

Corollary 3.1. *If $U \subset X$ is any subset, then $\overline{conv}(A) = \overline{conv(A)}$.*

Definition 3.2 (Balanced and absorbing set)**.** *A subset $A \subset X$ is balanced if $\lambda x \in A$ for any $|\lambda| \in [0,1]$. An absorbing set A satifies the property that if for all $x \in X$, we find $\varepsilon > 0$ with $\{tx \mid t \in [0,\varepsilon)\} \subset A$.*
We say A is absorbing at $x \in A$ if $A - x$ is an absorbing set or equivalently for any $y \in X$, there is $\varepsilon > 0$ such that $x + ty \in A$ for all $t \in [0,\varepsilon)$.

Remark 3.2. Any balanced or absorbing set necessarily contains the origin in X. A balanced set is then also symmetric about the origin. In fact, the line segment $[-x, x]$ is contained in C for all $x \in C$.

An example is the unit ρ-ball $\mathbb{B}_\rho = \{x \mid \rho(x) < 1\}$ in a vector space X, where $\rho\colon X \to [0,\infty)$ is a semi-norm. In fact, $\mathbb{B}_\rho$ is a convex balanced set, absorbing at each point. Conversely, the Minkowski functional ρ_C gives rise to a semi-norm for any such set C, where C becomes the unit ball with respect to ρ_C. This is the key ingredient for establishing the functional separation theorem.

Proposition 3.2 (Minkowski functional/gauge)**.** *Let X be a $\mathbb{K}$-vector space $C \subset X$ a nonempty convex, balanced set which is absorbing at each point $x \in C$. Then there exists a unique semi-norm $\rho_C\colon X \to [0,\infty)$ such that $C = \{x \mid \rho_C(x) < 1\}$.*

Proof. The desired semi-norm is defined via the following:

$$\rho_C\colon X \to [0,\infty), \quad \rho_C(x) := \inf\{t > 0 \mid x \in tC\}.$$

Since C absorbing and balanced, we have $X = \bigcup_{i\in\mathbb{N}} nC$. Thus for any $x \in C$, $\rho_C(x) < \infty$. We clearly have $\rho_C(0) = 0$ since $0 \in C$ and ρ_C is positive-semidefinite by construction. Now take without loss of generality $\lambda \neq 0$, since C balanced thus symmetric about the origin, we have:

$$\rho_C(\lambda x) = \inf\{t > 0 \mid \lambda x \in tC\} = \inf\{t > 0 \mid x \in \frac{t}{|\lambda|}C\} = |\lambda| \inf\{\frac{t}{|\lambda|} > 0 \mid x \in \frac{t}{|\lambda|}C\} = |\lambda|\rho_C(x).$$

It now remains to show sub-linearity. Let $x, y \in X$ and choose for fixed $\varepsilon > 0$, $t, \lambda > 0$ with

$$\rho_C(x) < t < \rho_C(x) + \varepsilon \text{ and } \rho_C(y) < \lambda < \rho_C(y) + \varepsilon \text{ and } \frac{x}{t}, \frac{y}{\lambda} \in C.$$

By convexity of C, we have:

$$\frac{x+y}{t+\lambda} = \frac{t}{t+\lambda}\frac{x}{t} + \frac{\lambda}{t+\lambda}\frac{y}{\lambda} \in C.$$

Thus by definition:

$$\rho_C(x+y) \leqslant t+\lambda < \rho_C(x)+\rho_C(y)+2\varepsilon \Rightarrow \rho_C(x+y) \leqslant \liminf_{\varepsilon\downarrow 0} \rho_C(x)+\rho_C(y)+2\varepsilon = \rho_C(x)+\rho_C(y).$$

Now it remains to show $C = \{x \mid \rho_C(x) < 1\}$. If $\rho_C(x) < 1$, we can choose $t \in (\rho_C(x), 1)$ with $\frac{x}{t} \in C$. By convexity, the line segment $[0, \frac{x}{t}]$, where $x \in [0, \frac{x}{t}]$ since $t < 1$, is contained in C. Thus $C \supset \{x \mid \rho_C(x) < 1\}$. On the other hand, let $x \in C$ with $\rho_C(x) > 0$(otherwise trivially $x \in \{y \mid \rho_C(y) < 1\}$). Since C absorbing at x, we can find $\varepsilon > 0$ such that:

$$x + t\frac{x}{\rho_C(x)} \in C \text{ for all } t \in [0, 2\varepsilon) \Rightarrow x + \varepsilon\frac{x}{\rho_C(x)} = (1+\frac{\varepsilon}{\rho_C(x)})x \in C,$$

which then shows $\rho_C(x) \leqslant \frac{1}{1+\frac{\varepsilon}{\rho_C(x)}} < 1$. The uniqueness statement follows from lemma 2.1. □

Remark 3.3. Any open subset $V \subset X$ of TVS is absorbing at each of its points.

