Answer Sheet

First Name Andrew	Last Name Zhang	Student Number: 1004996533

MIE334: Numerical Methods　　　　　　　　　　　　　　　　Assignment 2

Due: February 25th, 11:59 pm, 2021

(1)

a) $$\begin{bmatrix} 100009 & -5 & -4 \\ -5 & 130 & -25 \\ -4 & -25 & 49 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

b) $\frac{5}{10^5+9} * R1 + R2 \ and \ \frac{4}{10^5+9} * R1 + R3$:

$$\begin{bmatrix} 100009 & -5 & -4 \\ 0 & 129.99975 & -25.00020 \\ 0 & -25.00020 & 48.99984 \end{bmatrix}$$

$$R3 - \frac{-25.00020}{129.99975} * R2$$

$$\begin{bmatrix} 100009 & -5 & -4 \\ 0 & 129.99975 & -25.00020 \\ 0 & 0 & 44.19206 \end{bmatrix}$$

$$f_{21} = \frac{-5}{100009}, f_{31} = \frac{-4}{100009}, f_{32} = \frac{-25.0020}{129.99975}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.00005 & 1 & 0 \\ -0.00004 & -0.19231 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} = \begin{bmatrix} 100009 & -5 & -4 \\ 0 & 129.99975 & -25.00020 \\ 0 & 0 & 44.19206 \end{bmatrix}$$

$$[L]\vec{d} = \vec{b}$$

$$\vec{d} = \begin{bmatrix} 10 \\ 0.000500 \\ 0.00030 \end{bmatrix}$$

$$[U]\vec{x} = \vec{d}$$

$$\vec{x} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0.00009 \\ 0 \\ 0 \end{bmatrix}$$

c) $\vec{x} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0.000100 \\ 0.000006 \\ 0.000011 \end{bmatrix}$

d) $I_1 \colon E_a = \frac{|0.00009-0.00010|}{0.0001} \times 100\% = 10\%$

$I_2 \colon E_a = \frac{|0-0.000006|}{0.000006} \times 100\% = 100\%$

$I_3 \colon E_a = \frac{|0-0.000011|}{0.000011} \times 100\% = 100\%$

e) Yes, because the Frobenius norm of the matrix A is on the order of a thousand, meaning that the system is ill-conditioned and sensitive to round-off errors.

We can also see that in order to calculate f_{21}, f_{31} in matrix L, we must divide by an element that is on the order of 10^5 , meaning that f_{21}, f_{31} will be on the order of 10^{-5}. As a result, when we round off f_{21}, f_{31} to the fifth decimal place, we can only keep one digit of precision. The other digits are lost, contributing to the inaccuracy of the result.

(2)

a) The solution will not converge because the matrix is not diagonally dominant. The diagonal elements of the matrix are less than the absolute row sum of the non-diagonal elements. To fix it, we can simply swap the columns of the coefficient matrix so that the matrix becomes diagonally dominant.

$$A = \begin{bmatrix} 7 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -2 & 6 \end{bmatrix}$$

In this matrix, the diagonal elements are greater than the absolute row sum of the non diagonal elements. Therefore, the matrix will converge.

(b)

$$x_1 = \frac{-6-(-1)x_2 - x_3}{7}, x_2 = \frac{3 + x_1 + x_3}{4}, x_3 = \frac{9 - x_1 + 2x_2}{6}$$

$$First\ Iteration\ Calculations:$$

$$x_1^{(1)} = \frac{-6+0+0}{7} = -0.857143, \quad x_2^{(1)} = \frac{3-0.85714+0}{4} = 0.535714$$

$$x_3^{(1)} = \frac{9+0.857143+2(0.535714)}{6} = 1.821428$$

Iteration	x_1	ε_{a,x_1} (%)	x_2	ε_{a,x_2} (%)	x_3	ε_{a,x_3} (%)
0	0		0		0	
1	-0.857143	100%	0.535714	100%	1.821428	100%
2	-1.040816	17.65%	0.945153	43.32%	1.988520	8.40%
3	-1.006195	3.44%	0.995581	5.07%	1.999556	0.55%
4	-1.005683	0.56%	0.999748	0.42%	2.0000107	0.02%
5	-1.000038	0.05%	0.999993	0.03%	2.000004	0.00%

