MIE334 – Numerical Methods I

Lecture 14: Gauss-Seidel Method

Iterative Solutions: **Elimination Methods**

- The methods we have learned so far:
  - *Gauss Elimination*
  - *LU decomposition*

Iterative Solutions: **Elimination Methods**

- The methods we have learned so far:
  - *Gauss Elimination*
  - *LU decomposition*

- <u>Single-run</u> or <u>direct</u> methods for finding a solution:
  - ✓ Straightforward to use
  - ✓ Easy to code

Iterative Solutions: **Elimination Methods**

- The methods we have learned so far:
  - *Gauss Elimination*
  - *LU decomposition*

- Single-run or direct methods for finding a solution:
  - ✓ Straightforward to use
  - ✓ Easy to code
  - X Can introduce large round-off errors
  - X Not efficient at handling zero's in $[A]$ matrix

Iterative Solutions: **Elimination Methods**

- The methods we have learned so far:
  - *Gauss Elimination*
  - *LU decomposition*

- Single-run or direct methods for finding a solution:
  - ✓ Straightforward to use
  - ✓ Easy to code
  - X Can introduce large round-off errors
  - X Not efficient at handling zero's in $[A]$ matrix
  - X Matrices too big to invert in RAM

Iterative Solutions: **Elimination Methods**

- The methods we have learned so far:
  - *Gauss Elimination*
  - *LU decomposition*

- Single-run or direct methods for finding a solution:
  - ✓ Straightforward to use
  - ✓ Easy to code
  - X Can introduce large round-off errors
  - X Not efficient at handling zero's in $[A]$ matrix
  - X Matrices too big to invert in RAM

    Modest CFD problem of 100,000 nodes x 4 vars per node

    = 400,000 x 400,000 matrix x 32-bit float (4 bytes)

Iterative Solutions: **Elimination Methods**

- The methods we have learned so far:
  - *Gauss Elimination*
  - *LU decomposition*

- Single-run or direct methods for finding a solution:
  - ✓ Straightforward to use
  - ✓ Easy to code
  - X Can introduce large round-off errors
  - X Not efficient at handling zero's in $[A]$ matrix
  - X Matrices too big to invert in RAM

    Modest CFD problem of 100,000 nodes x 4 vars per node

    = 400,000 x 400,000 matrix x 32-bit float (4 bytes)

    *= 640,000,000,000 bytes = **640 Gb**! (Later: sparse matrices)*

Iterative Solutions: **Systems of Linear Equations**

- *Iterative Methods* can be applied which are similar in approach to the <u>Root finding methods</u>:

e.g. Fixed-point iteration, Bisection, Newton-Raphson

Iterative Solutions: **Systems of Linear Equations**

- *Iterative Methods* can be applied which are similar in approach to the Root finding methods:

e.g. Fixed-point iteration, Bisection, Newton-Raphson

Iterative Approach:

I) Initial guess of the solution

II) Calculate a new estimate of the solution

III) Repeat until stopping condition

Iterative Solutions: **Systems of Linear Equations**

- *Iterative Methods* can be applied which are similar in approach to the <u>Root finding methods</u>:

e.g. Fixed-point iteration, Bisection, Newton-Raphson

<u>Iterative Approach</u>:

I) Initial guess of the solution

II) Calculate a new estimate of the solution

III) Repeat until stopping condition

*** error is controlled by # of iterations ***

Iterative Methods: **Gauss-Seidel Method**

- Consider a system of 3 equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

Iterative Methods: **Gauss-Seidel Method**

- Consider a system of 3 equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

- If a_{kk} elements are non-zero:
  - $x_1 = \dfrac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$
  - $x_2 = \dfrac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$
  - $x_3 = \dfrac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$

Iterative Methods: **Gauss-Seidel Method**

- Consider a system of 3 equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

- If a_{kk} elements are non-zero:
  - $x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$ $\qquad x_1 = u(x_2, x_3)$
  - $x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$ $\qquad x_2 = v(x_1, x_3)$
  - $x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$ $\qquad x_3 = w(x_1, x_2)$

Iterative Methods: **Gauss-Seidel Method**

- Consider a system of 3 equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

- If a_{kk} elements are non-zero:
  - $x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$ $\qquad x_1 = u(x_2, x_3)$
  - $x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$ $\qquad x_2 = v(x_1, x_3)$
  - $x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$ $\qquad x_3 = w(x_1, x_2)$

- **What does this remind you of?**

Iterative Methods: **Gauss-Seidel Method**

- Consider a system of 3 equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

- If a_{kk} elements are non-zero:
  - $x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$ $\qquad x_1 = u(x_2, x_3)$
  - $x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$ $\qquad x_2 = v(x_1, x_3)$
  - $x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$ $\qquad x_3 = w(x_1, x_2)$

- **What does this remind you of?**
  - Guess x_2, x_3, and use to update guess for x_1 …

