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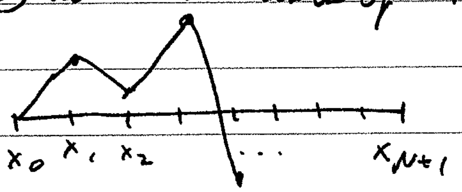
Recall: $\begin{cases} -(ku')' + cu = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$ Dirichlet problem

Finite element method : $a(u_h, v_h) = l(v_h)$

for all v_h the piecewise polynomial, $v_h(0) = v_h(1) = 0$.

2 Finite elements of 1^{st} order; we choose $p=1$.



$\dim(V_h) = N$ why? ← 2 endpts per line seg

degrees of freedom : $2(N+1)$

constraints : N (continuity)
$+$
2 (bdry cond)

$\therefore \ 2(N+1) - N - 2 = N$.

Write $u_h(x) := \sum_{i=1}^{N} \alpha_i \phi_i(x)$, with $\alpha_i \in \mathbb{R}$ and $\phi_1, \phi_2, \dots, \phi_N$ basis fnct.
We want ϕ_i to have small spt. Let's choose them as follows:

$$\phi_i(x) := \begin{cases} 0 & \text{on } (x_0, x_{i-1}) \cup (x_{i+1}, x_{N+1}) \\ \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{on } [x_{i-1}, x_i] \\ \frac{x - x_{i+1}}{x_i - x_{i+1}} & \text{on } [x_i, x_{i+1}] \end{cases} \quad spt(\phi_i) = [x_{i-1}, x_i] \cup [x_i, x_{i+1}]$$

note: $\frac{x - x_{i-1}}{x_i - x_{i-1}} = \frac{x - x_{i-1}}{h_{i-1}}$, $\frac{x - x_{i+1}}{x_i - x_{i+1}} = \frac{x - x_{i+1}}{-h_i}$

We call these the hat functions 

and they form a "nodal basis".

Now, just take $\alpha_j = u_h(x_j)$. This lets us interpolate.

Thus, $a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h$ is equivalent to
$a(u_h, \phi_i) = l(\phi_i)$ for $1 \le i \le N$.

Note that a is bilinear, so in fact

$$a\left(\sum_{j=1}^{N} \alpha_j \phi_j, \phi_i\right) = l(\phi_i)$$

$\Downarrow$ we can write

$$\sum_{j=1}^{N} \alpha_j a(\phi_j, \phi_i) = l(\phi_i) \quad \text{for } 1 \le i \le N$$

$$\Downarrow$$

$$A\vec{\alpha} = \vec{b}, \text{ where}$$

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \quad \text{and} \quad A_{ij} = a(\phi_j, \phi_i) \quad \text{and} \quad b = \begin{bmatrix} l(\phi_1) \\ \vdots \\ l(\phi_N) \end{bmatrix}$$

Let's look at the case $k=1$, $c=0$, i.e. the PDE is $-u'' = f$.

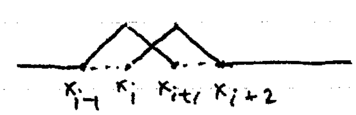
Well, $a(\phi_i, \phi_j) = \int_0^1 \phi_i' \cdot \phi_j'$. Recall that A is symmetric, so we can analyze this by first fixing i.

$$A_{ii} = \int_0^1 \phi_i' \cdot \phi_i' = \int_{x_{i-1}}^{x_{i+1}} (\phi_i')^2 = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h_{i-1}}\right)^2 + \int_{x_i}^{x_{i+1}} \left(\frac{-1}{h_i}\right)^2$$

$$= \frac{1}{h_{i-1}^2} \cdot \int_{x_{i-1}}^{x_i} 1 + \frac{1}{h_i^2} \int_{x_i}^{x_{i+1}} 1$$

$$= \frac{1}{h_{i-1}} + \frac{1}{h_i} \Leftarrow \text{if } \underbrace{h_i = h \text{ for all } i}_{\text{"uniform mesh"}}, \text{ then } A_{ii} = \frac{2}{h}.$$

$$A_{i,i+1} = \int_0^1 \phi_i' \cdot \phi_{i+1}'$$



$$= \int_{x_i}^{x_{i+1}} \phi_i' \cdot \phi_{i+1}'$$

unif. mesh.

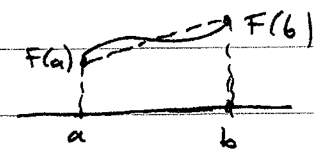
$$= \int_{x_i}^{x_{i+1}} \frac{-1}{h_i} \cdot \frac{1}{h_{i+1}} = \frac{-1}{h_i \cdot h_{i+1}} \cdot h_i = \frac{-1}{h_{i+1}} = \frac{-1}{h}$$

$A_{i,i+2}=0$, and $A_{i,k\geq i+2}=0$ as well. By symmetry,

$$A = \frac{1}{h}\cdot\begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}.$$

What is the RHS? $b_i = \int_0^1 f\cdot\phi_i = \int_{x_{i-1}}^{x_{i+1}} f\cdot\phi_i$.
Since we don't analytically need to know f, we use a numerical integration method for f. For example, employing the trapezoidal integration for f,

$$\int_a^b F \simeq \frac{b-a}{2}\cdot\left[F(a)+F(b)\right]$$



$$\therefore \int_{x_{i-1}}^{x_i} f\phi_i \simeq \frac{h}{2}\cdot\left[f(x_i)\cdot\underbrace{\phi_i(x_i)}_{1} + f(x_{i-1})\cdot\cancelto{0}{\phi_i(x_{i-1})}\right]$$
$$\simeq \frac{h}{2}\cdot f(x_i).$$

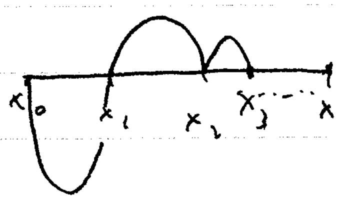
$$\therefore \int_{x_i}^{x_{i+1}} f\phi_i \simeq \frac{h}{2}\left[f(x_{i+1})\cdot\cancelto{0}{\phi_i(x_{i+1})} + f(x_i)\cdot\underbrace{\phi_i(x_i)}_{1}\right]$$
$$\simeq \frac{h}{2} f(x_i)$$

$\therefore b_i \simeq h\cdot f(x_i)$, since $b_i = \int_0^1 f\cdot\phi_i = \int_{x_{i-1}}^{x_{i+1}} f\cdot\phi_i$

This reveals: $$\frac{1}{h}\begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}\begin{bmatrix}\alpha_1 \\ \vdots \\ \vdots \\ \alpha_N\end{bmatrix} = h\cdot\begin{bmatrix} f(x_1) \\ \vdots \\ \vdots \\ f(x_n)\end{bmatrix}$$

as our system.

