a function Φ_i that satisfies $\Phi_i(x_j) = \delta_{ij}$. Second we define for each interval a function Ψ_i that vanishes at all nodes and takes the value one at the midpoint of (x_{i-1}, x_i).

$$1 \leq i \leq N, \quad \Phi_i(x) = \begin{cases} 0, \quad 0 \leq x \leq x_{i-1} \\ \frac{(x-x_{i-1})(x-x_{i-1/2})}{(x_i-x_{i-1})(x_i-x_{i-1/2})} \quad x_{i-1} \leq x \leq x_i \\ \frac{(x_{i+1}-x)(x-x_{i+1/2})}{(x_{i+1}-x_i)(x_i-x_{i+1/2})} \quad x_i \leq x \leq x_{i+1} \\ 0 \quad x_{i+1} \leq x \leq 1 \end{cases}$$

$$1 \leq i \leq N+1, \quad \Psi_i(x) = \begin{cases} 0, \quad 0 \leq x \leq x_{i-1} \\ \frac{(x-x_{i-1})(x-x_i)}{(x_{i-1/2}-x_{i-1})(x_{i-1/2}-x_i)} \quad x_{i-1} \leq x \leq x_i \\ 0 \quad x_i \leq x \leq 1 \end{cases}$$

4 Existence and uniqueness of finite element solution

The finite element solution $u_h \in V_h$ satisfies

$$\forall v \in V_h, \quad a(u_h, v) = \ell(v)$$

Since this is a linear problem in a finite-dimensional space, proving existence of the solution is equivalent to proving uniqueness of the solution. We will prove uniqueness. Assume that there are two solutions u_h^1 and u_h^2 and let $w_h = u_h^1 - u_h^2$. Using linearity of a and ℓ, we have

$$\forall v \in V_h, \quad a(w_h, v) = 0$$

Pick $v = w_h$. This gives

$$\int_0^1 k(w_h')^2 + \int_0^1 c(w_h)^2 = 0$$

Since $k \geq k_0 > 0$, we conclude that w_h is constant on each interval (x_i, x_{i+1}). Because w_h is continuous and $w_h(0) = 0$, we have that $w_h = 0$ everywhere.

5 A priori error estimate

We want to bound the error between the numerical solution and the true solution with respect to the following norm:

$$\|v\|_E = (a(v, v))^{1/2}$$

One can check that indeed it is a norm. It is called the *energy* norm.

The first important result is the *orthogonality equation*:

$$\forall v \in V_h, \quad a(u - u_h, v) = 0$$

The following lemma reduces the problem of finding a bound of the error to an approximation problem.

Lemma

$$\forall v \in V_h, \quad \|u - u_h\|_E \leq \|u - v\|_E$$

The proof uses Cauchy-Schwarz's inequality, that we now state. For any functions f, g that are square-integrable, we have

$$\int_0^1 fg \leq \left(\int_0^1 f^2\right)^{1/2} \left(\int_0^1 g^2\right)^{1/2}$$

This is Cauchy-Schwarz's inequality in the L^2 space (see definition in the next chapter). Cauchy-Schwarz's inequality in $\mathbb{R}^N$ is:

$$\forall a_i, b_i, \quad \sum_{i=1}^N a_i b_i \leq \left(\sum_{i=1}^N a_i^2\right)^{1/2} \left(\sum_{i=1}^N b_i^2\right)^{1/2}$$

We are now ready to prove the lemma. Using the orthogonality equation, we can write for all $v \in V_h$

$$\|u-u_h\|_E^2 = a(u-u_h, u-u_h) = a(u-u_h, u-u_h) + a(u-u_h, u_h-v) = a(u-u_h, u-v) \quad (8)$$

We expand

$$a(u - u_h, u - v) = \int_0^1 k(u' - u_h')(u' - v') + \int_0^1 c(u - u_h)(u - v)$$

