Determining a length of a knot by Lagrange polynomial interpolation compared to definite integral

By using Lagrange polynomial interpolation, how does the calculated length of a knot which is found by the integral of Lagrange polynomial equations compare to the theoretical length of a knot which is found by integral of parametric equations?

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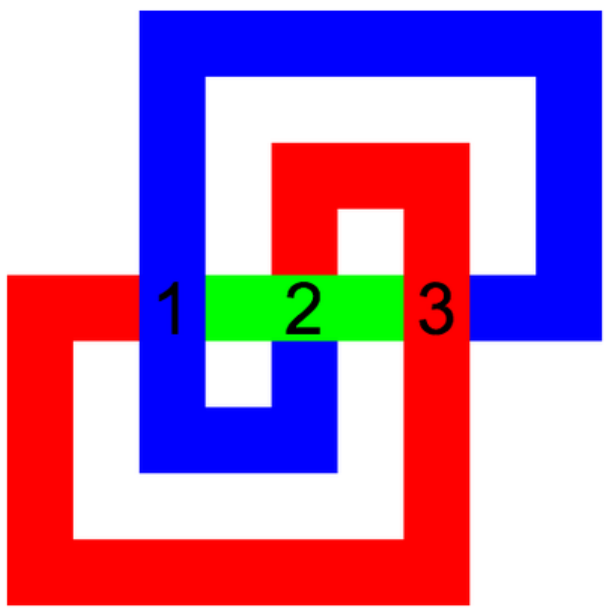
Introduction

A knot, in the area of mathematics, is defined as a "closed, non-self-intersecting curve that is embedded in three dimensions" (Knot). This can be simply made by making any sort of conventional knot with a string and connecting the two ends. Study of those mathematical knots, knot theory, still remains a difficult area of mathematics, as it is difficult to express knots with numerical values (Scharein). In order to assign a numerical identification for a certain knot, mathematicians have developed 'invariant points'. For example, crossing number is one type of invariant points and is the minimum number that the knot crosses itself from a 2-dimensional view (Lackenby). This can be shown by the following diagram (Crossing):



The diagram above is a three crossing knot, each shown by a number. However, invariant points still remain unsatisfactory as different knots can be attributed to the same crossing number. For example, the following list includes diagrams of different knots with 10-crossing.

Image (1) - To show the different variations of 10-crossing knot (Ricca)

One alternative method that I considered was by measuring the length of a mathematical knot. This can be done by applying integral calculus to parametric equations. Parametric equations are “a set of equations that express a set of quantities as explicit functions of a number of independent variables, known as ‘parameters’” (Parametric). The most well known knot, the trefoil knot, can be explained with the following parametric equations.

Equation (0) - Parametric equations of a trefoil knot (Stemkoski)

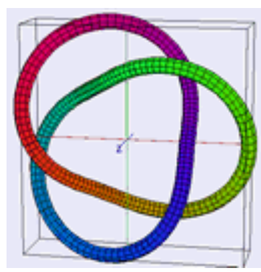
$$x = sin(t) + 2sin(2t)$$

$$y = cos(t) - 2cos(2t)$$

$$z = -sin(3t)$$

In the equations above, the parameter t is used to describe the x, y, z coordinates at a certain t. Therefore, when these equations are inserted into a three dimensional digital grapher, the following image will be shown.

Image (2) - 3-Dimensional image of a trefoil knot (Stemkoski)



Additionally, as the equations are trigonometric equations, using cos and sin, the parameter t only ranges from $0 \leq t \preccurlyeq 2\pi$ because cos and sin have a period of 2π meaning they will yield the same x, y, z value at $t = 0$ and $t = 2\pi$. To find the length using the parametric equations, integral calculus needs to be used which will be explained on page 8.

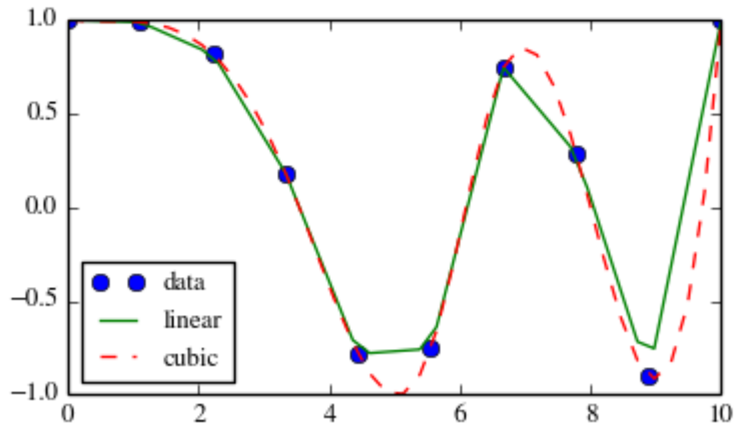
Parametric equations are used to visualize the “simplest” form of a knot-type such as having the minimum number of crossings. (Stemkoski) For example, the following image is still another diagram of the trefoil knot, but in an irregular form without using the parametric equations.

Image (3) - Image of a trefoil knot with more than 3 crossings (Stemkoski)

Unfortunately, not all knots have a parametric equation due to certain knots complexity. For the knots without parametric equations, an alternative method can be used by interpolating points along the knot, or alternatively known as 'connect-the-dots' method. Then, as long as I have a coordinate for the points, or 'dots', along the knot, I am able to create lines that can expresses a knot mathematically. To create the lines, I will be using Lagrange polynomial interpolation which will be explained on page 13.

Unfortunately, interpolating points does not exactly trace the desired image due to the difference in curvature. For example, the following image shows how the degree of the line, whether linear (2nd-degree) or cubic (3rd-degree), changes the overall image.

Image (4) - Image of a data interpolated by linear and cubic equations (Interpolation (scipy.interpolate))



One might simply ask to add more points closer together, which will obviously help trace the knot more accurately. However, to have a sensible process, I will set the number of points in a knot as a controlled variable.

More information on the degree and how to manipulate degree of the line in Lagrange Interpolating Polynomial will be further explained in page 13. However, due to the extent and the given time for the Extended Essay, I will be only using one degree but set up the procedure to allow further experiments into multiple degrees. Additionally, as I will use Microsoft Excel for my calculation which only gives a numerical value instead of an actual equation of the line, the integral calculus will have to be guided by Gaussian Quadrature which will be explained in page 16.

Therefore, my Extended Essay will focus on comparing and identifying the error of the length of a knot determined through the integral calculus of Lagrange

Interpolating Polynomial, which would not yield an exact value, against knot determined using integral calculus of the parametric equations, which would yield an exact value.

