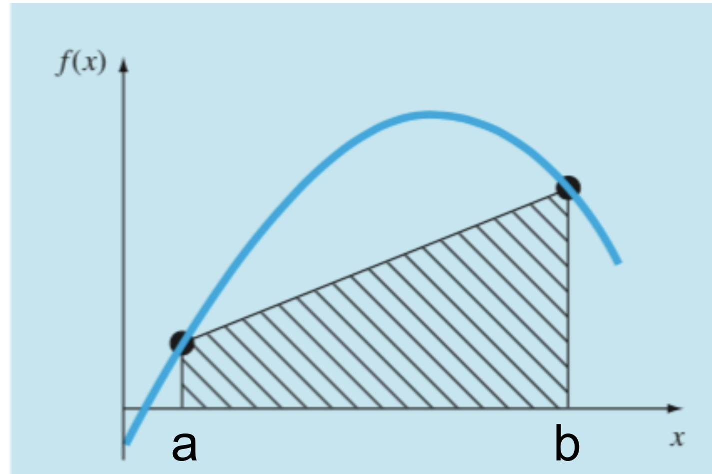
MIE334 – Numerical Methods I

Lecture 28: Gauss Quadrature *(C&C 22.4)*

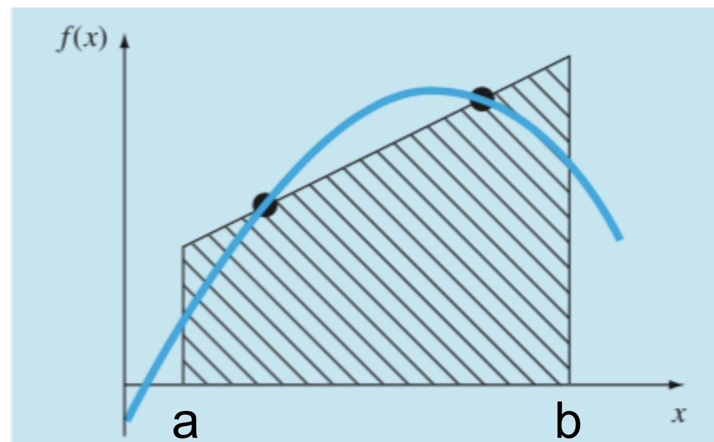
Gauss Quadrature: **Basic Idea**

- Trapezoid method fits a straight line to the **end points** of interval [a,b]



(a)

- What if we could find two points **within [a,b]** that work better?
  - Shifts line so over-estimation is better balanced by under-estimation



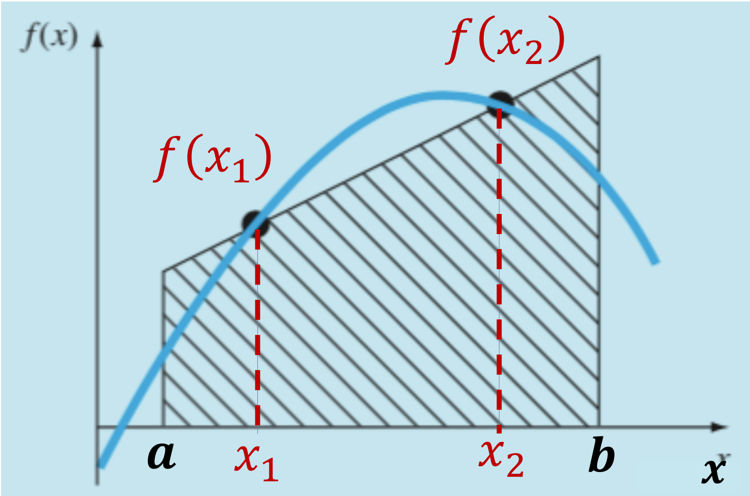
(b)