Using Cauchy-Schwarz's inequality we have

$$\int_0^1 k(u' - u_h')(u' - v') \leq \left(\int_0^1 k(u' - u_h')^2\right)^{1/2} \left(\int_0^1 k(u' - v')^2\right)^{1/2}$$

$$\int_0^1 c(u - u_h)(u - v) \leq \left(\int_0^1 c(u - u_h)^2\right)^{1/2} \left(\int_0^1 c(u - v)^2\right)^{1/2}$$

Therefore we have

$$a(u - u_h, u - v) \leq \|u - u_h\|_E \|u - v\|_E$$

Combining with (8) gives the result. $\square$

Taking infimum over all $v \in V_h$ and noting that $u_h \in V_h$, we have:

$$\|u - u_h\|_E = \inf_{v \in V_h} \|u - v\|_E$$

So the finite element solution yields the minimum energy norm. Next we need an approximation result (that is independent of the finite element solution). One can show that if V_h contains polynomials of degree p and if u is "smooth" enough, there is a constant C independent of h such that

$$\inf_{v \in V_h} \|u - v\|_E \leq Ch^k$$

This implies

$$\|u - u_h\|_E \leq Ch^k$$

The piecewise linear finite element method is a first order method (in the energy norm). Another convergence result in a weaker norm is:

$$\|u - u_h\|_{L^2(0,1)} \leq Ch^{k+1}$$

where $\|v\|_{L^2(0,1)} = (\int_0^1 v^2)^{1/2}$. So the piecewise linear finite element method is a second order method in the L^2 norm.

6 Non-homogeneous Dirichlet boundary conditions

We consider the two-point boundary value problem with modified boundary conditions

$$-\frac{d}{dx}\left(k(x)\frac{du}{dx}(x)\right) + c(x)u(x) = f(x), \quad 0 < x < 1$$
$$u(0) = \alpha, \quad u(1) = \beta$$

We decompose the finite element solution $u_h = u_0 + w_h$, with u_0 a piecewise polynomial function satisfying the Dirichlet boundary conditions, and w_h satisfying $w_h(0) = w_h(1) = 0$. We now construct u_0 for the case of piecewise linear finite element method. Let Φ_0 and Φ_{N+1} be defined as:

$$\Phi_0(x) = \begin{cases} \frac{x_1 - x}{h_0} & x_0 \leq x \leq x_1 \\ 0 & x_1 \leq x \leq 1 \end{cases} \qquad \Phi_{N+1}(x) = \begin{cases} 0, & 0 \leq x \leq x_N \\ \frac{x - x_N}{h_N} & x_N \leq x \leq x_{N+1} \end{cases}$$

Then

$$u_0(x) = \alpha\Phi_0(x) + \beta\Phi_{N+1}(x)$$

Second, we write w_h in the usual nodal basis function:

$$w_h(x) = \sum_{i=1}^{N} \alpha_i \Phi_i(x)$$

To solve for w_h, we solve the variational problem:

$$\forall v \in V_h, \quad a(w_h, v) = \ell(v) - a(u_0, v)$$

We see that the resulting matrix $\boldsymbol{A}$ is the same as for zero Dirichlet boundary conditions, and only the right-hand side changes. If we compute $a(u_0, \Phi_i)$ for $1 \leq i \leq N$, we obtain a new right-hand side with α/h added to the first component, and β/h added to the last component.

V. FUNCTIONAL SPACES

Convergence of finite element methods is obtained for problems with solutions not necessarily smooth. These solutions belong to particular Hilbert spaces, namely the Sobolev spaces.

1. Inner-product spaces

Definition An inner-product space is a real vector space X, in which an inner-product $(\cdot,\cdot)$ has been defined.