Remark 3.4. Note the Minkowski functional is also useful in the following sense: if C convex with $0 \in C$, then $\{x \mid \rho_C(x) < 1\} \subset C$, which can be deduced from the above proof.

Corollary 3.2 (LCS-characterisation)**.** *Let X be a Hausdorff TVS and denote by $\mathcal{U}$ the collection of all convex balanced sets in X. Then X locally convex if and only if $\mathcal{U}$ is a basis for the neighbourhood system at 0.*

The following is a construction of the finest locally convex topology on a vector space of countable dimension.

Proposition 3.3 (Direct limit topology)**.** *Let X be a vector space with $\dim X = \aleph_0$ and define:*

$$\mathcal{T} := \left\{ W \subset X \mid \forall x \in W, \exists \text{ convex, balanced } U \text{ s.t } \begin{cases} x+U \subset W \\ U \cap M \text{ open } \forall M \subset X \text{ s.t. } \dim M < \infty \end{cases} \right\}.$$

Then $(X, \mathcal{T})$ defines a LCS such that:

- *(i) $F \subset X$ closed $\Leftrightarrow$ $F \cap M$ closed for any $M \subset X$ with $\dim M < \infty$;*
- *(ii) $f\colon X \to Y$ continuous $\Leftrightarrow$ $f|_M$ continuous for any $M \subset X$ with $\dim M < \infty$;*
- *(iii) Y TVS and $T \in \mathcal{L}(X;Y) \Rightarrow T\colon X \to Y$ continuous mapping.*

Sketch proof. Check that translations and dilations indeed define homeomorphisms of $(X, \mathcal{T})$. The topology $\mathcal{T}$ is clearly Hausdorff and thus we can use the characterisation of locally convex TVS in Corollary 3.2. Using a countable basis $(x_i \mid i \in \mathbb{N})$ for X, we construct an increasing sequence of convex sets $C_N \subset U \cap \operatorname{span}(\{x_i | i \leqslant N\})$ for each open neighbourhood U of the origin. Then set:

$$D_N := C_1 + C_2 + \cdots + C_N \subset C_N \subset U \cap \operatorname{span}(\{x_i \mid i \leqslant N\}),$$

which is open and convex. Thus $U_0 = \bigcup_{i\in\mathbb{N}} D_i$ is an open convex neighbourhood of the origin contained, showing that $(X, \mathcal{T})$ is locally convex. Finite-dimensional subspaces in TVS remain closed by checking convergence of nets, in particular $F \cap M$ intersection of closed set thus closed. The second and third properties follow from the first property. □

4 Metrisability and normability in locally convex TVS

An interesting problem on locally convex topology, in particular for some function spaces, is whether the topology can be induced from a metric space structure. The following gives the explicit construction of a compatible metric when the topology is metrisable.

Theorem 4.1 (LCS-metrisability criterion)**.** *Let X be a $\mathbb{K}$-vector space and $\mathcal{P} = \{\rho_i \mid i \in \mathbb{N}\}$ a countable family of semi-norms on X such that $\bigcap_{i=1}^{\infty}\{x \mid \rho_i(x) = 0\} = \{0\}$. Define:*

$$\mathrm{d}\colon X \times X \to [0,\infty), \quad \mathrm{d}(x,y) := \sum_{i=1}^{\infty} 2^{-i}\frac{\rho_i(x-y)}{1+\rho_i(x-y)}.$$

Then d defines a metric on X and induces the locally convex topology $\mathcal{T}_{\mathcal{P}}$. In particular, a locally convex Hausdorff TVS is metrisable if and only if the generating family of semi-norms for the topology is countable.

Proof. We first verify that d indeed defines a metric and induces the topology on X. Symmetry of d follows from positive-semidefiniteness of semi-norms. $\mathrm{d}(x,x) = 0$ is clear since $\rho_i(0) = 0 \forall i$. On the other hand, $\mathrm{d}(x,y) = 0$ implies $\rho_i(x-y) = 0$ for all $i \in \mathbb{N}$. By assumption on the family of semi-norms, $x = y$. Now it remains to show triangle inequality. The semi-norms satisfy triangle inequality and assume positive distances in the sequel. We have $\rho_i(x-y) \leqslant \rho_i(x-z) + \rho_i(z-y)$ which gives $(\rho_i(x-z)+\rho_i(z-y))^{-1} \leqslant \rho_i(x-y)^{-1}$ and thus $(1+(\rho_i(x-z)+\rho_i(z-y))^{-1})^{-1} \leqslant (1+\rho_i(x-y)^{-1})^{-1}$ will imply:

$$\frac{\rho_i(x-y)}{1+\rho_i(x-y)} \leqslant \frac{\rho_i(x-z)+\rho_i(z-y)}{1+\rho_i(x-z)+\rho_i(z-y)} \leqslant \frac{\rho_i(x-z)}{1+\rho_i(x-z)} + \frac{\rho_i(z-y)}{1+\rho_i(z-y)} \forall i \in \mathbb{N}$$