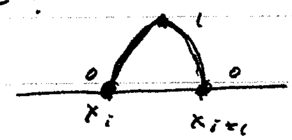
3 Finite elements of second order; $p = 2$.

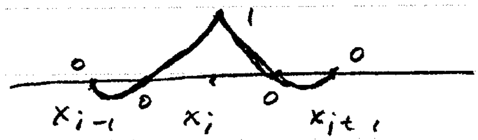


Now what is $\dim(V_h)$?

$$\Big[\underbrace{3\cdot(N+1)}_{\substack{\#\text{ param to def quadratic} \\ \times \#\text{ intervals}}}\Big] - \Big[\underbrace{N}_{\substack{\text{interior} \\ C^0\text{ cond}}} + \underbrace{2}_{\text{bdry values}}\Big] = 2N+1.$$

Which basis should we take now?

$N+1$ functions of the form 

N func of the form 

(we use them instead of the PWL hat func so that we're only dealing w/ quadratics, to get better control on convergence)

The "nodes" are now $\{x_i\} \cup \{x_{i+\frac{1}{2}}\}$.

4 Existence & Uniqueness of FEM soln.

(Uniqueness): Suppose u_1, u_2 are solns. Consider $w = u_1 - u_2$.

$$\begin{cases} \int_0^1 (k\, u_1' v' + c\cdot u_1 \cdot v) = \int_0^1 f\cdot v \\ \int_0^1 (k\cdot u_2' v' + c\cdot u_2\cdot v) = \int_0^1 f\cdot v \end{cases}$$

$$\Downarrow$$

$$\int_0^1 (k\cdot w'\cdot v' + c\cdot w\cdot v) = 0 \quad \text{for all } v \in V_h.$$

Pick $v = w$, then

$$\int_0^1 \big(\underbrace{k}_{\geq 0}\underbrace{(w')^2}_{\geq 0} + \underbrace{c}_{\geq 0}\underbrace{(w)^2}_{\geq 0}\big) = 0$$ but c might be 0...

$\therefore \int_0^1 k(w')^2 = \int_0^1 c(w)^2 = 0.$ so let's use $\int_0^1 k\cdot(w')^2 = 0$.

Now, $(w' = 0) \Rightarrow w$ is constant on each interval (x_i, x_{i+1}).
(Recall $v \in V_h$ is PWL!) Well, $w \in V_h$ now means each constant is the same. As w is 0 on the endpts, $w \equiv 0$ on $\{x_i\}$, and hence everywhere. $\square$

Here, proving uniqueness is equivalent to proving existence...
(Fundamental Theorem of Linear Algebra;
$A\vec{x} = \vec{b}$ has a unique soln also tells us
A is invertible so we can find $A^{-1}\vec{b} =: \vec{x}$ as a soln!
This is the crux of reformulating a problem as a
linear FD problem!)

5 Convergence

"Energy" norm of the error:

$$\left[\int_0^1 k(u' - u_h')^2 \, dx\right]^{\frac{1}{2}} =: \|u - u_h\|_E .$$

$\mathcal{L}^2$ norm of error:

$$\left[\int_0^1 (u - u_h)^2 \, dx\right]^{\frac{1}{2}} =: \|u - u_h\|_{\mathcal{L}^2} .$$

We can show that:

Lemma $\|u - u_h\|_E \le \|u - v_h\|_E$ for all $v_h \in V_h$,
i.e. $\|u - u_h\|_E = \min_{v_h \in V_h} \|u - v_h\|_E$. //

Result: u_h is a PW poly of degree p

$$\Rightarrow \begin{cases} \|u - u_h\|_E = O(h^p) \\ \|u - u_h\|_{\mathcal{L}^2} = O(h^{p+1}) . \end{cases}$$

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Convergence of FEM (in 1-D)

Thm $\|u-u_h\|_E \leq \min_{v_h \in V_h} \|u-v_h\|$.

Recall: Cauchy-Schwarz inequality (for $\mathcal{L}^2$)

Thm: $\forall f, g \in \mathcal{L}^2(\Omega)$, we have $\|f \cdot g\|_{\mathcal{L}^1} \leq \|f\|_{\mathcal{L}^2} \cdot \|g\|_{\mathcal{L}^2}$

and Cauchy-Schwarz inequality (for l_2)

Thm: $\forall \{a_i\}, \{b_i\} \in l_2(\mathbb{Z})$, we have $\left|\sum_{i=1}^{\infty} a_i b_i\right| \leq \left(\sum_{i=1}^{\infty} a_i^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{\infty} b_i^2\right)^{\frac{1}{2}}$

Remember that u_h satisfies

$$\int_0^1 [k u_h' v_h' + c u_h \cdot v_h] = \int_0^1 f v_h \quad \text{for all } v_h \in V_h.$$

We also have, for the exact soln u,

$$\int_0^1 [k \cdot u' \cdot v_h' + c \cdot u \cdot v_h] = \int_0^1 f \cdot v_h \quad \text{for all } v_h \in V_h$$

In other words, we have consistency of our PDE FD method!

i.e. $\int_0^1 [k(u-u_h)' v_h' + c \cdot (u-u_h) \cdot v_h] = 0$

$$\Updownarrow$$

$a(u-u_h, v_h) = 0$ for all $v_h \in V_h$.

These are the orthogonality equations.

Proof Assume $c = 0$. Then recall: $a(u,v) := \int_0^1 k \cdot u' v'$!

$$\|u-u_h\|_E^2 = a(u-u_h, u-u_h) + \underbrace{a(u-u_h, u_h-v_h)}_{=0 \text{ since } u_h - v_h \in V_h}$$

$$= a(u-u_h, u-v_h)$$

$$= \int_0^1 k \cdot (u-u_h)' (u-v_h)'$$

C-S

$$\leq \left[\int_0^1 (\sqrt{k} \cdot (u-u_h)')^2\right]^{\frac{1}{2}} \cdot \left[\int_0^1 (\sqrt{k} \cdot (u-v_h)')^2\right]^{\frac{1}{2}}$$

$$\leq \left[\int_0^1 k \cdot ((u-u_h)')^2\right]^{\frac{1}{2}} \cdot \left[\int_0^1 k \cdot ((u-v_h)')^2\right]^{\frac{1}{2}}$$

$$\leq \|u-u_h\|_E \cdot \|u-v_h\|_E. \quad \square$$

We've noted before that $\|u-u_h\|_E = \min_{v_h \in V_h} \|u - v_h\|_E = O(h^p)$.

