$$\boxed{\textit{Theorem:}\ \text{For } f \in C^1(M,\mathbb{R}),\ \text{we have } \tfrac{1}{2}\Delta\|\nabla f\|^2 = \|Hess(f)\|^2 + \langle \nabla f, \nabla(\Delta f)\rangle + Ric(grad(f), grad(f)).}$$

Proof: Throughout, ∇ will only be used to denote the covariant derivative, so that $grad(f) = (\nabla f)^\sharp$. Also, recall: $\sharp$ and $\flat$ are isomorphisms and isometries in the sense that $\langle X, Y\rangle = \langle X^\flat, Y^\flat\rangle$ and $\langle \omega, \eta\rangle = \langle \omega^\sharp, \eta^\sharp\rangle$; $\sharp$ and $\flat$ commute with ∇, as $(\nabla^{T^*M}\omega)^\sharp = \nabla^{TM}(\omega^\sharp)$ and $(\nabla^{TM}X)^\flat = \nabla^{T^*M}(X^\flat)$; and $\sharp$ and $\flat$ allow us to express such quantities: $\langle X, Y\rangle = X^\flat(Y)$, $\langle \omega, \eta\rangle = \omega(\eta^\flat)$, and $\omega(X) = \langle \omega^\sharp, X\rangle_{TM} = \langle \omega, X^\flat\rangle_{T^*M}$.

So, this statement is equivalent to: $\frac{1}{2}\Delta\|(\nabla f)^\sharp\|^2 = \|Hess(f)\|^2 + \langle (\nabla f)^\sharp, (\nabla(\Delta f))^\sharp\rangle + Ric((\nabla f)^\sharp, (\nabla f)^\sharp)$. This is what we'll show instead. Furthermore, we seek to prove a tensor identity, so we can fix a point $p \in M$ and utilize geodesic normal coordinates $\{\partial_i\}$ on M such that $\langle \partial_i, \partial_j\rangle = \delta_{ij}$ and $\nabla_{\partial_i}\partial_j|_p = 0$. We exploit the symmetry of $Hess(f)$ twice in this calculation.

$$\begin{aligned}
\frac{1}{2}\Delta\|(\nabla f)^\sharp\| &= \frac{1}{2}\sum_i \nabla_{\partial_i}\nabla_{\partial_i}\left\langle (\nabla f)^\sharp, (\nabla f)^\sharp\right\rangle = \sum_i \nabla_{\partial_i}\left\langle \nabla_{\partial_i}(\nabla f)^\sharp, (\nabla f)^\sharp\right\rangle \\
&= \sum_i \nabla_{\partial_i}\left[\left[\nabla_{\partial_i}(\nabla f)^\sharp\right]^\flat\left((\nabla f)^\sharp\right)\right] = \sum_i \nabla_{\partial_i}\left[\left[\nabla_{\partial_i}(\nabla f)\right]\left((\nabla f)^\sharp\right)\right] \\
&= \sum_i \nabla_{\partial_i}\left[\left[Hess(f)\right]\left(\partial_i, (\nabla f)^\sharp\right)\right] = \sum_i \nabla_{\partial_i}\left[\left[Hess(f)\right]\left((\nabla f)^\sharp\right)\partial_i\right] \\
&= \sum_i \nabla_{\partial_i}\left[\left[\nabla_{(\nabla f)^\sharp}(\nabla f)\right](\partial_i)\right] = \sum_i \nabla_{\partial_i}\left\langle \left[\nabla_{(\nabla f)^\sharp}(\nabla f)\right]^\sharp, \partial_i\right\rangle \\
&= \sum_i \nabla_{\partial_i}\left\langle \nabla_{(\nabla f)^\sharp}\left[(\nabla f)^\sharp\right], \partial_i\right\rangle \\
&= \sum_i \left\langle \nabla_{\partial_i}\nabla_{(\nabla f)^\sharp}\left[(\nabla f)^\sharp\right], \partial_i\right\rangle + \sum_i \cancelto{0}{\left\langle \nabla_{(\nabla f)^\sharp}\left[(\nabla f)^\sharp\right], \nabla_{\partial_i}\partial_i\right\rangle} \\
&= \sum_i \left\langle R(\partial_i, (\nabla f)^\sharp)\left[(\nabla f)^\sharp\right], \partial_i\right\rangle + \sum_i \left\langle \nabla_{(\nabla f)^\sharp}\nabla_{\partial_i}\left[(\nabla f)^\sharp\right], \partial_i\right\rangle + \sum_i \left\langle \nabla_{[\partial_i, (\nabla f)^\sharp]}\left[(\nabla f)^\sharp\right], \partial_i\right\rangle
\end{aligned}$$