Before stating my research question, I will be using the term *Calculated Value/Length* for an approximate value that was yielded through taking an indirect approach of finding the solution by interpolating points along the knot, and *Theoretical Value/Length* for an exact value that was yielded through taking a direct approach through parametric equations to find the solution.

The research question of this extended essay will be:

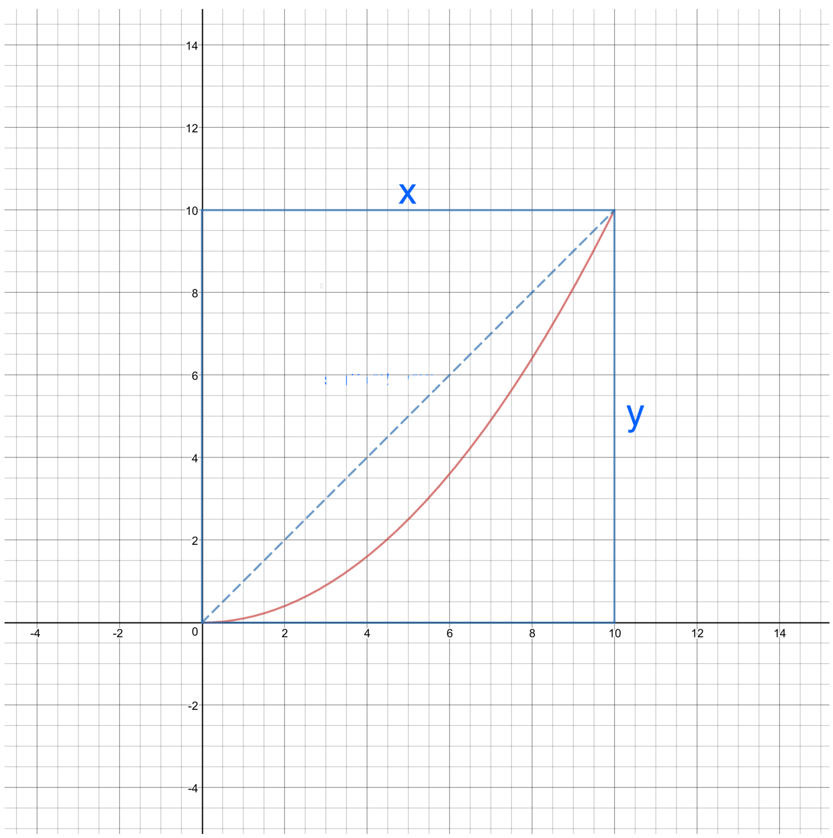
By using Lagrange polynomial interpolation, how does the calculated length of a knot which is found by the integral of Lagrange polynomial equations compare to the theoretical length of a knot which is found by integral of parametric equations?

For this investigation, I will use a unit circle, a zero crossing knot; trefoil knot, a three crossing knot; and a torus knot, a seven crossing knot, to perform my calculation on. An image of a torus knot can be found in appendix (Refer to Appendix A). Additionally, this essay will be first focusing on finding the theoretical length followed by determining the calculated length.

Determining the theoretical length by using integral of parametric equations

Before I begin, knowing how to find the theoretical length of a curve is crucial for my investigation. The method to approach this is through definite integral. To understand definite integral, we can take an example of finding the length of the following red curve that starts at the origin and ends at $(10,10)$.

Image (5) - Graph of $y = \frac{x^2}{10}$from $0 \leq x \leq 10$with a large rectangle (Source: Author's own)

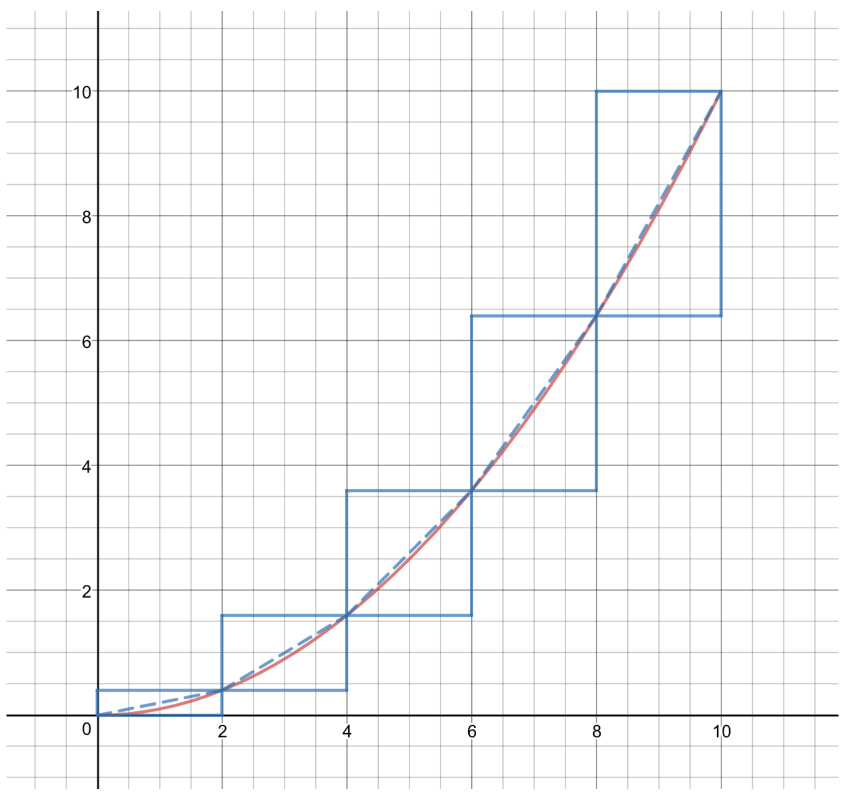


In this curve, I defined the distance between the start position and the end position of the curve with x and y. By using Pythagoras Theorem, I can determine the length of the dotted line by:

$$l = \sqrt{x^2 + y^2}$$

However, as it is shown, the dotted line does not exactly trace the red curve. Therefore, the dotted line will yield a length that is shorter than the length of the red line. In order to have a more accurate result, I created smaller rectangles.

Image (6) - Graph of $y = \frac{x^2}{10}$from $0 \leq x \leq 10$with smaller rectangles (Source: Author's own)



In this diagram, the smaller divisions of x and y per squares allowed the dotted line to trace the red line more accurately. If I continued to make smaller increments, I would label x as dx and y as dy, meaning small change of x or y, respectively. Therefore, I could rearrange the previous formula to the following:

$$l = \sqrt{dx^2 + dy^2}$$

However, the formula above is incorrect, as if there was a small change of x and y, it would be near 0. Therefore, I would take an integral of the curve with the range from $0 \leq x \leq 10$. This would mean that I would have a collection of small changes of $x$and y through $0 \leq x \leq 10$. The word integral means "to put together into a whole" (Hease). Therefore, I would have a collection of small changes of x and y that can be represented by the following.

