MIE334 – Numerical Methods I

Lecture 29: Gauss Quadrature in Higher Dimensions *(C&C: N/A)*

Gauss Quadrature: **Review of 1D**

I) Transform **coordinates** from $x \to r$:

$$x = c_0 + c_1 r = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right) r = \left(\frac{1-r}{2}\right) a + \left(\frac{1+r}{2}\right) b$$

II) Use this to transform **integral** from $x \to r$:

$$I = \int_a^b f(x)dx = \left(\frac{b-a}{2}\right) \int_{-1}^{1} f(x(r))dr = \left(\frac{b-a}{2}\right) \int_{-1}^{1} f(r)dr$$

III) Approximate transformed integral using n Gauss points:

$$\int_{-1}^{1} f(r)dr \cong \sum_{i=1}^{n} w_i f(r_i)$$

- n determined by polynomial order of $f(r)$

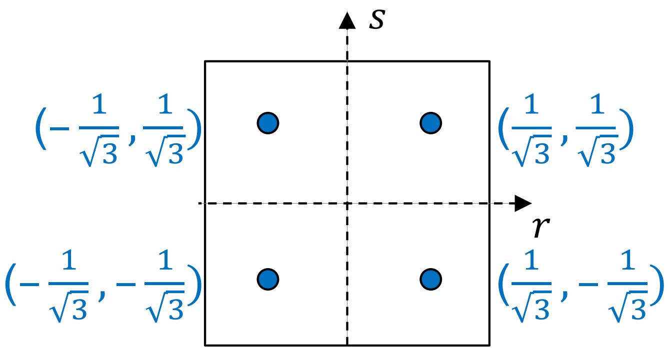
n	r_i	w_i
1	0	2
2	$\pm 1/\sqrt{3}$	1
3	0 $\pm\sqrt{3/5}$	8/9 5/9
⋮	⋮	⋮

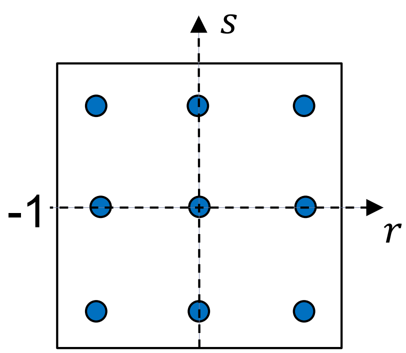
Gauss Quadrature: **2D and higher**

- For 2D (and 3D), simply extend points in all directions for unit square (cube):

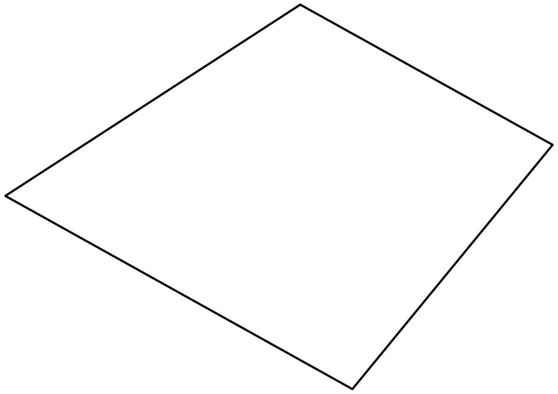
$$\int_{-1}^{1}\int_{-1}^{1} f(r,s)\,ds\,dr \cong \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j f(r_i,s_j)$$

$$\cong \sum_{k=1}^{n^2} w_k f(r_k,s_k)$$





- But what if domain is not a unit square or cube?



2D Integral Transformation: **The Jacobian**

- For 1D, we transformed the integral simply via:

$$dx = \left(\frac{dx}{dr}\right) dr$$

- For 2D and higher, we need to use the chain rule, because x and y are **both** functions of r and s:

$$dx = \left(\frac{\partial x}{\partial r}\right) dr + \left(\frac{\partial x}{\partial s}\right) ds$$

$$dy = \left(\frac{\partial y}{\partial r}\right) dr + \left(\frac{\partial y}{\partial s}\right) ds$$

- This can be expressed in matrix form as:

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = \begin{bmatrix} \partial x/\partial r & \partial x/\partial s \\ \partial y/\partial r & \partial y/\partial s \end{bmatrix} \begin{Bmatrix} dr \\ ds \end{Bmatrix} = [J] \begin{Bmatrix} dr \\ ds \end{Bmatrix}$$