Iterative Methods: **Gauss-Seidel Method**

- Gauss-Seidel Method:

I) Start with an initial guess of $\{X\}$ (typically $x_n = 0$)

Iterative Methods: **Gauss-Seidel Method**

- Gauss-Seidel Method:

I) Start with an initial guess of $\{X\}$ (typically $x_n = 0$)

II) Calculate new guess of x_1 using Eqn 1

Iterative Methods: **Gauss-Seidel Method**

- Gauss-Seidel Method:

I) Start with an initial guess of $\{X\}$ (typically $x_n = 0$)

II) Calculate new guess of x_1 using Eqn 1

III) Use this guess, plus previous guesses for $x_3 \cdots x_n$ to calculate a new guess for x_2 using Eqn 2

Iterative Methods: **Gauss-Seidel Method**

- Gauss-Seidel Method:

I) Start with an initial guess of $\{X\}$ (typically $x_n = 0$)

II) Calculate new guess of x_1 using Eqn 1

III) Use this guess, plus previous guesses for $x_3 \cdots x_n$ to calculate a new guess for x_2 using Eqn 2

IV) Continue until x_n, then loop back and start at x_1 again

Iterative Methods: **Gauss-Seidel Method**

- Gauss-Seidel Method:

I) Start with an initial guess of $\{X\}$ (typically $x_n = 0$)

II) Calculate new guess of x_1 using Eqn 1

III) Use this guess, plus previous guesses for $x_3 \cdots x_n$ to calculate a new guess for x_2 using Eqn 2

IV) Continue until x_n, then loop back and start at x_1 again

V) Repeat until we satisfy stopping condition

Gauss-Seidel: **Graphically**

Initialize: $x_{1..3}^{(0)} = 0$

Done!

Gauss-Seidel: **Graphically**

Initialize: $x_{1..3}^{(0)} = 0$

1st iteration (i=1)

$$x_1^{(1)} = \frac{b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}}{a_{11}}$$

Gauss-Seidel: **Graphically**

Initialize: $x_{1..3}^{(0)} = 0$

1st iteration (i=1)

$$x_1^{(1)} = \frac{b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}}{a_{11}}$$

$$x_2^{(1)} = \frac{b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}}{a_{22}}$$

Gauss-Seidel: **Graphically**

Initialize: $x_{1..3}^{(0)} = 0$

1st iteration (i=1)

$$x_1^{(1)} = \frac{b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}}{a_{11}}$$

$$x_2^{(1)} = \frac{b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}}{a_{22}}$$

$$x_3^{(1)} = \frac{b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}}{a_{33}}$$

Gauss-Seidel: **Graphically**

Initialize: $x_{1..3}^{(0)} = 0$

1st iteration (i=1)

$$x_1^{(1)} = \frac{b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}}{a_{11}}$$

$$x_2^{(1)} = \frac{b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}}{a_{22}}$$

$$x_3^{(1)} = \frac{b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}}{a_{33}}$$

Converged?

yes

Done!

Gauss-Seidel: **Graphically**

Initialize: $x_{1..3}^{(0)} = 0$

1st iteration (i=1)

$$x_1^{(1)} = \frac{b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}}{a_{11}}$$

$$x_2^{(1)} = \frac{b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}}{a_{22}}$$

$$x_3^{(1)} = \frac{b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}}{a_{33}}$$

Converged?

no

yes

Done!

2nd iteration (i=2)

$$x_1^{(2)} = \frac{b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)}}{a_{11}}$$

Gauss-Seidel: **Graphically**

Initialize: $x_{1..3}^{(0)} = 0$

1st iteration (i=1)

$$x_1^{(1)} = \frac{b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}}{a_{11}}$$

$$x_2^{(1)} = \frac{b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}}{a_{22}}$$

$$x_3^{(1)} = \frac{b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}}{a_{33}}$$

Converged?

no

yes

Done!

2nd iteration (i=2)

$$x_1^{(2)} = \frac{b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)}}{a_{11}}$$

$$x_2^{(2)} = \frac{b_2 - a_{21}x_1^{(2)} - a_{23}x_3^{(1)}}{a_{22}}$$

Gauss-Seidel: **Graphically**

Initialize: $x_{1..3}^{(0)} = 0$

1st iteration (i=1)

$$x_1^{(1)} = \frac{b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}}{a_{11}}$$

$$x_2^{(1)} = \frac{b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}}{a_{22}}$$

$$x_3^{(1)} = \frac{b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}}{a_{33}}$$

Converged?

no

yes

Done!

