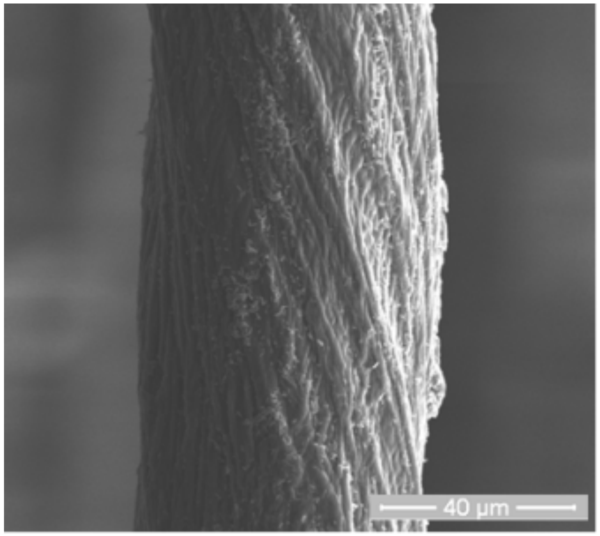
# MIE334 – Numerical Methods I

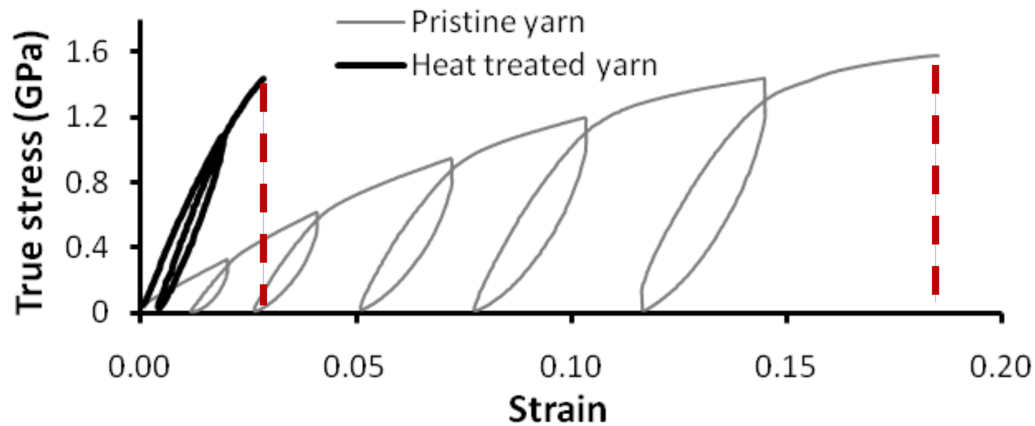
Lecture 25: Numerical Integration and the Trapezoidal Rule *(C&C 21.1)*

# Roadmap: C&C Part Six

- Newton-Cotes Integration (C&C 21)
  - Trapezoidal Rule (21.1)
  - Simpson 1/3 and 3/8 Rules (21.2)
  - Unequal Segments (21.3)
  - Multiple Integrals (21.5)
- Integrating Functions (C&C 22)
  - Romberg Integration (22.2)
  - Gauss Quadrature (22.4)
- Numerical Differentiation (C&C 23)
  - Higher-order Differentiation (23.1)
  - Richardson Extrapolation (23.2)

# Integration in Engineering: **Example – CNT Fibers**




- Integrated area under stress-strain curve gives fiber toughness (J/m$^3$):

$$\int_0^{\varepsilon_f} \sigma(\varepsilon)d\varepsilon$$

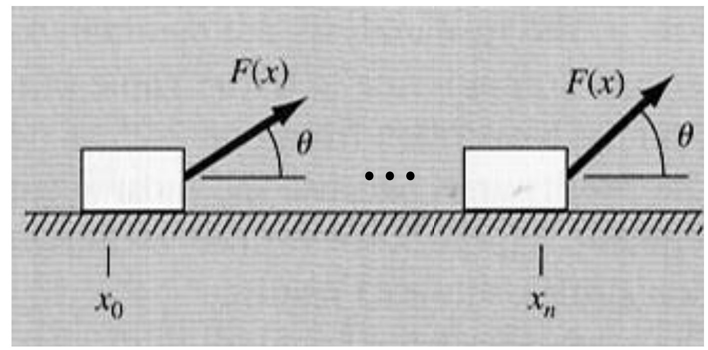
M. Naraghi et al., **ACS Nan4o**, (11) 6463-6476 (2010)