  - $[J]$ is called the **Jacobian** of the transformation

2D Integral Transformation

- To transform a multiple integral:

$$\iint f(x,y)dydx = \int_{-1}^{1}\int_{-1}^{1} f(x(r,s),y(r,s))|J|dsdr$$

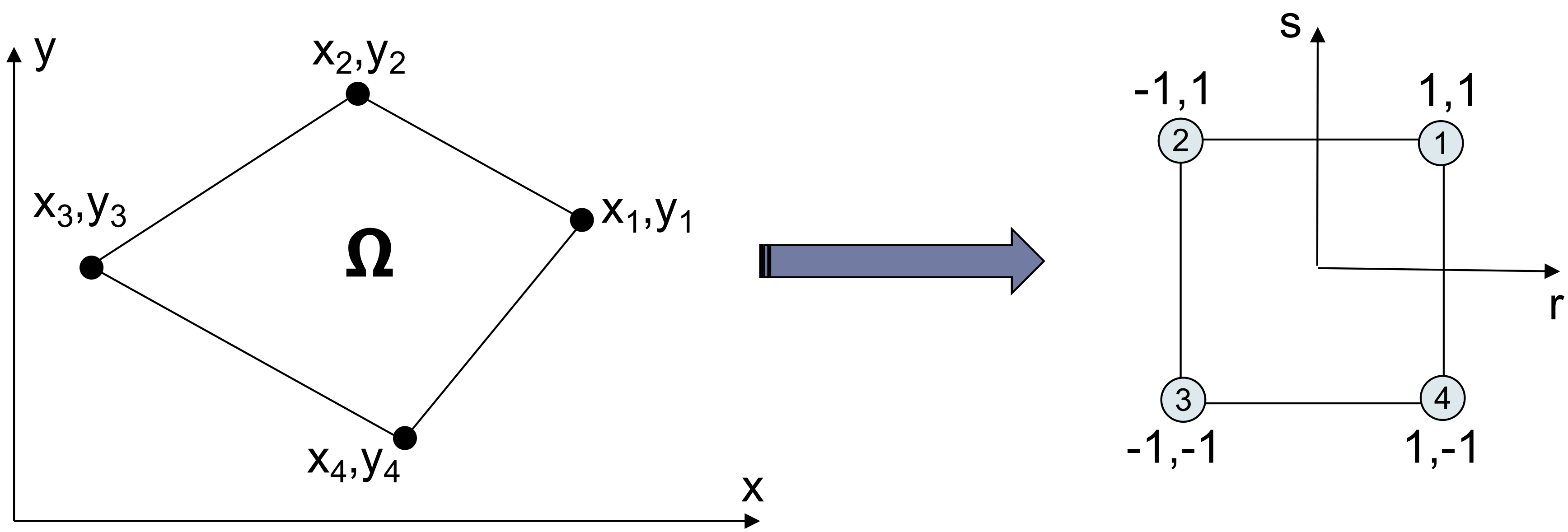
  - $|J|$ is the **determinant** of the Jacobian matrix.

- Compare this to 1D:

$$\int_{a}^{b} f(x)dx = \int_{-1}^{1} f(x(r))\left(\frac{b-a}{2}\right)dr$$

- Because the 1D "Jacobian" $\left(\frac{b-a}{2}\right)$ was a **constant**, we could move it outside of the integral:
  - Generally, however, $|J|$ may be a function of r, s
  - Can serve to increase the polynomial order of the integrand!