2nd iteration (i=2)

$$x_1^{(2)} = \frac{b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)}}{a_{11}}$$

$$x_2^{(2)} = \frac{b_2 - a_{21}x_1^{(2)} - a_{23}x_3^{(1)}}{a_{22}}$$

$$x_3^{(2)} = \frac{b_3 - a_{31}x_1^{(2)} - a_{32}x_2^{(2)}}{a_{33}}$$

Gauss-Seidel: **Graphically**

Initialize: $x_{1..3}^{(0)} = 0$

1st iteration (i=1)

$$x_1^{(1)} = \frac{b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}}{a_{11}}$$

$$x_2^{(1)} = \frac{b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}}{a_{22}}$$

$$x_3^{(1)} = \frac{b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}}{a_{33}}$$

Converged?

no

yes

2nd iteration (i=2)

$$x_1^{(2)} = \frac{b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)}}{a_{11}}$$

$$x_2^{(2)} = \frac{b_2 - a_{21}x_1^{(2)} - a_{23}x_3^{(1)}}{a_{22}}$$

$$x_3^{(2)} = \frac{b_3 - a_{31}x_1^{(2)} - a_{32}x_2^{(2)}}{a_{33}}$$

Converged?

no

yes

Done!

Gauss-Seidel: **Convergence Criteria**

- Check the approximate error after each loop:

$$|\epsilon_{a,i}| = \left|\frac{x_i^j - x_i^{j-1}}{x_i^j}\right| \times 100\%$$

x_i^j current estimate of x_i

x_i^{j-1} previous estimate of x_i

Gauss-Seidel: **Convergence Criteria**

- Check the <u>approximate error</u> after each loop:

$$|\epsilon_{a,i}| = \left|\frac{x_i^j - x_i^{j-1}}{x_i^j}\right| \times 100\%$$

x_i^j <u>current</u> estimate of x_i

x_i^{j-1} <u>previous</u> estimate of x_i

- Can check this for <u>all</u> variables $(x_1, x_2, \dots, x_n)$

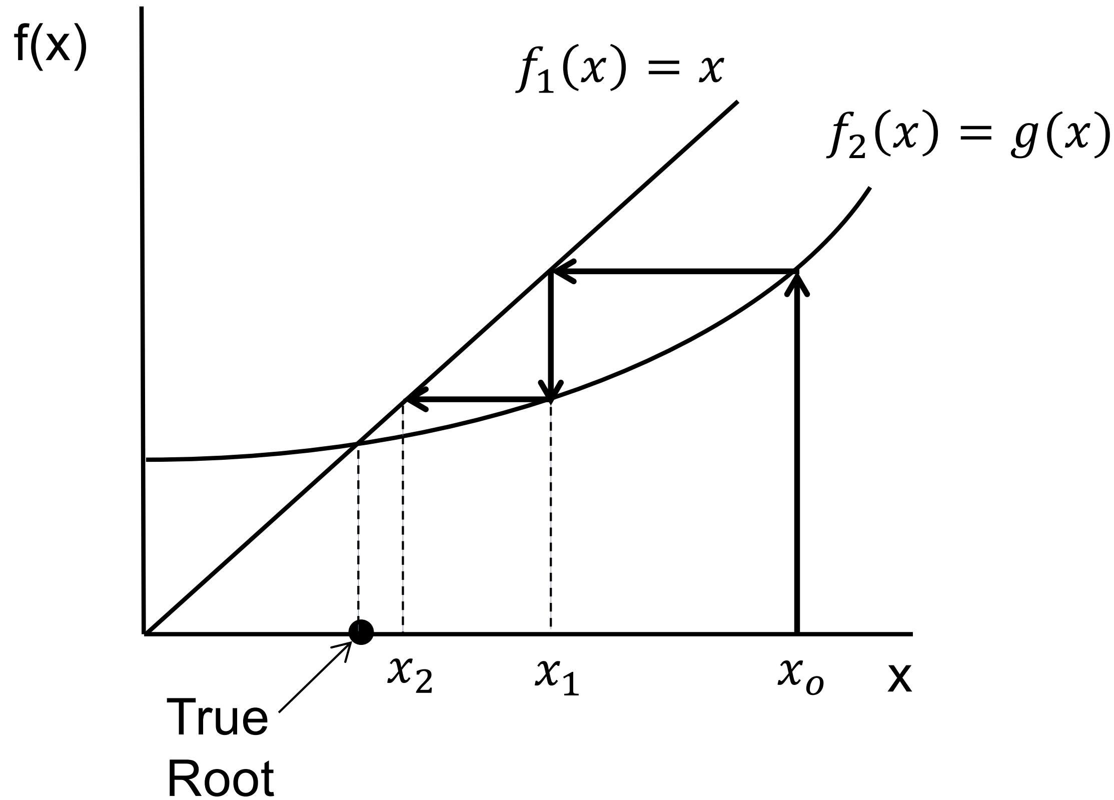
Gauss-Seidel: **Convergence Issues**

- G-S method shares the same issues as FPI:
  - Does not always converge
  - When it does, it can be slowly converging

Gauss-Seidel: **Convergence Criteria**

- Recall: fixed-point iteration

$$\underbrace{x}_{f_1(x)} = \underbrace{g(x)}_{f_2(x)}$$



<u>Convergence criteria</u>:

$$|g'(x)| < 1$$

Gauss-Seidel: **Convergence Criteria**

- For systems of non-linear multivariable equations $x = u(x, y),\ \ y = v(x, y)$ it can be shown that:

General convergence criteria:

$$\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right| < 1 \qquad \left|\frac{\partial v}{\partial x}\right| + \left|\frac{\partial v}{\partial y}\right| < 1$$

Gauss-Seidel: **Convergence Criteria**

Now consider G-S method for <u>two linear equations</u>:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}$$

$$x_1 = \frac{b_1 - a_{12}x_2}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}$$

Gauss-Seidel: **Convergence Criteria**

Now consider G-S method for two linear equations:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}$$

$$x_1 = \frac{b_1 - a_{12}x_2}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}$$

Can be expressed as:

$$u(x_1, x_2) = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 \qquad v(x_1, x_2) = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1$$

Gauss-Seidel: **Convergence Criteria**

Remember convergence criteria for $x = u(x, y), y = v(x, y)$:

$$\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right| < 1 \qquad \left|\frac{\partial v}{\partial x}\right| + \left|\frac{\partial v}{\partial y}\right| < 1$$

$$u(x_1, x_2) = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 \qquad v(x_1, x_2) = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1$$

Solve partial derivatives:

$$\frac{\partial u}{\partial x_1} = 0 \qquad \frac{\partial u}{\partial x_2} = -\frac{a_{12}}{a_{11}} \qquad \frac{\partial v}{\partial x_1} = -\frac{a_{21}}{a_{22}} \qquad \frac{\partial v}{\partial x_2} = 0$$

Substitute:

$0 + \left|\frac{a_{12}}{a_{11}}\right| < 1$ or $|a_{11}| > |a_{12}|$ $\qquad$ $\left|\frac{a_{21}}{a_{22}}\right| + 0 < 1$ or $|a_{22}| > |a_{21}|$

Gauss-Seidel: **Convergence Criteria**

In general: For a system of n equations

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

e.g. 3x3

$$\begin{bmatrix} \boldsymbol{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Gauss-Seidel: **Convergence Criteria**

<u>In general:</u> For a system of n equations

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

e.g. 3x3

$$\begin{bmatrix} \boldsymbol{a_{11}} & a_{12} & a_{13} \\ a_{21} & \boldsymbol{a_{22}} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Gauss-Seidel: **Convergence Criteria**

In general: For a system of n equations

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

e.g. 3x3

$$\begin{bmatrix} \boldsymbol{a_{11}} & a_{12} & a_{13} \\ a_{21} & \boldsymbol{a_{22}} & a_{23} \\ a_{31} & a_{32} & \boldsymbol{a_{33}} \end{bmatrix}$$

Gauss-Seidel: **Convergence Criteria**

<u>In general:</u> For a system of n equations

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

e.g. 3x3

$$\begin{bmatrix} \boldsymbol{a_{11}} & a_{12} & a_{13} \\ a_{21} & \boldsymbol{a_{22}} & a_{23} \\ a_{31} & a_{32} & \boldsymbol{a_{33}} \end{bmatrix}$$

- The <u>diagonal coefficients</u> in each equation must be larger than the sum of the absolute values of the other coefficients

Gauss-Seidel: **Convergence Criteria**

In general: For a system of n equations

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

e.g. 3x3

$$\begin{bmatrix} \boldsymbol{a_{11}} & a_{12} & a_{13} \\ a_{21} & \boldsymbol{a_{22}} & a_{23} \\ a_{31} & a_{32} & \boldsymbol{a_{33}} \end{bmatrix}$$

- The diagonal coefficients in each equation must be larger than the sum of the absolute values of the other coefficients
- Such as systems is called *diagonally dominant*

Gauss-Seidel: **Convergence Criteria**

<u>In general:</u> For a system of n equations

e.g. 3x3

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

$$\begin{bmatrix} \boldsymbol{a_{11}} & a_{12} & a_{13} \\ a_{21} & \boldsymbol{a_{22}} & a_{23} \\ a_{31} & a_{32} & \boldsymbol{a_{33}} \end{bmatrix}$$

- The <u>diagonal coefficients</u> in each equation must be larger than the sum of the absolute values of the other coefficients
- Such as systems is called <u>*diagonally dominant*</u>
- Condition is *sufficient* but *not necessary*

Gauss-Seidel: **Example**

- Solve the following systems of equations using the Gauss-Seidel Method:

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Gauss-Seidel: **Example**

- Solve the following systems of equations using the Gauss-Seidel Method:

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

<u>True solution</u>:

$$x_1 = 3$$
$$x_2 = -2.5$$
$$x_3 = 7$$

Gauss-Seidel: **Example**

- Solve the following systems of equations using the Gauss-Seidel Method:

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

True solution:

$$x_1 = 3$$
$$x_2 = -2.5$$
$$x_3 = 7 \;\; (7.002)$$

GE carrying 4 significant figures

Gauss-Seidel: **Example – Check Convergence**

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

- Check convergence criterion:
  - $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