1. $\forall u, v \in X, \quad (u, v)$ is a real number.
2. $\forall u, v \in X, \quad (u, v) = (v, u)$
3. $\forall u, v \in X, \forall \alpha \in \mathbb{R}, \quad (\alpha u, v) = \alpha(u, v)$
4. $\forall u \in X, \quad (u, u) > 0$ if $u \neq 0$
5. $\forall u, v, w \in X, \quad (u + v, w) = (u, w) + (v, w)$

An example of inner-product spaces is $X = \mathbb{R}^n$ with

$$\forall u = (u_i)_i, v = (v_i)_i, \quad (u, v) = \sum_{i=1}^{n} u_i v_i$$

Another important example is the space $X = L^2(\Omega)$ for any domain $\Omega \subset \mathbb{R}^n$.

$$L^2(\Omega) = \{u \text{ Lebesgue measurable } : \int_\Omega u^2 < \infty\}$$

Lemma: An inner-product space is a normed space with the norm:

$$\forall u \in X, \quad ||u|| = (u, u)^{1/2}$$

This norm satisfies the following properties

1. $||u|| > 0$ if $u \neq 0$
2. $||\alpha u|| = |\alpha| ||u||, \quad \alpha \in \mathbb{R}$
3. $||u + v|| \leq ||u|| + ||v||$
4. Pythagoeran law: If $(u, v) = 0$ then $||u + v||^2 = ||u||^2 + ||v||^2$
5. Cauchy-Schwarz's inequality:

$$|(u, v)| \leq ||u|| ||v||$$

Let us prove Cauchy-Schwarz's inequality, which is a classic result. Pick $\lambda \in \mathbb{R}$ and write

$$0 \leq (u - \lambda v, u - \lambda v) = \|u\|^2 - 2\lambda(u,v) + \lambda^2\|v\|^2$$

Now choose $\lambda = (u,v)$:

$$0 \leq \|u\|^2 - 2(u,v)^2 + (u,v)^2\|v\|^2$$

First assume that $\|v\| = 1$. From the inequality above we have

$$0 \leq \|u\|^2 - (u,v)^2$$

Rearranging this inequality and taking square root gives:

$$|(u,v)| \leq \|u\|$$

which is Cauchy-Schwarz's inequality in that particular case. Next, for a general v, define $w = v/\|v\|$ (possible if $v \neq 0$), which has norm equal to one. Apply the result just obtained:

$$|(u, \frac{v}{\|v\|})| \leq \|u\|$$

Multiply both sides by $\|v\|$ to obtain the result.

The space L^2 is equipped with the inner-product and norm:

$$(u,v) = \int_\Omega uv, \quad \|u\|_{L^2(\Omega)} = \Big(\int_\Omega u^2\Big)^{1/2}$$

Definition If $(u,v) = 0$, we say that u and v are orthogonal and we write $u \perp v$. Let $V \subset X$. If $u \in X$ is orthogonal to all the elements in V, we simply write $u \perp V$.

It is easy to show the following:

$$u = 0 \text{ if and only if } (u,v) = 0 \forall v \in X$$

2. Hilbert spaces

Definition A Hilbert space is a complete inner-product space.

Let us recall the definition of Cauchy sequences, and complete space.

Definition A sequence $(u_n)_{n\geq 0}$ in a normed space is said to be Cauchy if

$$\forall \epsilon > 0, \quad \exists N_0 \quad \forall n \geq N_0, \forall m \geq N_0, \quad \|u_n - u_m\| \leq \epsilon$$

In other words

$$\lim_{n\to\infty} \sup_{i,j\geq n} \|u_i - u_j\| = 0$$

Definition If every Cauchy sequence in the space X is convergent to a point in X, then the space is said to be complete.

The space $L^2(\Omega)$ introduced in the previous section is an Hilbert space.

3. Dual spaces

Definition Let X be an inner-product space and let $\Phi : X \to \mathbb{R}$ be a linear mapping. We say the Φ is a linear functional. In addition, Φ is called a bounded linear functional if

$$\sup_{u \in X, \, \|u\|=1} |\Phi(u)| < \infty$$

The norm of Φ is defined as

$$\|\Phi\| = \sup_{u \in X, \, \|u\|=1} |\Phi(u)|$$

The space of all bounded linear functionals is called the dual space of X and it is denoted by X'.

Lemma A linear functional is continuous if and only if it is bounded.