Quadrilateral Element: **Transformation**

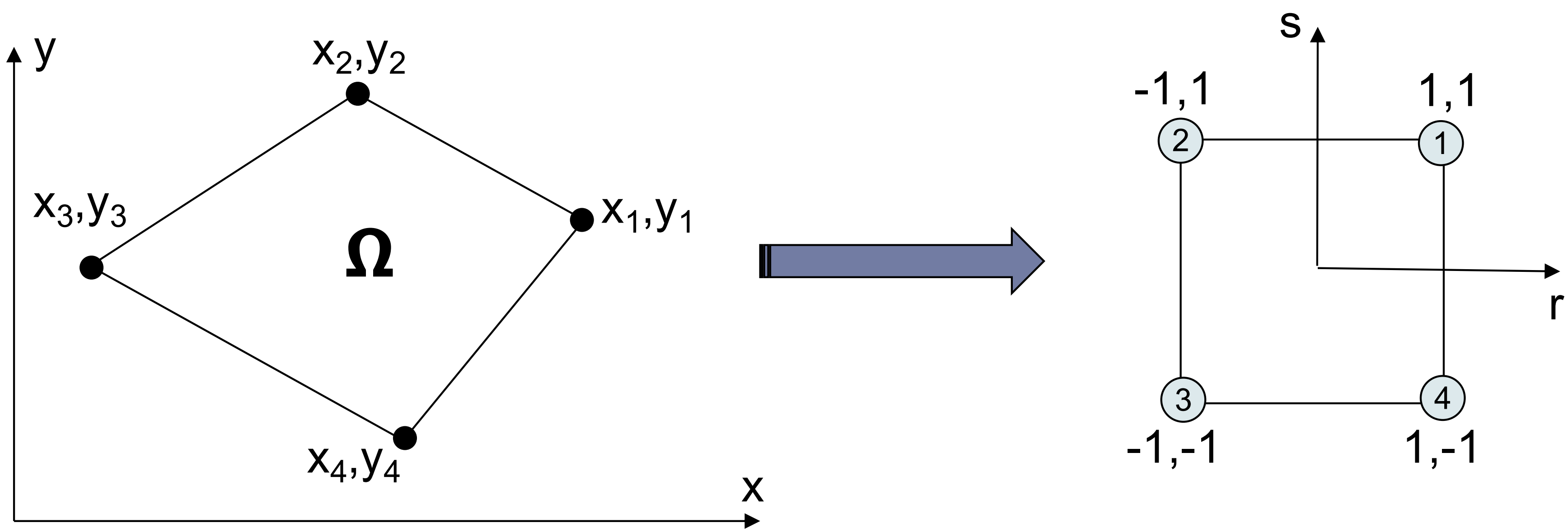


$$x = c_1 + c_2 r + c_3 s + c_4 rs$$

$$x_1 = c_1 + c_2 + c_3 + c_4$$
$$x_2 = c_1 - c_2 + c_3 - c_4$$
$$x_3 = c_1 - c_2 - c_3 + c_4$$
$$x_4 = c_1 + c_2 - c_3 - c_4$$

$$c_1 = \left(x_1 + x_2 + x_3 + x_4\right)/4$$
$$c_2 = \left(x_1 - x_2 - x_3 + x_4\right)/4$$
$$c_3 = \left(x_1 + x_2 - x_3 - x_4\right)/4$$
$$c_4 = \left(x_1 - x_2 + x_3 - x_4\right)/4$$

Quadrilateral Element: **Shape Functions**



$$x = \underbrace{\frac{1}{4}(1+r)(1+s)}_{\phi_1(r,s)}x_1 + \underbrace{\frac{1}{4}(1-r)(1+s)}_{\phi_2(r,s)}x_2 + \underbrace{\frac{1}{4}(1-r)(1-s)}_{\phi_3(r,s)}x_3 + \underbrace{\frac{1}{4}(1+r)(1-s)}_{\phi_4(r,s)}x_4$$

Like Lagrange polynomials, but in 2D

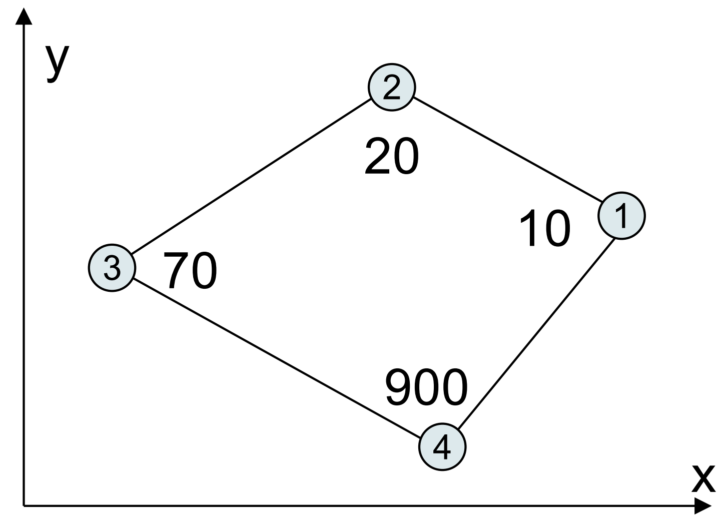
$$x = \sum_{i=1}^{4}\phi_i(r,s)x_i \qquad y = \sum_{i=1}^{4}\phi_i(r,s)y_i$$

ϕ_i are called shape functions

Quadrilateral Element: **Example**

Compute average pressure given nodal pressures solved by FEM:

$$\overline{P} = \frac{\iint P(x,y)dydx}{\iint dydx}$$



I) Transform coordinates:

$$x = c_1 + c_2 r + c_3 s + c_4 rs$$

where:

$$c_1 = (x_1 + x_2 + x_3 + x_4)/4 = 1.85$$

$$c_2 = (x_1 - x_2 - x_3 + x_4)/4 = 0.70$$

$$c_3 = (x_1 + x_2 - x_3 - x_4)/4 = 0.55$$

$$c_4 = (x_1 - x_2 + x_3 - x_4)/4 = -0.10$$

i	$x_i\ [m]$	$y_i\ [m]$	$P_i\ [kPa]$
1	**3.0**	1.6	10
2	**1.8**	2.3	20
3	**0.5**	1.4	70
4	**2.1**	0.5	900

Quadrilateral Element: **Example**

- So x-transformation is:

$$x = 1.85 + 0.7r + 0.55s - 0.1rs$$

- Repeat for y:

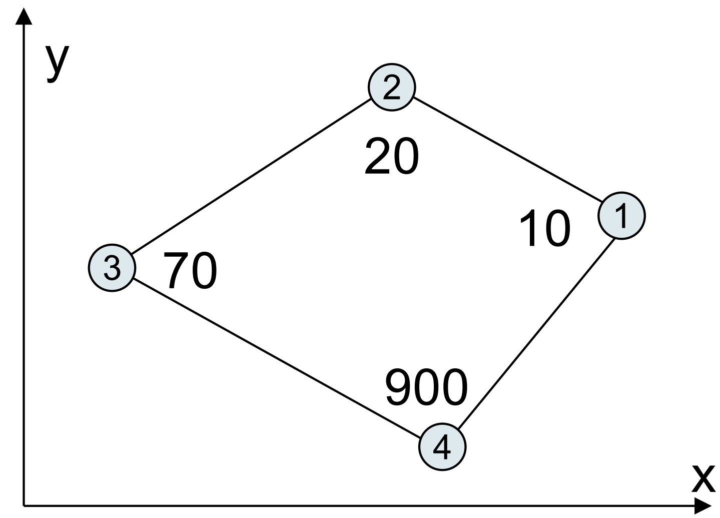
$$y = d_1 + d_2 r + d_3 s + d_4 rs$$

where:

$$d_1 = (y_1 + y_2 + y_3 + y_4)/4 = 1.45$$
$$d_2 = (y_1 - y_2 - y_3 + y_4)/4 = -0.40$$
$$d_3 = (y_1 + y_2 - y_3 - y_4)/4 = 0.50$$
$$d_4 = (y_1 - y_2 + y_3 - y_4)/4 = 0.05$$

- So y-transformation is:

$$y = 1.45 - 0.4r + 0.5s + 0.05rs$$



i	$x_i\ [m]$	$y_i\ [m]$	$P_i\ [kPa]$
1	3.0	**1.6**	10
2	1.8	**2.3**	20
3	0.5	**1.4**	70
4	2.1	**0.5**	900

Quadrilateral Element: **Example**

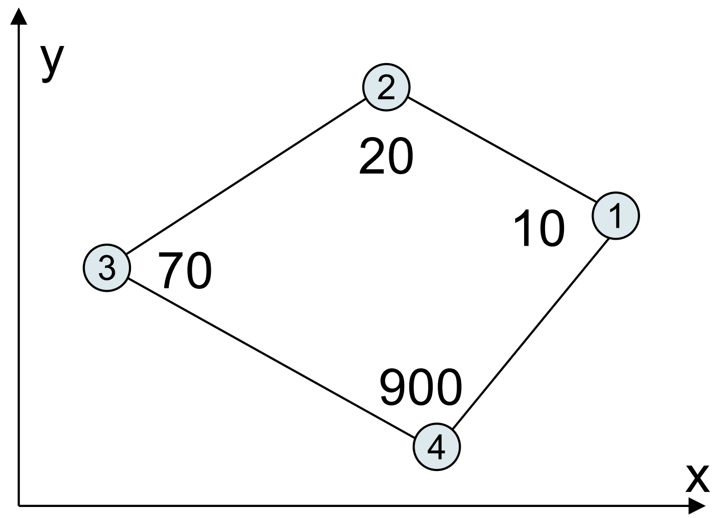
II) Transform Integral:

$$\int_{-1}^{1}\int_{-1}^{1} P(x(r,s),y(r,s))|J|dsdr$$

with:

$$x = 1.85 + 0.7r + 0.55s - 0.1rs$$

$$y = 1.45 - 0.4r + 0.5s + 0.05rs$$



- Construct Jacobian:

$$[J] = \begin{bmatrix} \partial x/\partial r & \partial x/\partial s \\ \partial y/\partial r & \partial y/\partial s \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 - 0.1s & 0.55 - 0.1r \\ -0.4 + 0.05s & 0.5 + 0.05r \end{bmatrix}$$

i	$x_i\ [m]$	$y_i\ [m]$	$P_i\ [kPa]$
1	3.0	1.6	10
2	1.8	2.3	20
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Quadrilateral Element: **Example**

- What about $P(x, y)$?

