Curve fitting: **Polynomial Interpolation**

Two points ($n = 2$)	Three points ($n = 3$)	Four points ($n = 4$)
straight line:	quadratic:	cubic function:
$y = a_0 + a_1 x$	$y = a_0 + a_1 x + a_2 x^2$	$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

- Unique polynomial allows predictions of intermediate values

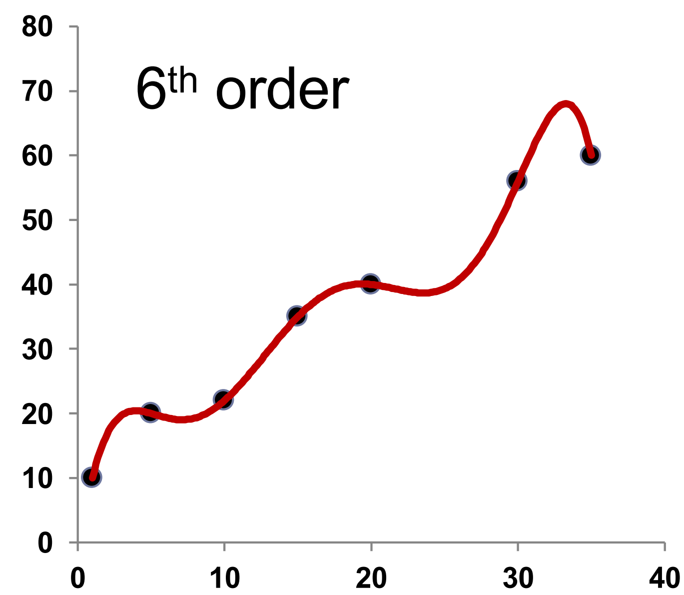
Curve fitting: **Polynomial Interpolation**

- Polynomial Interpolation:

  I) Finding the unique n^{th}-order polynomial for a set of (n+1) data-points

  II) Using it to estimate new, intermediate points

- Must be careful when using high-order polynomial interpolations!
  - **What is our other option?**

