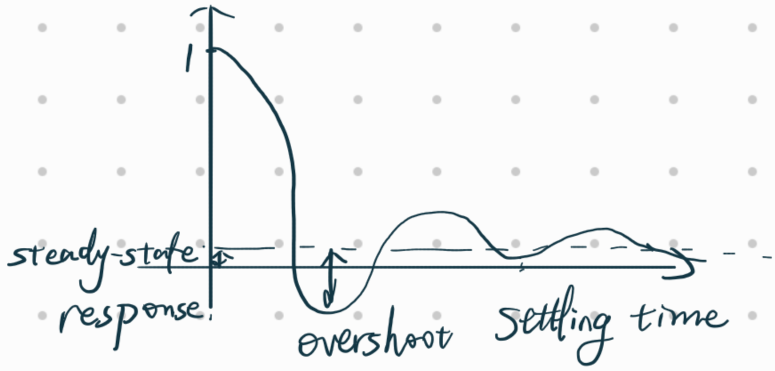
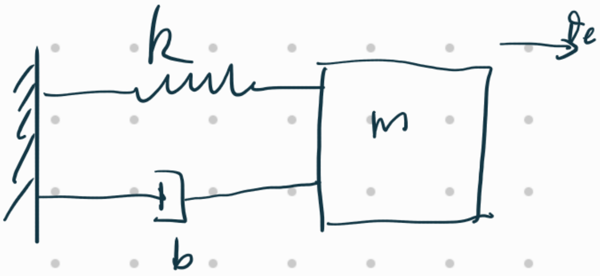
Desired motion: $\theta_d(t)$.

actual motion: $\theta(t)$

error: $\theta_e(t) = \theta_d(t) - \theta(t)$





$m\ddot{\theta}_e + b\dot{\theta}_e + k\theta_e = 0$

LCCDE:

$$a_p \theta_e^{(p)} + a_{p-1}\theta_e^{(p-1)} + \cdots + a_2\ddot{\theta}_e + a_1\dot{\theta}_e + a_0\theta_e = \begin{cases} 0 & : \text{homogeneous} \\ c & : \text{non-homogen.} \end{cases}$$

Let $x_1 = \theta_e$

$x_2 = \dot{x}_1 = \dot{\theta}_e$ $\qquad x = [x_1 \ x_2 \ \cdots \ x_p]^T$

$x_p = \dot{x}_{p-1} = \theta_e^{(p-1)}$

$\dot{x}_p = -a_0' x_1 - a_1' x_2 - \cdots - a_{p-1}' x_p$

$\dot{x}(t) = A x(t) \rightarrow x(t) = e^{At} x(0)$

$$A = \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & & & & \\ 0 & 0 & \cdots & & 1 \\ -a_0' & -a_1' & \cdots & & -a_{p-1}' \end{bmatrix} \in \mathbb{R}^{p \times p}$$

eig(A) = roots of eq. $\det(sI - A) = s^p + a_{p-1}' s^{p-1} + \cdots + a_2' s^2 + a_1' s + a_0 = 0$

A necessary condition for stability is that all a_i' are positive!

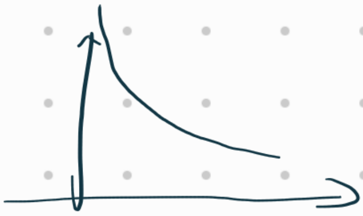
Consider $m\ddot{\theta}_e + b\dot{\theta}_e + k\theta_e = 0$

$m = 0$: $\dot{\theta}_e + \frac{k}{b}\theta_e = 0$

$\theta_e(t) = e^{-t/\tau}\theta_e(0)$, $\tau = \frac{b}{k}$.