  $\int_{-1}^{1}\int_{-1}^{1} P(x(r,s), y(r,s))|J|dsdr$

- Nodal pressures were actually solved by FEM using an **isoparametric** element:

$$x = \underbrace{\frac{1}{4}(1+r)(1+s)}_{\phi_1(r,s)}x_1 + \underbrace{\frac{1}{4}(1-r)(1+s)}_{\phi_2(r,s)}x_2 + \underbrace{\frac{1}{4}(1-r)(1-s)}_{\phi_3(r,s)}x_3 + \underbrace{\frac{1}{4}(1+r)(1-s)}_{\phi_4(r,s)}x_4$$

$$x = \sum_{i=1}^{4}\phi_i(r,s)x_i \qquad y = \sum_{i=1}^{4}\phi_i(r,s)y_i \qquad P = \sum_{i=1}^{4}\phi_i(r,s)P_i$$

- So just like x and y, can transform $P(x, y) \rightarrow P(r, s)$ via:

  $P = e_1 + e_2 r + e_3 s + e_4 rs$

Quadrilateral Element: **Example**

- Repeat for P:

  $P = e_1 + e_2 r + e_3 s + e_4 rs$

  where:

  $e_1 = (P_1 + P_2 + P_3 + P_4)/4$ = 250

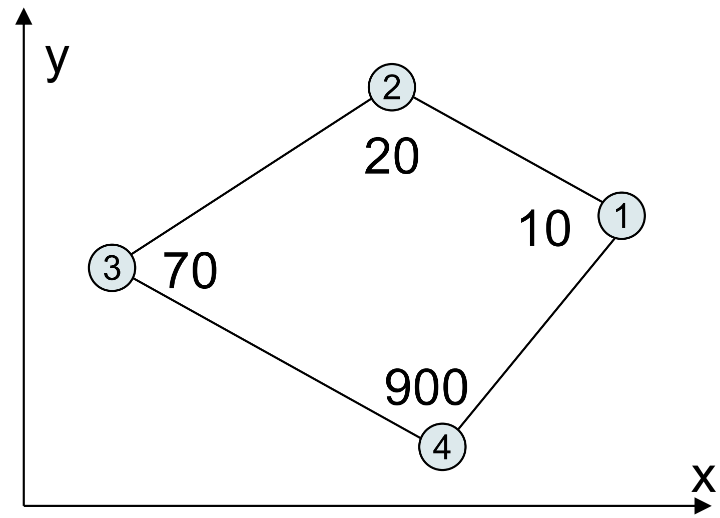
  $e_2 = (P_1 - P_2 - P_3 + P_4)/4$ = 205

  $e_3 = (P_1 + P_2 - P_3 - P_4)/4$ = -235

  $e_4 = (P_1 - P_2 + P_3 - P_4)/4$ = -210

- So $P(x, y) \rightarrow P(r, s)$:

  $P = 250 + 205r - 235s - 210rs$



i	$x_i\ [m]$	$y_i\ [m]$	$P_i\ [kPa]$
1	3.0	1.6	**10**
2	1.8	2.3	**20**
3	0.5	1.4	**70**
4	2.1	0.5	**900**

Quadrilateral Element: **Example**

- Remember our goal:

$$\overline{P} = \frac{\iint P(x,y)dydx}{\iint dydx} = \frac{\int_{-1}^{1}\int_{-1}^{1} P(r,s)|J(r,s)|dsdr}{\int_{-1}^{1}\int_{-1}^{1} |J(r,s)|dsdr}$$

- Now we have:

$$P(r,s) = 250 + 205r - 235s - 210rs$$

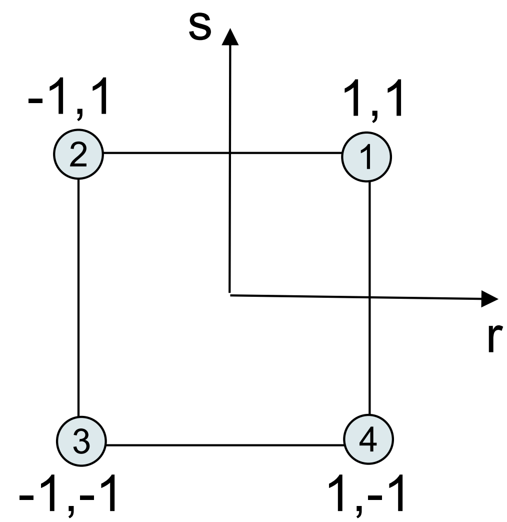
$$|J(r,s)| = \begin{vmatrix} 0.7 - 0.1s & 0.55 - 0.1r \\ -0.4 + 0.05s & 0.5 + 0.05r \end{vmatrix}$$

  - Both are linear polynomials in r and s

- Perform Gauss quadrature:

$$\overline{P} \cong \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j P(r_i,s_j)|J(r_i,s_j)|}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j |J(r_i,s_j)|}$$

  - **What to choose for n?**



i	$x_i\ [m]$	$y_i\ [m]$	$P_i\ [kPa]$
1	3.0	1.6	**10**
2	1.8	2.3	**20**
3	0.5	1.4	**70**
4	2.1	0.5	**900**

Quadrilateral Element: **Example**

$$\overline{P} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j P(r_i, s_j) |J(r_i, s_j)|}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j |J(r_i, s_j)|}$$

