Theorem: The Euler-Lagrange equations for the energy functional are $\tau(u) = 0$, where $\tau(u) = \tau(u)^i \partial_{x^i}$ is given by

Proof: In any local calculation, let's adopt coordinates x^i with Latin indices for the domain M and coordinates y^i with Greek indices for the target N. Let $u_t : M \times [0,1] \to N$ be a geodesic variation of $u = u_0$, e.g., $u_t(x) = exp_{u(x)}(t\psi)$ where $\psi = \psi^\alpha \partial_{y^\alpha}$ is a vector field along u. We compute:

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} E(u_t) &= \left.\frac{d}{dt}\right|_{t=0} \int_M \frac{1}{2}\|Du_t\|^2 dvol_M = \left.\frac{d}{dt}\right|_{t=0} \int_M \frac{1}{2}\left\langle Du_t, Du_t\right\rangle dvol_M \\
&= \frac{1}{2}\int_M \frac{d}{dt}\left.\left\langle \frac{\partial u_t^\alpha}{\partial x^i}\partial_{y^\alpha}\otimes dx_t^i, \frac{\partial u_t^\beta}{\partial x^j}\partial_{y^\beta}\otimes dx_t^j\right\rangle\right|_{t=0} dvol_M \\
&= \int_M \left.\left\langle \nabla_{\partial_t}\left[\partial_{u_t^\alpha}\otimes dx_t^i\right], \partial_{u_t^\beta}\otimes dx_t^j\right\rangle\right|_{t=0} dvol_M \\
&= \int_M \left.\left\langle \left[\nabla_{\partial_t} du_t^i\right]\otimes\partial_{y^i} + du_t^i\otimes\left[\nabla_{\partial_t}\partial_{y^i}\right], du_t^j\otimes\partial_{y^j}\right\rangle\right|_{t=0} dvol_M \\
&= \int_M \left.\left\langle \cancel{\left[\nabla_{\partial_t} du_t^i\right]}\otimes\partial_{y^i} + du_t^i\otimes\left[\nabla_{\partial_t}\partial_{y^i}\right], du_t^j\otimes\partial_{y^j}\right\rangle\right|_{t=0} dvol_M
\end{aligned}
$$

□