# Integration in Engineering: **Example - Work**

- $Work = Force \times displacement$

- In real Engineering problems the force typically varies:

$$W = \int_{x_0}^{x_n} F(x)dx$$




- Need to solve an integral!
  - e.g. variable force acting on block

$$W = \int_{x_0}^{x_n} F(x)\cos[\theta(x)]dx$$

- If $F(x)$ & $\theta(x)$ are simple functions, can be solved analytically:
  - e.g. $F(x) = 5x^2 + 2x - 1$

# Integration in Engineering: **Example - Work**

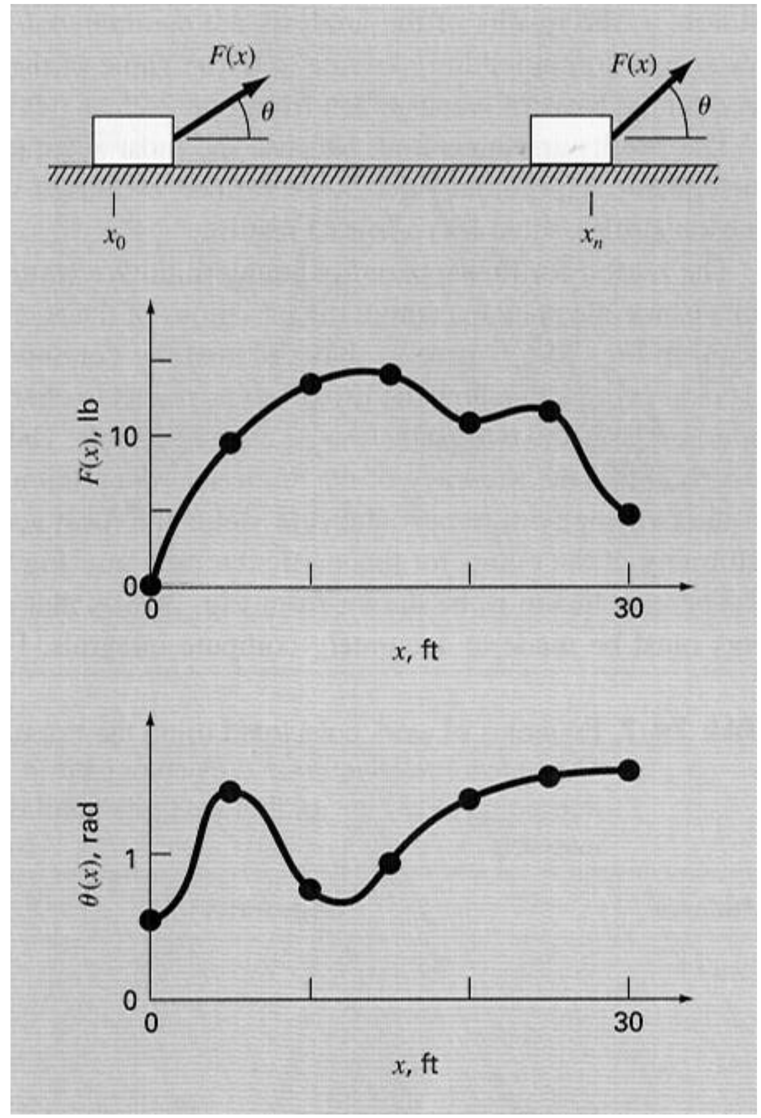
- $W = \int_{x_0}^{x_n} F(x)\cos[\theta(x)]dx$
  - If $F(x)$ & $\theta(x)$ are more complicated we need Numerical Methods to solve the integral

- Two situations arise:

  **I)** Known but complicated functions:

  - e.g. $F(x) = \frac{x^3}{e^x - 1}$

  **II)** Data points only



# Numerical Integration: **General Idea**

- Replace a complicated function, $f(x)$, with an approximation that is easer to integrate:

  $$I = \int_a^b f(x)dx \cong \int_a^b f_{approx}(x)dx$$

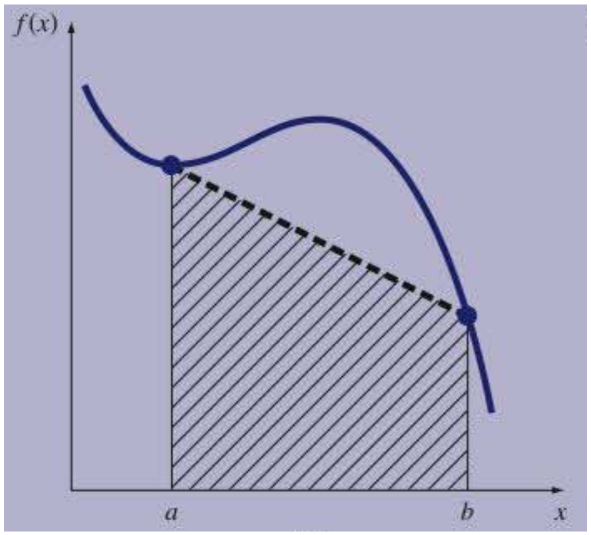
  - What to choose for $f_{approx}(x)$?

- **Newton-Cotes** methods use interpolating polynomials:

  $$f_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

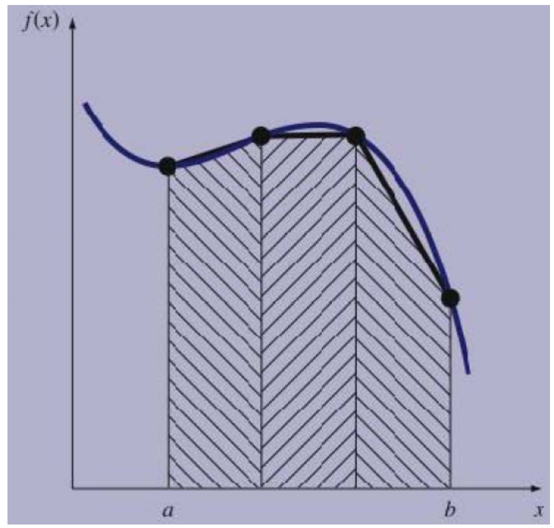
- Polynomials are easy to integrate
  - Simple formulas when data points are equally spaced

# Numerical Integration: **Graphically**

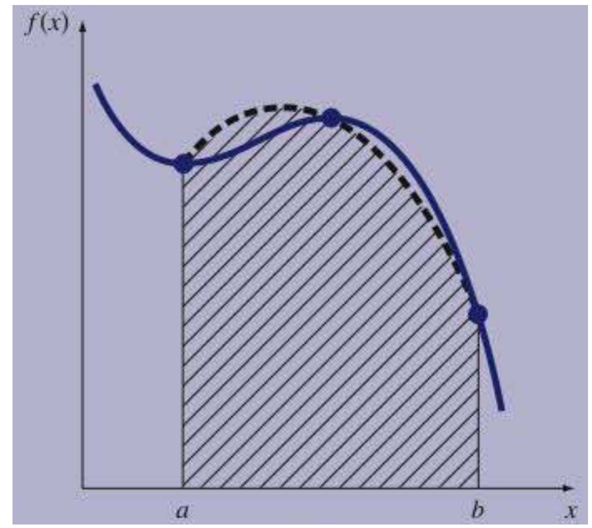



Trapezoidal Rule

1st-order polynomial




Multisegment TR




Simpson's Rules

Higher-order poly.

# Trapezoidal Rule

- Newton-Cotes formula with 1st-order (linear) interpolating polynomial, e.g.