Gauss pts.	Numerator w x P x |J|	Denominator w x |J|	$\overline{P}$
1 x 1	570.000000000000	2.280000000000	250.000000000000
2 x 2	592.916666666667	2.280000000000	260.051169590643
3 x 3	592.916666634620	2.280000000001	260.051169576435
	diff. in 11th sig. digit	*diff. in 13th sig. digit*	*diff. in 11th sig. digit*

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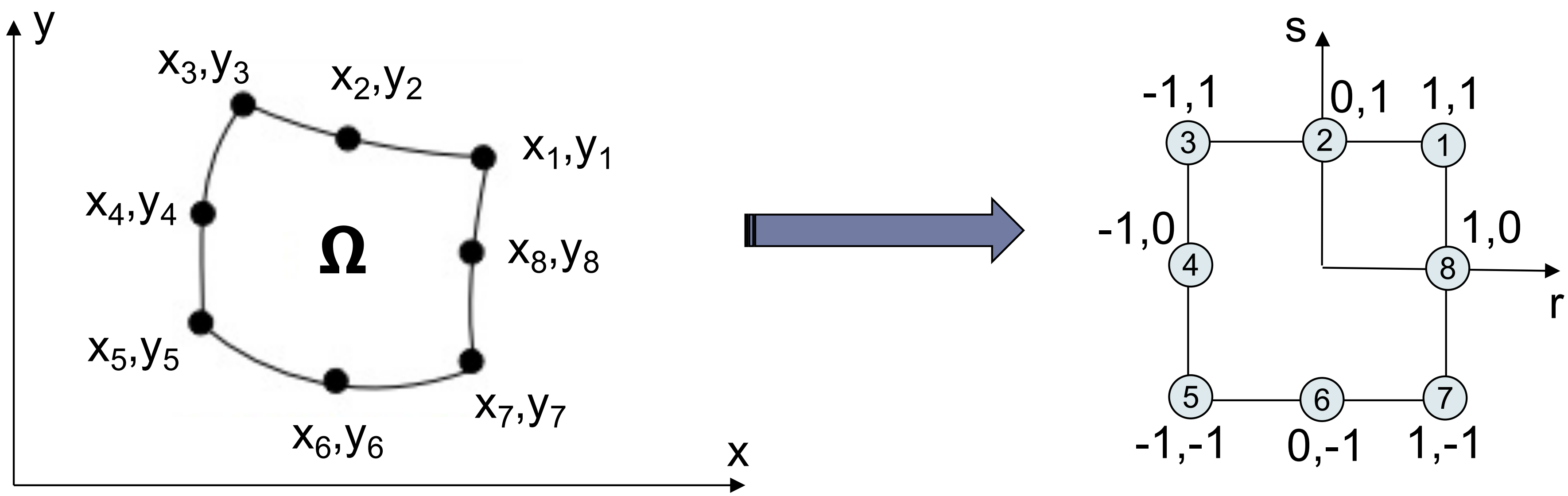
2D Quadrature: **Quadratic Quadrilateral Element**



$$x = c_1 + c_2 r + c_3 s + c_4 rs + c_5 r^2 + c_6 s^2 + c_7 r^2 s + c_8 s^2 r$$

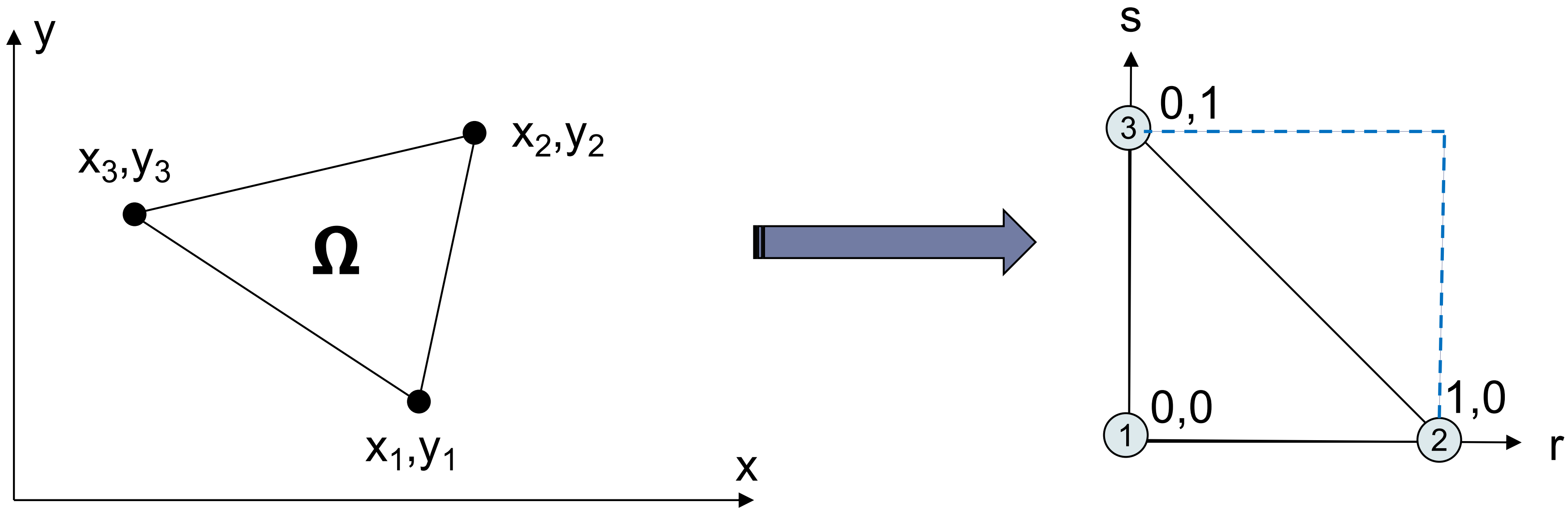
$$\phi_1 = \frac{rs}{4}(r-1)(s-1) \qquad \phi_2 = \frac{s}{2}(s-1)(1-r^2)$$