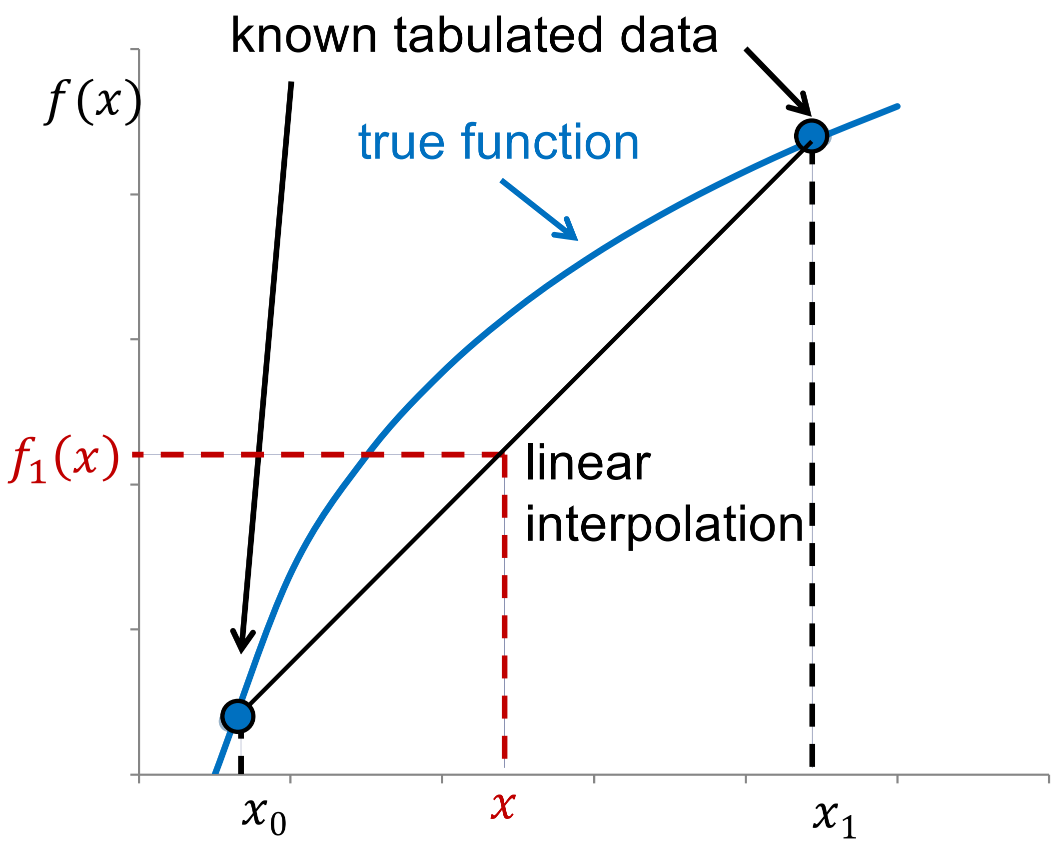


Curve fitting: **Polynomial Interpolation**

- We can express the general form of the polynomial in multiple ways:
  - **Newton Polynomials**
  - Lagrange Polynomials

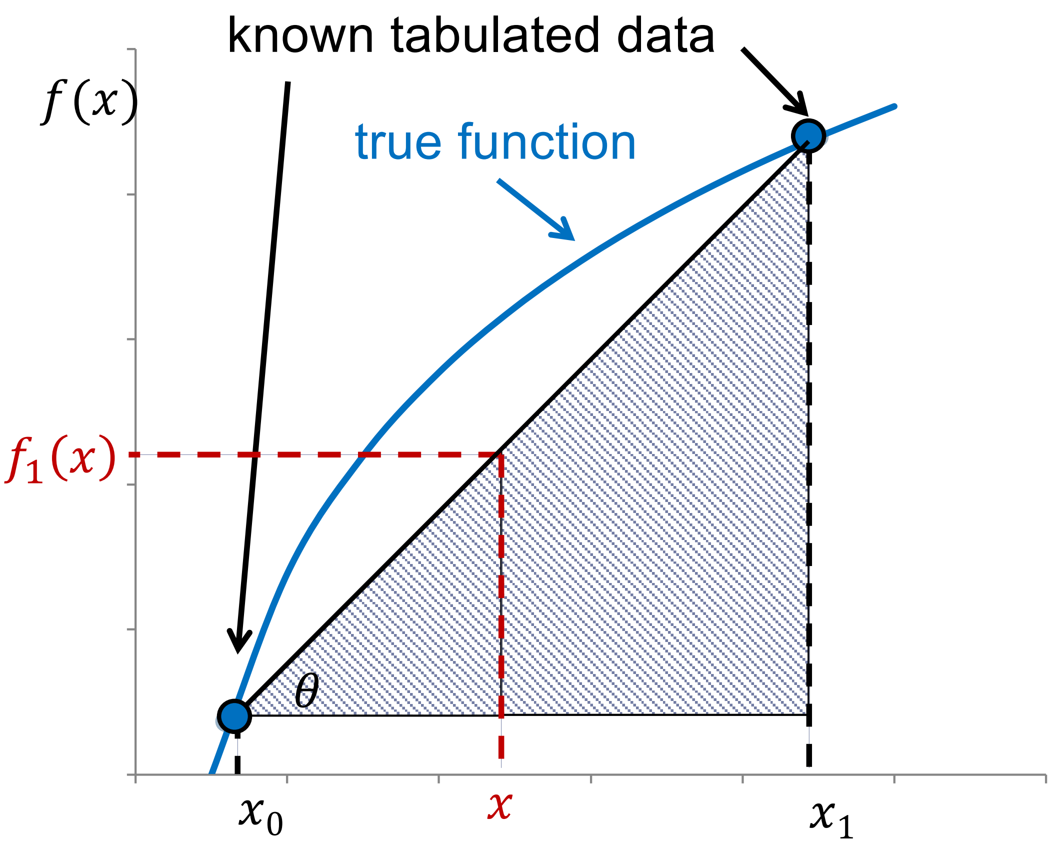
Curve fitting: **1st order Newton's Polynomial**