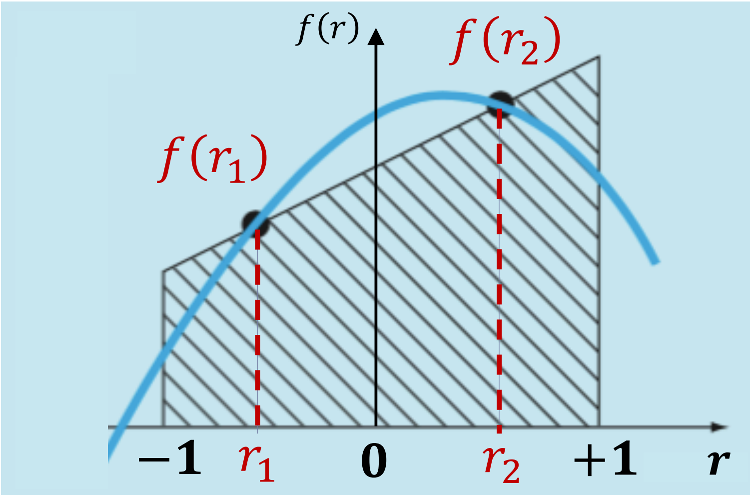
Gauss Quadrature: **Definitions**

- Assume following form (2 pts):

$$I = \int_a^b f(x) \cong \boxed{w_1 f(x_1) + w_2 f(x_2)}$$

- Four unknowns:
  - Two **locations** (x_1, x_2)
  - Two **weights** (w_1, w_2)
  - Need four conditions/equations
- To simplify, assume integral goes from $[-1, +1]$
  - Will show later how transform back to $[a, b]$
  - $r \equiv$ **reference** coord system





Gauss Quadrature: **Finding Coords/Weights**

$$I = \int_{-1}^{1} f(r)dr \cong w_1 f(r_1) + w_2 f(r_2)$$

- Start by assuming I can be calculated **exactly** for a **constant** function, $f(r) = 1$:

$$I = \int_{-1}^{1} (1)dr = [r]_{-1}^{1} = [1 - (-1)] = 2$$

$$I = w_1(1) + w_2(1) = w_1 + w_2 \qquad w_1 + w_2 = 2 \quad \textbf{[EQ1]}$$

- And also **exactly** for **linear** function, $f(r) = r$:

$$I = \int_{-1}^{1} (r)dr = \left[\frac{r^2}{2}\right]_{-1}^{1} = \left[\frac{1}{2} - \frac{1}{2}\right] = 0$$

$$I = w_1(r_1) + w_2(r_2) \qquad w_1 x_1 + w_2 x_2 = 0 \quad \textbf{[EQ2]}$$

Gauss Quadrature: **Finding Coords/Weights**

$$I = \int_{-1}^{1} f(r)dr \cong w_1 f(r_1) + w_2 f(r_2)$$

- …also **exactly** for a **quadratic** function, $f(r) = r^2$:

$$I = \int_{-1}^{1} (r^2)dr = \left[\frac{r^3}{3}\right]_{-1}^{1} = \left[\frac{1}{3} - \frac{-1}{3}\right] = \frac{2}{3}$$

$$I = w_1(r_1^2) + w_2(r_2^2) = w_1 + w_2 \qquad w_1 r_1^2 + w_2 r_2^2 = \frac{2}{3} \quad \textbf{[EQ3]}$$

- and then **exactly** for a **cubic** function, $f(r) = r^3$:

$$I = \int_{-1}^{1} (r^3)dr = \left[\frac{r^4}{4}\right]_{-1}^{1} = \left[\frac{1}{4} - \frac{1}{4}\right] = 0$$

$$I = w_1(r_1^3) + w_2(r_2^3) = w_1 + w_2 \qquad w_1 r_1^3 + w_2 r_2^3 = 0 \quad \textbf{[EQ4]}$$