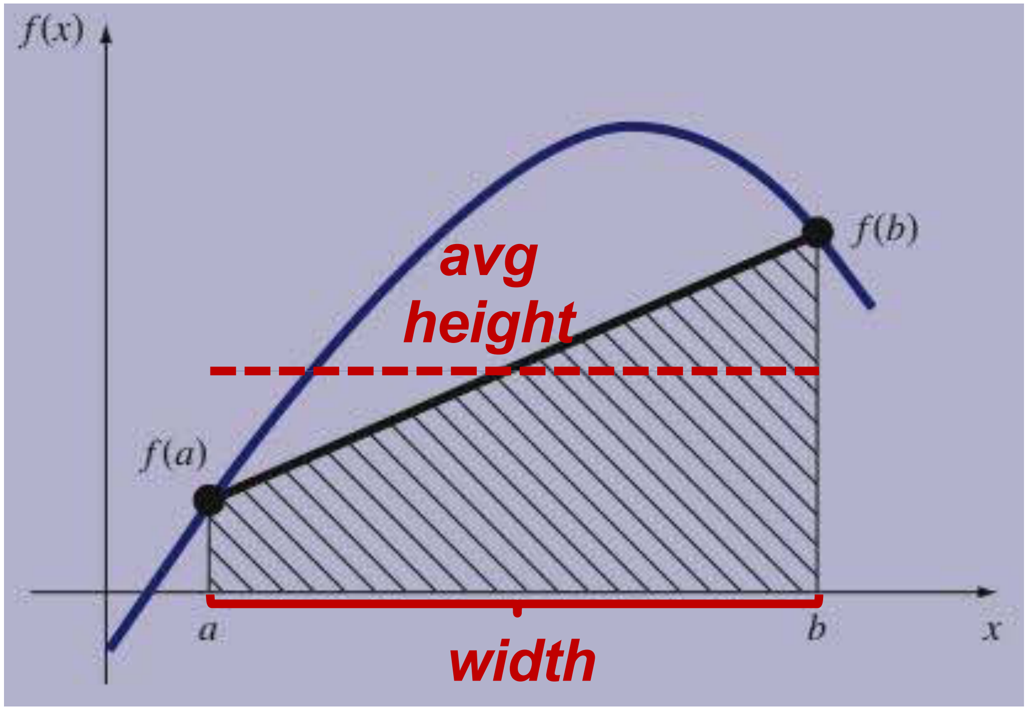
$$f_1(x) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x - x_0)$$

- Trapezoidal rule:

$$I \cong \int_a^b f_1(x)dx$$

- Substituting:

$$\boxed{I \cong \underbrace{(b-a)}_{\text{width}}\underbrace{\left(\frac{f(a)+f(b)}{2}\right)}_{\text{avg height}}}$$



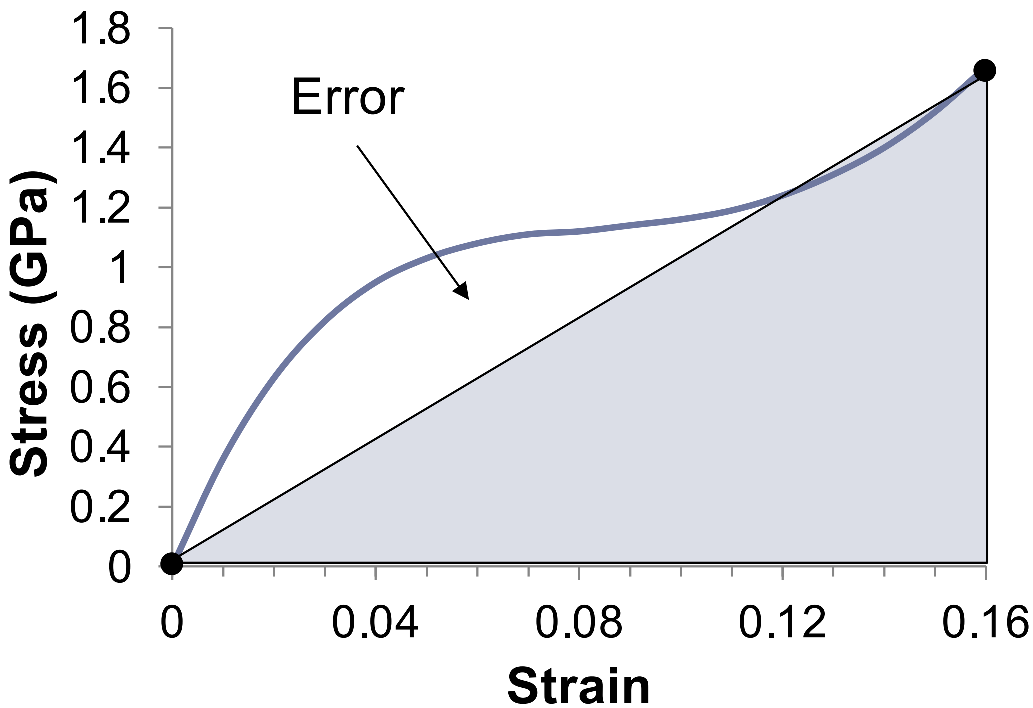
# Newton-Cotes Integration

- All Newton-Cotes formulae can be expressed in this general format:

$$\boxed{I \cong (b - a) \times (average\ height)}$$

- We will later determine formulae for the average height of other interpolating polynomials