- Simplest form of Newton's Polynomial – Straight Line



Curve fitting: **1st order Newton's Polynomial**

- Simplest form of Newton's Polynomial – Straight Line



- We can use similar triangles (shaded) to get the following:

$$\frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Interpolation estimate at x

Curve fitting: 1st order Newton's Polynomial

- Re-arranging this relation:

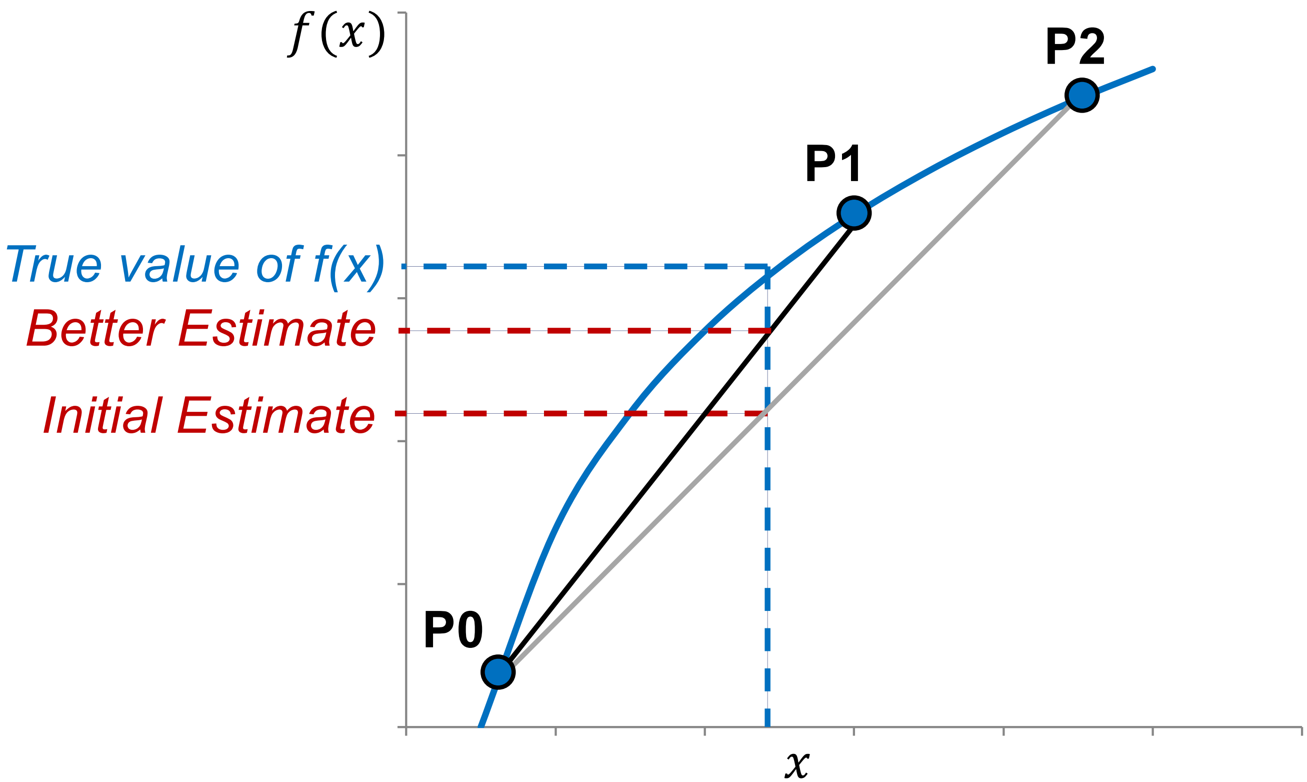
$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