(c)

$$x_1 = \frac{-6-(-1)x_2 - x_3}{7}, x_2 = \frac{3 + x_1 + x_3}{4}, x_3 = \frac{9 - x_1 + 2x_2}{6}$$

$$First\ Iteration\ Calculations:$$

$$x_1^{*(1)} = \frac{-6+0+0}{7} = -0.857143, \quad x_1^{(1)} = (1.2)x_1^{*(1)} + (1-1.2)(0) = -1.028572$$

$$x_2^{*(1)} = \frac{3-1.028572+0}{4} = 0.492857, \quad x_2^{(1)} = (1.2)x_2^{*(1)} + (1-1.2)(0) = 0.591428$$

$$x_3^{*(1)} = \frac{9+1.028572+2(0.591428)}{6} = 1.868571, \quad x_3^{(1)} = (1.2)x_3^{*(1)} = 2.242286$$

Iteration	x_1	ε_{a,x_1} (%)	x_2	ε_{a,x_2} (%)	x_3	ε_{a,x_3} (%)
0	0		0		0	
1	-1.028572	100.00%	0.591428	100.00%	2.242286	100.000%
2	-1.10586122	6.99%	1.122641633	47.32%	2.021771755	-10.91%
3	-0.96153578	-15.01%	0.993542467	-12.99%	1.985369791	-1.83%
4	-1.00629181	4.45%	0.9950149	0.15%	2.002190365	0.84%
5	-0.99997172	-0.63%	1.001662614	0.66%	2.000221316	-0.10%
6	-0.99975858	-0.02%	0.999806299	-0.19%	1.999829972	-0.02%
7	-1.00005234	0.03%	0.999972029	0.02%	2.000033286	0.01%

The number of iterations needed when applying an overrelaxation factor is more than the number of iterations needed without the factor. The reason that it converges slower is because the overrelaxation factor is too large and it causes the variables to the correct values, thus increasing the number of iterations needed to be within the stopping criterion.

(d) Pseudocode:

```
X1 =0
X2 =0
X3 =0
Ea1=1
Ea2=1
Ea3=1

do:
   X1_old = X1
   X2_old = X2
   X3_old = X3

   X1 = (-6+X2-X3)/7
   X2 = (3+X1+X3)/4
   X3 = (9-X1+2*X2)/6

   Ea1 = abs(X1_old-X1)/X1
   Ea2 = abs(X2_old-X2)/X2
   Ea3 = abs(X3_old-X3)/X3

while(Ea1 and Ea2 and Ea3 all > 0.001)
```

(3)

a)

$$C5:\quad Q_{25} \times c_2 = Q_{51} \times c_5$$
$$C6:\quad Q_{36} \times c_3 = Q_{64} \times c_6$$
$$C7:\quad Q_{47} \times c_4 = Q_{73} \times c_7$$

$$Q_{51} = \frac{Q_{25} \times c_2}{c_5} = \frac{4c_2}{5}$$
$$Q_{64} = \frac{Q_{36} \times c_3}{c_6} = 2c_3$$
$$Q_{73} = \frac{10 \times c_4}{12} = \frac{5c_4}{6}$$

b)

$$3Q_{01} + Q_{51}c_5 = Q_{12}c_1$$
$$Q_{12}c_1 + Q_{64}c_6 = c_2(Q_{25} + NQ_{23})$$
$$c_2NQ_{23} + Q_{73}c_7 = c_3(Q_{36} + Q_{34} + Q_{33})$$
$$Q_{34}c_3 + Q_{04}c_{04} = Q_{47}c_4$$

$$3 + 8c_2 = 5c_1$$
$$5c_1 + 8c_3 = 18c_2$$
$$10c_2 + 10c_4 = 34c_3$$
$$15c_3 + 6 = 10c_4$$

$Ax = B$:

$$\begin{bmatrix} -5 & 8 & 0 & 0 \\ 5 & -18 & 8 & 0 \\ 0 & 10 & -34 & 10 \\ 0 & 0 & 15 & -10 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \\ -6 \end{bmatrix}$$

c)

$$e_2 = -1, e_3 = -1, e_4 = -0.5769$$
$$f_1 = -5, f_2 = -10, f_3 = -26, f_4 = -4.2308$$
$$g_2 = 8, g_3 = 8, g_4 = 10$$

$$[L]\vec{d} = \vec{b}$$
$$d_1 = b_1 = -3$$
$$d_2 = b_2 - d_1e_2 = 0 - (-3)(-1) = -3$$
$$d_3 = -3$$
$$d_4 = -7.7308$$

$$[U]\vec{x} = \vec{d}$$
$$x_4 = \frac{d_4}{f_4} = 1.8273$$
$$x_3 = \frac{d_3 - g_3x_4}{f_3} = 0.8182$$
$$x_2 = 0.9545$$
$$x_1 = 2.1273$$

$$\vec{x} = \begin{bmatrix} 2.1273 \\ 0.9545 \\ 0.8182 \\ 1.8273 \end{bmatrix}$$

d)

$$\mathrm{N} = 48, \mathrm{c}_2 = 0.01989$$