2% settling time: $\frac{\theta_e(t)}{\theta_e(0)} = 0.02$

$t = 3.91\tau$



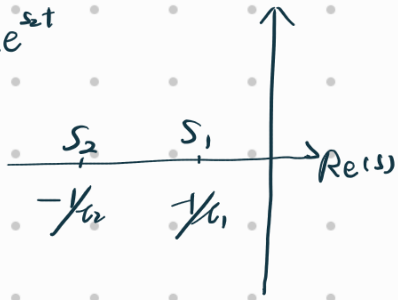
$m \neq 0$. $\omega_n = \sqrt{\frac{k}{m}}$, $\zeta = \frac{b/m}{2\sqrt{k/m}} = \frac{b}{\sqrt{2km}}$

$\ddot{\theta}_e(t) + 2\zeta\omega_n\dot{\theta}_e(t) + \omega_n^2 = 0$

$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

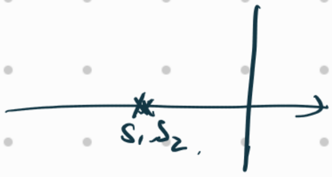
$\zeta > 1$, overdamped $\theta_e(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$



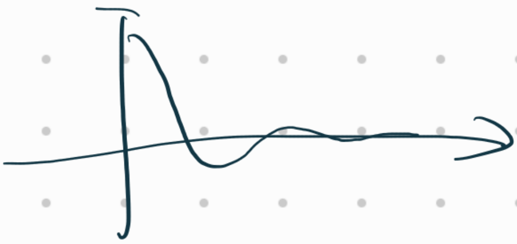
$\zeta = 1$, critically damped

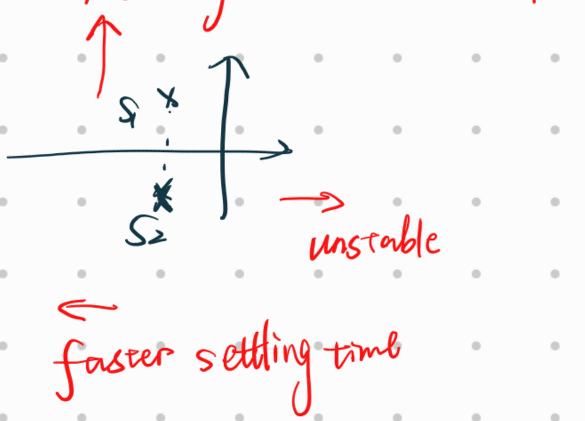
$\theta_e(t) = (c_1 + c_2 t)e^{st}$

$s_1 = s_2$



$\zeta > 1$, overdamped





Closed-loop Control

P - control: $\dot{\theta}(t) = K_p(\theta_d - \theta(t)) = K_p\theta_e(t)$

e.g. setpoint control: $\theta_d(t) = 0$

$\dot{\theta}_e(t) = \dot{\theta}_d(t) - \dot{\theta}(t) = -\dot{\theta}(t)$

$\dot{\theta}_e(t) = -K_p\theta_e(t) \Rightarrow \dot{\theta}_e(t) + K_p\theta_e(t) = 0$

$\tau = 1/K_p$

K_p limits!

Large K_p => huge vibrations; actuator non-linear.

PI control:

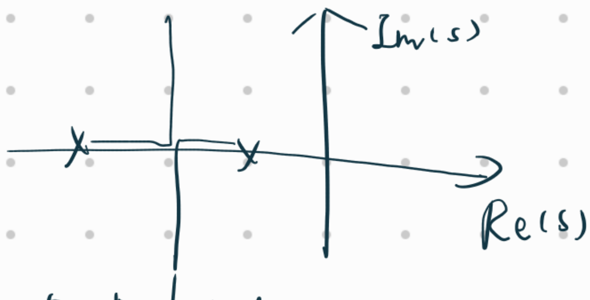
$\dot{\theta}(t) = K_p\theta_e(t) + K_i\int_0^t \theta_e(t)\,dt$