- $f_1(x)$ - identifies that this is a 1st-order polynomial

- Slope term: Approximation of $f'(x)$; it is called a finite-divided-difference (**fdd**) approximation
  - *The size of the interval between points may affect our interpolation quality significantly!*

Curve fitting: 1st order Newton's Polynomial

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$



	x	$f(x)$
P0	1.2	0.18
P1	6	1.79
P2	9	2.19

Newton's Polynomial: **Quadratic Interpolation**

- Most of the error in the last example comes from approximating a **curve** with a line using **two** points (x_0, x_1)

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$= b_0 + b_1\,(x - x_0)$$

- If **three** data points are available (x_0, x_1, x_2), we can introduce curvature via a 2nd-order term:
  - $f_2(x) = b_0 + b_1\,(x - x_0) + b_2(x - x_0)(x - x_1)$
  - **Where is x_2 in this formula?**
    - How to calculate b_0, b_1, b_2?

Newton's Polynomial: **Quadratic Interpolation**

- This form is different, but equivalent to the general polynomial form we saw earlier:

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$
$$= b_0 + b_1 x - b_1 x_0 + b_2 x^2 - b_2 x_0 x - b_2 x_1 x + b_2 x_0 x_1$$

- Group terms:

$$f_2(x) = (b_0 - b_1 x_0 + b_2 x_0 x_1) + (b_1 - b_2 x_0 - b_2 x_1)x + (b_2)x^2$$
$$= a_0 + a_1 x + a_2 x^2$$

- Still, how do we find b_0, b_1, b_2?

Newton's Polynomial: **Quadratic Interpolation**

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

- **Step I)** estimate b_0 directly from $f(x_0)$ – it is the function value at Point 0:

  $$b_0 = f(x_0)$$

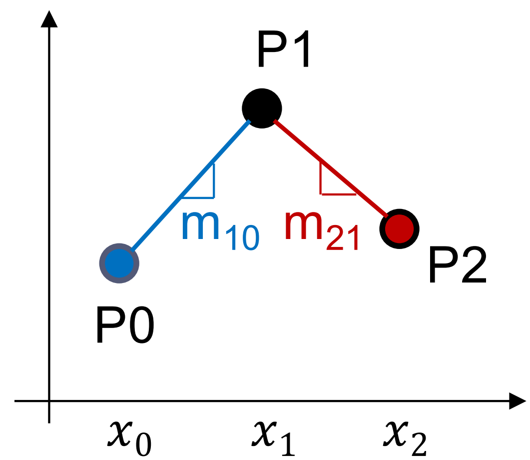
- **Step II)** estimate b_1 as the slope of the line connecting Points 0 and 1:

  - $b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$

Newton's Polynomial: **Quadratic Interpolation**

- **Step III)** estimate b_2 as the <u>curvature</u> from Points 0 to 2, through Point 1

$$b_2 = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}$$

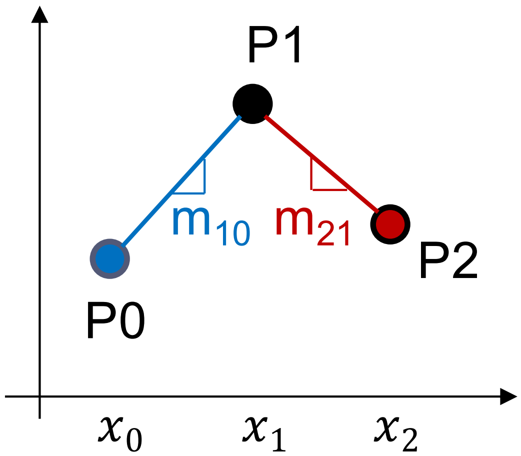


- Similar to the finite-divided difference approximation of the 2nd derivative with uniform point spacing, h