Each of these three summands can be computed as follows:

$$\sum_i \left\langle R(\partial_i, (\nabla f)^\sharp)\left[(\nabla f)^\sharp\right], \partial_i\right\rangle = Ric\left((\nabla f)^\sharp, (\nabla f)^\sharp\right) \text{ by definition of the Ricci tensor.}$$

$$\begin{aligned}
\sum_i \left\langle \nabla_{(\nabla f)^\sharp}\nabla_{\partial_i}\left[(\nabla f)^\sharp\right], \partial_i\right\rangle &= \sum_i \nabla_{(\nabla f)^\sharp}\left\langle \nabla_{\partial_i}\left[(\nabla f)^\sharp\right], \partial_i\right\rangle - \sum_i \cancelto{0}{\left\langle \nabla_{\partial_i}\left[(\nabla f)^\sharp\right], \nabla_{(\nabla f)^\sharp}\partial_i\right\rangle} \\
&= \sum_i (\nabla f)^\sharp\left\langle \nabla_{\partial_i}\left[(\nabla f)^\sharp\right], \partial_i\right\rangle = \sum_i (\nabla f)^\sharp\left[\left[\nabla_{\partial_i}\left((\nabla f)^\sharp\right)^\flat\right](\partial_i)\right] \\
&= \sum_i (\nabla f)^\sharp\left[\left[\nabla_{\partial_i}(\nabla f)\right](\partial_i)\right] = \nabla f)^\sharp\left[\sum_i \left[\left[\nabla_{\partial_i}(\nabla f)\right](\partial_i)\right]\right] \\
&= (\nabla f)^\sharp(\Delta f) = \left[\nabla(\Delta f)\right]\left((\nabla f)^\sharp\right) \\
&= \left\langle (\nabla(\Delta f))^\sharp, (\nabla f)^\sharp\right\rangle
\end{aligned}$$

$$
\begin{aligned}
\sum_i \left\langle \nabla_{[\partial_i,(\nabla f)^\sharp]}\left[(\nabla f)^\sharp\right], \partial_i \right\rangle &= \sum_i \left[\nabla_{[\partial_i,(\nabla f)^\sharp]}\left[(\nabla f)^\sharp\right]\right]^\flat(\partial_i) = \sum_i \left[\nabla_{[\partial_i,(\nabla f)^\sharp]}(\nabla f)\right](\partial_i) \\
&= \sum_i [Hess(f)]\left(\left[\partial_i,(\nabla f)^\sharp\right], \partial_i\right) \\
&= \sum_i [Hess(f)]\left(\nabla_{\partial_i}\left[(\nabla f)^\sharp\right] - \cancelto{0}{\nabla_{(\nabla f)^\sharp}\partial_i}, \partial_i\right) \\
&= \sum_i [Hess(f)]\left(\partial_i, \nabla_{\partial_i}\left[(\nabla f)^\sharp\right]\right) = \sum_i \left[\nabla_{\partial_i}(\nabla f)\right]\left(\nabla_{\partial_i}\left[(\nabla f)^\sharp\right]\right) \\
&= \sum_i \left\langle \nabla_{\partial_i}(\nabla f), \left(\nabla_{\partial_i}\left[(\nabla f)^\sharp\right]\right)^\flat \right\rangle = \sum_i \left\langle \nabla_{\partial_i}(\nabla f), \nabla_{\partial_i}(\nabla f)\right\rangle \\
&= \sum_i \frac{1}{\langle dx^i, dx^i\rangle}\left\langle dx^i \otimes \nabla_{\partial_i}(\nabla f), dx^i \otimes \nabla_{\partial_i}(\nabla f)\right\rangle \\
&= \sum_i \left\langle dx^i \otimes \nabla_{\partial_i}(\nabla f), dx^i \otimes \nabla_{\partial_i}(\nabla f)\right\rangle \\
&= \sum_{i,j} \left\langle dx^i \otimes \nabla_{\partial_i}(\nabla f), dx^j \otimes \nabla_{\partial_j}(\nabla f)\right\rangle \text{ by orthogonality of } \{dx^i\} \\
&= \sum_j \left\langle \sum_i \left(dx^i \otimes \nabla_{\partial_i}(\nabla f)\right), dx^j \otimes \nabla_{\partial_j}(\nabla f)\right\rangle \\
&= \left\langle \nabla^2 f, \sum_j \left(dx^j \otimes \nabla_{\partial_j}(\nabla f)\right)\right\rangle \\
&= \left\langle \nabla^2 f, \nabla^2 f\right\rangle = \|Hess(f)\|^2
\end{aligned}
$$

Summing these three quantities yields the desired result.

$\square$