$\dot{\theta}_e(t) + K_p\theta_e(t) + K_i\int_0^t \theta_e(t)\,dt = c.$

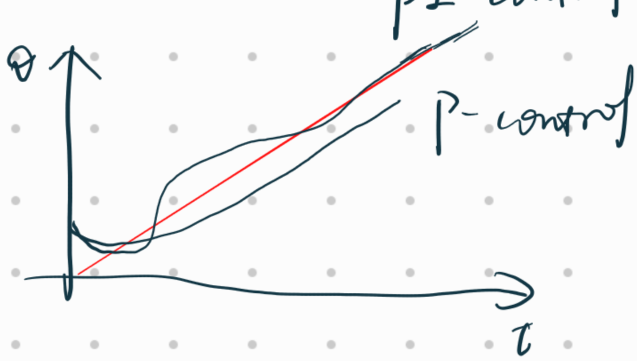
$\ddot{\theta}_e + K_p\dot{\theta}_e + K_i\theta_e = 0$. $\zeta = K_p/2\sqrt{K_i}$

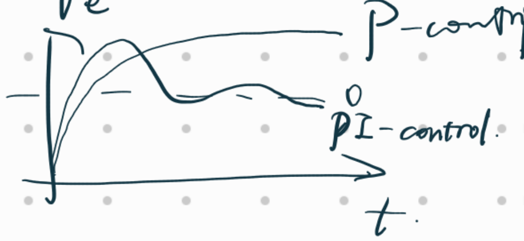
$\omega_n = \sqrt{K_i}$

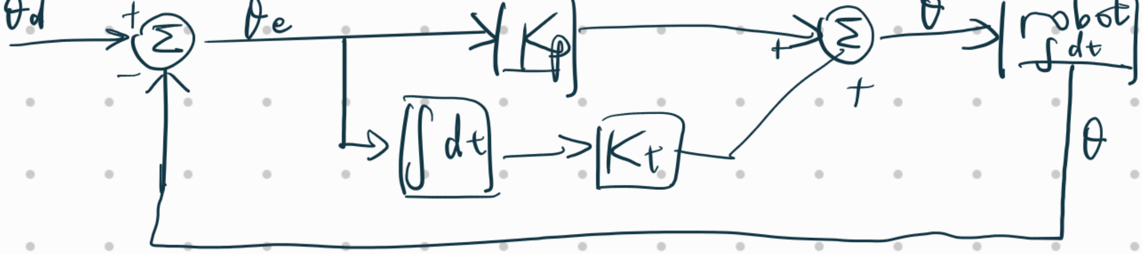
$s_{1,2} = -\frac{K_p}{2} \pm \sqrt{\frac{K_p^2}{4} - K_i}$



root locus







$$\dot\theta(t) = \dot\theta_d(t) + K_p\theta_e(t) + K_i\int_0^t \theta_e(t)\,dt.$$

: Fast-forward + PI control.

$$K_i\theta_e(t) + K_p\dot\theta_e(t) + \ddot\theta_e(t) = 0$$

Multi-joint.

Task-space control w/ velocity inputs.

$$\mathcal{V}_b(t) = [Ad_{X^{-1}X_d}]\mathcal{V}_d(t) + K_pX_e(t) + K_i\int_0^t X_e(t)\,dt$$

↳ Adjoint transformation.

$$\dot\theta = J_b^{\dagger}(\theta)\mathcal{V}_b$$

↳ pinv of Jacobian.

$X := X_{sb}$. actual ee frame

$X_d := X_{sd}$. desired ee frame.

$X_e \neq X_d - X$: makes no sense to subtract in $SE(3)$.

$$[X_e] = \text{logm}(X^{-1}X_d)$$

$$V_d = \text{logm}[X_{sb}^{-1}X_{sd}] = \text{logm}[X^{-1}X_d]$$

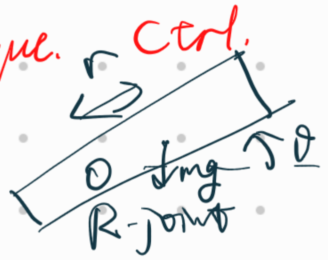
Alternatively,
$X = (R, p)$:

$$\begin{bmatrix} \omega_b(t) \\ \dot p(t) \end{bmatrix} = \begin{bmatrix} R^T(t)R_d(t) & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \omega_d(t) \\ \dot p_d(t) \end{bmatrix} + K_pX_e(t) + K_i\int_0^t X_e(t)\,dt$$

$$X_e(t) = \begin{bmatrix} \omega_e(t) \\ p_d(t) - p(t) \end{bmatrix} \qquad [\omega_e] = \log(R^TR_d)$$

Control w/ force and torques inputs.