Newton's Polynomial: **Quadratic Interpolation**

- **Step III)** estimate b_2 as the <u>curvature</u> from Points 0 to 2, through Point 1

$$b_2 = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}$$



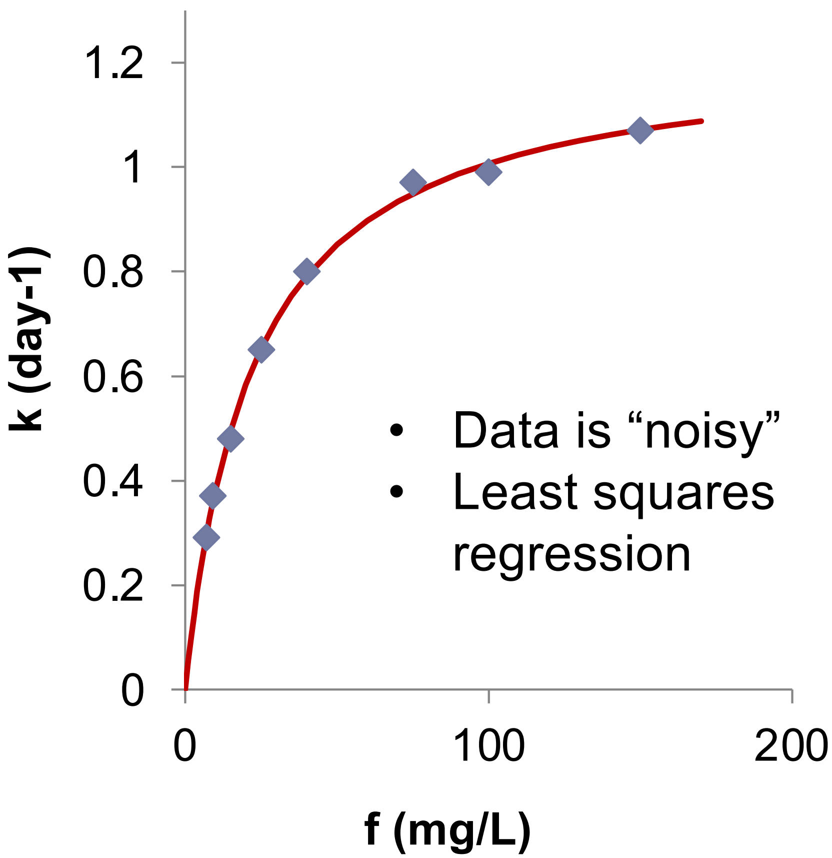
- Similar to the finite-divided difference approximation of the 2nd derivative with uniform point spacing, h

$$b_2 = \frac{\frac{f(x_2)-f(x_1)}{h} - \frac{f(x_1)-f(x_0)}{h}}{2h} = \frac{f(x_2)-2f(x_1)+f(x_0)}{2h^2}$$