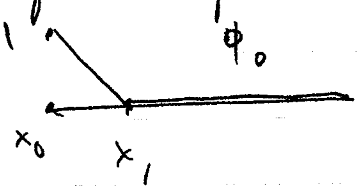
when using a degree p polynomial

We can also show that $\|u-u_h\|_{\mathcal{L}^2} = O(h^{p+1})$. ← to check code, run on a known soln which is a degree p polynomial, as $O(h^p) = C \cdot h^p$ where $C = u^{(p+1)}(\xi)$.

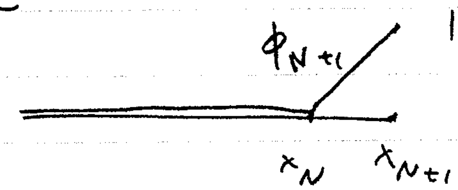
6 Non-homogeneous boundary condition.

$$-(ku')' + c \cdot u = f$$
$$u(0) = \alpha$$
$$u(1) = \beta.$$

Again, let $V_h := \{C^0, \text{PW-degree } p \text{ poly } \underline{\text{and}}\ v_h(0) = v_h(1) = 0\}$.
Now, $u_h \notin V_h$! Nevertheless, we prefer this b/c integration by parts still kills bdy terms (when evaluating v_h on the boundary). To account for this, we study $u_h - \alpha\phi_0 - \beta\phi_{N+1} \in V_h$, where ϕ_0 and ϕ_{N+1} are the "half-hat" fncs looking like



and



Call $w_h := u_h - \alpha\phi_0 - \beta \cdot \phi_{N+1} \in V_h$.
Define $u_{Dirichlet} = u_D := \alpha\,\phi_0(x) + \beta\,\phi_{N+1}(x)$, so that $w_h = u_h - u_D$, i.e. $u_h = w_h + u_D$. (It's important that $w_h \in V_h$!)
so $\int_0^1 [(k \cdot w_h' v_h') + c \cdot w_h \cdot v_h] = \int_0^1 f \cdot v_h - \int_0^1 (k u_D' v_h' + c\, u_D v_h)$.

Now, $A\vec{w_h} = \vec{b} + \vec{u}_D$ ↙ Dirichlet data!

Function spaces

To understand the FEM, we need to know about Sobolev spaces.

1 Inner product spaces

defn X is an inner product space if it is equipped with an inner product, i.e. a map $(\cdot,\cdot): X \times X \to \mathbb{R}$ satisfying

- $\forall u, v \in X$, $(u,v) = (v,u)$ symmetry
- $\forall \alpha \in \mathbb{R}$, $(\alpha \cdot u, v) = \alpha \cdot (u,v)$ linearity
- $\forall u, v, w \in X$, $(u+v, w) = (u,w) + (v,w)$.
- $\forall u \in X$, $(u,u) \geq 0$ w/ equality iff $u = 0$. non-degeneracy.

ex/ ($X = \mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ = dot product)

ex/ $X = \mathcal{L}^2(\Omega)$, $\|f\|_{\mathcal{L}^2} \rightsquigarrow (f,g) = \int_\Omega f \cdot g$.

Lemma An inner product induces a norm, i.e. to every inner product space $(X, (\cdot,\cdot))$, we can associate the normed space $(X, \|\cdot\|_{(\cdot,\cdot)})$.

defn X is a normed space if it is equipped with a norm, i.e. a map $\|\cdot\|: X \to \mathbb{R}$ satisfying

- $\forall u \in X$, $\|u\| \geq 0$ w/ equality iff $u = 0$
- $\forall \alpha \in \mathbb{R}, \forall u \in X$, $\|\alpha \cdot u\| = |\alpha| \cdot \|u\|$
- $\forall u, v \in X$, $\|u+v\| \leq \|u\| + \|v\|$.

Lemma If $\|u\| := (u,u)^{\frac{1}{2}}$, then we also have the Pythagorean theorem: $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ if $(u,v) = 0$.

and also the Cauchy-Schwarz inequality

Lemma $|(u,v)| \le \|u\| \cdot \|v\|$ for all $u,v \in X$.

defn A *Hilbert space* is a complete inner product space.

(Recall definitions of Cauchy seqns, completeness, etc.)

3 Dual spaces

defn Given $(X, (\cdot,\cdot))$ an inner product space,
$\phi : X \to \mathbb{R}$ is a *bounded linear functional* if

- $\phi(u+v) = \phi(u) + \phi(v)$, $\forall u,v \in X$
- $\phi(\alpha \cdot u) = \alpha \cdot \phi(u)$, $\forall \alpha \in \mathbb{R}, u \in X$
- $\sup_{\|u\|=1} \|\phi(u)\| < \infty$.

defn X^* is the *dual space* of X if $(X, (\cdot,\cdot))$ is an inner prod space and $X^* := \{f \mid f \text{ is a bdd linear fncl on } X\}$.

ex/ $X = \mathcal{L}^2(\Omega)$ and $u_0 \in \mathcal{L}^2(\Omega)$,
then $(u_0)^* \in (\mathcal{L}^2(\Omega))^*$, where
$|(u_0^*)(v)| := |\int_\Omega u_0 \cdot v| < \|u_0\|_{\mathcal{L}^2} \cdot \|v\|_{\mathcal{L}^2}$ by C-S

Thm We can map $X \leftrightarrow X^*$ by the association above.

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4 Weak derivatives

$\mathcal{C}_0^\infty(\Omega) := \{ f \in \mathcal{C}^\infty(\Omega) \mid \mathrm{spt}(f) \subset\subset \Omega \}$.

Note: $f \in \mathcal{C}_0^\infty(\Omega) \Rightarrow f^{(k)}(a) = f^{(k)}(b) = 0$ when $\Omega \supset [a,b] \supset \mathrm{spt}(f)$.

Pick $u \in \mathcal{C}^2(\Omega)$; $\forall \phi \in \mathcal{C}_0^\infty(\Omega)$ we have

$$\int_\Omega \frac{\partial u}{\partial x} \cdot \phi = -\int_\Omega u \cdot \frac{\partial \phi}{\partial x} + \int_{\partial\Omega} u \cdot \overset{=0}{\phi} \cdot \nu_x$$

$$\int_\Omega \frac{\partial^2}{\partial x^2} u \cdot \phi = -\int_\Omega \left(\frac{\partial}{\partial x} u\right)\left(\frac{\partial}{\partial x}\phi\right) = (-1)^2 \cdot \int_\Omega u \cdot \left(\left(\frac{\partial}{\partial x}\right)^2 \phi\right), \text{ etc.}$$

defn We call g the (partial) weak derivative of u wrt x if $\forall \phi \in \mathcal{C}_0^\infty(\Omega)$, we have

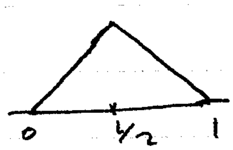
$$\int_\Omega g \cdot \frac{\partial \phi}{\partial x} = -\int_\Omega u \cdot \phi.$$ ← we require $g \in \mathcal{L}^1_{loc}(\Omega) := \{ h \mid \mathcal{L}^1_K(K) \ni h, \forall K \subset\subset \Omega \}$

For the most part, we will denote g by $\frac{\partial u}{\partial x}$ as well.