Gauss-Seidel: **Example – Check Convergence**

$$\begin{bmatrix} \boxed{3 \quad -0.1 \quad -0.2} \\ 0.1 \quad 7 \quad -0.3 \\ 0.3 \quad -0.2 \quad 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

- Check convergence criterion:
  - $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

Row 1:

- $|a_{11}| = 3$

Gauss-Seidel: **Example – Check Convergence**

$$\begin{bmatrix} \boxed{3 \quad -0.1 \quad -0.2} \\ 0.1 \quad 7 \quad -0.3 \\ 0.3 \quad -0.2 \quad 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

- Check convergence criterion:
  - $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

<u>Row 1:</u>

- $|a_{11}| = 3$
- $|a_{12}| + |a_{13}| = 0.3$

Gauss-Seidel: **Example – Check Convergence**

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ \boxed{0.1 \quad 7 \quad -0.3} \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

- Check convergence criterion:
  - $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

Row 1:
- $|a_{11}| = 3$
- $|a_{12}| + |a_{13}| = 0.3$

Row 2:
- $|a_{22}| = 7$
- $|a_{21}| + |a_{23}| = 0.4$

Gauss-Seidel: **Example – Check Convergence**

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ \boxed{0.3 \quad -0.2 \quad 10} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

- Check convergence criterion:
  - $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

<u>Row 1:</u>
- $|a_{11}| = 3$
- $|a_{12}| + |a_{13}| = 0.3$

<u>Row 2:</u>
- $|a_{22}| = 7$
- $|a_{21}| + |a_{23}| = 0.4$

<u>Row 3:</u>
- $|a_{33}| = 10$
- $|a_{31}| + |a_{32}| = 0.5$

Gauss-Seidel: **Example – Check Convergence**

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

- Check convergence criterion:
  - $|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$

Row 1:
- $|a_{11}| = 3$
- $|a_{12}| + |a_{13}| = 0.3$

Row 2:
- $|a_{22}| = 7$
- $|a_{21}| + |a_{23}| = 0.4$

Row 3:
- $|a_{33}| = 10$
- $|a_{31}| + |a_{32}| = 0.5$

[A] is diagonally dominant!

Gauss-Seidel: **Example – Solve**

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3}$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

Gauss-Seidel: **Example – Solve**

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3}$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

- Initial guess: $x_i^{(0)} = 0$

Gauss-Seidel: **Example – Solve**

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3}$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

- Initial guess: $x_i^{(0)} = 0$

$$x_1^{(1)} = \frac{7.85 + 0.1\boxed{0} + 0.2\boxed{0}}{3} = 2.617$$

$$x_2^{(1)} = \frac{-19.3 - 0.1\boxed{2.617} + 0.3\boxed{0}}{7} = -2.795$$

$$x_3^{(1)} = \frac{71.4 - 0.3\boxed{2.617} + 0.2\boxed{-2.795}}{10} = 7.006$$

Gauss-Seidel: **Example – Solve**

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3}$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

- Initial guess: $x_i^{(0)} = 0$

$$x_1^{(1)} = \frac{7.85 + 0.1\boxed{0} + 0.2\boxed{0}}{3} = 2.617$$

$$x_2^{(1)} = \frac{-19.3 - 0.1\boxed{2.617} + 0.3\boxed{0}}{7} = -2.795$$

$$x_3^{(1)} = \frac{71.4 - 0.3\boxed{2.617} + 0.2\boxed{-2.795}}{10} = 7.006$$

$$x_1^{(2)} = \frac{7.85 + 0.1\boxed{-2.795} + 0.2\boxed{7.006}}{3} = 2.991$$

$$x_2^{(2)} = \frac{-19.3 - 0.1\boxed{2.991} + 0.3\boxed{7.006}}{7} = -2.500$$

$$x_3^{(2)} = \frac{71.4 - 0.3\boxed{2.991} + 0.2\boxed{-2.500}}{10} = 7.000$$

Gauss-Seidel: **Example – Solve**

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3}$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

- Initial guess: $x_i^{(0)} = 0$

$$x_1^{(1)} = \frac{7.85 + 0.1\boxed{0} + 0.2\boxed{0}}{3} = 2.617$$

$$x_2^{(1)} = \frac{-19.3 - 0.1\boxed{2.617} + 0.3\boxed{0}}{7} = -2.795$$

$$x_3^{(1)} = \frac{71.4 - 0.3\boxed{2.617} + 0.2\boxed{-2.795}}{10} = 7.006$$

$$x_1^{(2)} = \frac{7.85 + 0.1\boxed{-2.795} + 0.2\boxed{7.006}}{3} = 2.991$$

$$x_2^{(2)} = \frac{-19.3 - 0.1\boxed{2.991} + 0.3\boxed{7.006}}{7} = -2.500$$

$$x_3^{(2)} = \frac{71.4 - 0.3\boxed{2.991} + 0.2\boxed{-2.500}}{10} = 7.000$$

… MIE334_Lecture_14_ExGS.xlsx

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 0.1 & 7 & -0.3 \\ 3 & -0.1 & -0.2 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -19.3 \\ 7.85 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 0.1 & 7 & -0.3 \\ 3 & -0.1 & -0.2 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -19.3 \\ 7.85 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