Then the metric triangle inequality follows from summing over $i \in \mathbb{N}$. Note also the metric is invariant under translations. In particular, it suffices to show the two topologies agree around the origin. We will use the fact that any topology generated by a countable family of semi-norms is first-countable. In particular, topology of a first-countable space is characterised by its convergent sequences. It then suffices to show the following claim. **Claim:** $\mathrm{d}(x_k, 0) \to 0$ if and only if $\rho_i(x_k) \to 0 \forall i \in \mathbb{N}$.
For each $i \in \mathbb{N}$, we have by positive-semidefiniteness of semi-norms, $\rho_i(x_k) \leqslant \frac{\rho_i(x_k)}{1+\rho_i(x_k)} \leqslant 2^i \mathrm{d}(x_k, 0)$. Suppose first $\mathrm{d}(x_k, 0) \to 0$. Choose an arbitrary open neighbourhood $U \subset X$ of the origin. The above inequality shows that for k sufficiently large, x_k is contained in any ρ_i-ball of arbitrarily small radius. Thus $x_k \to 0$ as a net in $(X, \mathcal{T}_{\mathcal{P}})$. The other direction is essentially the same.
Now by above any locally convex TVS generated by a countable family of semi-norms is necessarily metrisable. Assume on the contrary that X is a metrisable LCS with metric d. Define $U_N = \{x \mid d(x,0) < \frac{1}{N}\} \in \mathcal{T}_X$. Then we can choose continuous semi-norms $(p_i)_{i=1}^k(N)$ on X such that:

$$\bigcap_{i=1}^{k(N)} \{x \mid p_i(x) < \varepsilon_i\} \subset U_N \text{ for some } \varepsilon_1, \ldots, \varepsilon_k(N) > 0.$$

Set $\rho_N := \varepsilon_1^{-1} p_1 + \cdots + \varepsilon_k(N)^{-1} p_k(N)$. Then $(\rho_N \mid N \in \mathbb{N})$ defines a countable family of continuous semi-norms such that $x \in U_N$ for all $\rho_N(x) < 1$. If $x_i \to 0$ in X, then $\rho_N(x_i) \to 0$ for any N. Conversely, suppose $\rho_N(x_i) \to 0$ for all N. Choose N large such that $N > \varepsilon^{-1}$. Then for all i sufficiently large, $\rho_N(x_i) < 1$. In particular, $x_i \in U_N$ for all i large and thus $d(x_i, 0) < \frac{1}{N} < \varepsilon$. The topology on metric spaces, or more generally first-countable spaces, is characterised by convergence of sequences, which thus gives the equivalence of topologies. ☐

Example 4.1. *Let X be a Tychonoff space, i.e. X is completely regular(closed subsets and singletons can be separated by bounded continuous functions) Hausdorff topological space. Then we can equip $C^0(X;\mathbb{K})$ with a family of semi-norms $\rho_K(f) := \sup_K |f|$ for each compact subset $K \subset X$, where the resultant topology is locally convex.*
The space of continuous functions $C^0(X;\mathbb{K})$ is metrisable if and only if $X = \bigcup_{i=1}^\infty K_i$ for an increasing sequence of compacts sets. Indeed, any compact subset is contained in K_k for some $k \in \mathbb{N}$. In the special case where X is locally compact, it suffices to assume X is σ-compact to deduce metrisability of $C^0(X;\mathbb{K})$.

Definition 4.1 (Fréchet space)**.** *A Hausdorff topological vector space X is called a Fréchet space if its topology is metrisable with a translation-invariant metric d such that (X, d) is a complete metric space.*

Let X be a normed space and a set B is bounded if and only if $\sup\{\|x\|_X \mid x \in B\} < \infty$. This equivalence gives the following intuitive notion of boundedness in a general topological vector space.