  - So b_2 is like an approximate f'' term… 🤔

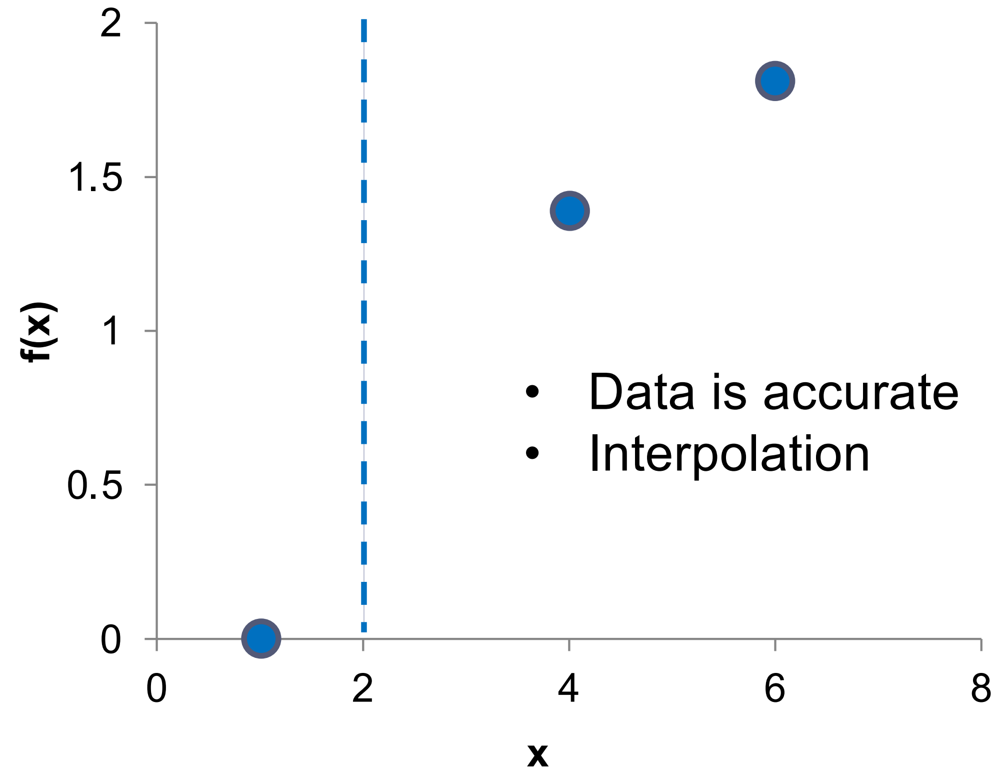
Newton's Polynomial: **Example**

Previous Example:
Yeast in Beer production



- Data is "noisy"
- Least squares regression

What if instead?

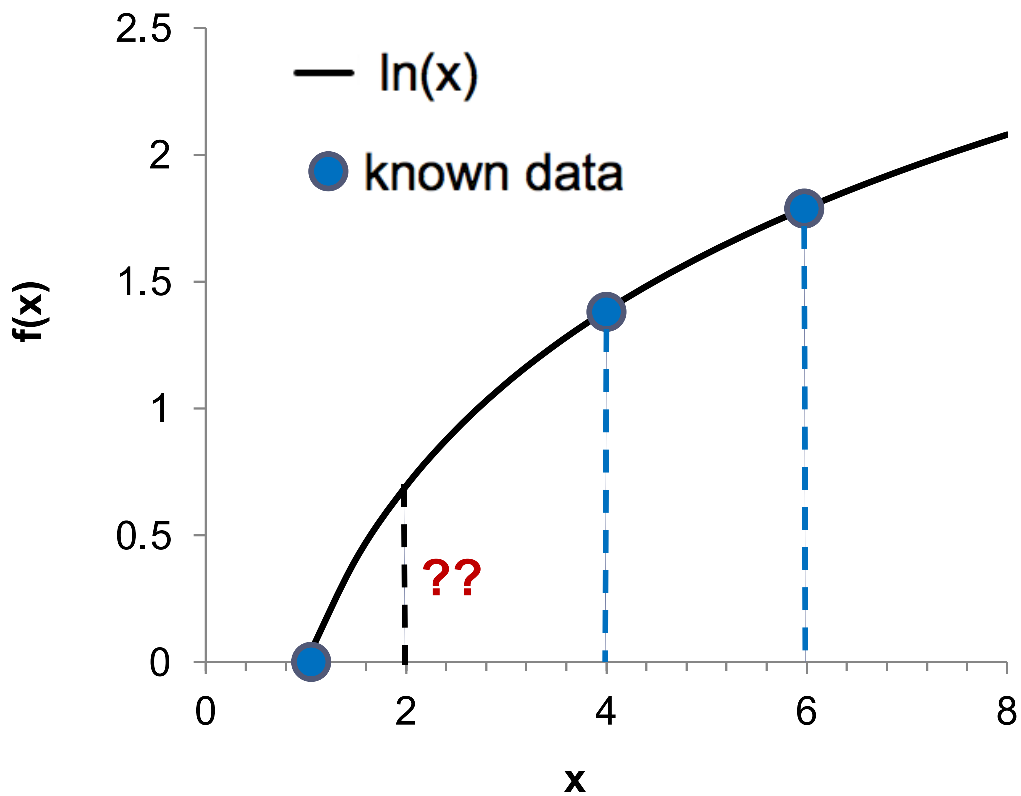


- Data is accurate
- Interpolation

- **Find estimate of f(x) at x = 2**

Newton's Polynomial: **Example 18.1 & 18.2 – ln(x)**

True function: f(x) = ln(x)



Use interpolation to estimate f(2)

x	$f(x)$
1	0
4	1.386294
6	1.791759

- Use data calculated from true function to "test" interpolation fitting
- In real cases we <u>don't know</u> true function!

