CAAM 452: Homework 2
Posted online on January 30
Due February 13 in class
Printout of codes are to be included and also uploaded in owlspace.

Problem 1 (20 points)

The MATLAB script `poisson.m` solves the Poisson problem on the unit square $(0,1)^2$ with a grid size $h_x = h_y = h = 1/(N+1)$, using the 5-point Laplacian. It is set up to solve a test problem for which the exact solution is $u(x,y) = \exp(x + y/2)$, using Dirichlet boundary conditions and the right hand side $f(x,y) = 1.25 \exp(x + y/2)$.

1. Test this script by performing a grid refinement study to verify that it is second order accurate: consider $h = 1/10, 1/20, 1/40, 1/80$.
2. Modify the script so that it works on a rectangular domain $[a_x, b_x] \times [a_y, b_y]$, but still with $h_x = h_y = h$. Test your modified script on the domain $(0,1) \times (1,3)$ and use the values $h = 1/10, 1/20, 1/40, 1/80$. Give the errors and convergence rates.

Problem 2 (20 points)

1. Consider the finite difference method:
$$\frac{U_{i+2} - U_{i+1} - U_{i-1} + U_{i-2}}{3h^2} = f(x_i)$$
to solve the problem $u'' = f$ on a uniform grid $x_i = ih$ of the unit interval. What is the order of the local truncation error? Is this finite difference method consistent?
2. Consider the finite difference method:
$$\frac{U_{i+1,j} - U_{i,j+1} - U_{i,j-1} + U_{i-1,j}}{h^2} = f(x_i, y_j)$$
to solve the problem $u_{xx} - u_{yy} = f$ on a uniform grid $x_i = ih, y_j = jh$ of the unit square. What is the order of the local truncation error? Is this finite difference method consistent?

Problem 3 (30 points)

For any $\epsilon > 0$, we want to solve the following problem

$$-\epsilon u''(x) + u'(x) = 0, 0 < x < 1$$
$$u(0) = 0, \quad u(1) = 0$$

using the finite difference method based on the following approximations:

$$u''(x_i) \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}, \quad u'(x_i) \approx \frac{u(x_{i+1}) - u(x_{i-1})}{2h}$$

Write the linear system we obtain for this finite difference. Assume that there are N interior nodes, and that $x_i = ih$. Write the code and obtain the numerical solution for $h = 1/50$. Plot the numerical solution and the exact solution for the following values of ϵ: 1, 0.1, 0.01, 0.001. Describe the results. The exact solution is

$$u(x) = \frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1}$$

Problem 4 (30 points)

(a) Implement the finite element method using continuous piecewise linears for solving

$$-u''(x) = f(x), 0 < x < 1$$
$$u(0) = 0, \quad u(1) = 0$$

(b) Verify your code on the following two examples

1. The exact solution is $u(x) = x(1-x)$.
2. The exact solution is $u(x) = x(1-x)e^{-x^2}$

Consider a sequence of uniform meshes with $h = 1/4, 1/8, 1/16, 1/32$. Denote by u_h the finite element solution. For each mesh, plot the numerical solution and compute the following errors:

$$err0 = (\int_0^1 (u - u_h)^2 dx)^{1/2}, \quad err1 = (\int_0^1 (u' - u_h')^2 dx)^{1/2}$$

Obtain the numerical convergence rates for $err0$ and $err1$. Hint: to compute the errors, write the integral as a sum of integrals over each subinterval and use a quadrature rule (trapezoid rule for $err0$ and midpoint rule for $err1$). For instance if $h = 1/(N+1)$ and the grid nodes are $x_i = ih$, we can write

$$\int_0^1 (u - u_h)^2 dx = \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} (u - u_h)^2 dx \approx \sum_{i=1}^{N+1} \frac{h}{2}((u(x_{i-1}) - u_h(x_{i-1}))^2 + (u(x_i) - u_h(x_i))^2)$$