Definition 4.2 (Bounded subsets in TVS)**.** *A subset $B \subset X$ of a TVS X is said to be bounded if for any $U \in \mathcal{T}_X$ with $0 \in U$, there exists $\varepsilon > 0$ such that $\varepsilon B \subset U$.*

Theorem 4.2 (Kolmogorov's normability criterion)**.** *Let X be a Hausdorff topological $\mathbb{K}$-vector space. Then X is normable if and only if X admits a nonempty bounded convex neighbourhood of the origin.*

Proof. If X normable with $\|\cdot\|_X$ generating its topology, then the collection $\{\mathbb{B}(0,r) \mid r \in (0,\infty)\}$ of open balls forms a basis of open neighbourhoods around the origin which are convex and bounded.
On the contrary, let U be a convex bounded neighbourhood of the origin, which admits a balanced open subset $V_0 \subset U$ with $0 \in V_0$. Then necessarily $W := conv(V_0)$ remains a balanced, convex neighbourhood of 0 contained in U by its convexity. Furthermore, W is the convex hull of an open set, thus defines an open set itself. In particular the open subset W is absorbing at each of its points. We claim that the corresponding Minkowski functional ρ_W defines a norm on X which generates the topology on X. Indeed, the Minkowski functional of any convex, balanced set, absorbing at each of its points, defines a semi-norm. If $x \neq 0$, we can find $A \subset X$ open neighbourhood of the origin such that $x \notin A$. By boundedness of W, we can find $\varepsilon > 0$ with $\varepsilon W \subset A$. In particular, $x \notin \varepsilon W$. By convexity, we necessarily have $x \notin \delta W$ for any $\delta \in (0,\varepsilon)$, giving us $\rho_W(x) \geqslant \varepsilon > 0$. Thus the Minkowski functional for W gives rise to a norm, for which we now denote $\rho_W(\cdot) = \|\cdot\|$.

The collection of open balls $\{\mathbb{B}(0,r) \mid r \in (0,\infty)\}$ forms a neighbourhood basis for the norm topology near the origin. Now any open ball around zero is a $\mathcal{T}_X$-neighbourhood since $\frac{r}{2}W \subset \mathbb{B}(0,r)$ for any $r \in (0,\infty)$. On the other hand, any $\mathcal{T}_X$-neighbourhood V_0 of the origin satisfies $\delta W = \mathbb{B}(0,\delta) \subset V_0$ for $r > 0$ sufficiently small by boundedness of W. Hence $\mathcal{T}_X = \mathcal{T}_{\|\cdot\|}$. □

My intuition for the normability criterion is by verifying that the Minkowski functional of the open unit ball(which is clearly bounded) coincides with the ambient norm. To conclude this section, we state some useful facts about bounded subsets in Hausdorff TVS, whose proofs are routine and hence omitted.

Proposition 4.1 (Properties of bounded subsets)**.** *Let X be a Hausdorff topological vector space. Then:*

(i) $B \subset X$ *bounded subset* $\Rightarrow \bar{B}$ *bounded subset of* X*;*

(ii) any finite union of bounded sets is bounded;

(iii) every compact set is bounded;

(iv) $B \subset X$ *bounded* $\Leftrightarrow \forall (x_i) \in B, (\lambda_i) \in \mathfrak{c}_0,\ \lambda_i x_i \to 0$ *in* X*;*

(v) continuous image of bounded subset in TVS is bounded;

(vi) X *locally convex* $\Rightarrow B \subset X$ *bounded if and only if* $\forall \rho$ *continuous semi-norm on* X*,* $\sup\{\rho(x) \mid x \in B\}$*;*

(vii) X *Fréchet space,* $B \subset X$ *bounded* $\Rightarrow \operatorname{diam} B < \infty$*;*

(viii) translation of bounded sets remain bounded;

5 Geometric consequences of the Hahn-Banach theorem

In a Euclidean space, for instance $\mathbb{R}^n$, any two disjoint convex sets can be separated by a hyperplane, which can be proven using an induction argument. The generalisation, in the setting of a locally convex Hausdorff TVS, is a geometric consequence of the celebrated Hahn-Banach theorem. The following are some prototypical results.

Definition 5.1 (Affine subspace and hyperplanes)**.** *Let X be a Hausdorff topological $\mathbb{K}$-vector space. An affine subspace in X is a subset $Y \subset X$ such that $Y - y \subset X$ is a linear subspace for any $y \in Y$.*
A hyperplane P in X is a maximal proper affine subspace in X, i.e. $P \subsetneq X$ whereas if $P \subset A \subset X$ for some $A \subset X$, then either $A = P$ or $A = X$.

Remark 5.1. Affine subspaces can interpreted precisely as the translation of linear subspaces, which corresponds to the infinite-dimensional generalisation of planes in Euclidean spaces. On the other hand, the definition of a hyperplane is consistent with the intuition that any hyperplane should have co-dimension 1. In fact, any translation of a hyperplane remains a hyperplane in a TVS.