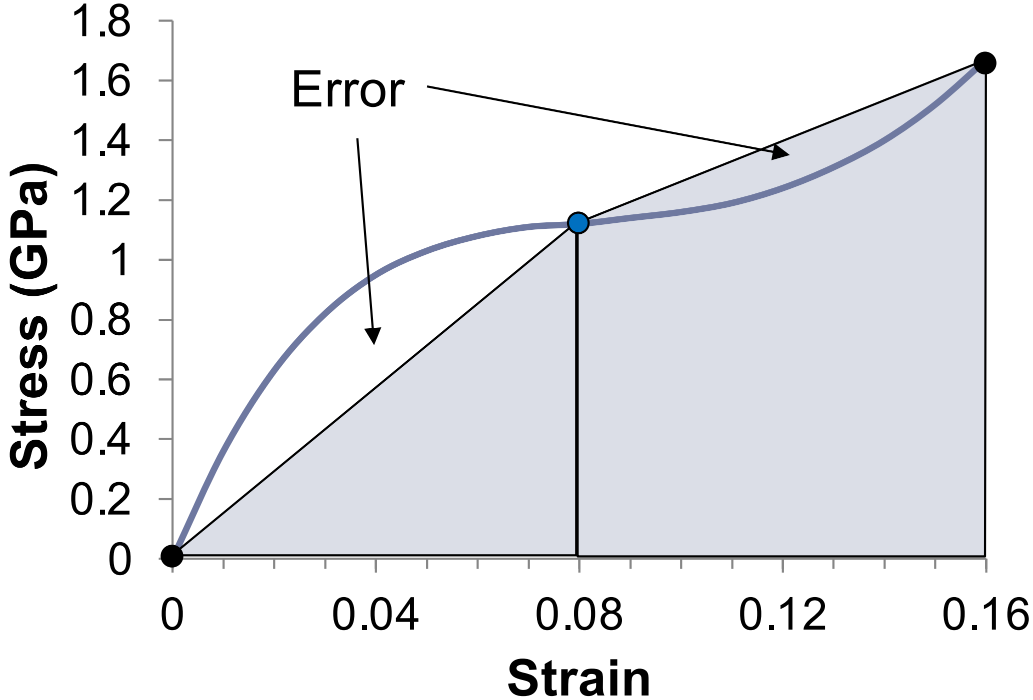
# Trapezoidal Rule: **Example – 1 Segment**




| ***Strain (--)*** | ***Stress (GPa)*** |
|---|---|
| 0 | 0 |
| 0.16 | 1.67 |

- $I \cong (0.16 - 0)\left(\frac{0+1.67}{2}\right)$
  $= 0.134$ GJ/m³ ($\varepsilon_t = 21\%$)