Gauss Quadrature: **Finding Coords/Weights**

- Now four (non-linear) equations, four unknowns:

$$w_1 + w_2 = 2 \qquad w_1 r_1 + w_2 r_2 = 0$$

$$w_1 r_1^2 + w_2 r_2^2 = \frac{2}{3} \qquad w_1 r_1^3 + w_2 r_2^3 = 0$$

- Solving:

$$r_1 = -\frac{1}{\sqrt{3}} \cong -0.577350 \qquad w_1 = 1$$

$$r_2 = +\frac{1}{\sqrt{3}} \cong +0.577350 \qquad w_2 = 1$$

Gauss **points** Gauss **weights**

- Known as a **two-point** Gauss-Legendre formula:

$$I = \int_{-1}^{1} f(r)dr \cong w_1 f(r_1) + w_2 f(r_2) \cong f\left(-1/\sqrt{3}\right) + f\left(1/\sqrt{3}\right)$$

Gauss Quadrature: **Two-point formula**

$$I \cong f\left(-1/\sqrt{3}\right) + f\left(1/\sqrt{3}\right)$$

- Advantages:
  - Perfect accuracy for polynomials up to n=3!
  - Requires only **two** function evaluations
    - Compare to multipoint methods like TR, SR1/3, etc.
- Disadvantages:
  - Defined for interval [-1,1]
    - But can **transform** integrals that are [a,b]

Gauss Quadrature: **Transformations**

- Need to transform integral from $x \in [a, b]$ to $r \in [-1,1]$:

  $\int_a^b f(x)dx = \int_{-1}^{1} [??]dr$

- Consider simple linear transformation from $x \rightarrow r$:

  $x = c_0 + c_1 r$

- Plug in known coordinates:

  $a = c_0 + c_1(-1) = c_0 - c_1$

  $b = c_0 + c_1(1) = c_0 + c_1$

- Solve for c_0 and c_1:

  $c_0 = \frac{b+a}{2}$ $\qquad$ $c_1 = \frac{b-a}{2}$

Gauss Quadrature: **Transformations**

- So, transformation from $x \rightarrow r$ is:

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right) r \quad \text{or} \quad x = \left(\frac{1-r}{2}\right) a + \left(\frac{1+r}{2}\right) b$$

Newton poly. form *Lagrange poly. form*

- Don't forget to transform dx:

$$dx = \left(\frac{dx}{dr}\right) dr = \left(\frac{b-a}{2}\right) dr$$

- Substitute into original integral:

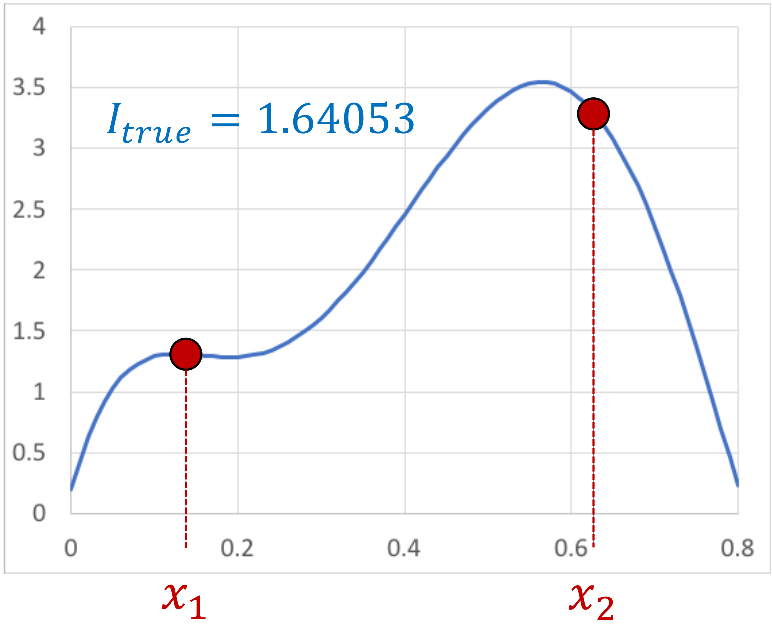
$$I = \int_a^b f(x)dx = \int_{-1}^{1} [??] dr = \int_{-1}^{1} f(x(r)) \left(\frac{b-a}{2}\right) dr$$