Equation (1) - Integral to showcase the length in two dimension

$$l = \int_0^{10} \sqrt{dx^2 + dy^2}$$

However, in our parametric equations of the knot, there is a defined x, y, and z value as it is a three-dimensional object. Therefore, it would look like the following:

$$l = \int_0^{10} \sqrt{dx^2 + dy^2 + dz^2}$$

Even so, this formula is still incorrect. In the previous integral of the curve, in equation (1), it was possible as the value of y was determined by the value of x. However, the values for x, y, and z cannot be determined without the use of a parameter. This is because a certain x-coordinate does not yield a single pair of y- and z-coordinates. This is when parametric equations come into use. In parametric equations, x, y, and z coordinates are defined in terms of a parameter t. Additionally, I have mentioned that in our trigonometric parametric equations the range will be $0 \leq t \leqslant 2\pi$. Therefore, the use the parameter t and its boundary can represented by the following:

$$l = \int_0^{2\pi} \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt} dt$$

I have not changed anything to the formula as I divided by dt but also multiplied by dt. This could be simplified further on:

$$l = \int_0^{2\pi} \frac{\sqrt{dx^2 + dy^2 + dz^2}}{\sqrt{dt^2}} dt$$

$$l = \int_0^{2\pi} \sqrt{\frac{dx^2 + dy^2 + dz^2}{dt^2}} dt$$

$$l = \int_0^{2\pi} \sqrt{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}} dt$$

Equation (2) - Integral to showcase the length in three dimensions

$$l = \int_0^{2\pi} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} dt$$

By simplifying, $\frac{dx}{dt}$ was shown, which is also the derivative of x respect to t. Therefore, the length of a parametric equation is determined as the square root of the sum of the squares of derivatives of x , y , z with respect to t . For example, a circle's parametric equations are given by the following:

$$x = cos(t)$$

$$y = sin(t)$$

The derivatives of the two equations can be represented by following:

$$\frac{dx}{dt} = -sin(t)$$

$$\frac{dy}{dt} = cos(t)$$

Using formula (2), which is the formula to find the length using definite integral, it could be shown by the following. The following could be entered in a TI-84 and be calculated (Refer to Appendix B):

$$l = \int_0^{2\pi} \sqrt{(-sin(t))^2 + (cos(t))^2} dt = 2\pi$$

This method is known as the definite integral or, in my investigation, the theoretical length of a circle. The same method is repeated to gather the theoretical length of knots:

	Circle	Trefoil Knot	Torus Knot
Parametric Equation	$x = cos(t)$ $y = sin(t)$	$x = sin(t) + 2sin(2t)$ $y = cos(t) - 2cos(2t)$ $z = -sin(3t)$	$x = cos(3t)\times(3 + cos(4x))$ $y = sin(3t)\times(3 + cos(4x))$ $z = sin(4t)$
Derivative	$\frac{dx}{dt} = -sin(t)$ $\frac{dy}{dt} = cos(t)$	$\frac{dx}{dt} = cos(t) + 4cos(2t)$ $\frac{dy}{dt} = -sin(t) + 4sin(2t)$ $\frac{dz}{dt} = -3cos(3t)$	$\frac{dx}{dt} = (-3sin(3x))\times(3 + cos(4x)) + cos(3t)\times(-4sin(4x))$ $\frac{dy}{dt} = (3cox(3x))\times(3 + cos(4x)) + sin(3t)\times(-4sin(4x))$ $\frac{dz}{dt} = 4cos(4t)$
Theoretical Length	6.283185307	28.82628995	62.13269572

Determining the calculated length a curve by using integral of Lagrange Interpolating Polynomial

So far, I have investigated the definite length or the theoretical length of a knot. However, by using Lagrange Interpolation and Gaussian Quadrature, I can calculate a calculated length of a knot.

Lagrange Interpolation

Lagrange Interpolating Polynomial is a function of a line that passes *n* number of points with a degree of $n-1$(Lagrange Interpolating). For example, if I was given 3 points, (x_1, y_1), (x_2, y_2), and (x_3, y_3), Lagrange Interpolating Polynomial will be a quadratic line, which is an order of 2, that goes through all three points. In order to do so, Lagrange Polynomial uses a base function, l_j, that gives a one for their corresponding x but 0 for other values of x. For example, with the 3 points, (x_1, y_1), (x_2, y_2), and (x_3, y_3), there will be three base functions. The first base function will yield to a value of 1 for x_1 and 0 for x_2 and x_3. This can be organized into a table:

	l_1	l_2	l_3
x_1	1	0	0
x_2	0	1	0
x_3	0	0	1

With this base function, I would multiply the corresponding y value after each of the base function. This means that if the base function yielded a 1, the result of the polynomial yield in the corresponding y value. This can be shown with the following function:

$$f(x) = l_1 y_1 + l_2 y_2 + l_3 y_3$$

This formula can be described by this situation:

In a function f(x), this can be described as x is the key to yield a certain answer y. However, the three base functions are locks that only open to certain keys. For example a lock, l_1, will only open if x_1 key was inserted. However, when the lock opens, a multiplier is waiting to make 1 into y_1. If the lock was not open, the multiplier would not get an input and will not form anything.

In conclusion, Lagrange Polynomial can be expressed using the following mathematical formula

Equation (3) - Formula of Lagrange Polynomial (Lagrange Interpolating)

$$P(x) \; = \sum_{j=1}^{n} l_j(x) y_j$$

In this formula, the $\sum$ notation means to sum up. In this formula, it shows that the line P(x) that passes through an n number of points is the sum of all of the base functions multiplied by their multipliers, y_j. This can be expressed by the following:

$$P(x) = l_1 y_1 + l_2 y_2 + \ldots + l_n y_n$$

Additionally, l_j can be represented by following:

$$l_j(x) = \prod_{k=1, k \neq j}^{n} \frac{x - x_k}{x_j - x_k}$$

(Lagrange Interpolating)

In this formula, l_j is the base function previously discussed. The $\prod$ notation represents the product of all of the terms in a range. Therefore, this will be the product of $\frac{x-x_k}{x_j-x_k}$ when k starts at 1 and increases with the increment of 1, but cannot be j, until it reaches n. This can be expanded and explained with the following where j is 1, 2, or n:

$$l_1(x) = \frac{(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}$$

$$l_2(x) = \frac{(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}$$

$$l_n(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}$$

As I would need multiply all of the base functions with their corresponding y_j and be added together, asked the function 3, it can be written out as the following:

$$P(x) = \frac{(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}y_1 + \frac{(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}y_2 + \ldots + \frac{(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}y_n$$