Thm If $u \in \mathcal{C}^1(\Omega)$, then g is in fact the classical $\frac{\partial u}{\partial x}$.

If α is a multi-index, then we can also define the weak derivative of u wrt x^α, i.e. $D^\alpha u$; here, $D^\alpha u$ coincides with $\frac{\partial u}{\partial x^\alpha}$ if $u \in \mathcal{C}^{|\alpha|}(\Omega)$.

defn $D^\alpha u$ is defined by $\int (D^\alpha u)\phi = (-1)^{|\alpha|} \cdot \int_\Omega u (D^\alpha \phi)$.

ex/ 

the hat fnc doesn't have a strong derivative on $[0,1]$, since $\frac{1}{2}$ is a pt of discontinuity for its derivative.

Well, $-\int_0^1 u\phi' = -\int_0^{\frac12} u\phi' - \int_{\frac12}^1 u\phi'$

$$= +\int_0^{\frac12} (1)\cdot\phi + \int_{\frac12}^1 (-1)\cdot\phi = \int_0^1 g\phi$$

$\therefore g(x) = \begin{cases} 1 & x \in [0,\frac12) \\ -1 & x \in (\frac12, 1] \end{cases}$. $+ \underbrace{\tfrac12 \phi(\tfrac12) - 0 + 0\phi(1) - \tfrac12\phi(\tfrac12)}_{=0}$

5 **Sobolev Spaces**

where v' is merely a weak derivative

$\mathcal{L}^2(\Omega) := \{v \mid \int_\Omega v^2 < \infty\}$

$H_2^1(\Omega) := \{v \mid \int_\Omega v^2 < \infty$ and $\int_\Omega (v')^2 < \infty\}$ ← Sobolev space.

More generally, $H_p^k(\Omega) := \{v \mid D^\alpha v \in \mathcal{L}^p(\Omega), \forall |\alpha| \le k\}$.

↑ understood to be the weak derivatives!

Lemma: $H^1(\Omega)$ is a normed space, with $\|v \in H^1(\Omega)\|_{H^1(\Omega)} := \left[\int_\Omega v^2 + \int_\Omega \nabla v \cdot \nabla v\right]^{\frac{1}{2}}$

Note: $H_2^k(\Omega)$ has norm $\|v\|_{H_2^k(\Omega)} := \sum_{|\alpha|=0}^{2} \|D^\alpha v\|_{\mathcal{L}^2}$

defn: $H_2^1(\Omega)_0 := \{v \in H_2^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\}$.

Thm: A norm on $H_2^1(\Omega)_0$ is given by $\|v\|_{H_2^1(\Omega)_0} := \left[\int_\Omega \nabla v \cdot \nabla v\right]^{\frac{1}{2}}$.
(This was the Energy norm!)

(we drop the 2 in H_2^k from now on!)

Thm (Green's Theorem) $\forall u \in H^2(\Omega), \forall v \in H^1(\Omega)$

$$-\int_\Omega \Delta u \cdot v = \int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} (\nabla u \cdot n) v.$$

↳ in 2-D ...

$$-\int_\Omega \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) v = \int_\Omega \left(\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y}\right) - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot v.$$

If K is a matrix and we have ~~a~~ PDE:

$$-\nabla \cdot K(\vec{x}) \nabla u(\vec{x}) = f,$$

then we can utilize

$$-\int_\Omega (\nabla \cdot K \nabla u) v = \int_\Omega K \nabla u \cdot \nabla v - \int_{\partial\Omega} K \frac{\partial u}{\partial n} \cdot v.$$

Thm (Poincaré Inequality) $\forall v \in H^1_0(\Omega)$,
$\exists C > 0$ s.t. $\|v\|_{L^2(\Omega)} \leq C \cdot \|\nabla v\|_{L^2(\Omega)}$

<u>Variational Problems</u>

FEM belongs to the class of variational problems
(whereas FDM doesn't!)

defn For X a Hilbert space, we look for $u \in X$
s.t. $a(u,v) = l(v)$ for all $v \in X$. Here,
$a: X \times X \to \mathbb{R}$ is bilinear and
$v: X \to \mathbb{R}$ is linear.
This set-up is called a <u>variational problem</u>.

ex/ Elliptic Variational problems
$\Omega \subset \mathbb{R}^d$, $\partial\Omega = \Gamma_N \cup \Gamma_D \leftarrow (|\Gamma_D| > 0)$
on which we prescribe Neumann & Dirichlet bdy cond

$$\underbrace{-\nabla \cdot (K \nabla u)}_{\text{diffusion term}} + \underbrace{b \cdot \nabla u}_{\substack{\text{convection term} \\ (b = \text{convection direction})}} + \underbrace{c \cdot u}_{c \geq 0} = f \text{ in } \Omega$$

K will be symmetric, positive defn

(Green's Thm hold is K is symmetric)

with $\begin{cases} u = 0 & \text{on } \Gamma_D \\ K \nabla u \cdot n = g & \text{on } \Gamma_N. \end{cases}$

Assume $f \in L^2(\Omega)$, $u \in H^2(\Omega)$, and take

$$X = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D \}$$

Step 1: multiply by $v \in X$.

Step 2: Apply Green's Theorem

$$-\int_\Omega (\nabla \cdot (K \nabla u)) v + \int_\Omega b \cdot \nabla u \cdot v + \int_\Omega c \cdot u \cdot v = \int_\Omega f v$$

$$\Rightarrow \int_\Omega (K \nabla u) \cdot \nabla v - \int_{\partial\Omega} ((K \nabla u) \cdot n) v + \int_\Omega b \cdot \nabla u \cdot v + \int_\Omega c \cdot u \cdot v = \int_\Omega f v$$

$$\Rightarrow \int_\Omega (K \nabla u) \cdot \nabla v - \left[\int_{\Gamma_D} \overset{0}{} + \int_{\Gamma_N} \right] + \int_\Omega b \cdot \nabla u \cdot v + \int_\Omega c \cdot u \cdot v = \int_\Omega f v$$

$$\Rightarrow \int_\Omega (K \nabla u) \cdot \nabla v - \int_{\Gamma_N} g \cdot v + \int_\Omega b \cdot \nabla u \cdot v + \int_\Omega c \cdot u \cdot v = \int_\Omega f v$$

$$\therefore \underbrace{\int_\Omega (K \nabla u \cdot \nabla v + b \cdot \nabla u + c \cdot u \cdot v)}_{\overset{!!}{a(u,v)}} = \underbrace{\int_{\Gamma_N} g \cdot v + \int_\Omega f \cdot v}_{\overset{\shortparallel}{l(v)}}$$

Check that a is bilinear, l is linear.