Lemma 5.1 (Functional-hyperplane correspondence)**.** *Let X be a Hausdorff topological $\mathbb{K}$-vector space and $P \subset X$ a subset. Then the following statements are equivalent:*

(i) P is a hyperplane in X;

(ii) $L = P - x_0$ remains a hyperplane for any arbitrary $x_0 \in P$;

(iii) $P = \{f = \lambda\}$ for some $f \in X^\# \setminus \{0\}$ and $\lambda \in \mathbb{K}$.

Proof. $(i) \Rightarrow (ii)$: Set $L = P - x_0$ for some arbitrary fixed $x_0 \in P$. Then L defines a proper linear subspace since P is hyperplane. In particular, L is closed under any linear operations, thus $L - y = L$ remains a linear subspace for any $y \in L$. Now it remains to show that L is a maximal affine subspace. Take any affine subspace $A \subsetneq X$ with $L = P - x_0 \subset A$, which is equivalent to $P \subset A + x_0 \subsetneq X + x_0 = X$. Thus, L inherits maximality.
$(ii) \Rightarrow (iii)$: Suppose any translate of P remains a hyperplane. Fix $x_0 \in P$ and set $L = P - x_0$ which defines a hyperplane. Since L is maximal proper, we can pick any arbitrary $z \in X \setminus L$ such that $X = L + \mathbb{K} \cdot z$. Define the linear functional:

$$\phi\colon X = L + \mathbb{K} \cdot z \to \mathbb{K}, \quad \phi(x + \lambda z) := \lambda.$$

Clearly, $L = \ker \phi$ by maximality. By linearity, we can recover P by translation, i.e. $P = \{\phi = \phi(x_0)\}$.
$(iii) \Rightarrow (i)$: Let $f \in X^\# \setminus \{0\}$ and $P = \{f = \lambda\}$ for some $\lambda \in \mathbb{K}$. By linearity of f, backwards translation of P by any

element in P is equal to $\ker f$, which always defines a linear subspace. Since $f \not\equiv 0$, we can find $z \in X$ such that $f(z) \neq 0$. Now $(P+z) \cap P = \emptyset$, thus P is a proper affine subspace. If $\ker f$ defines a hyperplane, then note $P - x_0 = \ker f$ for any $x_0 \in P$, whence P would also define a hyperplane. Indeed, fix any $v \in X \setminus \ker f$, we have for any $x \in X$, $f(x - \frac{f(x)}{f(v)}v) = 0$, which implies $x - \frac{f(x)}{f(v)}v \in \ker f$ and therefore $X = \ker f + \mathbb{K} \cdot z$. This shows that $P = \{f = \lambda\}$ defines a hyperplane in X. □

Theorem 5.1 (Disjoint hyperplane from convex set)**.** *Let X be a Hausdorff TVS and $C \subset X$ non-empty open convex subset such that $0 \notin C$. Then there exists closed hyperplane P such that $P \cap C = \emptyset$.*

Proof. We will exploit the correspondence between linear functionals and hyperplanes here, which reduces the problem to finding a suitable $f \in X^* \setminus \{0\}$. Note any $\mathbb{C}$-vector space is also $\mathbb{R}$-linear. If the result holds for $\mathbb{R}$-vector spaces, we can set $F(x) = f(x) - if(ix)$ such that $f = \Re(F)$ and $C \cap \ker f = \emptyset$. Then $P = \ker F = \ker f \cap i \ker f$ is a closed hyperplane disjoint from C.
Now it suffices to show the $\mathbb{R}$-linear case. For any $x_0 \in C$, the translated and dilated subset $V = x_0 - C$ is an open convex neighbourhood of the origin. The Minkowski functional ρ_V defines a non-negative continuous sublinear functional(thus a semi-norm) such that $V = \{x \mid \rho_V(x) < 1\}$. In particular, we can deduce $\rho_V(x_0) \geqslant 1$ since $x_0 \notin V$. Consider the following:

$$f_0 \colon X_0 := \operatorname{span}_{\mathbb{R}}(\{x_0\}) \to \mathbb{R}, \quad f(tx_0) = t\rho_V(x_0).$$

Depending on the parity of the constant $t \in \mathbb{R}$, we have:

$$\begin{cases} t \geqslant 0 & \Rightarrow f_0(tx_0) = t\rho_V(x_0) = \rho_V(tx_0); \\ t < 0 & \Rightarrow f_0(tx_0) = -|t|\rho_V(x_0) = -\rho_V(tx_0) < 0 \leqslant \rho_V(tx_0). \end{cases} \Rightarrow f \leqslant \rho_V \text{ on } X_0.$$

The analytic form of Hahn-Banach theorem then gives rise to an extension $f \in X^* \setminus \{0\}$ controlled by the same semi-norm, i.e. $f|_{X_0} \equiv f_0$ on X_0 and $f \leqslant \rho_V$ on the whole space X. Now we can set $P := \{f = 0\}$. Then, for any $x \in C$, since $x_0 - x \in V$, we have:

$$f(x_0) - f(x) = f(x_0 - x) \leqslant \rho_V(x_0 - x) < 1 \Rightarrow f(x) > f(x_0) - 1 = \rho_V(x_0) - 1 \geqslant 0 \Rightarrow P \cap C = \emptyset.$$