- And so, finally:

$$I = \left(\frac{b-a}{2}\right) \int_{-1}^{1} f(r)\, dr \cong \boxed{\left(\frac{b-a}{2}\right) \left[f(x(-1/\sqrt{3})) + (x(1/\sqrt{3})\right]}$$

Gauss Quadrature: **Example**

$$I = \int_0^{0.8} [0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5]dx$$



- Transform $x \rightarrow r$:

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right) r$$
$$= \left(\frac{0.8+0}{2}\right) + \left(\frac{0.8-0}{2}\right) r$$
$$= 0.4 + 0.4r$$

- Two Gauss points, in x coordinates:

$$x_1 = x(r_1) = 0.4 + 0.4\left(-\frac{1}{\sqrt{3}}\right) = 0.1691$$
$$x_2 = x(r_2) = 0.4 + 0.4\left(+\frac{1}{\sqrt{3}}\right) = 0.6309$$

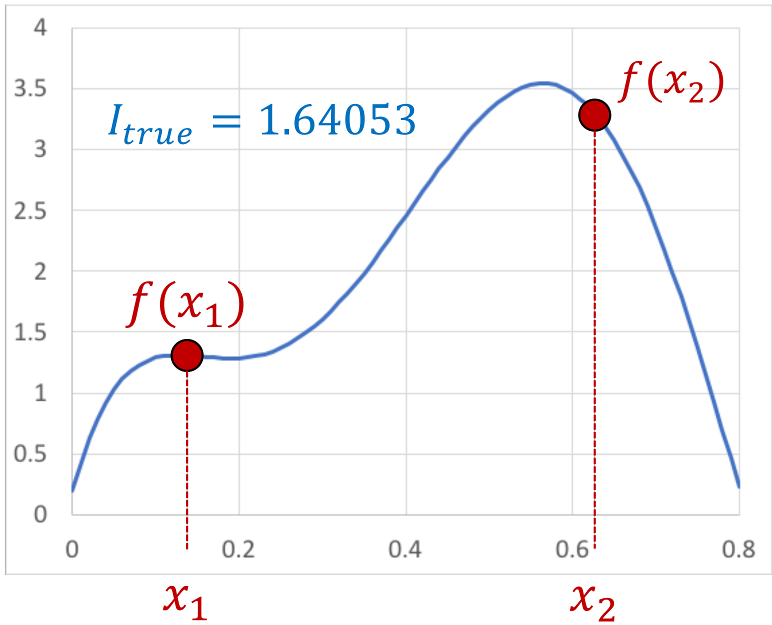
Gauss Quadrature: **Example**

$$I = \int_0^{0.8} [0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5] dx$$

- Evaluate $f(x)$ at Gauss points:

  $f(x_1) = f(0.1691) = 1.292$

  $f(x_2) = f(0.6309) = 3.265$



- Evaluate Gauss formula:

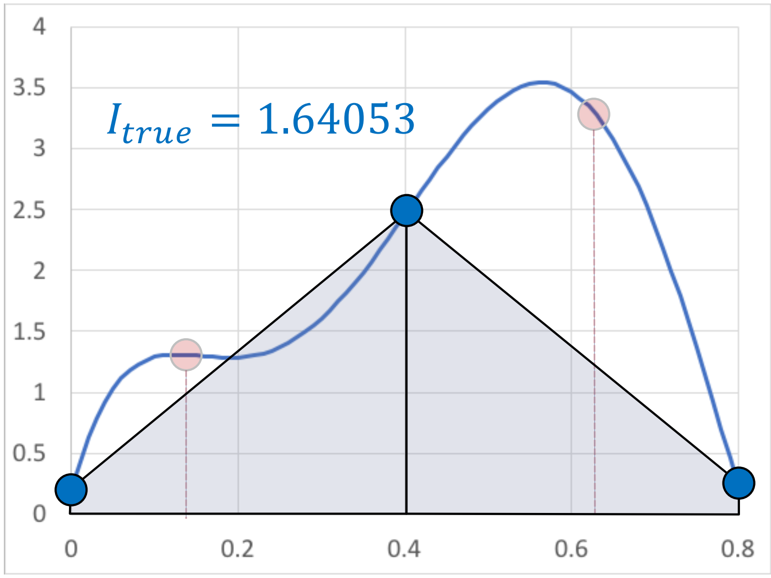
$$I \cong \left(\frac{b-a}{2}\right)[f(x_1) + f(x_2)]$$