Torque. Ctrl.



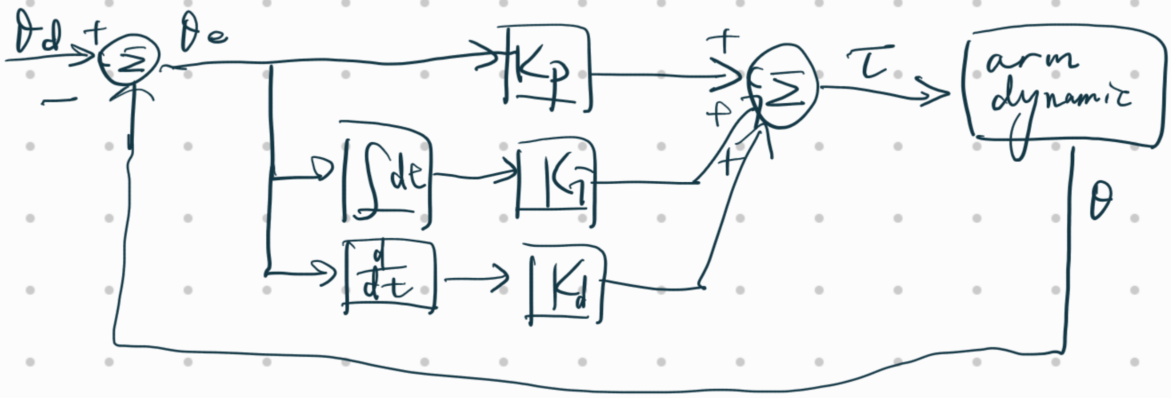
$$\tau = M\ddot{\theta} + mgr\cos\theta + b\dot{\theta}$$

M ↳ inertia (scalar)

$b\dot{\theta}$ ↳ friction term

PID control:

$$\tau = K_p\theta_e + K_i\int\theta_e(t)dt + K_d\dot{\theta}_e$$



Setpoint PD control w/ $g=0$.

$$\tau = K_p\theta_e + 0 + K_d\dot{\theta}_e$$

$$\tau = M\ddot{\theta} + b\dot{\theta}$$

$$\dot{\theta}_d = \ddot{\theta}_d = 0$$

$$\Rightarrow M\ddot{\theta} + b\dot{\theta} = K_p\theta_e + K_d\dot{\theta}_e$$

$$\ddot{\theta}_e + \frac{(b+K_d)}{M}\dot{\theta}_e + \frac{K_p}{M}\theta_e = 0.$$

$$\zeta = \frac{b+K_d}{2\sqrt{MK_p}}, \quad w_n = \sqrt{\frac{K_p}{M}}$$

Stable $\Rightarrow$ $Re(s) < 0$.

$$\begin{cases} b+K_d > 0 \\ K_p > 0. \end{cases}$$

$$s^2 + 2\zeta w_n s + w_n^2 = 0$$

$$s = -\zeta w_n \pm w_n\sqrt{\zeta^2 - 1}$$

$$w = w_n\sqrt{1-\zeta^2}$$

Setpoint PD control w/ $g > 0$.

$\tau = K_p \theta_e + 0 + K_d \dot{\theta}_e$ — control

$\tau = M\ddot{\theta} + mgr\cos\theta + b\dot{\theta}$ — dynamics.

$\dot{\theta}_d = \ddot{\theta}_d = 0$.