However, this formula only works in two-dimensional diagrams. In order to incorporate x, y, and z, it would be presented by the following formula:

$$P(x,y,z) = \sum_{j=1}^{n} l_j x_j y_j z_j$$

Again, it would need to use the parameter t to create three functions, each representing x, y, and z in terms of t. Therefore, it can be expressed with this formula:

$$P_x(t) = \sum_{j=1}^{n} l_j(t) x_j$$

$$P_y(t) = \sum_{j=1}^{n} l_j(t) y_j$$

$$P_z(t) = \sum_{j=1}^{n} l_j(t) x_j$$

l_j is defined similarly as before but used with the parameter t as shown with the following:

$$l_j(t) = \prod_{k=1, k \neq j}^{n} \frac{t - t_k}{t_j - t_k}$$

This can also be expanded and explained by the following functions with j being 1, 2, or n:

$$l_1(t) = \frac{(t - t_2)(t - t_3)\ldots(t - t_n)}{(t_1 - t_2)(t_1 - t_3)\ldots(t_1 - t_n)}$$

$$l_2(t) = \frac{(t - t_1)(t - t_3)\ldots(t - t_n)}{(t_2 - t_1)(t_2 - t_3)\ldots(t_2 - t_n)}$$

$$l_n(t) = \frac{(t - t_1)(t - t_2)\ldots(t - t_{n-1})}{(t_n - t_1)(t_n - t_2)\ldots(t_n - t_{n-1})}$$

Therefore, the three formulas can be presented by the following:

$$P_x(t) = \frac{(t - t_2)(t - t_3)\ldots(t - t_n)}{(t_1 - t_2)(t_1 - t_3)\ldots(t_1 - t_n)} x_1 + \frac{(t - t_1)(t - t_3)\ldots(t - t_n)}{(t_2 - t_1)(t_2 - t_3)\ldots(t_2 - t_n)} x_2 + \cdots + \frac{(t - t_1)(t - t_2)\ldots(t - t_{n-1})}{(t_n - t_1)(t_n - t_2)\ldots(t_n - t_{n-1})} x_n$$

$$P_y(t) = \frac{(t - t_2)(t - t_3)\ldots(t - t_n)}{(t_1 - t_2)(t_1 - t_3)\ldots(t_1 - t_n)} y_1 + \frac{(t - t_1)(t - t_3)\ldots(t - t_n)}{(t_2 - t_1)(t_2 - t_3)\ldots(t_2 - t_n)} y_2 + \ldots + \frac{(t - t_1)(t - t_2)\ldots(t - t_{n-1})}{(t_n - t_1)(t_n - t_2)\ldots(t_n - t_{n-1})} y_n$$

$$P_z(t) = \frac{(t - t_2)(t - t_3)\ldots(t - t_n)}{(t_1 - t_2)(t_1 - t_3)\ldots(t_1 - t_n)} z_1 + \frac{(t - t_1)(t - t_3)\ldots(t - t_n)}{(t_2 - t_1)(t_2 - t_3)\ldots(t_2 - t_n)} z_2 + \ldots + \frac{(t - t_1)(t - t_2)\ldots(t - t_{n-1})}{(t_n - t_1)(t_n - t_2)\ldots(t_n - t_{n-1})} z_n$$

Gaussian Quadrature

While trying to take the integral of the Lagrange Interpolating Polynomial, as there is a limitation in using excel which only produces a numerical value without a variable, the necessity arose to use Gaussian Quadrature. Gaussian Quadrature is a rule to be able to approximate a definite integral of a function by giving an n number of points (Legendre-Gauss). The Gaussian Quadrature will only produce a result if the function is between -1 and 1 (Legendre-Gauss). The reasoning behind how this produces an accurate approximation is beyond the scope of the essay. However, I can utilize this method to provide an approximation for the length of my function.

Gaussian Quadrature says to find the integral at a certain point and multiply by the weight. The sum of all of the product of integral and the weight will be the length of the curve. The weights of the corresponding point are given with the following table.

Table (1) - Points and Weights that correspond for a set number of points (Legendre-Gauss)		
Number of points, n	Points, x_i	Weights, w_i
1	0	2
2	$\pm\sqrt{\frac{1}{3}}$	1
3	0	$\frac{8}{9}$
	$\pm\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18-\sqrt{30}}{36}$
5	0	$\frac{128}{225}$
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{\frac{10}{7}}}$	$\frac{322+13\sqrt{70}}{900}$
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{\frac{10}{7}}}$	$\frac{322-13\sqrt{70}}{900}$

This table continues on with additional number of points.

Application

In order to apply mathematics into our context, I would have to make some number of equidistant points along the curve of the knots. I have decided to use 60 points, as I will be able to use Lagrange Interpolation with 3, 4, 5, 6, or 7 points, each creating polynomials with degree of 2, 3, 4, 5, and 6, respectively. The polynomials that I will create have to be head-to-toe, meaning that the endpoint of the first polynomial will be the head of the second polynomial. Additionally, the tail of the last polynomial have to be the same as the head for the first polynomial. For example, if I decide to use three points, I will create a polynomial with the points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$. The next polynomial I will create will consist of the points $(x_3, y_3, z_3), (x_4, y_4, z_4), (x_5, y_5, z_5)$, and the last polynomial will be $(x_{59}, y_{59}, z_{59}), (x_{60}, y_{60}, z_{60}), (x_1, y_1, z_1)$. If the complete head-to-toe train was complete by coming back to (x_1, y_1, z_1), I will end up with 30 polynomials using 3 points, 20 polynomials using 4 points, 15 polynomials using 5 points, 12 polynomials using 6 points, and 10 polynomials using 7 points. Additionally, the sum of the length of all of these polynomials will yield to my calculated value of the entire knot.

For my first interpolation, I have decided to use three points. The base functions and its derivatives can be defined by the following equations:

	Table (2) - Interpolation when $n = 3$(Source: Author's Own)
Base function	$l_1(t) = \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)}$………(4.1) $l_2(t) = \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)}$………(4.2) $l_3(t) = \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}$………(4.3)
Parametric functions of the polynomial	$P_x(t) = \frac{(t-t_2)(t-t_3)...(t-t_n)}{(t_1-t_2)(t_1-t_3)...(t_1-t_n)} x_1 +$ $\frac{(t-t_1)(t-t_3)...(t-t_n)}{(t_2-t_1)(t_2-t_3)...(t_2-t_n)} x_2 + \ldots + \frac{(t-t_1)(t-t_2)...(t-t_{n-1})}{(t_n-t_1)(t_n-t_2)...(t_n-t_{n-1})} x_n$………(5.1) $P_y(t) = \frac{(t-t_2)(t-t_3)...(t-t_n)}{(t_1-t_2)(t_1-t_3)...(t_1-t_n)} y_1 +$ $\frac{(t-t_1)(t-t_3)...(t-t_n)}{(t_2-t_1)(t_2-t_3)...(t_2-t_n)} y_2 + \ldots + \frac{(t-t_1)(t-t_2)...(t-t_{n-1})}{(t_n-t_1)(t_n-t_2)...(t_n-t_{n-1})} y_n$………(5.2) $P_z(t) = \frac{(t-t_2)(t-t_3)...(t-t_n)}{(t_1-t_2)(t_1-t_3)...(t_1-t_n)} z_1 +$ $\frac{(t-t_1)(t-t_3)...(t-t_n)}{(t_2-t_1)(t_2-t_3)...(t_2-t_n)} z_2 + \ldots + \frac{(t-t_1)(t-t_2)...(t-t_{n-1})}{(t_n-t_1)(t_n-t_2)...(t_n-t_{n-1})} z_n$………(5.3)
Derivative of base function	$l_1'(t) = \frac{2t-(t_2+t_3)}{(t_1-t_2)(t_1-t_3)}$………(6.1) $l_2'(t) = \frac{2t-(t_1+t_3)}{(t_2-t_1)(t_2-t_3)}$………(6.2) $l_1'(t) = \frac{2t-(t_1+t_2)}{(t_3-t_1)(t_3-t_1)}$………(6.3)
Derivative of x, y, and z	$\frac{dx}{dt} = l_1'(t)x_1 + l_2'(t)x_2 + l_3'(t)x_3$………(7.1) $\frac{dy}{dt} = l_1'(t)y_1 + l_2'(t)y_2 + l_3'(t)y_3$………(7.2) $\frac{dz}{dt} = l_1'(t)z_1 + l_2'(t)z_2 + l_3'(t)z_3$………(7.3)

*The number inside the bracket is used to number the equation

As the range of my function is 2π, I have divided 2π by 60, having 60 equidistant points from 0 to 2π. For my first polynomial I have used the first three points which are $\frac{0\pi}{30}, \frac{1\pi}{30}, \frac{2\pi}{30}$. Then with those points, I have created three base functions from formulas (4.1), (4.2), and (4.3) that yielded a 1 with those corresponding points. Then using formulas (5.1), (5.2), and (5.3), I have created three parametric equations that go through all three points. After, I have used formulas (6.1), (6.2), and (6.3) to find the derivatives of each of the base functions to have the derivatives of the parametric functions. To find the derivatives of the parametric functions, I have used formulas (7.1), (7.2), and (7.3). Using the derivatives of the parametric functions, I have used formula (2) to find the length (Refer to Appendix C).

In my process, I have found out that, due to the limitations of using Excel for my calculation, I am unable to create an equation in respect to x, y, or z, as I only received a numerical value. As I cannot take the definite integral of a numerical value, I would have to use Gaussian Quadrature.

Before I begin using Gaussian Quadrature, I had to change my range from $\frac{0\pi}{30} \leq t \leqslant \frac{2\pi}{30}$ to$-1 \leqslant u \leqslant 1$, as Gaussian Quadrature only works within the range of -1 and 1. I have decided to use a different variable for my parameters to avoid any confusion. To make this transition, I used $y = ax + b$, a linear equation. Therefore, it could be expressed with the following formula:

$$-1 = \frac{0\pi}{30}a + b$$

$$1 = \frac{2\pi}{30}a + b$$

By doing the calculations it yielded the following value:

$$a = \frac{30}{\pi}$$

$$b = -1$$

Therefore, the transition from u to t can be shown by following:

Equation (8) - Equation to find out $u$in terms of t

$$t = \frac{u+1}{\frac{30}{\pi}}$$

I would have to use formula (8) every time I used the parameter u, as our functions were defined in terms of t.

Next, I have decided to use 5 points Gaussian Quadrature. To not confuse this with the number of points I have decided with the Lagrange Interpolation, these are 5 points chosen within our polynomial interpolating three points, that is used to estimate a definite integral. However, I could produce more points that will lead to a higher accuracy, however, 5 was a reasonable amount of points without being too tedious.

After being unable to proceed from the Lagrange Interpolating Polynomial which is evident from Appendix C and I have used Gaussian Quadrature from that process and multiplied their corresponding weights as it was shown in Table 1. After, I have added all of the integrals of the five points, this yielded to a value of 0.209439092 (Refer to appendix D).

Before I continue to find the length of a circle, I was able to check how accurate I was. The distance I have calculated can also be defined as the arc length from $\frac{0\pi}{30} \leq t \preccurlyeq \frac{2\pi}{30}$.

Arc length is defined by the following formula:

$$arc\ length = \theta r$$

In our example, $\theta$is $\frac{2\pi}{30}$ as it is the angle between $\frac{0\pi}{30}$and $\frac{2\pi}{30}$. Additionally, r, meaning radius, will be 1 as it is a unit circle. Therefore, the arc length should be $\frac{2\pi}{30}$ which can be explain numerically as 0.2094395102. The difference between the theoretical value and the calculated value, found from Appendix D, can be shown by following:

$$difference = 0.2094395102 - 0.209439092 = 0.0000004182$$

As the percent difference can be given by the following formula:

$$percent\ difference = \frac{theoretical - calculated}{theoretical} \times 100$$

I can also calculate the percent difference:

$$percent\ difference\ = \frac{0.0000004182}{0.2094395102} * 100 = 0.0001996757916\%$$

By having an accuracy of 0.0001996757916% shows a high accuracy of the arc length and can be used to infer that there is no mathematical error in my calculation. Next, I have repeated the following step 30 times, each with different parameters that are head to toe (Refer to Appendix E). The addition of all of the length of the 30 polynomials came out to be the following:

$$calculated\ length\ = 6.283172757$$

As I have performed the theoretical length of the knot previously, I could compare the two value to find a percent difference:

$$percent\ difference\ = \frac{6.283185307 - 6.283172757}{6.283185307} * 100 = 0.0001997394536\%$$

