CAAM 452: Homework 4
Posted online on March 18
Due March 27 in class (give printout of codes and upload in owlspace)
This homework is pledged: you cannot discuss the problems with anyone but the instructor. You can have access to the lecture notes and textbooks. Write the pledge on the first page of the homework.

Problem 1 (30 points)

Let $\Omega \subset I\!R^n$ with $\partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$ and $\Gamma_0 \neq \emptyset$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let α be a constant. Give a variational formulation of the problem

$$-\Delta u = f, \quad \text{in} \quad \Omega \tag{1}$$

$$u = 0, \quad \text{in} \quad \Gamma_0 \tag{2}$$

$$\alpha u + \frac{\partial u}{\partial n} = 0, \quad \text{on} \quad \Gamma_1 \tag{3}$$

Under what conditions does a solution to the variational problem exist?

You may use the following facts: Let $X = H^1(\Omega) \cap \{u : u = 0 \quad \text{on} \quad \Gamma_0\}$. Poincaré's inequality says that there is a constant $C_p > 0$ such that

$$\forall v \in X : \quad \|v\|_{L^2(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)}$$

There is also another constant $C_t > 0$ such that

$$\forall v \in X : \quad \|v\|_{L^2(\Gamma_1)} \leq C_t\|v\|_{H^1(\Omega)}$$

The boundary condition (3) is called Robin boundary condition, or mixed boundary condition.

Problem 2 (30 points)

Let E be a quadrilateral element with vertices $A_i(x_i, y_i), i = 1, \ldots, 4$ (ordered counterclockwise). Let $\hat{E}$ be the reference element with vertices $\hat{A}_1(-1,-1)$, $\hat{A}_2(1,-1)$, $\hat{A}_3(1,1)$, $\hat{A}_4(-1,1)$. Let F_E be the mapping that maps $\hat{E}$ onto E such that $\hat{A}_i$ is mapped onto A_i.

(a) Give a formula for the mapping F_E.

(b) Compute $\det(B_E)$ where B_E is the jacobian matrix of F_E.

(c) Assume, in addition, that E is a rectangle such that the vertical sides are parallel to the y-axis and the horizontal sides parallel to the x-axis. Show that $\det(B_E)$ never vanishes.

Problem 3 (40 points)

Let $\hat{E}$ be the reference triangle element with vertices $\hat{A}_i$, $i = 1, 2, 3$, as defined in class. Let $\hat{\Phi}_i$ be the linear local basis function, for $i = 1, 2, 3$. Let E be any triangle with vertices $A_i(x_i, y_i)$ for $i = 1, 2, 3$ such that the vertex $\hat{A}_i$ is mapped onto A_i. Let Φ_i be the corresponding linear basis functions defined on E, i.e.

$$\Phi_i(x, y) = \hat{\Phi}_i(\xi, \eta) = \hat{\Phi}_i(F_E^{-1}(x, y))$$

(a) Show that $\int_E \nabla\Phi_i \cdot \nabla\Phi_j$ can be written as a linear combination of the terms $\int_{\hat{E}} \frac{\partial\hat{\Phi}_i}{\partial\xi} \frac{\partial\hat{\Phi}_j}{\partial\eta}$, $\int_{\hat{E}} \frac{\partial\hat{\Phi}_i}{\partial\xi} \frac{\partial\hat{\Phi}_j}{\partial\xi}$, $\int_{\hat{E}} \frac{\partial\hat{\Phi}_i}{\partial\eta} \frac{\partial\hat{\Phi}_j}{\partial\eta}$, $\ldots$.

(b) Using part (a), write a code that computes the local stiffness matrix $(\int_E \nabla\Phi_i \cdot \nabla\Phi_j)_{i,j}$ defined on any triangle E. Test your code for the following cases: (i) $E = \hat{E}$ and (ii) E is a triangle with vertices $A_1(0, 0)$, $A_2(2, 1)$, $A_3(1, 1)$.