- Check convergence criterion:
  - $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} \boxed{0.1 \quad 7 \quad -0.3} \\ 3 \quad -0.1 \quad -0.2 \\ 0.3 \quad -0.2 \quad 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -19.3 \\ 7.85 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

- Check convergence criterion:
  - $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

Row 1:
- $|a_{11}| = 0.1$
- $|a_{12}| + |a_{13}| = 7.3$

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 0.1 & 7 & -0.3 \\ \boxed{3 \quad -0.1 \quad -0.2} \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -19.3 \\ 7.85 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

- Check convergence criterion:
  - $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

Row 1:
- $|a_{11}| = 0.1$
- $|a_{12}| + |a_{13}| = 7.3$

Row 2:
- $|a_{22}| = 0.1$
- $|a_{21}| + |a_{23}| = 3.2$

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 0.1 & 7 & -0.3 \\ 3 & -0.1 & -0.2 \\ \boxed{0.3 \quad -0.2 \quad 10} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -19.3 \\ 7.85 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

- Check convergence criterion:
  - $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

Row 1:
- $|a_{11}| = 0.1$
- $|a_{12}| + |a_{13}| = 7.3$

Row 2:
- $|a_{22}| = 0.1$
- $|a_{21}| + |a_{23}| = 3.2$

Row 3:
- $|a_{33}| = 10$
- $|a_{31}| + |a_{32}| = 0.5$

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 0.1 & 7 & -0.3 \\ 3 & -0.1 & -0.2 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -19.3 \\ 7.85 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

- Check convergence criterion:
  - $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

Row 1:
- $|a_{11}| = 0.1$
- $|a_{12}| + |a_{13}| = 7.3$

Row 2:
- $|a_{22}| = 0.1$
- $|a_{21}| + |a_{23}| = 3.2$

Row 3:
- $|a_{33}| = 10$
- $|a_{31}| + |a_{32}| = 0.5$

No longer diagonally dominant!

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 0.1 & 7 & -0.3 \\ 3 & -0.1 & -0.2 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -19.3 \\ 7.85 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

$$x_1 = \frac{-19.3 - 7x_2 + 0.3x_3}{0.1}$$

$$x_2 = \frac{7.85 - 3x_1 + 0.2x_3}{-0.1}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 0.1 & 7 & -0.3 \\ 3 & -0.1 & -0.2 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -19.3 \\ 7.85 \\ 71.4 \end{Bmatrix}$$

$$x_1 = \frac{-19.3 - 7x_2 + 0.3x_3}{0.1}$$

$$x_2 = \frac{7.85 - 3x_1 + 0.2x_3}{-0.1}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

- Initial guess: $x_i^{(0)} = 0$

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 0.1 & 7 & -0.3 \\ 3 & -0.1 & -0.2 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -19.3 \\ 7.85 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

$$x_1 = \frac{-19.3 - 7x_2 + 0.3x_3}{0.1}$$

$$x_2 = \frac{7.85 - 3x_1 + 0.2x_3}{-0.1}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

- Initial guess: $x_i^{(0)} = 0$

$$x_1^{(1)} = \frac{-19.3 - 7\boxed{0} + 0.3\boxed{0}}{0.1} = -193$$

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 0.1 & 7 & -0.3 \\ 3 & -0.1 & -0.2 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -19.3 \\ 7.85 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

$$x_1 = \frac{-19.3 - 7x_2 + 0.3x_3}{0.1}$$

$$x_2 = \frac{7.85 - 3x_1 + 0.2x_3}{-0.1}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

- Initial guess: $x_i^{(0)} = 0$

$$x_1^{(1)} = \frac{-19.3 - 7\boxed{0} + 0.3\boxed{0}}{0.1} = -193$$

$$x_2^{(1)} = \frac{-19.3 - 0.1\boxed{-193} + 0.3\boxed{0}}{-0.1} = -5869$$

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 0.1 & 7 & -0.3 \\ 3 & -0.1 & -0.2 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -19.3 \\ 7.85 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

$$x_1 = \frac{-19.3 - 7x_2 + 0.3x_3}{0.1}$$

$$x_2 = \frac{7.85 - 3x_1 + 0.2x_3}{-0.1}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