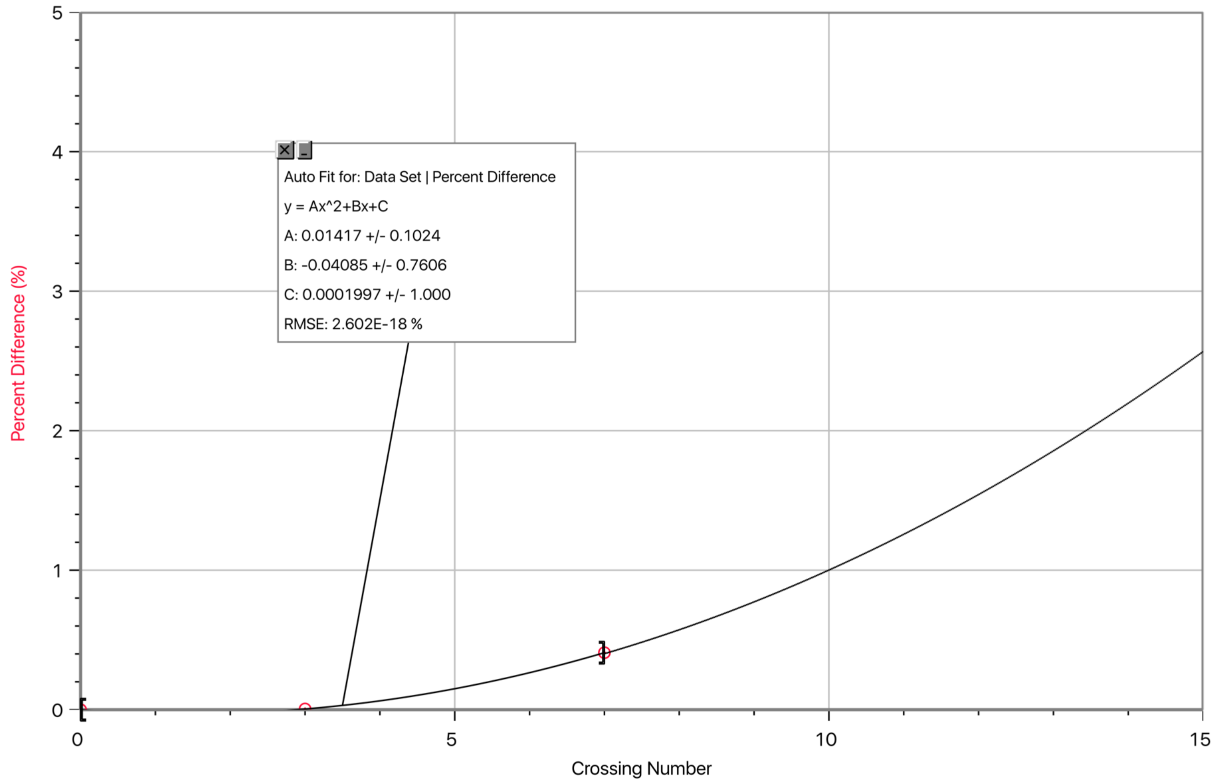
The same step can be repeated to the following results:

	Unit circle	Trefoil knot	Torus knot
Theoretical Length	6.283185307	28.82628995	62.13269572
Calculated Length	6.283172757	28.82478498	61.87872069
Difference	0.000012550	0.001504971	0.253975027
Percent Difference	0.0001997394536 %	0.005220827941%	0.4087622902%

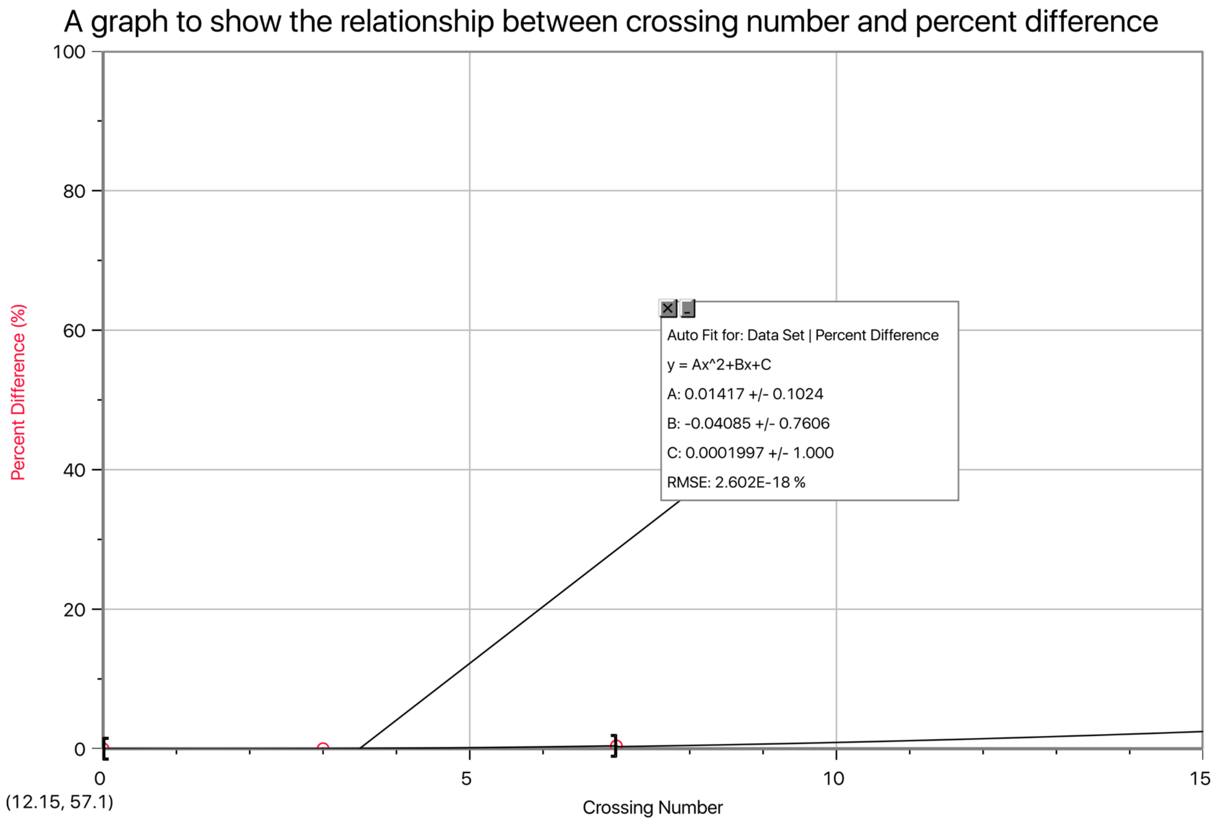
Conclusion

The three knots can be represented by crossing number as 0, 3, and 7, respectively. With all of the data, it shows to have a growing percent difference in terms of crossing number. This can be represented with this graph.