$$\phi_3 = \frac{rs}{4}(r+1)(s-1) \qquad \phi_4 = \frac{r}{2}(r+1)(1-s^2)$$

$$\phi_5 = \frac{rs}{4}(r+1)(s+1) \qquad \phi_6 = \frac{s}{2}(s+1)(1-r^2)$$

$$\phi_7 = \frac{rs}{4}(r-1)(s+1) \qquad \phi_8 = \frac{r}{2}(r-1)(1-s^2)$$

2D Quadrature: **Triangular Element**

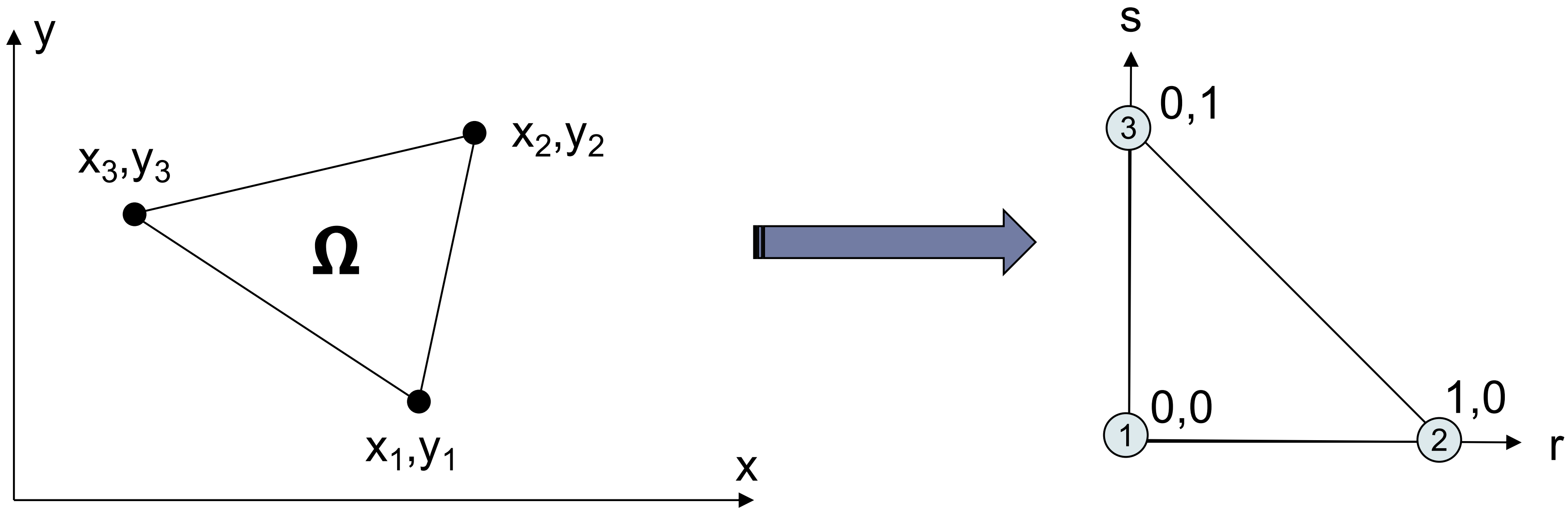


$$\iint_{\Omega} f(x,y)\,dy\,dx \qquad = \int_0^1 \int_0^{1-r} f(x(r,s),y(r,s))\,|J|\,ds\,dr$$