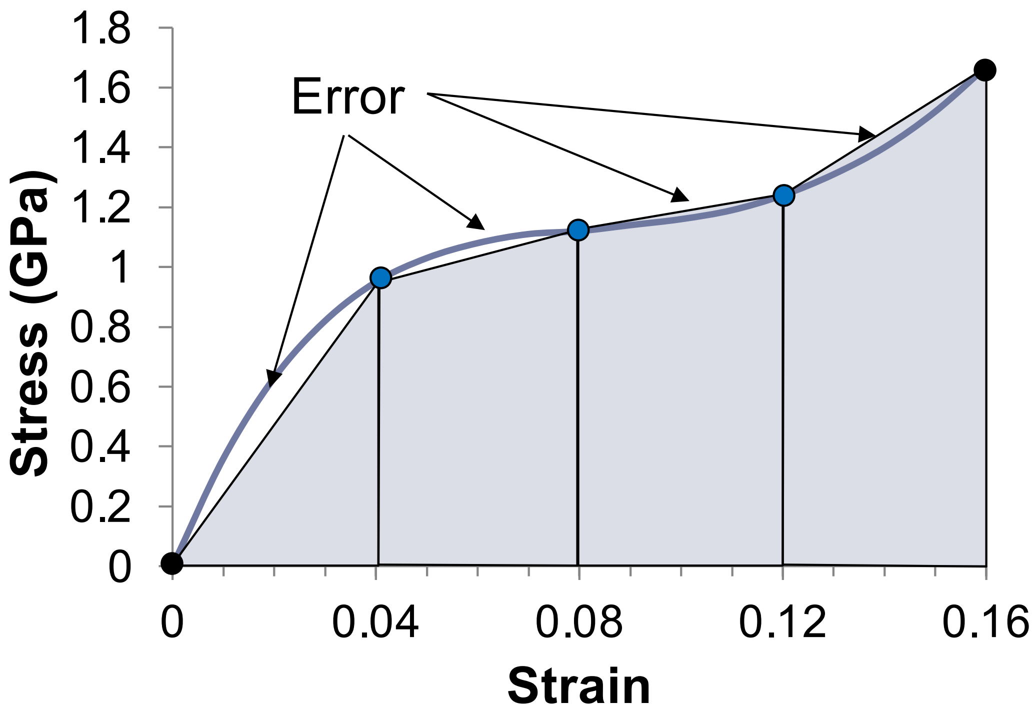
# Trapezoidal Rule: **Example – 2 Segments**




| Strain (--) | Stress (GPa) |
|---|---|
| 0 | 0 |
| 0.08 | 1.12 |
| 0.16 | 1.67 |

- $I \cong (0.08-0)\left(\frac{0+1.12}{2}\right) + (0.16-0.08)\left(\frac{1.12+1.67}{2}\right)$
  $= 0.156$ GJ/m$^3$ ($\varepsilon_t = 8\%$)

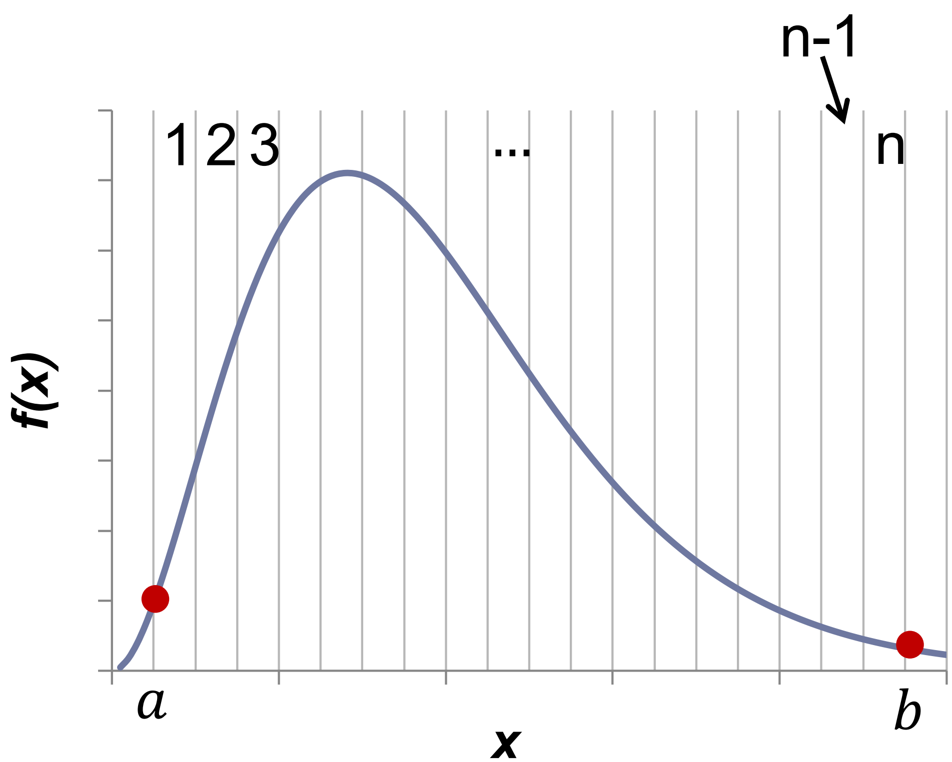
# Trapezoidal Rule: **Example – 4 Segments**




| *Strain (--)* | *Stress (GPa)* |
|---|---|
| 0 | 0 |
| 0.04 | 0.95 |
| 0.08 | 1.12 |
| 0.12 | 1.24 |
| 0.16 | 1.67 |