□

Corollary 5.1. *If $Y \subset X$ is a closed affine subspace with $Y \cap C = \emptyset$, then we can find a closed affine hyperplane P such that $Y \subset P$ and $P \cap C = \emptyset$*

Proof. Assume without loss of generality(i.e. via translation) $Y \subset X$ is a closed linear subspace. We have

$$\pi_Y^{-1}(\pi_Y(C)) = \{y + C \mid y \in Y\}, \quad \text{where } \pi_Y \colon X \to X/Y,\ x \longmapsto x + Y.$$

In particular, $\pi_Y C$ is convex and open with respect to the quotient topology. By assumption, $0 \notin \pi_Y C$. Thus by Theorem 5.1 there exists closed hyperplane $\mathcal{P} \subset X/Y$ with $\mathcal{P} \cap \pi_Y C = \emptyset$. Then we can simply use $P = \pi_Y^{-1}(\mathcal{P})$, which is a closed hyperplane satisfying $P \cap C = \pi_Y^{-1}(\mathcal{P} \cap \pi_Y C) = \emptyset$ and $Y \subset P$. □

Recall any hyperplane in a Hausdorff topological $\mathbb{K}$-vector space is given by the level set of a linear functional $f \colon X \to \mathbb{K}$. The result below shows the geometric correspondence between linear functionals and hyperplanes.

Proposition 5.1 (Closed-dense duality of hyperplanes)**.** *A hyperplane $H = \{f = \lambda\}$ with $f \in X^{\#} \setminus \{0\}$ in a Hausdorff topological vector space is closed if and only if $f \in X^*$, and H dense if and only if f is discontinuous.*

Proof. Since translations are homeomorphisms of TVS, we may as well assume $\lambda = 0$. If $f \in X^*$, H is clearly closed by continuity. On the contrary, if H is dense, f cannot be continuous since f is not identically zero.
Recall from lemma 1.2: continuity in a topological space is characterised by convergence of nets. Suppose H is closed and let $\langle x_i \rangle$ be a net in H such that $x_i \to 0$. Fix $u \in X$ with $f(u) = 1$. Assume for contradiction that $f(x_i) \not\to 0$, which then implies without loss of generality, $|f(x_i)| > \varepsilon$ for all i for some $\varepsilon > 0$. Now consider the net with elements given by $y_i := u - \frac{f(u)}{f(x_i)} x_i \in H = \{f = 0\}$. Since $\sup_i \frac{f(u)}{f(x_i)} \leqslant \frac{1}{\varepsilon}$ and $x_i \to 0$, we necessarily have $y_i \to u$, thus giving $u \in H$ by closedness, which is a contradiction.
For f discontinuous, we find net $\langle x_i \rangle$ and $\varepsilon > 0$ such that $x_i \to 0$ while $|f(x_i)| > \varepsilon \forall i$. Take any $x \in X$ and construct net $z_i := x - \frac{f(x)}{f(x_i)} x_i \in H$ with $z_i \to x$ as above. In particular, $x \in \bar{H} \Rightarrow \bar{H} = X$. □

An advantage for geometric discussion in locally convex Hausdorff TVS is that any non-zero continuous $\mathbb{R}$-linear functional, the hyperplane $\ker f$ disconnects the space.

Proposition 5.2 (Functional disconnection)**.** *Let X be a Hausdorff topological $\mathbb{R}$-vector space. Then*

(i) $G \in \mathfrak{T}_X$ connected $\Rightarrow G$ is arcwise connected

(ii) $f \in X^* \setminus \{0\} \Rightarrow X \setminus \ker f$ has precisely two connected components.

Proof. (i) Any path-connected Hausdorff space is necessarily arcwise connected since the image of a path is closed. Now the claim is equivalent to local path-connectedness. Indeed, denote the reachable set for a fixed $x \in G$ by:

$$\Gamma_x = \{y \in G \mid \exists \gamma\colon [0,1] \to G \text{ continuous}, \gamma(0) = x, \gamma(1) = y\}.$$