$$x = c_1 + c_2 r + c_3 s$$

$$y = d_1 + d_2 r + d_3 s$$

Triangular Element: **Transformation**

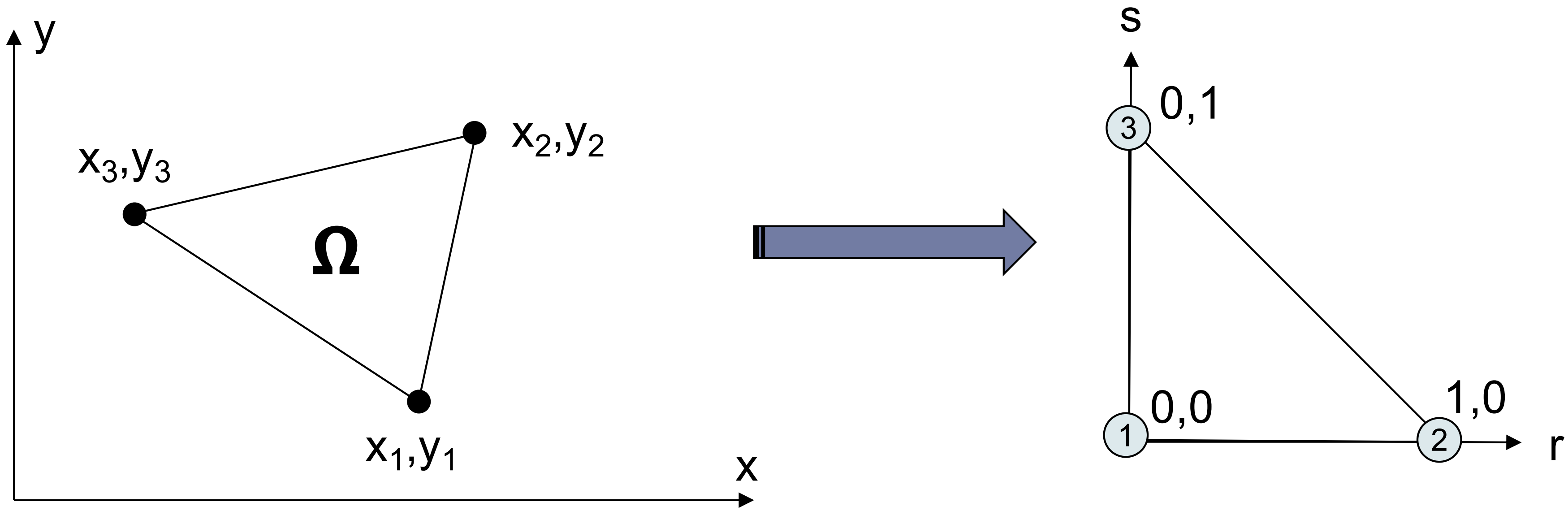


$$x = x_1 + \left(x_2 - x_1\right)r + \left(x_3 - x_1\right)s \qquad = \left(1 - r - s\right)x_1 + (r)x_2 + (s)x_3$$

$$y = y_1 + \left(y_2 - y_1\right)r + \left(y_3 - y_1\right)s \qquad = \underbrace{\left(1 - r - s\right)}_{\phi_1(r,s)}y_1 + \underbrace{(r)}_{\phi_r(r,s)}\,y_2 + \underbrace{(s)}_{\phi_3(r,s)}\,y_3$$

- Transform arbitrary triangle to reference coordinates

Triangular Element: **Jacobian and Area**



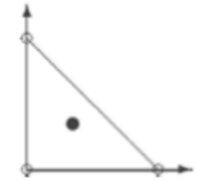
$$J = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \Rightarrow |J| = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$$

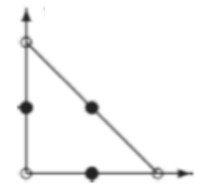
|J| is independent of r,s!

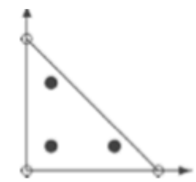
$$A = \iint dy\,dx = |J| \int_0^1 \int_0^{1-r} ds\,dr = \frac{1}{2}|J|$$

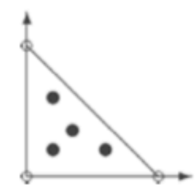
Triangular Element: **Gauss Points**

# points, n	Poly. order	Points, x_i	Weights, w_i
1	Bi-Linear	(1/3,1/3)	1
3	Bi-Quadratic	(0,1/2) (1/2, 0) (1/2, 1/2)	1/3 1/3 1/3
3	Bi-Quadratic	(1/6,1/6) (2/3, 1/6) (1/6, 2/3)	1/3 1/3 1/3
4	Bi-Cubic	(1/3, 1/3) (1/5, 3/5) (1/5, 1/5) (3/5, 1/5)	-27/48 25/48 25/48 25/48









$$\iint_{\Omega} f(x,y)\,dy\,dx = |J| \sum_{i=1}^{n} w_i f(r_i, s_i)$$

Finite Element Libraries…