CAAM 452: Homework 3
Posted online on March 7
Due March 18 in class
Printout of codes are to be included and codes are to be uploaded in owlspace.

Problem 1 (20 points)

Let X_h be a finite element space with basis functions $\phi_1, \ldots, \phi_N$. Let u_h be the finite element solution satisfying the variational problem:

$$\forall v_h \in X_h \quad a(u_h, v_h) = \ell(v_h)$$

Show that the following two statements are equivalent.

$$(i) \quad \forall v_h \in X_h \quad a(u_h, v_h) = \ell(v_h)$$
$$(ii) \quad \forall 1 \le i \le N \quad a(u_h, \phi_i) = \ell(\phi_i)$$

Problem 2 (20 points)

Let X be an inner-product space with inner-product $(\cdot, \cdot)$. Define

$$\forall v \in X, \quad \|v\| = (v, v)^{1/2}$$

(a) Show that for any $u, v \in X$

$$\forall \alpha \in I\!R, \ \|\alpha v\| = |\alpha| \|v\|$$
$$\|u + v\| \le \|u\| + \|v\|$$
$$\text{if } (u, v) = 0, \text{ then } \|u + v\|^2 = \|u\|^2 + \|v\|^2$$

(b) Let $\{\Phi_1, \ldots, \Phi_n\}$ be a basis for a subspace $Y \subset X$, and let $\boldsymbol{G}$ be the matrix defined by: $G_{ij} = (\Phi_j, \Phi_i)$. Show that $\boldsymbol{G}$ is positive definite: i.e. show that $x \cdot \boldsymbol{G}x > 0$ for all $x \in I\!R^n$ with $x \neq 0$. Deduce that $\boldsymbol{G}$ is non-singular.

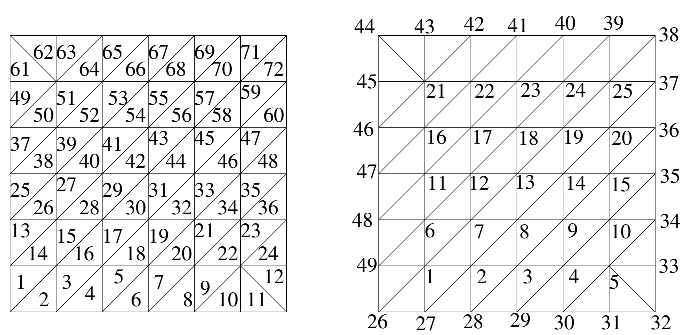
Problem 3 (30 points)

Let $f : [a, b] \to I\!R$ that is $\mathcal{C}^1$ (i.e. f is continuous, and f' is continuous). Assume that $f(a) = f(b) = 0$. Prove that

$$\|f\|_{L^2(a,b)} \le \frac{b - a}{\sqrt{2}} \|f'\|_{L^2(a,b)}$$

Problem 4 (30 points)

Write a code that generates automatically a structured finite element mesh with $2N^2$ triangles as given in the figure. Note that $N = 6$ in the example of the figure. Upload the code in owlspace.



TRIANGLES numbers   NODES numbers

The code should generate two arrays `triangle`, `nodes`.

- `nodes(k,1)` is the x coordinate of the node number k.
- `nodes(k,2)` is the y coordinate of the node number k.
- `nodes(k,3)` is equal to -1 if the node is constrained (i.e. on the boundary of the domain) and $+1$ if the node is a free node (i.e. interior to the domain). Note that the free nodes are numbered first (from 1 to 25 on the figure).
- `triangle(k,i)` is the global number of the local node number i of element k, for $i = 1, 2, 3$. The local numbering is done counterclockwise, starting with the node at the square angle of the triangle.

Test your code for $N = 2$ and $N = 4$: give the entries of the arrays described above and plot the grids.