There has to be ss error for control to hold it stationary.

sol 1: setpoint $\boxed{\text{PID control}}$, $g > 0$.

$$\tau = K_p \theta_e + K_i \int \theta_e dt + K_d \dot{\theta}_e$$

$$\tau = M\ddot{\theta} + \underbrace{mgr\cos\theta}_{=:\tau_{dist}} + b\dot{\theta}$$

$\dot{\theta}_d = \ddot{\theta}_d = 0$.

$$\Rightarrow \quad M\ddot{\theta}_e + (b+K_d)\dot{\theta}_e + K_p\theta_e + K_i \int \theta_e(t) dt = \tau_{dist}$$

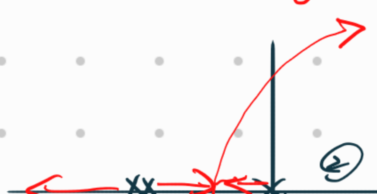
$$M\dddot{\theta}_e + (b+K_d)\ddot{\theta}_e + K_p\dot{\theta}_e + K_i\theta_e = 0$$

$$s^3 + \frac{b+K_d}{M}s^2 + \frac{K_p}{M}s + \frac{K_i}{M} = 0$$

For $Re(s) < 0$ for all roots,

$$\begin{cases} K_d > -b \\ K_p > 0 \\ \frac{(b+K_d)K_p}{M} > K_i > 0 \end{cases}$$

In PID, choosing too large K_i can result in instability!



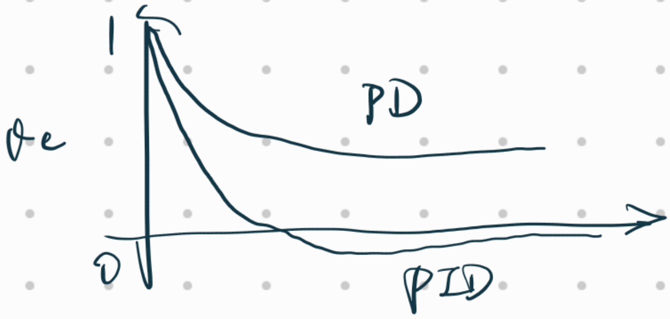
② Set K_p, K_d const.

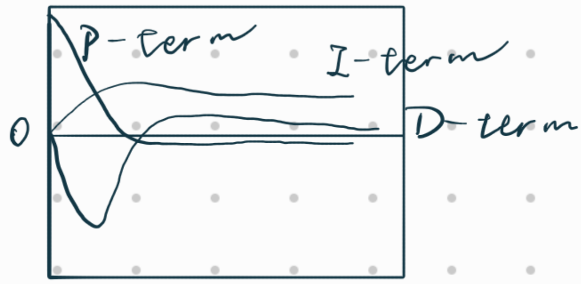
Set $K_i = \varepsilon \Rightarrow$ 3rd root.

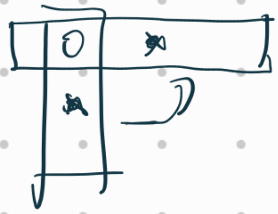
① Set $K_i = 0$. (PD)

Set critical damping.

Ki improves ss response but can worsen transient response.







$\tau = M\ddot{\theta} + h(\theta, \dot{\theta})$

PID feedback control: $\tau = K_p\theta_e + K_i \int \theta_e dt + K_d \dot{\theta}_e$

Dynamic model: $\{\tilde{M}, \tilde{h}\}$

Feedforward control: $\tau = \tilde{M}\ddot{\theta}_d + \tilde{h}(\theta_d, \dot{\theta}_d)$

Combining two: Computed torque control

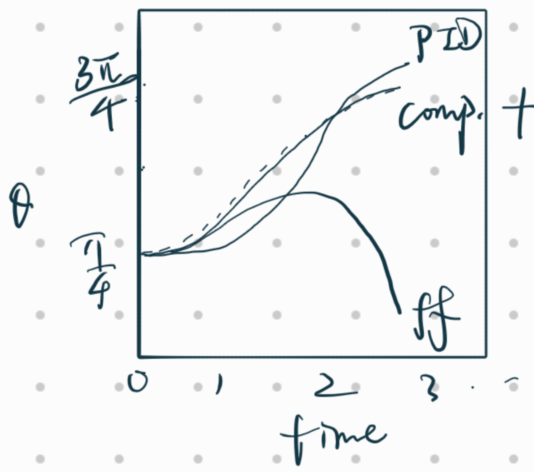
$$\tau = \tilde{M}\underbrace{\left(\ddot{\theta}_d + K_p\theta_e + \cancel{K_i \int \theta_e dt} + K_d\dot{\theta}_e\right)}_{\text{Commanded } \ddot{\theta}} + \tilde{h}(\theta, \dot{\theta})$$