The reachable set is clearly nonempty(since we can connect x to itself via the constant path). If we can show that the reachable set Γ_x is open for each $x \in G$, we can then write G as a disjoint union:

$$\Gamma_x \cup \big(\bigcup_{y \notin \Gamma_x} \Gamma_y\big) = G,$$

where the disjointness follows from the fact that reachablility induces an equivalence relation on points in G, contradicting the connectedness of G unless $G = \Gamma_x$. For this purpose, it suffices to show that G is locally star-shaped. Again, it suffices to assume without loss of generality $0 \in G$ and simply consider the geometry near the origin. We will now show that for any open neighbourhood U of the origin, there exists a further subset $V \subset U$ with $0 \in V$ and $\varepsilon V \subset V$ for all $\varepsilon \in [0,1]$. We know that $U_0 = \{(t,x) \in \mathbb{R} \times U \mid tx \in U\}$ is an open neighbourhood of the point $(0, \mathbf{0}_X) \in \mathbb{R} \times X$, which then admits a open rectangle $(-\delta, \delta) \times W \subset U_0$ containing the origin. Now we can define $V = \bigcup_{t \in [0,\delta)} tW$, which is the desired open neighbourhood of 0 such that $\varepsilon V = \bigcup_{t \in [0,\varepsilon\delta)} tW \subset \bigcup_{t \in [0,\delta)} tW = V$ for any $\varepsilon \in [0,1]$. Then V is star-shaped at 0.
(ii) This follows from $X \setminus \ker f = f^{-1}((-\infty, 0)) \cup f^{-1}((0,\infty))$, which is disjoint union of two open sets. ☐

The following definitions, motivated by the above result, describe the separation of topological vector spaces.

Definition 5.2 (Half-spaces)**.** *Let X be a Hausdorff topological $\mathbb{R}$-vector space. A subset $S \subset X$ is an open half-space if there exists $f \in X^*$ such that $S = \{f > \lambda\}$ for some $\lambda \in \mathbb{R}$. A closed half-space $S \subset X$ is given by $S = \{f \geqslant \lambda\}$ for some $f \in X^*$ and $\lambda \in \mathbb{R}$.*

Definition 5.3 (Separation in TVS)**.** *Let X be a Hausdorff topological $\mathbb{R}$-vector space and $A, B \subset X$ subsets. We say A and B are*

(i) strictly separated if they are contained in two disjoint open half-spaces;

(ii) separated if they are contained in two closed half-spaces whose intersection is at most a closed hyperplane.

Remark 5.2. It should be intuitively clear that the closure of any open half-space is a closed half-space and interior of a closed hal-space is an open half-space. In particular, we can rephrase the separation conditions using continuous linear functionals on X.

Theorem 5.2 ($\mathbb{R}$-functional separation theorem)**.** *Let X be a Hausdorff topological $\mathbb{R}$-vector space and $A, B \subset X$ disjoint convex sets with $A \in \mathfrak{T}_X$. Then there exists $\varphi \in X^*$ and $\lambda \in \mathbb{R}$ such that:*

$$\varphi(a) < \lambda \leqslant \varphi(b) \text{ for any } a \in A, b \in B.$$

If we assume further that B is open in X, then A and B are strictly separated by φ.

Proof. Set $C = A - B = \{a - b \mid a \in A, b \in B\} = \bigcup_{b \in B}(A - b)$, which is clearly open(since translations are homeomorphisms) and convex(from convexity of A and B). Note by disjointness, $0 \notin C$, whence we can find a closed hyperplane $P = \{\varphi = 0\} \subset X$ disjoint from C by Theorem 5.1. By linearity of φ, we have $\varphi(C)$ is a convex subset of $\mathbb{R} \setminus \{0\}$. In particular, φ has a sign on the convex set C and we can assume without loss of generality that $\varphi > 0$ on C. Then for any $a - b \in C$, we have:

$$0 < \varphi(a - b) = \varphi(a) - \varphi(b) \Leftrightarrow \varphi(b) < \varphi(a) \to \sup_{b \in B} \varphi(b) \leqslant \lambda \leqslant \inf_{a \in A} \varphi(a),$$

for some constant $\lambda \in \mathbb{R}$. However, we notice that $\varphi \not\equiv 0$, which implies we can choose some $y \in X$ with $\varphi(y) = 1$. Then since A is open, for any $x \in A$ and $\varepsilon > 0$ sufficiently small, $x \pm \varepsilon y \in A$, which shows that $f(A)$ is open. In particular, by convexity, $f(A)$ is an open interval and we have $\sup_{b \in B} \varphi(b) \leqslant \lambda < \varphi(a)$ for any $a \in A$. If B is open, we can then use the same argument to deduce strict separation. ☐

Lemma 5.2. *Let X be a Hausdorff topological vector space, $K \subset X$ compact subset, and $V \subset X$ an open neighbourhood of K. Then there exists an open neighbourhood U of the origin such that $K + U \subset V$.*