Let X be an inner-product space. Fix $u \neq 0$ in X. Define the mapping:

$$\forall v \in X, \quad \Phi(v) = (u, v)$$

Then $\Phi \in X'$ and one can show that $\|\Phi\| = \|u\|$.

An important classic result for bounded linear functionals on Hilbert spaces is the Riesz representation theorem.

Riesz representation theorem Let X be a Hilbert space and let $\Phi \in X'$. Then there is a unique $u \in X$ such that

$$\forall v \in X, \quad \Phi(v) = (u, v)$$

So we can identify X with its dual space X'.

As a consequence, if X is a Hilbert space, its dual space X' is also a Hilbert space.

4. Weak derivatives

Definitions A subset $K \subset \mathbb{R}^n$ is compact if and only if K is closed and bounded. Let $\Omega \subset \mathbb{R}^n$ be an open domain. Let $u : \Omega \to \mathbb{R}$. The support of u is

$$\operatorname{supp}(u) = \overline{\{x \in \Omega : \, u(x) \neq 0\}}$$

Let $C_0^\infty(\Omega)$ be the set of all functions that are infinitely differentiable in Ω and that have compact support. We note that all functions in $C_0^\infty(\Omega)$ and their partial derivatives of any order are zero on the boundary of Ω.

An example of such $\mathcal{C}_0^\infty(\Omega)$ function is given by ϕ_{x_0}, given a point $x_0 \in \Omega$ and $r > 0$ such that the ball of radius r and origin x_0, denoted by $B_r(x_0)$ belongs to Ω.

$$\phi_{x_0}(x) = \begin{cases} e^{-(r^2-|x-x_0|^2)^{-1}}, & \text{in } B_r(x_0) \\ 0 & \text{elsewhere} \end{cases}$$

Definition We say that u is locally integrable on Ω if for any $K \subset \Omega$, with K compact, $\int_K |u| < \infty$. We write $u \in L^1_{loc}(\Omega)$.

Next, we define the partial weak derivatives, that generalize the definition of classical (strong) partial derivatives. Assume $u \in \mathcal{C}^1(\Omega)$ and pick $\phi \in \mathcal{C}_0^\infty(\Omega)$. Using integration by parts and the fact that $\phi = 0$ on the boundary $\partial\Omega$, we have:

$$\int_\Omega \frac{\partial u}{\partial x}\phi = -\int_\Omega u\frac{\partial \phi}{\partial x}$$

Definition Let u be locally integrable on Ω. The weak partial derivative of u with respect to x is the function $g \in L^1_{loc}(\Omega)$ that satisfies:

$$\forall \phi \in \mathcal{C}_0^\infty(\Omega), \quad \int_\Omega g\phi = -\int_\Omega u\frac{\partial \phi}{\partial x}$$

If, in addition $u \in \mathcal{C}^1(\Omega)$, then the weak derivative g is equal to the strong derivative $\frac{\partial u}{\partial x}$. Therefore we abuse the notation and denote g by $\frac{\partial u}{\partial x}$. Similarly if n, m are integers, the weak partial derivative of order $n+m$ of u is the function $g \in L^1_{loc}(\Omega)$ that satisfies:

$$\forall \phi \in \mathcal{C}_0^\infty(\Omega), \quad \int_\Omega g\phi = (-1)^{n+m}\int_\Omega u\frac{\partial^{n+m}\phi}{\partial x^n \partial y^m}$$

We also abuse the notation and write:

$$g = \frac{\partial^{n+m}u}{\partial x^n \partial y^m}$$

These definitions exist for all partial derivatives of any order.