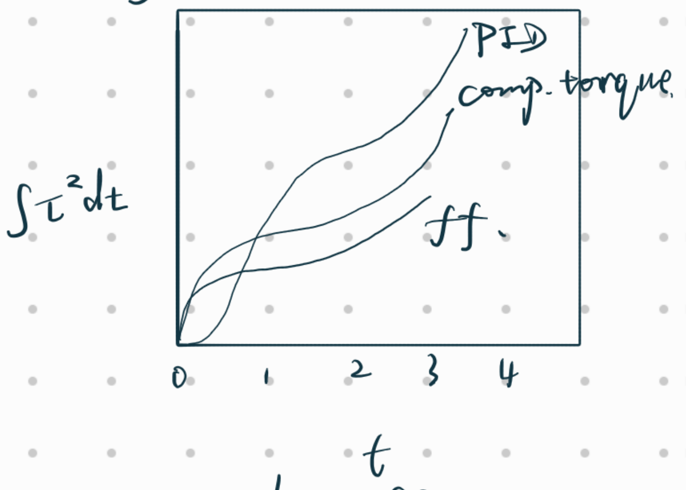
Instability from I-control ⇒ Ki term can be X.

error dynamics: $\ddot{\theta}_e = \ddot{\theta}_d - \ddot{\theta}$

$\ddot{\theta}_e = -K_d\dot{\theta}_e - \dots$

$$\ddot{\theta}_e = -K_d \dot{\theta}_e - K_p \theta_e - K_i \int \theta_e \, dt$$





Better precision w/ less control effort.

single joint:

$$\tau = \tilde{M}\left(\ddot{\theta}_d + K_p\theta_e + K_i \int \theta_e dt + K_d \dot{\theta}_e\right) + \tilde{h}(\theta, \dot{\theta})$$

multi-joint:

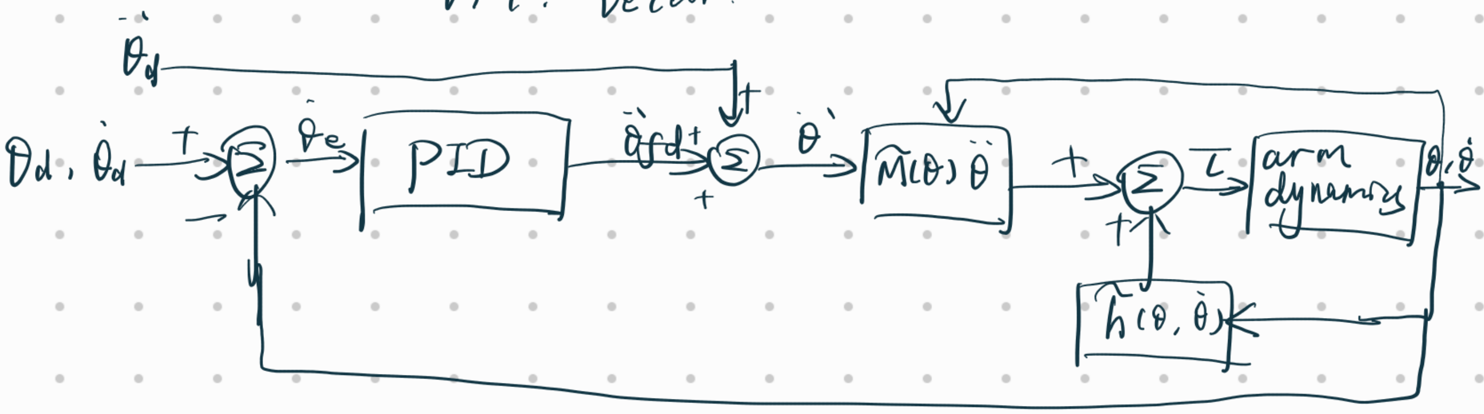
$$\tau = \underline{\tilde{M}(\theta)}\left(\ddot{\theta}_d + K_p\theta_e + K_i \int \theta_e dt + K_d \dot{\theta}_e\right) + \underline{\tilde{h}(\theta, \dot{\theta})}$$