- Initial guess: $x_i^{(0)} = 0$

$$x_1^{(1)} = \frac{-19.3 - 7\boxed{0} + 0.3\boxed{0}}{0.1} = -193$$

$$x_2^{(1)} = \frac{-19.3 - 0.1\boxed{-193} + 0.3\boxed{0}}{-0.1} = -5869$$

$$x_3^{(1)} = \frac{71.4 - 0.3\boxed{-193} + 0.2\boxed{-5869}}{10} = -104.4$$

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 0.1 & 7 & -0.3 \\ 3 & -0.1 & -0.2 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -19.3 \\ 7.85 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

$$x_1 = \frac{-19.3 - 7x_2 + 0.3x_3}{0.1}$$

$$x_2 = \frac{7.85 - 3x_1 + 0.2x_3}{-0.1}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

- Initial guess: $x_i^{(0)} = 0$

$$x_1^{(1)} = \frac{-19.3 - 7\boxed{0} + 0.3\boxed{0}}{0.1} = -193$$

$$x_2^{(1)} = \frac{-19.3 - 0.1\boxed{-193} + 0.3\boxed{0}}{-0.1} = -5869$$

$$x_3^{(1)} = \frac{71.4 - 0.3\boxed{-193} + 0.2\boxed{-5869}}{10} = -104.4$$

… MIE334_Lecture_14_ExGSbad.xlsx

Gauss-Seidel: **Example w/ Swapped Rows**

$$\begin{bmatrix} 0.1 & 7 & -0.3 \\ 3 & -0.1 & -0.2 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -19.3 \\ 7.85 \\ 71.4 \end{Bmatrix}$$

True solution: $x_1 = 3, x_2 = -2.5, x_3 = 7$

$$x_1 = \frac{-19.3 - 7x_2 + 0.3x_3}{0.1}$$

$$x_2 = \frac{7.85 - 3x_1 + 0.2x_3}{-0.1}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

- Initial guess: $x_i^{(0)} = 0$

$$x_1^{(1)} = \frac{-19.3 - 7\boxed{0} + 0.3\boxed{0}}{0.1} = -193$$

$$x_2^{(1)} = \frac{-19.3 - 0.1\boxed{-193} + 0.3\boxed{0}}{-0.1} = -5869$$

$$x_3^{(1)} = \frac{71.4 - 0.3\boxed{-193} + 0.2\boxed{-5869}}{10} = -104.4$$

Unlike direct solution methods, order of equations can matter for iterative methods.

… MIE334_Lecture_14_ExGSbad.xlsx

Gauss-Seidel: **Improved Convergence**

Can improve convergence via <u>relaxation</u>

- After each current estimate of x_i is found, modify it by weighted average of current and previous estimate:

$$x_i^{curr} = \lambda x_i^{curr*} + (1-\lambda) x_i^{prev}$$

Gauss-Seidel: **Improved Convergence**

Can improve convergence via relaxation

- After each current estimate of x_i is found, modify it by weighted average of current and previous estimate:

$$x_i^{curr} = \lambda x_i^{curr*} + (1-\lambda) x_i^{prev}$$

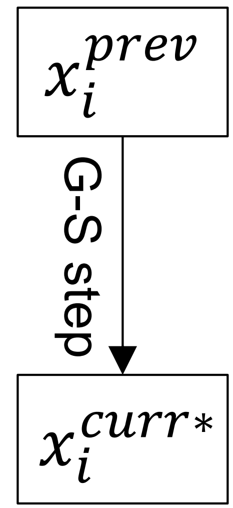
x_i^{prev}	estimate from previous iteration
x_i^{curr*}	estimate from current G-S step
x_i^{curr}	estimate after relaxation
λ	relaxation factor

Gauss-Seidel: **Improved Convergence**

Can improve convergence via <u>relaxation</u>

- After each current estimate of x_i is found, modify it by weighted average of current and previous estimate:

$$x_i^{curr} = \lambda x_i^{curr*} + (1-\lambda) x_i^{prev}$$



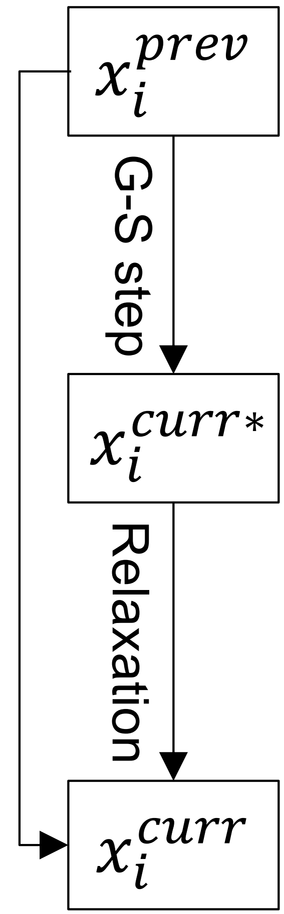
x_i^{prev}	estimate from previous iteration
x_i^{curr*}	estimate from current G-S step
x_i^{curr}	estimate after relaxation
λ	relaxation factor