A graph to show the relationship between crossing number and percent difference

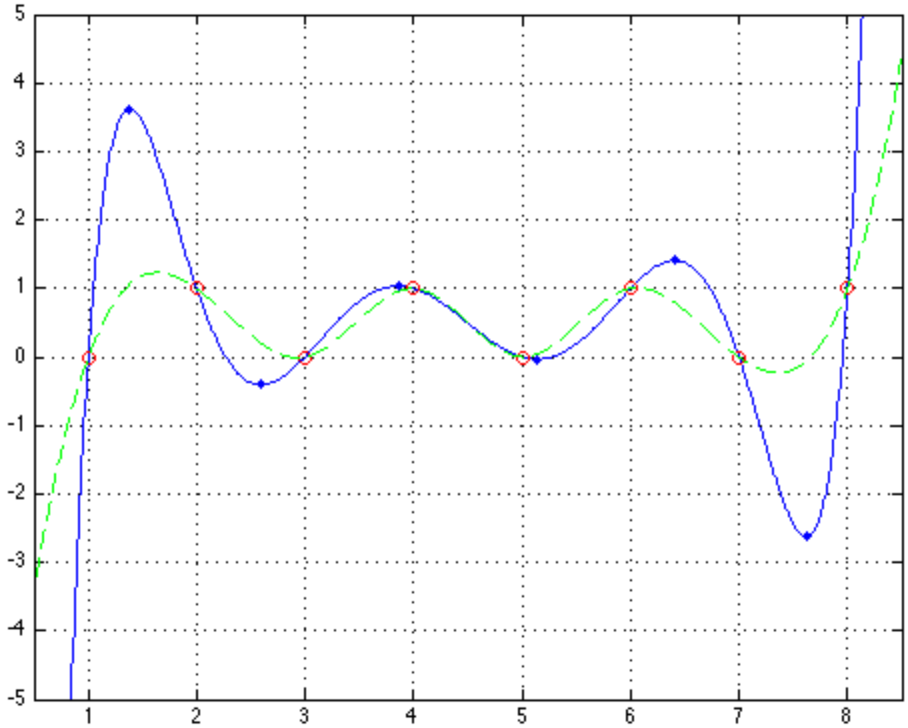


This graph is represented with a quadratic line and shows that it has a rapid increase of percent difference. However, this function could be looked again with y-axis scaled to be from 0% to 100%:



This graph shows the percent difference only rising up to approximately 5% by 15 crossing. 5% is still a relatively low percent uncertainty. The possible source of error can come from the degree of the Lagrange polynomial or the Gaussian quadrature.

In Lagrange polynomial, I have only gathered 60 points along the curve. The 60 points does not exactly trace out the curve and will, therefore, be more accurate with a large number of points. Additionally, when using Lagrange polynomial, I am connecting the points with a hypothetical line that I think best depicts the line. This degree of the line can alter the hypothetical line created as a higher degree will have more turning points, which described as curvatures. While one may think that having a higher degree will yield a more accurate polynomial, an example can be found that shows otherwise (Ellis):



In this diagram, the blue polynomial shows the interpolation of all 8 points at once, therefore, creating a 7th degree polynomial. However, the green line shows a collection of lesser degree polynomial that was used to connect 8 points, in an head-to-toe method. This can be one source of error as the hypothetical line will not match the actual curve of the knot. Additionally, it is impossible to say which degree is best suiting as it depends on the curve. However, the calculated value of the knot, when it was done with three points, was lower than the theoretical value. If this was the only source of error, it can be inferred that the knot had more curvature than what a quadratic polynomial would hold.

Secondly, the other source of error might be from Gaussian Quadrature. Gaussian Quadrature is an estimation of the definite integral of a line. Therefore, it would not perfectly yield to the theoretical value. Furthermore, having more points to use when estimating might help us to get a more accurate data, but I have decided to control the variable to only have 5 points. Although, Gaussian Quadrature will be a source of error, the understanding of how the mathematical component works is above my skill level, therefore, unable to analyse the possible source of error.

Thirdly, it was evident that as the crossing number increased the length also increased but the percent uncertainty increased. I believe this is because the amount of curvature in a knot increased. This is evident from the .gif files that I have created of a 3-dimensional model of a trefoil knot and a torus knot (Refer to Appendix F). Also, as I had a defined range of parameter t, all of the knots would fit in a set volume of a cube. However, if the length was longer, it would mean that it occupied more volume by having more loops within the set cube, therefore increasing the curvature. When I used Lagrange polynomial with three points, the difference will increase with more curvature, as quadratic line does not have a lot of curvature.

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Author's own (For the purpose of anonymity, the student's name has been withheld.

Graph of $y = \frac{x^2}{10}$from $0 \leq x \leq 10$with a large rectangle

Graph of $y = \frac{x^2}{10}$from $0 \leq x \leq 10$with smaller rectangles

Table of Interpolation when $n = 3$

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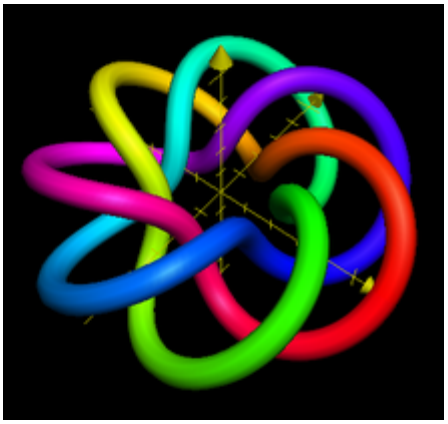
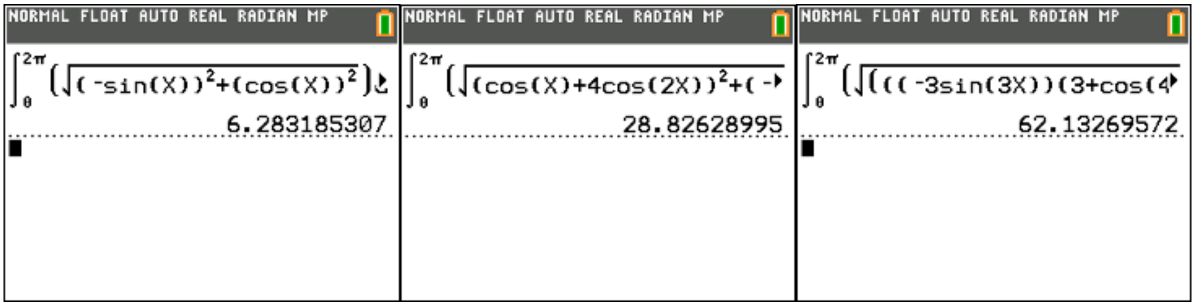
Appendix

Appendix A - Image of a torus knot

 (Torus)