$$\cong \left(\frac{0.8-0}{2}\right)[1.292 + 3.265]$$

$$\cong 1.823 \quad (\varepsilon_t = -11\%)$$ ***note****: overestimation*

Gauss Quadrature: **Compared to TR**

$$I = \int_0^{0.8} [0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5]dx$$



- Trapezoid Rule (n=1)

  $f(0) = 0.2$ and $f(0.8) = 0.232$

  $I \cong \left(\frac{0.8-0}{2}\right)(0.2 + 0.232)$

  $\cong 0.1728\ (\varepsilon_t = 89\%)$

- Trapezoid Rule (n=2)

  $f(0.4) = 2.456$

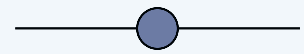
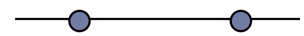
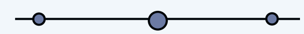
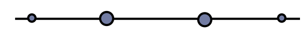
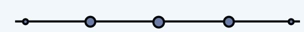
  $I \cong \left(\frac{0.8-0}{2}\right)(0.2 + (2)2.456 + 0.232)$

  $\cong 1.069\ (\varepsilon_t = 35\%)$

Gauss Quadrature: General Formula

- Can improve accuracy by using more Gauss points:
  - $I = \int_a^b f(x) \cong \sum_{i=1}^n w_i f(x_i)$
- As before, points typically tabulated for $r \in [-1,1]$, so:
  - $I = \int_1^{-1} f(x(r)) \left(\frac{b-a}{2}\right) dr \cong \left(\frac{b-a}{2}\right) \sum_{i=1}^n w_i f(x(r_i))$
- Point locations and weights can be derived by adding more constraints, i.e., exactness for higher polynomials

Gauss Quadrature: **Higher order points**

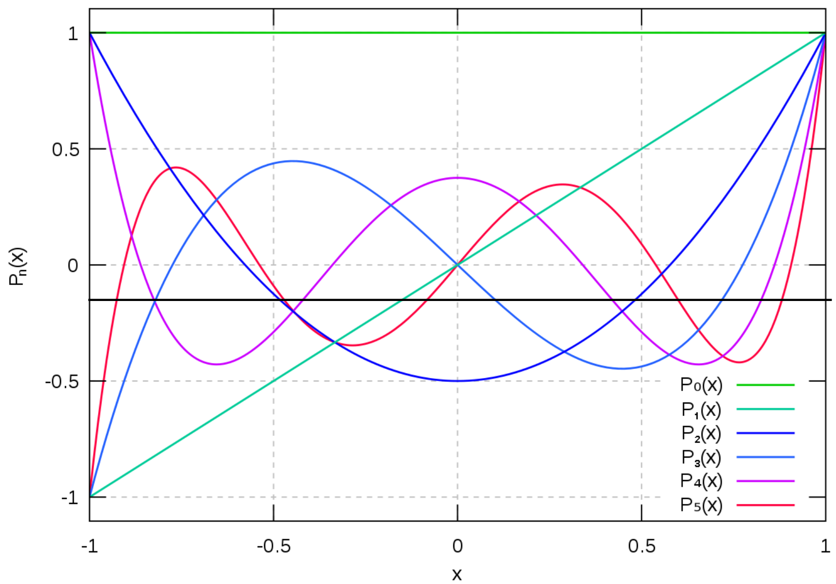
# of points, n	Point locations, r_i		Weights, w_i		Visualization
1	0	0	2	2	
2	$\pm\frac{1}{\sqrt{3}}$	±0.57735	1	1	
3	0	0	$\frac{8}{9}$	0.888889	
	$\pm\sqrt{\frac{3}{5}}$	±0.774597	$\frac{5}{9}$	0.555556	
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	±0.339981	$\frac{18+\sqrt{30}}{36}$	0.652145	
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	±0.861136	$\frac{18-\sqrt{30}}{36}$	0.347855	
5	0	0	$\frac{128}{225}$	0.568889	
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{\frac{10}{7}}}$	±0.538469	$\frac{322+13\sqrt{70}}{900}$	0.478629	
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{\frac{10}{7}}}$	±0.90618	$\frac{322-13\sqrt{70}}{900}$	0.236927	