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Recall: we want to find $u \in X$ s.t. $\forall v \in X$ $a(u,v) = l(v)$ is the variational problem formulation; with FEM, we restrict to a finite dim'l subspace (e.g., with PW linear, PW polynomials).

We solve PDE's with the psychology that the solution is weak; this allows us to try the soln against test function via the PDE, i.e. PDE $\rightsquigarrow \int (PDE)\cdot v \rightsquigarrow$ Green's Thm.

defn Dirichlet BC's are called "essential" for the FEM and Neumann BC's are called "natural" for the FEM.

/Note: $a(u,v) = \int_\Omega K \nabla u \cdot \nabla v + (b \cdot \nabla u)\cdot v + c \cdot u \cdot v$
$l(v) = \int_\Omega f \cdot v + \int_{\Gamma_N} g \cdot v$

ex/ Pure Neumann problem

$$-\Delta u = f \text{ in } \Omega$$
$$\nabla u \cdot n = g \text{ on } \partial\Omega \quad \leftarrow \text{i.e. } \frac{\partial u}{\partial n} = g.$$

These BC ~~impose~~ (on f, g):

$$\int_\Omega f + \int_{\partial\Omega} g = 0$$

since $-\int_\Omega \Delta u\, v = \int_\Omega f v$ and $\int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot v = \int_\Omega f \cdot v$

Thus, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v$
$l(v) = \int_\Omega f \cdot v + \int_{\partial\Omega} g \cdot v$

Note also: a soln is not unique, as $u + C$ (for any constant C) is also a soln.
So, we don't want to use simply $X := H^1(\Omega)$.
Instead, we take
$$X := \{[v] \mid v \in H^1(\Omega)\}$$
where $[v] := \{w \in H^1(\Omega) \mid w - v \equiv C \text{ for some } C\}$.
An alternative is to fix $u(\vec{x}_0) = $ given or $\int_e u = $ given.

ex/ Stokes problem: $-\mu \Delta \vec{u} + \nabla p = \vec{f}$ ← vector eqn

$\vec{u} :=$ fluid velocity $\quad \nabla \cdot \vec{u} = 0$ ← incompressible, scalar eqn

$p :=$ fluid pressure $\quad \vec{u} = 0$ on $\partial\Omega$

$\mu > 0$ fluid viscosity $\quad$ (seek $\vec{u}, p$).

Here, we abuse notation:
$$\Delta \vec{u} = \Delta \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} := \begin{pmatrix} \Delta u_1 \\ \vdots \\ \Delta u_n \end{pmatrix}.$$

Let's find $\vec{u} \in (H_0^1(\Omega))^d$ and $p \in L_0^2(\Omega) := \{q \in L^2(\Omega) \mid \int_\Omega q = 0\}$ such that:
$$\int_\Omega \mu \nabla \vec{u} : \nabla \vec{v} - \int_\Omega p \cdot \nabla \cdot \vec{v} = \int_\Omega \vec{f} \cdot \vec{v}$$
$$\int_\Omega (\nabla \cdot \vec{u}) q = 0$$
for all $\vec{v} \in (H_0^1(\Omega))^d$ and $q \in L_0^2(\Omega)$.

Recall: if $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, then:

$$\nabla \vec{u} := \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{pmatrix} \quad \text{and also}$$

if $A, B \in M_{n \times n}(\mathbb{R})$, then

$$A : B = \sum_{i,j} A_{ij} \cdot B_{ij} = \mathrm{Tr}(A B^T)$$

3 Existence and uniqueness of solns to variational problems

Thm (Lax–Milgram 1954) ← baby version, generalized by Babuška

X a Hilbert space with norm $\|\cdot\|_X$. Suppose:

(i) $a(\cdot,\cdot)$ is bounded, i.e. $\exists M > 0$ s.t. $\forall u, v \in X$, $a(u,v) \leq M \cdot \|u\|_X \|v\|_X$

(ii) $a(\cdot,\cdot)$ is coercive, i.e. $\exists \alpha > 0$ s.t. $\forall v \in X$, $a(v,v) \geq \alpha \cdot \|v\|^2$

(iii) $\ell(\cdot)$ is bounded, i.e. $\ell \in X^*$, and $\exists m > 0$ s.t. $\forall v \in X$, $|\ell(v)| \leq m \cdot \|v\|$

Then there is a unique $u \in X$ that satisfies:

$\forall v \in X$, $a(u,v) = \ell(v)$ <u>and</u>

$\|u\|_X \leq \frac{1}{\alpha} \cdot \|\ell\|_{X^*}$.

Since this establishes
existence, uniqueness, and stability,

defn — we say the problem is <u>well-posed</u>.

(i) and (iii) are usually satisfied

(ii) is not always likely, depending on the PDE.

ex $\begin{cases} -\Delta u = f \\ u = 0 \end{cases} \rightsquigarrow X = H_0^1(\Omega)$

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v}_{a(u,v)} = \underbrace{\int_\Omega f v}_{\ell(v)} \qquad \|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{\mathcal{L}^2(\Omega)}$$

← subscript 0 means $\|u\|_{\mathcal{L}^2(\partial\Omega)} = 0$

(Note: $\|u\|_{\cancel{H_0^1}(\Omega)} \leq C_p \|\nabla u\|_{L^2(\Omega)}$ by Poincaré inequality!)

(i): $\int_\Omega \nabla u \cdot \nabla v \overset{?}{\leq} M \cdot \|u\|_{\mathcal{L}^2(\Omega)} \cdot \|v\|_{\mathcal{L}^2(\Omega)}$

yes, by Cauchy-Schwarz!

(ii): do $a(v,v) \geq \alpha \cdot \|v\|^2_{\mathcal{L}^2(\Omega)}$ for all v?

yes, since $a(v,v) = \|v\|^2_{\mathcal{L}^2(\Omega)}$!

(iii): $|\ell(v)| \leq m \cdot \|v\|_{\mathcal{L}^2(\Omega)}$ for all v?

yes, since $|\ell(v)| = |\int_\Omega f v|$

$\leq \|f\|_{\mathcal{L}^2(\Omega)} \cdot \|v\|_{\mathcal{L}^2(\Omega)}$

Poincaré! $\quad \leq \cancel{K} \cdot C_p \|\nabla v\|_{\mathcal{L}^2(\Omega)}$

simply ~~take~~ $K \cdot C_p(\Omega) =: m$.