Proof. Denote by $\mathcal{U}_0$ the collection of open neighbourhoods of the origin and equip $\mathcal{U}_0$ with the partial order $\preceq$ by reverse inclusion, i.e. $U_1 \preceq U_2$ if $U_1 \supset U_2$. Suppose by contradiction that $K + U \not\subset V$ for any $U \in \mathcal{U}_0$. This then gives us two nets $\langle x_U \mid U \in \mathcal{U}_0\rangle \in K$ and $\langle y_U \mid U \in \mathcal{U}_0\rangle$ with $y_U \in U$ and $x_U + y_U \in X \setminus V$ for any $U \in \mathcal{U}_0$. By our choice of ordering, we necessarily have $y_U \to 0$ and thus $\langle y_U\rangle$ clusters at the origin. The compactness of K gives us the existence of a cluster point $x \in K$ for the net $\langle x_U\rangle$. Note now any neighbourhood of x is the translation of a neighbourhood around the origin by x and vice versa. In particular, $\langle x_U + y_U\rangle$ clusters at x, which implies that $x \in X \setminus V = \overline{X \setminus V}$. This leads to a contradiction since $K \subset V$. ☐

Remark 5.3. The compactness assumption on K is necessary; it is not sufficient to assume K is closed.

Theorem 5.3 (Closed-compact functional separation)**.** *Let X be a locally convex Hausdorff topological $\mathbb{R}$-vector space and $A, B \subset X$ disjoint closed convex subsets with B compact. Then A and B are strictly separated.*

Proof. Lemma 5.2 gives an open neighbourhood U_0 of the origin such that $B + U_0 \subset X \setminus A$. By local convexity, we can find a continuous semi-norm $\rho\colon X \to [0, \infty)$(e.g. Minkowski functional for U_0) such that $\{x \mid \rho(x) < 1\} \subset U_0$(e.g. via scaling the semi-norm). Set $U := \{x \mid \rho(x) < \frac{1}{2}\} \in \mathcal{T}_X$. We clearly have $(A + U) \cap (B + U) = \emptyset$. Indeed, otherwise we can find $x \in (A + U) \cap (B + U)$ such that:

$$x = a + u = b + v \text{ for some } u, v \in U, a \in A, b \in B \Rightarrow a = b + (v - u), \text{ where } \rho(v - u) \leqslant \rho(v) + \rho(u) < 1,$$

which then implies $v - u \in U_0$ giving us $a \in B + U_0 \subset X \setminus A$, a contradiction. Now we have:

$$A + U = \bigcup_{a \in A} a + U \quad \text{and} \quad B + U = \bigcup_{b \in B} b + U,$$

which are thus open subsets containing A and B respectively. Recall that balls with respect to semi-norms are convex, from which it follows that $A + U$ and $B + U$ are both convex sets. Thus we can apply the usual functional separation theorem 5.1 to deduce the desired result. ☐

Remark 5.4. Consider the ambient space $X = \mathbb{R}^2$ with disjoint closed convex subsets $A = \{(x, y) \mid y \leqslant 0\}$ and $B = \{(x, y) \mid y \geqslant x^{-1} > 0\}$. Then A and B cannot be strictly separated in $\mathbb{R}^2$. In particular, it is necessary to assume the compactness of one of the disjoint closed convex sets.

Corollary 5.2. *Any singleton can be strictly separated from a disjoint closed convex set in a Hausdorff $\mathbb{R}$-LCS.*

Corollary 5.3 (Closed convex hull)**.** *Let X be a locally convex Hausdorff $\mathbb{R}$-TVS and $A \subset X$. Then: $\overline{conv}(A)$ is the intersection of all closed half-spaces containing A.*

Corollary 5.4. $\mathrm{clin}_{\mathbb{R}}(A) = \overline{\mathrm{span}_{\mathbb{R}}(A)} = \bigcap\{P \subset X \textit{ closed hyperplane } \mid P \supset A\}$.

Theorem 5.4 ($\mathbb{C}$-functional separation theorem)**.** *Let X be a locally convex Hausdorff TVS and $A, B \subset X$ disjoint closed convex subsets. If B is compact in X, then there exists $f \in X^*$ and constants $\lambda \in \mathbb{R}$, $\varepsilon > 0$ such that:*

$$\Re f(a) \leqslant \lambda < \lambda + \varepsilon \leqslant \Re f(b) \quad \textit{for any } a \in A, b \in B.$$

Corollary 5.5. *Let $Y \subset X$ be a linear subspace. Then $Y \subset X$ is a dense subspace if and only if for $f \in X^*$, $f \equiv 0$ on Y implies $f \equiv 0$ on X.*

Corollary 5.6 (Distance functional)**.** *Let $Y \subset X$ be a closed subspace and $x \in X \setminus Y$. Then there exists $\phi \in X^*$ such that $f|_Y \equiv 0$ and $f(x) = 1$.*

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