Example Let us compute the weak partial derivative of u with respect to x for u defined as:

$$\forall (x,y) \in (0,1)^2, \quad u(x,y) = \begin{cases} x & 0 < x \leq 0.5 \\ 1-x & 0.5 < x < 1 \end{cases}$$

Clearly u is not differentiable w.r.t. x in the classical sense since the partial derivative for $x = 0.5$ and any y is not uniquely defined. Let $\phi \in \mathcal{C}_0^\infty(\Omega)$. We compute

$$-\int_\Omega u\frac{\partial\phi}{\partial x} = -\int_0^1\int_0^{1/2} x\frac{\partial\phi}{\partial x} - \int_0^1\int_{1/2}^1 (1-x)\frac{\partial\phi}{\partial x}$$

$$= \int_0^1\int_0^{1/2}\phi - \int_0^1 0.5\phi(0.5,y) - \int_0^1\int_{1/2}^1\phi + \int_0^1 0.5\phi(0.5,y) = \int_\Omega g\phi$$

with g defined by:

$$\forall (x,y) \in (0,1)^2, \quad g(x,y) = \begin{cases} 1 & 0 < x \leq 0.5 \\ -1 & 0.5 < x < 1 \end{cases}$$

Clearly $g \in L^1_{loc}(\Omega)$ and thus we conclude that g is the weak partial derivative of u with respect to x.

5. Sobolev spaces

We recall the L^2 space:

$$L^2(\Omega) = \{v \text{ measurable } \int_\Omega v^2 < \infty\}$$

We define the Sobolev space $H^1(\Omega)$. Let us denote the gradient of u by $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$ for two-dimensional domains. The partial derivatives are here **weak**.

$$H^1(\Omega) = \{v \in L^2(\Omega),\, \nabla u \in L^2(\Omega)\}$$

In other words, $\frac{\partial u}{\partial x} \in L^2(\Omega)$ and $\frac{\partial u}{\partial y} \in L^2(\Omega)$.
In one-dimensional domains, this simply gives:

$$H^1(a,b) = \{v \in L^2(a,b),\, u' \in L^2(a,b)\}$$

If we increase the order of weak partial derivatives, we obtain other Sobolev spaces.

$$H^2(\Omega) = \{v \in H^1(\Omega),\, \frac{\partial^2 u}{\partial x^n \partial y^m} \in L^2(\Omega) : 0 \leq n+m \leq 2\}$$

Similarly we define by $H^k(\Omega)$ the space of functions $v \in L^2(\Omega)$ such that all partial weak derivatives of order up to k belong to $L^2(\Omega)$. Clearly we have:

$$k \geq 3, \quad H^k(\Omega) \subset H^{k-1}(\Omega) \subset H^2(\Omega) \subset H^1(\Omega) \subset L^2(\Omega)$$

Sobolev spaces are Hilbert spaces. For instance $H^1(\Omega)$ is equipped with the inner-product and norm:

$$(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv), \quad \|u\|_{H^1(\Omega)} = \Big(\int_\Omega ((\nabla u)^2 + u^2)\Big)^{1/2}$$

For PDEs with Dirichlet boundary conditions, we also need the Sobolev space:

$$H^1_0(\Omega) = \{v \in H^1(\Omega),\, v = 0 \text{ on } \partial\Omega\}$$

The equation $v = 0$ on $\partial\Omega$ has to be understood in the sense of traces. This is a technical and delicate topic, that we do not cover in class. For the space $H_0^1(\Omega)$, one can show that this space is Hilbert equipped with the inner-product and norm:

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v, \quad \|u\|_{H_0^1(\Omega)} = \left(\int_\Omega (\nabla u)^2 \right)^{1/2}$$

6. Green's theorem, Poincaré's inequality

Let $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$. We have the following Green's theorem:

$$-\int_\Omega (\Delta u) v = \int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} (\nabla u \cdot n) v$$

where n is the unit outward normal vector to Ω. If $\boldsymbol{K}$ is a symmetric positive definite matrix, we also have:

$$-\int_\Omega (\nabla \cdot (\boldsymbol{K} \nabla u)) v = \int_\Omega \boldsymbol{K} \nabla u \cdot \nabla v - \int_{\partial\Omega} (\boldsymbol{K} \nabla u \cdot n) v$$

Poincaré's inequality states that there is a constant $C_P > 0$ such that

$$\forall v \in H_0^1(\Omega), \quad \|v\|_{L^2(\Omega)} \leq C_P \|\nabla v\|_{L^2(\Omega)} = \|v\|_{H_0^1(\Omega)}$$

VI. VARIATIONAL PROBLEMS

Finite element methods belong to the class of discrete variational problems. In this chapter, we study the theory for general variational problems. Variational problems (or formulations) are also called *weak problems.*

1. Definition

Let X, V be two Hilbert spaces. Let $a : X \times V \to \mathbb{R}$ be a bilinear form and let $\ell : V \to \mathbb{R}$ be a linear form. The variational problem is to find $u \in X$ such that

$$\forall v \in V, \quad a(u,v) = \ell(v)$$

Note that in class, most of the cases have $X = V$.