4 Discrete variational problem

Find $u_h \in X_h$ s.t. $\forall v_h \in X_h$, we have $a(u_h, v_h) = \ell(v_h)$.

Now, X_h is finite dim'l! For example,

$X_h := \{v \in \mathcal{C}^q(\overline{\Omega}) \mid v \text{ is PW poly of degree } p\}$

← with $\|\cdot\|_X|_{X_h}$ norm

Note: since $X_h \subset X$, Lax-Milgram applies to X_h.

Thus, we obtain u_h and also u as solutions (to their respective problems).

$$\forall v \in X,\ a(u,v) = \ell(v)$$

$$\forall v_h \in X_h,\ a(u_h, v_h) = \ell(v_h)$$

We have an error bound by approximation theory:

Thm $\|u-u_h\|_X \le \left(1+\frac{M}{\alpha}\right)\cdot \inf_{w_h \in X_h} \|u-w_h\|_X$

So, when X_h is of degree p polys,
$u-w_h$ (as w_h ranges over X_h)
can eliminate the p^{th} order Taylor approx of u!
Hence, $\|u-w_h\|$ will be $O(h^{p+1})$ w/ constant
controlled by $u^{(p+1)}$.

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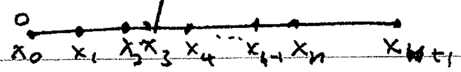
(Derivation of Galerkin method in 1-D)



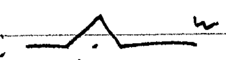
Let $0=x_0, x_1, \dots, x_{N+1}=1$ be a mesh of $[0,1]$,
and denote $K_i := (x_{i-1}, x_i)$ for $i=1,2,\dots,N+1$.
Define $h := \max_i (x_i - x_{i-1})$ and also

$$h_i := \begin{cases} \min(x_{i+1}-x_i,\ x_i - x_{i-1}) & \text{for } i=1,2,\dots,N \\ x_{i+1}-x_i & \text{if } i=0 \\ x_i - x_{i-1} & \text{if } i=N+1 \end{cases}$$

Let $V_{DG} := \{ v \mid v|_{K_i} \in \mathbb{P}_r(K_i) \text{ for } i=1,2,\dots,N+1 \}$
↑ polynomials of degree r on K_i

Let also
$$[w(x_i)] := \begin{cases} -w|_{K_{i+1}}(x_i) & \text{if } i=0 \\ w|_{K_i}(x_i) - w|_{K_{i+1}}(x_i) & \text{if } i=1,2,3,\dots,N \\ w|_{K_i}(x_i) & \text{if } i=N+1 \end{cases}$$
(defined by limits if w is the weak derivative)

and
$$\{w(x_i)\} := \begin{cases} \frac{1}{2}\left[w'|_{K_i}(x_i) + w'|_{K_{i+1}}(x_i)\right] & \text{if } i=1,2,\dots,N \\ w'|_{K_{i+1}}(x_i) & \text{if } i=0 \\ w'|_{K_i}(x_i) & \text{if } i=N+1 \end{cases}$$

e.g.


$\{w(x_i)\} = \frac{1}{2}((-1)+(1)) = 0$

Consider the problem of finding u s.t.

$$\begin{cases} -u''(x) = f(x) & \text{for } 0<x<1 \\ u(0) = \alpha \\ u(1) = \beta \end{cases}$$

Let $v \in V_{DG}$. Then:

$$-\int_0^1 u''(x)\cdot v(x)\,dx = \int_0^1 f(x)\cdot v(x)\,dx$$

(Note that V_{DG} is not a ~~subspace~~ of $H^1([0,1])$!)

$$\Rightarrow -\sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} u''(x)\cdot v(x)\,dx = \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} f(x)\cdot v(x)\,dx$$

$$\Rightarrow \sum_{i=1}^{N+1}\left[\int_{x_{i-1}}^{x_i} u'(x)\cdot v'(x)\,dx - u'|_{K_i}(x)\cdot v|_{K_i}(x)\Big|_{x_{i-1}}^{x_i}\right] = \sum_{i=1}^{N+1}\int_{x_{i-1}}^{x_i} f(x)\cdot v(x)\,dx$$

$$\therefore \sum_{i=1}^{N+1}\left[\int_{x_{i-1}}^{x_i} u'(x)\cdot v'(x)\,dx\right] - \sum_{i=1}^{N+1}\left(u'|_{k_i}(x)\cdot v|_{k_i}(x)\right]\Big|_{x_{i-1}}^{x_i} = \sum_{i=1}^{N+1}\int_{x_{i-1}}^{x_i} f(x)\cdot v(x)\,dx$$

Note that $\sum_{i=1}^{N+1}\left[u'|_{k_i}(x)\cdot v|_{k_i}(x)\right]\Big|_{x_{i-1}}^{x_i}$

$$= \sum_{i=1}^{N+1}\left[u'|_{k_i}(x_i)\cdot v|_{k_i}(x_i) - u'|_{k_i}(x_{i-1})\cdot v|_{k_i}(x_{i-1})\right]$$

$$= -u'|_{k_1}(x_0)\cdot v|_{k_1}(x_0) + \sum_{i=1}^{N}\left[u'|_{k_i}(x_i)\cdot v|_{k_i}(x_i) - u'|_{k_{i+1}}(x_i)\cdot v|_{k_{i+1}}(x_i)\right]$$

$$+ u'|_{k_{N+1}}(x_{N+1})\cdot v|_{k_{N+1}}(x_{N+1}).$$

Assuming continuity of u', we get that (the middle summand)

$$\sum_{i=1}^{N}\left[u'|_{k_i}(x_i)\cdot v|_{k_i}(x_i) - u'|_{k_{i+1}}(x_i)\cdot v|_{k_{i+1}}(x_i)\right]$$

$$= \sum_{i=1}^{N} u'(x_i)\cdot\left[v|_{k_i}(x_i) - v|_{k_{i+1}}(x_i)\right]$$

not approx!

$$= \sum_{i=1}^{N}\{u(x_i)\}\cdot[v(x_i)].$$

Accounting for the defns of $\{\cdot\}$ & $[\cdot]$ at endpts,

$$\int_0^1 f(x)v(x)\,dx = \sum_{i=1}^{N+1}\int_{x_{i-1}}^{x_i} u'(x)v'(x)\,dx - \sum_{i=0}^{N+1}\{u'(x_i)\}\cdot[v(x_i)].$$

How do we find a u which doesn't jump too much?
We penalize jumps by adding a term on the RHS:

$$\sum_{i=1}^{N}\frac{\sigma}{h_i}[u(x_i)]\cdot[v(x_i)] + \frac{\sigma}{h_0}(u(x_0)-\alpha)\cdot v(x_0) + \frac{\sigma}{h_{N+1}}(u(x_{N+1})-\beta)\cdot v(x_{N+1})$$

where σ is the penalty parameter.
Adding this term makes the problem well-posed!
This discontinuous Galerkin method is called the "incomplete interior ~~elliptic~~ penalty Galerkin".