Gauss-Seidel: **Improved Convergence**

Can improve convergence via relaxation

- After each current estimate of x_i is found, modify it by weighted average of current and previous estimate:

$$x_i^{curr} = \lambda x_i^{curr*} + (1-\lambda) x_i^{prev}$$



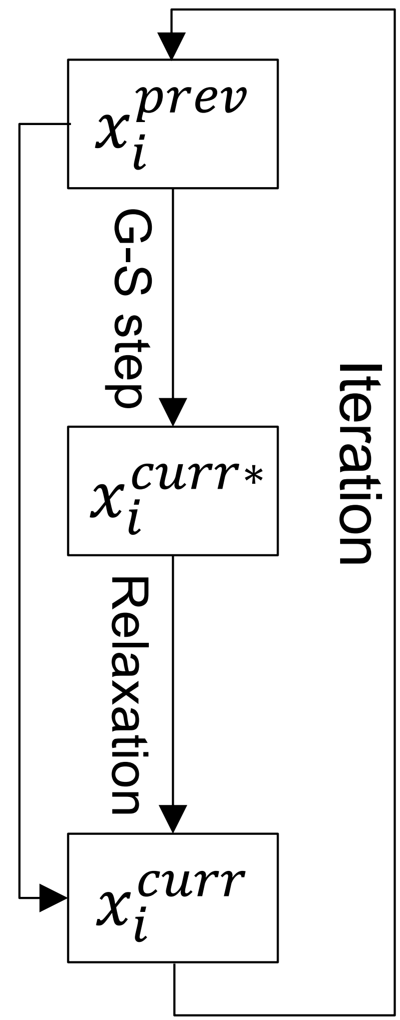
x_i^{prev}	estimate from previous iteration
x_i^{curr*}	estimate from current G-S step
x_i^{curr}	estimate after relaxation
λ	relaxation factor

Gauss-Seidel: **Improved Convergence**

Can improve convergence via <u>relaxation</u>

- After each current estimate of x_i is found, modify it by weighted average of current and previous estimate:

$$x_i^{curr} = \lambda x_i^{curr*} + (1-\lambda) x_i^{prev}$$



x_i^{prev}	estimate from previous iteration
x_i^{curr*}	estimate from current G-S step
x_i^{curr}	estimate after relaxation
λ	relaxation factor

Gauss-Seidel: **Relaxation**

$$x_i^{curr} = \lambda x_i^{curr*} + (1-\lambda)x_i^{prev}$$

- The value of λ is a weighting factor: $0 < \lambda < 2$
- It controls the relaxation process:

Gauss-Seidel: **Relaxation**

$$x_i^{curr} = \lambda x_i^{curr*} + (1 - \lambda) x_i^{prev}$$

- The value of λ is a weighting factor: $0 < \lambda < 2$

- It controls the relaxation process:
  - For $\lambda = 1$, the result is unmodified

Gauss-Seidel: **Relaxation**

$$x_i^{curr} = \lambda x_i^{curr*} + (1-\lambda) x_i^{prev}$$

- The value of λ is a weighting factor: 0 < λ < 2
- It controls the relaxation process:
  - For $\lambda = 1$, the result is unmodified
  - For $0 < \lambda < 1$, we have *under-relaxation*

Gauss-Seidel: **Relaxation**

$$x_i^{curr} = \lambda x_i^{curr*} + (1-\lambda) x_i^{prev}$$

- The value of λ is a weighting factor: 0 < λ < 2
- It controls the relaxation process:
  - For $\lambda = 1$, the result is unmodified
  - For $0 < \lambda < 1$, we have *under-relaxation*
  - For $1 < \lambda < 2$, we have *over-relaxation*

Gauss-Seidel: **Under-relaxation**

$$x_i^{curr} = \lambda x_i^{curr*} + (1-\lambda) x_i^{prev}$$

- Under-relaxation: For $0 < \lambda < 1$
  - Weighted average between *previous* and *current* estimate

Gauss-Seidel: **Under-relaxation**

$$x_i^{curr} = \lambda x_i^{curr*} + (1-\lambda)x_i^{prev}$$

- Under-relaxation: For $0 < \lambda < 1$
  - Weighted average between *previous* and *current* estimate
  - Designed to make a non-convergent system converge
  - Damps out oscillations

Gauss-Seidel: **Over-relaxation**

$$x_i^{curr} = \lambda x_i^{curr*} + (1-\lambda) x_i^{prev}$$

- Over-relaxation: For $1 < \lambda < 2$
  - Extra weight placed on *current result*

Gauss-Seidel: **Over-relaxation**

$$x_i^{curr} = \lambda x_i^{curr*} + (1-\lambda)x_i^{prev}$$

- Over-relaxation: For $1 < \lambda < 2$
  - Extra weight placed on *current result*
  - Assumes iterations are moving in the right direction but at slow rate
  - Designed to accelerate the convergence

Gauss-Seidel: **Choice of λ**

$$x_i^{curr} = \lambda x_i^{curr*} + (1-\lambda) x_i^{prev}$$

- The selection of λ is typically found empirically (trial and error)
- Very problem specific