2. Examples

2.1. Elliptic problems with Dirichlet boundary

Let $\Omega \subset \mathbb{R}^d$ and assume its boundary is partitioned into a Neumann part and a Dirichlet part: $\partial\Omega = \overline{\Gamma}_N \cup \overline{\Gamma}_D$, with $\Gamma_N \cap \Gamma_D = \emptyset$. Assume that the Dirichlet part has positive measure: $|\Gamma_D| > 0$. The general elliptic problem is

$$\begin{aligned} -\nabla \cdot \boldsymbol{K}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f, &\quad \text{in} \quad \Omega \\ u = 0, &\quad \text{on} \quad \Gamma_D \\ \boldsymbol{K}\nabla u \cdot \boldsymbol{n} = g, &\quad \text{on} \quad \Gamma_N \end{aligned}$$

Formally, we multiply the PDE by a function v, integrate over Ω, and use Green's theorem for the first term:

$$\int_\Omega \boldsymbol{K}\nabla u \cdot \nabla v + \int_\Omega (\boldsymbol{b} \cdot \nabla u) v + \int_\Omega cuv - \int_{\Gamma_N \cup \Gamma_D} \boldsymbol{K}\nabla u \cdot \boldsymbol{n} v = \int_\Omega fv$$

Next, we use the boundary conditions and assume that $v = 0$ on Γ_D. The equation reduces to:

$$\int_\Omega \boldsymbol{K}\nabla u \cdot \nabla v + \int_\Omega (\boldsymbol{b} \cdot \nabla u) v + \int_\Omega cuv = \int_\Omega fv + \int_{\Gamma_N} gv$$

For this equation to be well defined, each term should be finite. This implies that we need the following assumptions:

- The matrix $\boldsymbol{K} = (K_{ij})_{i,j}$ is such that $K_{i,j} \in L^\infty(\Omega)$. In addition, we will assume that $\boldsymbol{K}$ is symmetric positive definite.
- $\boldsymbol{b} \in (L^\infty(\Omega))^d$
- $c \in L^\infty(\Omega)$

- $f \in L^2(\Omega), \quad g \in L^2(\Gamma_N)$
- $u, v \in H^1(\Omega)$

Therefore we consider the space

$$X = V = \{v \in H^1(\Omega) : \ v = 0 \text{ on } \Gamma_D\}$$

and the forms

$$a(u,v) = \int_\Omega \boldsymbol{K}\nabla u \cdot \nabla v + \int_\Omega (\boldsymbol{b} \cdot \nabla u) v + \int_\Omega cuv$$
$$\ell(v) = \int_\Omega fv + \int_{\Gamma_N} gv$$

The weak solution $u \in V$ of the elliptic problem satisfies:

$$\forall v \in V, \quad a(u,v) = \ell(v)$$

Recall that the derivatives in the definition of a are all weak derivatives. A strong solution of the elliptic problem is a function $u \in \mathcal{C}^2(\Omega) \cup \mathcal{C}^1(\overline{\Omega})$ satisfying the PDE and boundary conditions pointwisely for every point in Ω. The input data $\boldsymbol{K}, \boldsymbol{b}, c, f, g$ are assumed to be smooth in the classical sense.

Remark: What is the relationship between the weak and the strong solutions? Clearly a strong solution is also a weak solution. The converse is not true in general. If u is a weak solution and if we assume that $u \in H^2(\Omega)$, then u satisfies the PDE and boundary conditions in the L^2 sense, namely almost everywhere in Ω.