Note that the RHS (thought of as a bilinear operator on u and v) is not symmetric (no $\{v\}[u']$ term, etc.).

So, we ~~impose~~: note that
$$0=\sum_{i=0}^{N+1}\{v'(x_i)\}\cdot[u(x_i)] + \alpha\cdot v'(x_0) - \beta\cdot v'(x_{n+1}).$$
~~We~~ can simply add multiples of this to the FEM:

Let $a_{DG}(w,v) := \sum_{i=1}^{n+1}\int_{x_{i-1}}^{x_i} w'(x)\, v'(x)\,dx$ ← ε is the "symmetrization parameter"
$$- \sum_{i=0}^{n+1}\{w'(x)\}\cdot[v(x_i)] + \varepsilon\cdot\sum_{i=0}^{n+1}\{v'(x_i)\}\cdot[w(x_i)]$$
$$+ \sum_{i=1}^{N}\frac{\sigma}{h_i}[w(x_i)]\cdot[v(x_i)] + \frac{\sigma}{h_0}w(x_0)v(x_0) + \frac{\sigma}{h_{n+1}}w(x_{n+1})v(x_{n+1})$$
and $l(v) = \int_0^1 f(x)\cdot v(x)\,dx + \frac{\sigma}{h_0}\cdot\alpha\cdot v(x_0) + \frac{\sigma}{h_{n+1}}\cdot\beta\cdot v(x_{n+1})$
$$- \varepsilon\cdot\alpha V'(x_0) + \varepsilon\beta v'(x_{n+1}).$$

Then: our problem is (approximated) by
$a_{DG}(u,v) = l(v)$ for all $v \in V_{DG}$.
↑ since u cts ⇒ many terms disappear!

We can obtain a discontinuous Galerkin approx to u by finding a $u_{DG} \in V_{DG}$ such that
$a_{DG}(u_{DG},v) = l(v)$ for all $v \in V_{DG}$.

When $\varepsilon=-1$, we call it the "symmetric interior penalty Galerkin (SIPG)" ← (then $a_{DG}(w,v) = a_{DG}(v,w)$)
When $\varepsilon = 0$, we call it the "incomplete interior penalty Galerkin (IIPG)" (SIPG and IIPG need $\sigma >> 0$ for well-posedness)
When $\varepsilon = 1$, we call it the "non-symmetric interior penalty Galerkin (NIPG)" ← developed by Riviere!
↑ can use any $\sigma > 0$ for well-posedness!

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(Using notation from last time.)

We measure the error in the energy norm:

$$\|v\|_E = \left(\sum_{j=1}^{N+1} \int_{x_{j-1}}^{x_j} (v'(x))^2\, dx\right)^{\frac{1}{2}}$$

... but this isn't actually a norm on V_{DG}!

We have to modify it! (accounting for potential discty @ x_i's)

$$\|v\|_E := \left[\sum_{j=1}^{N+1} \int_{x_{j-1}}^{x_j} (v'(x))^2\, dx + \sum_{j=0}^{N+1} \frac{\sigma}{h_j} [v(x_j)]^2\right]^{\frac{1}{2}}$$

Note that the approx soln only has the bdy condition weakly enforced, i.e. the bdy value might not be correct.

It can be shown that

$\|u - u_{DG}\|_E = O(h^r)$ where r is as in $\mathbb{P}^r(K_i)$.

and even

$$\|u - u_{DG}\|_E = \begin{cases} O(h^{r+1}) \text{ if } \varepsilon = -1, \text{ i.e. in SIPG case} \\ O(h^r) \text{ if } \varepsilon = 1 \text{ or } \varepsilon = 0, \text{ i.e. (NIPG or IIPG)} \end{cases}$$

SIPG seems better, but remember that we need $\sigma \gg 0$.

NIPG resolves this by working for any $\sigma > 0$, but the error rate of convergence is only $O(h^r)$.

Consider the PDE:

$$\begin{cases} -\nabla \cdot (K \nabla u) = f \text{ in } \Omega \subset \mathbb{R}^d \text{ (for } d = 2 \text{ or } 3) \\ \qquad u = g \text{ on } \Gamma = \partial\Omega \end{cases}$$

where K is symmetric and there is a uniform upper and lower bound on the eigenvalues of K on Ω, i.e. $\exists K_1, K_2 \in \mathbb{R}$ s.t., $\forall v \in \mathbb{R}^d$, $K_1 v^T v \le v^T K v \le K_2 v^T v$.

and $f \in L^2(\Omega)$, $g \in H^1(\Omega)$.

The weak form of this problem is to find $u \in H^1(\Omega)$ s.t. $u = g$ on Γ and, for all $v \in H_0^1(\Omega)$, we have

$$\int_\Omega (K\nabla u)\cdot\nabla v \, dx = \int_\Omega f v \, dx$$

Let $\tilde{g} \in H^1(\Omega)$ be such that $\tilde{g}|_{\partial\Omega} = g$ on $\Gamma = \partial\Omega$. We can then write $u = \tilde{g} + \tilde{u}$, where $\tilde{u} \in H_0^1(\Omega)$ is such that

$$a(\tilde{u}, v) = l(v) \quad \text{for all } v \in H_0^1(\Omega)$$

where the symmetric bilinear form

$$a(w,v) = \int_\Omega (K\nabla w)\cdot\nabla v \, dx$$

and the linear

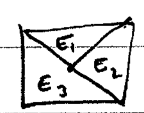
$$l(\cdot) : H_0^1(\Omega) \to R$$

$$l(v) = \int_\Omega f v \, dx - \int_\Omega (K\nabla \tilde{g})\cdot\nabla v \, dx.$$

Since $a(w,v) \le K_2 \cdot \|\nabla w\|_{L^2(\Omega)} \cdot \|\nabla v\|_{L^2(\Omega)}$
$\le K_2 \cdot \|w\|_{H^1(\Omega)} \cdot \|v\|_{H^1(\Omega)}$
and also $a(v,v) \ge K_1 \cdot \|v\|^2_{L^2(\Omega)} \ge K_1 \cdot C \cdot \|v\|^2_{L^2(\Omega)}$

Lax-Milgram applies. ∎

Now, let $\mathcal{P}$ be a partition of Ω.
Let $V_{DG} = \{ v \mid v|_E \in P_{r_E}(E) \text{ for all } E \in \mathcal{P} \}$
where r_E is a non-negative integer.
Discontinuous Galerkin methods shine here:

e.g., $\mathcal{P}$ = 

← would be very difficult to enforce cty along bdry's!