Newton's Polynomial: **Example 18.1 & 18.2 – ln(x)**

I) Estimate $f(2)$ using 1st order polynomial:

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

	x	$f(x)$
x_0	1	0
	4	1.386294
x_1	6	1.791759

- Need 2 points: try $x = 1$ and $x = 6$ first

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$= 0 + \frac{1.791759 - 0}{6 - 1}(x - 1)$$

$$= 0 + 0.358352(x - 1)$$

- Substitute $x = 2$:

$$f_1(2) = 0.358352$$

$$\varepsilon_t = \frac{\ln(2) - f_1(2)}{\ln(2)}$$
$$= \frac{0.693147 - 0.358352}{0.693147}$$
$$= \mathbf{48\%}$$

Newton's Polynomial: **Example 18.1 & 18.2 – ln(x)**

I) Estimate $f(2)$ using 1st order polynomial:

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

	x	$f(x)$
x_0	1	0
x_1	4	1.386294
	6	1.791759

- Need 2 points: try $x = 1$ and $x = 4$ now

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$= 0 + \frac{1.386294 - 0}{4 - 1}(x - 1)$$

$$= 0 + 0.462098(x - 1)$$

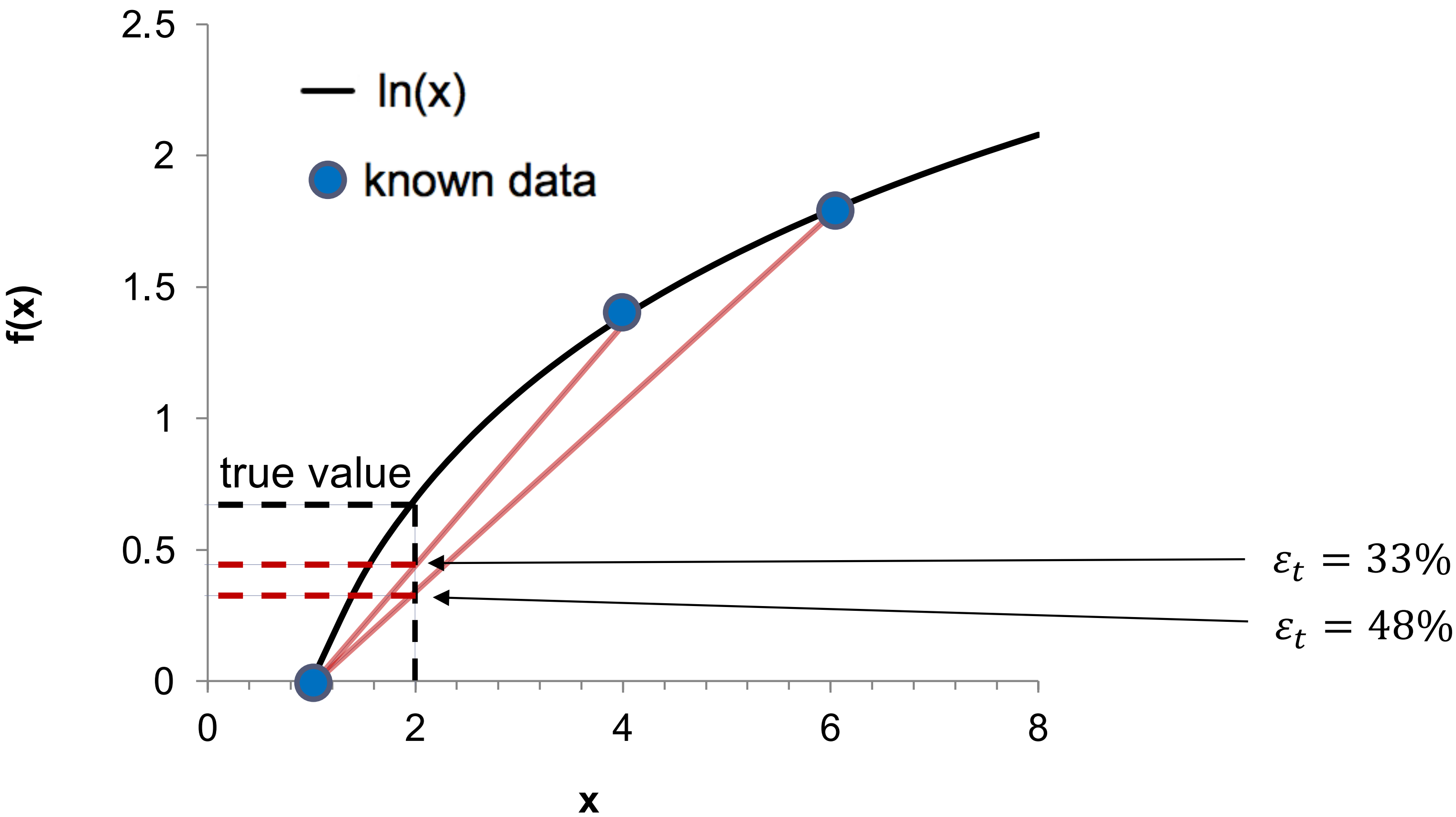
- Substitute $x = 2$:

$$f_1(2) = 0.462098$$

$$\varepsilon_t = \frac{\ln(2) - f_1(2)}{\ln(2)} = \frac{0.693147 - 0.462098}{0.693147} = \mathbf{33\%} \text{ vs. } 48\%$$

Newton's Polynomial: **Example 18.1 & 18.2 – ln(x)**

<u>Result I</u>: Smaller interval gives a better estimate



Newton's Polynomial: **Example 18.1 & 18.2 – ln(x)**

II) Estimate $f(2)$ using 2nd order polynomial:

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

	x	$f(x)$
x_0	1	0
x_1	4	1.386294
x_2	6	1.791759

- Need all 3 points

$$b_0 = f(x_0) = \mathbf{0}$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \mathbf{0.462098}$$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} = \frac{\frac{1.791759 - 1.386294}{6 - 4} - 0.462098}{6 - 1}$$

$$= \frac{0.202732 - 0.462098}{6 - 1}$$

Newton's Polynomial: **Example 18.1 & 18.2 – ln(x)**

II) Estimate $f(2)$ using 2nd order polynomial:

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

	x	$f(x)$
x_0	1	0
x_1	4	1.386294
x_2	6	1.791759

- Need all 3 points

$$b_0 = f(x_0) = \mathbf{0}$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \mathbf{0.462098}$$

$$b_2 = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0} = \frac{\frac{1.791759 - 1.386294}{6 - 4} - 0.462098}{6-1}$$

$$= \frac{0.202732 - 0.462098}{6-1} = \mathbf{-0.051873}$$

Newton's Polynomial: **Example 18.1 & 18.2 – ln(x)**

- Combining:

$$f_2(x) = 0.462098(x-1) - 0.051873\,(x-1)(x-4)$$

- Substitute $x = 2$:

$$\begin{aligned} f_2(2) &= 0.462098(2-1) - 0.051873\,(2-1)(2-4) \\ &= 0.462098 + 0.103746 \\ &= 0.565844 \end{aligned}$$

- Calculate error
  - $\varepsilon_t = \frac{\ln(2) - f_2(2)}{\ln(2)} = \frac{0.693147 - 0.565844}{0.693147} = \mathbf{18\%}\ \text{vs.}\ 33\%$

Newton's Polynomial: **Example 18.1 & 18.2 – ln(x)**

<u>Result II</u>: Curvature reduces the error further