gen. mass matrix.

Coriolis, possibly w/ fraction terms.

K_p, K_i, K_d: matrix

θ, τ: vector.



Note: if dynamic model is bad, performance may be worse than PID only.

Computation expensive for $\tilde{h}(\theta, \dot{\theta})$.

Task space computed torque ctrl.

$$\mathcal{F}_b = \Lambda(\theta)\dot{\mathcal{V}}_b + \eta(\theta, \mathcal{V}_b)$$

dyn. model: $\{\tilde{\Lambda}, \tilde{\eta}\}$.

$$\tau = \tilde{M}(\theta)\left(\ddot{\theta}_d + K_p\theta_e + K_i\int\theta_e\,dt + K_d\dot{\theta}_e\right) + \tilde{h}(\theta, \dot{\theta})$$

c.f.

$$\mathcal{F}_b = \tilde{\Lambda}(\theta)\left(\dot{\mathcal{V}}_d + K_pX_e + K_i\int X_e\,dt + K_d\mathcal{V}_e\right) + \eta(\theta, \mathcal{V}_b)$$

$[X_e] = \log m[X^{-1}X_d]$ $\quad$ $\mathcal{V}_e = [Ad_{X^{-1}X_d}]\mathcal{V}_d - \mathcal{V}_b$.

$$\tau = J_b^T(\theta)\mathcal{F}_b.$$

Force ctrl.

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta) + J^T(\theta)\mathcal{F}_{tip}.$$

robot slow in most cases →

$$\tau = \underbrace{\tilde{g}(\theta)}_{\text{compensate gravity}} + \underbrace{J^T(\theta)\mathcal{F}_{tip}}_{\text{defy joint wrench}}$$

— force control w/o end effector force feedback.

feedback: on θ, $\mathcal{F}_{tip}$.

↳ force-torque sensor.

$$\tau = \tilde{g}(\theta) + J^T(\theta)\left(\mathcal{F}_d + K_{fp}\mathcal{F}_e + K_{fi}\int\mathcal{F}_e(t)\,dt\right)$$

$\dot{\mathcal{F}}$ is not used: ① torque sensor noisy → LP filter

② lack of dynamics btw joint forces, torques, wrenches

Hybrid Motion-Force Control

$$\mathcal{F}_b = \underbrace{\Lambda(\theta)\dot{\mathcal{V}}_b + \eta(\theta, \mathcal{V}_b)}_{\mathcal{F}_{motion}} + \underbrace{A^T(\theta)\lambda}_{\mathcal{F}_{tip}}$$

$A(\theta)\mathcal{V}_b = 0$, $A(\theta) \in \mathbb{R}^{k\times 6}$ ... Pfaffian constraints.

Let $P(\theta)\mathcal{F}_b = \mathcal{F}_{motion}$

$(I - P(\theta))\mathcal{F}_b = \mathcal{F}_{tip}$

$\Rightarrow$ $P = I - A^T(A\Lambda^{-1}A^T)^{-1}A\Lambda^{-1} \in \mathbb{R}^{6\times 6}$

$\mathrm{rank}(P) = 6 - k$. rank of A. (# of constraints)

$\Rightarrow$

Hybrid motion-force control:

$\mathcal{F}_b = (I - P(\theta)) \times$ force ctrler output

$+ P(\theta) \times$ task-space motion ctrler output.

Implementation: estimate constraints