exactly integrates polynomial up to *2n-1*
(see also C&C Table 22.1)

$$E_t = \frac{2^{2n+1}[n!]^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi)$$

Gauss Quadrature: **Legendre polynomials**

- Point **locations** are roots of Legendre polynomials, $P_n(x)$:



n	$\widetilde{P}_n(x)$
0	1
1	$2x - 1$
2	$6x^2 - 6x + 1$
3	$20x^3 - 30x^2 + 12x - 1$
4	$70x^4 - 140x^3 + 90x^2 - 20x + 1$
5	$252x^5 - 630x^4 + 560x^3 - 210x^2 + 30x - 1$

- $P_n(x)$ are exact solutions of Legendre ODE on [-1,1]

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

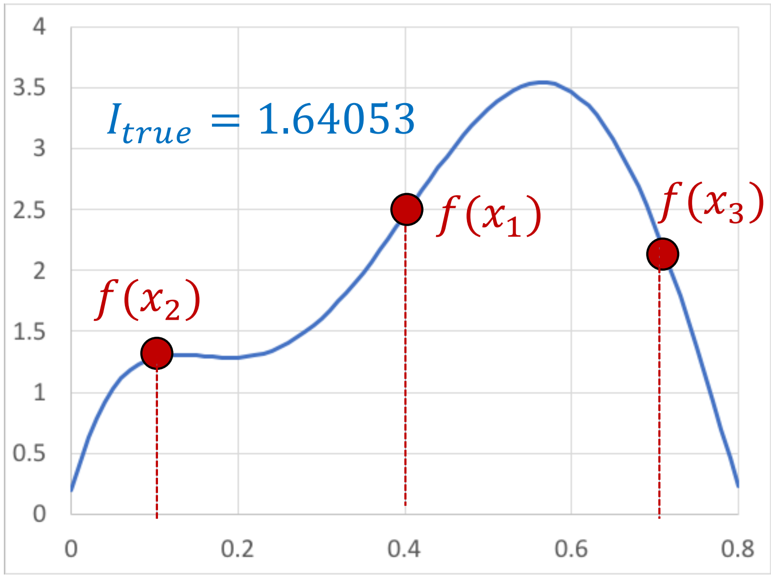
- Derives from heat conduction PDE in spherical cords
  - MIE563…

Gauss Quadrature: **Example**

$$I = \int_0^{0.8} [0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5]dx$$

- Three-point Gauss formula:

	$i=1$	$i=2$	$i=2$
r_i	0	$-\sqrt{3/5}$	$+\sqrt{3/5}$
x_i	0.4	0.09016	0.7098
$f(x_i)$	2.456	1.266	2.188
w_i	8/9	5/9	5/9



- Function values:

$$I \cong \left(\frac{b-a}{2}\right) \sum_{i=1}^{n} w_i f(x_i)$$

$$\cong \left(\frac{0.8-0}{2}\right)\left[\frac{8}{9}(2.456) + \frac{5}{9}(1.266) + \frac{5}{9}(2.188)\right] = \mathbf{1.638}$$

Gauss Quadrature: **Summary**

- Ideal when you know $f(x)$
  - Not suitable for tabulated data, unlike TR or SR

- Generally requires fewer points (i.e., $f(x)$ evaluations) compared to TR or SR

- Concept extends easily to 2D and 3D
  - Central to Finite Element Methods
  - Next lecture…