Remark: The boundary condition $u = 0$ is imposed in the space V; it is called an *essential* boundary condition. The Neumann boundary condition is then called *natural.*

2.2. Elliptic problems with pure Neumann boundary

Let us consider the simplified elliptic problem

$$-\Delta u = f, \quad \text{in} \quad \Omega$$
$$\nabla u \cdot \boldsymbol{n} = g, \quad \text{on} \quad \partial\Omega$$

Using a similar approach as in the previous section, we can formally obtain a variational formulation defined by

$$V = X = H^1(\Omega)$$
$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v$$
$$\ell(v) = \int_\Omega fv + \int_{\partial\Omega} gv$$

This problem does not have a solution for any f, g. Indeed, if we choose $v = 1$, the weak problem reduces to

$$\int_\Omega f + \int_{\partial\Omega} g = 0$$

This condition is called the *compatibility condition* and it should be satisfied by the data in order for the problem to make sense. From now on, we assume this condition is satisfied.

Next, we observe that if u is a weak solution, then $u + C$ where C is any constant, is also a solution. Thus, the weak solution is not unique and the weak formulation is not well-posed. To overcome this difficulty, we replace the space $H^1(\Omega)$ by the quotient space

$$V = H^1(\Omega)/\mathbb{R} = \{[v],\, v \in H^1(\Omega)\}$$

where $[v]$ is an equivalence class:

$$[v] = \{w \in H^1(\Omega)\, w - v = C \in \mathbb{R}\}$$

In other words, two functions in V are equal if they differ by a constant. We then define new forms:

$$a([u],[v]) = \int_\Omega \nabla u \cdot \nabla v$$

$$\ell([v]) = \int_\Omega fv + \int_{\partial\Omega} gv$$

In this case, this new weak problem is well posed.

2.3. Stokes problem

Stokes equations characterize fluid flows that are not turbulent. Let $\boldsymbol{u}(x) \in \mathbb{R}^d$ be the fluid velocity and $p(x) \in \mathbb{R}$ the fluid pressure at a point $x \in \Omega$. Let $\mu > 0$ be the fluid viscosity. The Stokes equations are:

$$\begin{aligned} -\mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, &\quad \text{in} \quad \Omega \\ \nabla \cdot \boldsymbol{u} = 0, &\quad \text{in} \quad \Omega \\ \boldsymbol{u} = \boldsymbol{0}, &\quad \text{on} \quad \partial\Omega \end{aligned}$$

Recall that $\Delta\boldsymbol{u}$ is a vector with components Δu_j. The weak problem is defined by the following spaces

$$X = V = (H^1(\Omega))^d$$

$$Q = L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$$

and the forms

$$\forall \boldsymbol{u}, \boldsymbol{v} \in V, \quad a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mu \sum_{i,j} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}$$

$$\forall \boldsymbol{v} \in V, \forall q \in Q, \quad b(\boldsymbol{v}, q) = \int_\Omega q \nabla \cdot v$$

$$\forall \boldsymbol{v} \in V, \quad \ell(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}$$

The weak solution $(\boldsymbol{u}, p) \in V \times Q$ satisfies:

$$\forall \boldsymbol{v} \in V, \quad a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \ell(\boldsymbol{v})$$
$$\forall q \in Q, \quad b(\boldsymbol{u}, q) = 0$$

Note that this can be rewritten as a standard variational problem given in Section 1. Define the bilinear form

$$\mathcal{A}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - b(\boldsymbol{u}, q)$$

and the form

$$\mathcal{L}((\boldsymbol{v}, q)) = \ell(\boldsymbol{v})$$

then we can write: find $(\boldsymbol{u}, p) \in \boldsymbol{X} = \boldsymbol{V} \times Q$ such that

$$\forall (\boldsymbol{v}, q) \in \boldsymbol{X}, \quad \mathcal{A}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = \mathcal{L}((\boldsymbol{v}, q))$$

3. Existence and uniqueness of weak solutions

Well-posedness of the weak problem can be obtained by applying the Lax-Milgram theorem, stated below.