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(Returning to studying FEM for elliptic PDE)

Recall: PDE $\rightarrow a(u,v) = \ell(v) \rightsquigarrow a(u_h, v_h) = \ell(v_h)$.

$$\begin{cases} -\nabla \cdot (K \nabla u) + c \cdot u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Find $u_h \in X_h = \{ \mathcal{C}^0 \text{ pw poly of degree } p \text{ vanishing on } \partial\Omega \}$

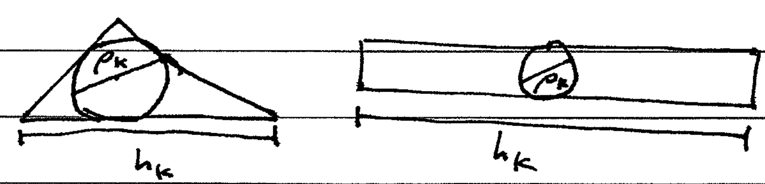
s.t. $\forall v_h \in X_h$, $\underbrace{\int_\Omega (K \nabla u_h \cdot \nabla v_h + c\, u_h \cdot v_h)\, dx}_{a(u_h, v_h)} = \underbrace{\int_\Omega f \cdot v_h}_{\ell(v_h)}$

1 Finite element meshes

We partition Ω into a finite # of "elements", i.e. triangles, quadrilaterals, tetrahedra, hexaedra, prisms, etc. We name τ_h the mesh, and so $\tau_h = \bigcup_K K$, $h_K = \operatorname{diam}(K) = \max_{x,y \in K} \|x-y\|$, $h := \max_{K \in \tau_h} h_K$.

We still would like $\|u - u_h\|_E = \mathcal{O}(h^p)$ as $h \to 0$.

defn For $K \in \tau_h$, let ρ_K denote the diameter of the largest ball inscribed in K.



defn A family of meshes $\{\tau_h\}_h$ is shape regular if there is a constant $\theta > 0$ so that, $\forall h > 0$, $\forall K \in \tau_h$, we have $\frac{h_K}{\rho_K} \leq \theta$.

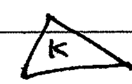
This shape regularity is necessary for $\|u - u_h\| = \mathcal{O}(h^p)$!

defn A family of meshes $\{\mathcal{T}_h\}_{h>0}$ is *quasi-uniform* if it is shape regular and if $\exists\, m > 0$ so that $\forall h > 0$, $\forall K \in \mathcal{T}_h$, we have $m \cdot h \le h_K$. It is *uniform* if $h_K = h$ for all $K \in \mathcal{T}_h$.

2 Finite element spaces $\mathbb{P}_p(K) :=$ space of deg p poly on K

ex/ $\mathbb{P}_1(K) = \text{span}(1, x, y)$

ex/ $\mathbb{P}_2(K) = \text{span}(1, x, y, x^2, xy, y^2)$.



Exception: when K is a rectangle, we denote:

$\mathbb{Q}_1(K) = \text{span}\{1, x, y, xy\}$

$\mathbb{Q}_2(K) = \text{span}\{1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2\}$

$\mathbb{Q}_p(K) = \mathbb{P}_p(K) \otimes \mathbb{P}_p(K)$ is of dimension $(p+1)^2$.

and also $\mathbb{P}_p(K)$ has dimension $\frac{(p+1)(p+2)}{2}$.

$X_h := \{ v \in C(\Omega) \mid \forall K \in \mathcal{T}_h$, we have $v|_K \in \mathbb{P}_p(K)$ and $v|_{\partial\Omega} = 0 \}$.

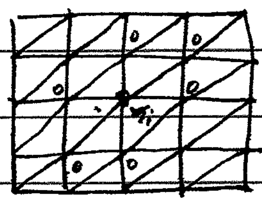
defn There is a set of points in Ω s.t. any $v \in X_h$ is uniquely determined by its values at these points. These are called *nodes*. If ϕ_i is a func which is equal to 1 at one node and 0 at the other nodes, it is a *nodal basis function*.

When the elements are triangles or quadrilaterals, the nodes are simply the interior points, and $\dim(X_h) = \#$ interior vertices.

when using pw degree 1

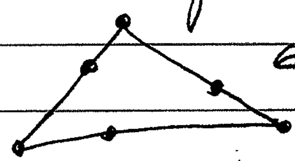
Lemma $X_h = \{ v \mid \forall K \in \tau_h,\ v|_K \in P_1(K)$ and v is cts at the nodes and $v|_{\partial\Omega} = 0 \}$.

ex/



nodal function ϕ_{n_i} is only nonzero on the elements containing the node n_i

When dealing with pw quadratics, we have instead



← some point on each edge, in addition to vertices.

For consistency, we pick the midpoints of edges. In other words, when dealing with triangles, the nodes are the interior vertices and also the midpoints of interior edges. If the mesh is made of a rectangle, we need also the barycenters of K.

3 Linear system

$X_h = \text{span}\{\phi_i \mid 1 \le i \le N\}$ where N = # nodes.

expand: $u_h = \sum_{j=1}^{N} \alpha_j \phi_j(\vec{x})$

note: $\phi_j(\text{node}_i) = \delta_{ij} \Rightarrow \alpha_j = u_h(\text{node}_j)$

Now, $\forall v_h \in X_h$, we have

$$\sum_{j=1}^{N} \alpha_j \left(\int_\Omega K \cdot \nabla\phi_j \cdot \nabla v_h + c\phi_j \cdot v_h \right) = l(v_h) = \int_\Omega f v_h$$

$$\iff \sum_{j=1}^{N} \alpha_j \left(\int_\Omega K \cdot \nabla\phi_j \cdot \nabla\phi_i + c \cdot \phi_j \cdot \phi_i \right) = \int_\Omega f \cdot \phi_i \text{ for all } i.$$

$(A+B)\vec{\alpha} = \vec{b}$ ← is ~~the~~ goal representation

where $\vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}$, $b_i = \int_\Omega f \cdot \phi_i$,

$A_{ij} = \int_\Omega K \cdot \nabla \phi_j \cdot \nabla \phi_i$ is the stiffness matrix,

and $B_{ij} = \int_\Omega c \cdot \phi_j \cdot \phi_i$ is the mass matrix.