- $I \cong (0.04)\left(\frac{0+0.95}{2}\right) + (0.04)\left(\frac{0.95+1.12}{2}\right) + (0.04)\left(\frac{1.12+1.24}{2}\right) + (0.04)\left(\frac{1.24+1.67}{2}\right) = 0.166\ \text{GJ/m}^3 \ \ (\varepsilon_t = 2\%)$

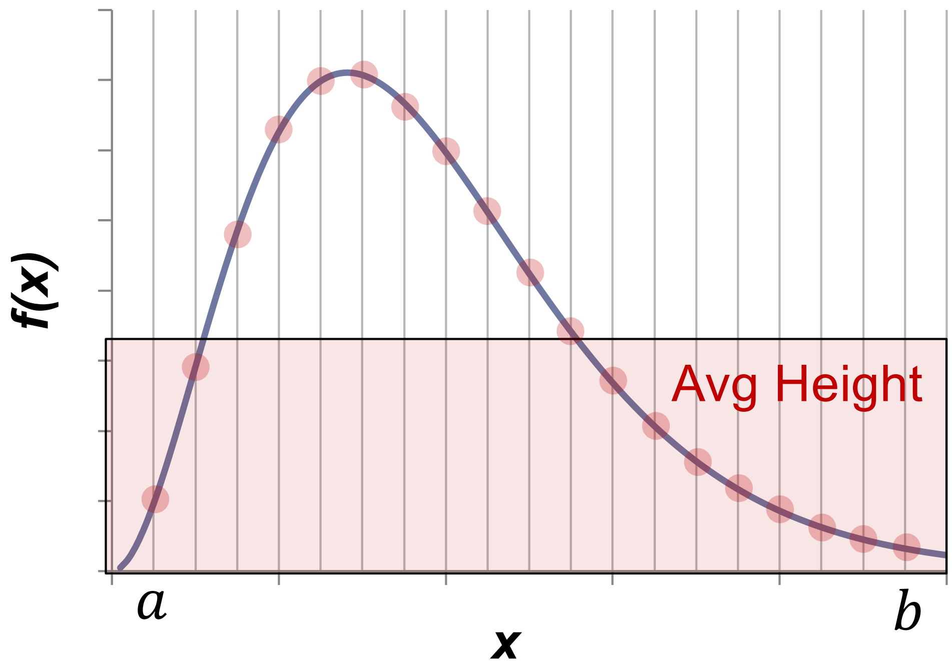
# Trapezoidal Rule: **Multi-segment**




$$e.g.\ f(x) = \frac{x^3}{e^x - 1}$$

- Break the interval $(a, b)$ into $n$ equally spaced segments:
  - $I = \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{b} f(x)dx$

# Trapezoidal Rule: **Multi-segment**




Avg Height:

$$\frac{\begin{array}{l} \text{Segment 1: } \frac{f(x_0)+f(x_1)}{2} \\ + \\ \text{Segment 2: } \frac{f(x_1)+f(x_2)}{2} \\ + \\ \text{Segment 3: } \frac{f(x_2)+f(x_3)}{2} \\ \vdots \\ \text{Segment n-1: } \frac{f(x_{n-2})+f(x_{n-1})}{2} \\ + \\ \text{Segment n: } \frac{f(x_{n-1})+f(x_n)}{2} \end{array}}{n}$$

General Form:

$$I = (b-a)\left(\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}\right)$$

# Trapezoidal Rule: **Multi-segment**

$$I = \underbrace{(b-a)}_{\text{width}} \underbrace{\left(\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}\right)}_{\text{overall average height}}$$

$$I = \left(\frac{b-a}{n}\right)\left(\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2}\right)$$

$$= \boxed{\underbrace{(h)}_{\text{point spacing}} \times \underbrace{\left(\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2}\right)}_{\text{weighted sum of fcn values}}}$$