Lax-Milgram Theorem

Let X be a Hilbert space with norm $\|\cdot\|_X$. Let $a : X \times X \to \mathbb{R}$ be a bilinear form and let $\ell : X \to \mathbb{R}$ be a linear form. Assume that

1. The form a is bounded: there is a constant $M > 0$ such that

$$\forall u, v \in X, \quad a(u, v) \leq M \|u\|_X \|v\|_X$$

2. The form a is coercive: there is a constant $\alpha > 0$ such that

$$\forall v \in X, \quad a(v, v) \geq \alpha \|v\|_X^2$$

3. The form ℓ is continuous, i.e. $\ell \in X'$

Then there exists a unique $u \in X$ satisfying

$$\forall v \in X, \quad a(u, v) = \ell(v)$$

In addition, the bound holds:

$$\|u\|_X \leq \frac{1}{\alpha}\|\ell\|_{X'}$$

In that case we say the weak problem is well-posed.

Coercivity of a is sometimes hard to obtain, or even impossible. A more general Lax-Milgram theorem replaces the coercivity condition by the following inf-sup condition (also called the LBB condition):

$$\inf_{u \in X} \sup_{v \in X} \frac{|a(u, v)|}{\|u\|_X \|v\|_X} \geq \alpha > 0$$

One can check that Lax-Milgram's theorem can be applied to the weak problems defined in Section 2. We now give an example by considering the weak problem: find $u \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v$$

First we check continuity of a, by applying Cauchy-Schwarz's inequality:

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

We know that $\|v\|_{H_0^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$, so we have continuity of a with the constant $M = 1$.

Second, we check coercivity of a. This is trivial:

$$a(v, v) = \int_\Omega \nabla v \cdot \nabla v = \|v\|^2_{H_0^1(\Omega)} \geq \|v\|^2_{H_0^1(\Omega)}$$

and the coercivity constant is $\alpha = 1$.

Third, we check continuity of ℓ.

$$\ell(v) = \int_\Omega f v \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

Using Poincaré's inequality, we have:

$$\|v\|_{L^2(\Omega)} \leq C_P \|v\|_{H_0^1(\Omega)}$$

for some constant $C_P > 0$. Therefore, we have

$$\ell(v) \leq C_P \|f\|_{L^2(\Omega)} \|v\|_{H_0^1(\Omega)}$$

which proves boundedness of ℓ and also proves that

$$\|\ell\|_{X'} \leq C_P \|f\|_{L^2(\Omega)}$$

We then conclude there is a unique weak solution u. In addition, the bound holds:

$$\|u\|_{H_0^1(\Omega)} \leq C_P \|f\|_{L^2(\Omega)}$$

4. Discrete weak problems

Let $X_h \subset X$ be a finite dimensional subspace of the Hilbert space X. The discrete variational problem is: find $u_h \in X_h$ such that

$$\forall v_h \in X_h, \quad a(u_h, v_h) = \ell(v_h)$$

This is a finite-dimensional problem, and we can apply Lax-Milgram by replacing X by X_h. In other words, if we have continuity and coercivity of a in X_h and continuity of ℓ in X_h, then the discrete weak problem is well-posed.

Of course, if the assumptions of Lax-Milgram's theorem hold for X, then they also hold for X_h. In that case, we have an error bound:

$$\|u - u_h\|_X \leq \frac{M}{\alpha} \inf_{w_h \in X_h} \|u - w_h\|_X$$

The proof follows the same derivation as in 1D:

$$\alpha \|u - u_h\|_X^2 \leq a(u - u_h, u - u_h)$$

Since $a(u - u_h, v_h) = 0$ for all v_h, we have

$$\alpha \|u - u_h\|_X^2 \leq a(u - u_h, u - v_h) \leq M \|u - u_h\|_X \|u - v_h\|_X$$

Thus we have

$$\|u - u_h\|_X \leq \frac{M}{\alpha} \|u - v_h\|_X$$