# Trapezoidal Rule: **Example – Multi-segment**

| Strain (--) | Stress (GPa) |
|---|---|
| 0 | 0 |
| 0.01 | 0.360916 |
| 0.02 | 0.62823 |
| 0.03 | 0.818887 |
| 0.04 | 0.948535 |
| 0.05 | 1.031519 |
| 0.06 | 1.080888 |
| 0.07 | 1.10839 |
| 0.08 | 1.124476 |
| 0.09 | 1.138295 |
| 0.1 | 1.1577 |
| 0.11 | 1.189243 |
| 0.12 | 1.238176 |
| 0.13 | 1.308455 |
| 0.14 | 1.402734 |
| 0.15 | 1.522369 |
| 0.16 | 1.667416 |

$$I = (h) \times \left( \frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2} \right)$$

| n | h | Integral | eps_t | error reduction |
|---|---|---|---|---|
| 1 | 0.16 | 0.133393 | 21.13% | |
| 2 | 0.08 | 0.156655 | 7.38% | 2.9 |
| 4 | 0.04 | 0.165796 | 1.98% | 3.7 |
| 8 | 0.02 | 0.168289 | 0.50% | 3.9 |
| 16 | 0.01 | 0.168925 | 0.13% | 4.0 |

**MIE334_Lecture_25_ExTR.xlsx**

$$\sigma(\varepsilon) = -5413\varepsilon^4 + 3149\varepsilon^3 - 558.69\varepsilon^2 + 41.369\varepsilon$$

# Trapezoidal Rule: **Pseudocode**

$$I = (h) \times \left( \frac{f(x_0) + f(x_1)}{2} \right)$$

$$I = (h) \times \left( \frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2} \right)$$

```
FUNCTION Trap (h, f0, f1)
  Trap = h * (f0 + f1)/2
END Trap
```

```
FUNCTION Trapm (h, n, f)
  sum = f0
  DOFOR i = 1, n - 1
    sum = sum + 2 * fi
  END DO
  sum = sum + fn
  Trapm = h * sum / 2
END Trapm
```

# Trapezoidal Rule: **Error Analysis**

- Recall for single segment TR:

$$I \cong \int_a^b f_1(x)dx$$

- where:

$$f_1(x) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x - x_0)$$

- Recall truncation error for $n^{th}$- order Newton poly.

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1) \ldots (x - x_n)$$

- For $1^{st}$-order, truncation error simplifies to:

$$R_1 = \frac{f''(\xi)}{2}(x - x_0)(x - x_1)$$

# Trapezoidal Rule: **Error Analysis**

- Integral including error term:

$$I = \int_a^b [f_1(x) + R_1(x)]dx = \underbrace{\int_a^b f_1(x)dx}_{\text{TR}} + \underbrace{\int_a^b R_1(x)dx}_{\text{truncation error}}$$

- Error term simplifies to:

$$E = \int_a^b \frac{f''(\xi)}{2}(x - \overset{a}{x_0})(x - \overset{b}{x_1})dx = \boxed{-\frac{f''(\xi)}{12}(b-a)^3}$$

- For multisegment TR, remember:

$$h = \frac{b-a}{n}$$

- So error for **any one segment**, $i$:

$$E_i = -\frac{f''(\xi_i)}{12}(h)^3 = -\frac{f''(\xi_i)}{12}\left(\frac{b-a}{n}\right)^3$$

# Trapezoidal Rule: **Error Analysis**

- Now sum up errors for all segments:

$$E = \sum_{i=1}^{n} -\frac{f''(\xi_i)}{12}\left(\frac{b-a}{n}\right)^3$$

$$= -\frac{1}{12}\left(\frac{b-a}{n}\right)^3 \sum_{i=1}^{n} f''(\xi_i)$$

$$= -\frac{1}{12}\frac{(b-a)^3}{n^2}\frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

$$\boxed{= -\frac{1}{12}\frac{(b-a)^3}{n^2}\overline{f''}}$$

- So if $n$ is doubled, error goes **down** by ~ 4x
  - Just as we saw!

# Trapezoidal Rule: **Summary**

- **Advantages:**
  - Simple to apply and code
  - Multisegment TR can work over large intervals
  - Works best with flat, slowly changing functions/data

- **Disadvantages:**
  - Large $n$ can require many function evaluations/data points
  - Slow-ish (quadratic) reduction in error with more segments
  - With too many segments, round-off errors can accumulate