Appendix B - Calculations of the theoretical length using TI-84



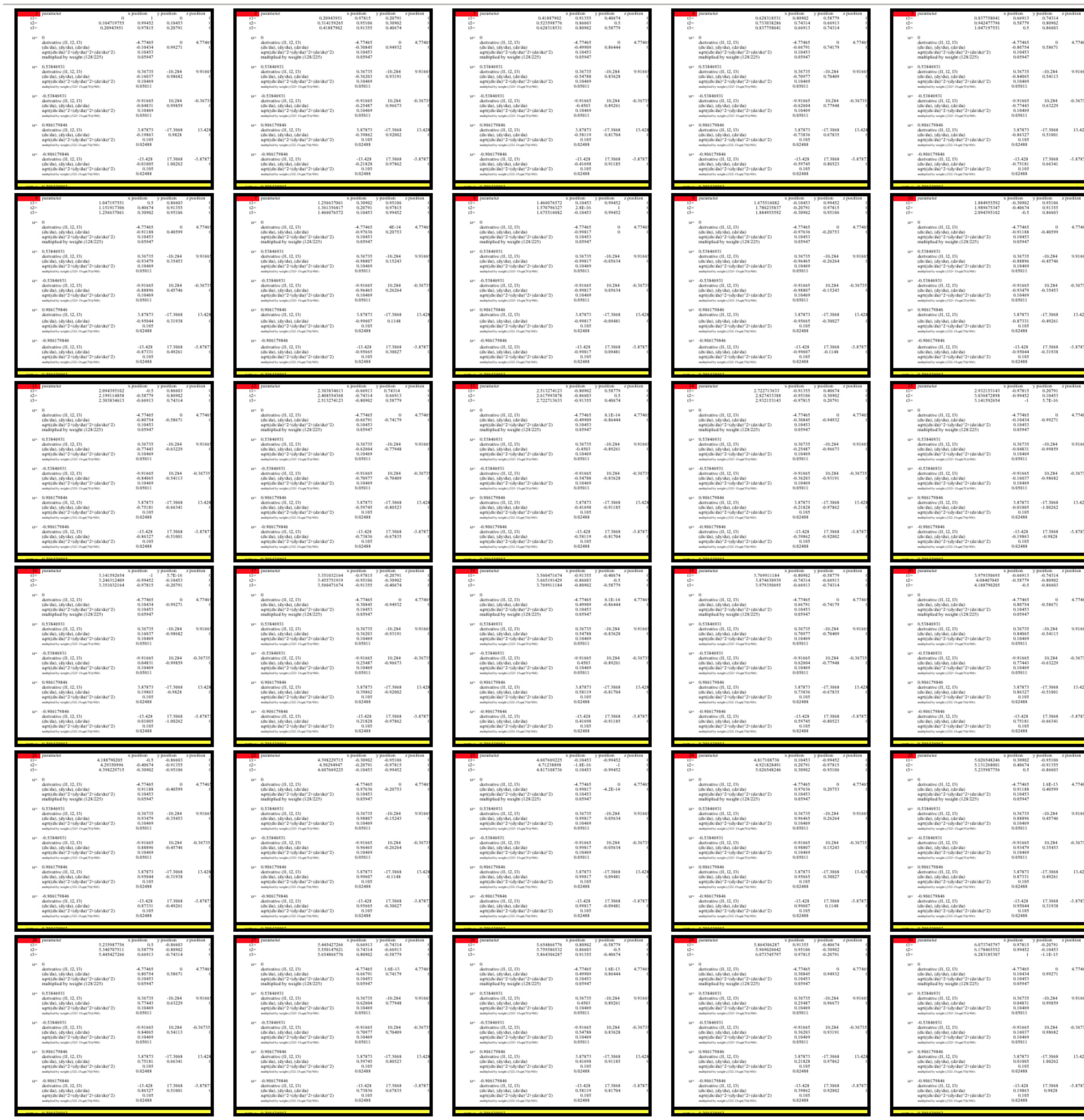
Appendix C - Calculations done using Excel

1	parameter	x position	y position	z position	
t1=	0	1	0	0	
t2=	0.104719755	0.9945219	0.10452846	0	
t3=	0.20943951	0.9781476	0.20791169	0	
t=	0				
	l1, l2, l3	1	0	0	base function
	x(t), y(t), z(t)	1	0	0	
	derivative (l1, l2, l3)	-14.323945	19.0985932	-4.7746483	
	(dx/dt), (dy/dt), (dz/dt)	-0.0002866	1.0036414	0	
	sqrt((dx/dt)^2+(dy/dt)^2+(dz/dt)^2)	1.00364144			

Appendix D - Calculations for Gaussian Quadrature

1	parameter	x position	y position	z position	
t1=	0	1	0	0	
t2=	0.104719755	0.9945219	0.1045285	0	
t3=	0.20943951	0.9781476	0.2079117	0	
t=	0				
	l1, l2, l3	1	0	0	base function
	x(t), y(t), z(t)	1	0	0	
	derivative (l1, l2, l3)	-14.323945	19.098593	-4.7746483	
	(dx/dt), (dy/dt), (dz/dt)	-0.0002866	1.0036414	0	
	sqrt((dx/dt)^2+(dy/dt)^2+(dz/dt)^2)	1.0036414			
	in order to make 0<t<2pi/30 to -1<u<1 for guassian, u=(30/pi)t-1, t=(u+1)/(30/pi)				
u=	0				
	l1, l2, l3	-0.08	0.96	0.12	
	x(u), y(u), z(u)	0.9921187	0.1252967	0	
	derivative (l1, l2, l3)	-4.7746483	0	4.7746483	
	(dx/du), (dy/du), (dz/du)	-0.1043375	0.9927052	0	
	sqrt((dx/du)^2+(dy/du)^2+(dz/du)^2)	0.1045285			
	multiplied by weight (128/225)	0.0594651			
u=	0.53846931				
	l1, l2, l3	-0.0650855	0.2840079	0.7810776	
	x(u), y(u), z(u)	0.9813758	0.1920821	0	
	derivative (l1, l2, l3)	0.3673549	-10.284006	9.9166514	
	(dx/du), (dy/du), (dz/du)	-0.1603658	0.9868164	0	
	sqrt((dx/du)^2+(dy/du)^2+(dz/du)^2)	0.1046948			
	multiplied by weight ((322+13sqrt(70))/900)	0.0501099			
u=	-0.53846931				
	l1, l2, l3	0.3226124	0.8009384	-0.1235508	
	x(u), y(u), z(u)	0.9983123	0.0580332	0	
	derivative (l1, l2, l3)	-9.9166514	10.284006	-0.3673549	
	(dx/du), (dy/du), (dz/du)	-0.0483093	0.998594	0	
	sqrt((dx/du)^2+(dy/du)^2+(dz/du)^2)	0.1046948			
	multiplied by weight ((322+13sqrt(70))/900)	0.0501099			
u=	0.906179846				
	l1, l2, l3	0.1850118	-0.6574395	1.4724276	
	x(u), y(u), z(u)	0.9714254	0.2374138	0	
	derivative (l1, l2, l3)	3.8787318	-17.30676	13.428028	
	(dx/du), (dy/du), (dz/du)	-0.1986264	0.982795	0	
	sqrt((dx/du)^2+(dy/du)^2+(dz/du)^2)	0.1049989			
	multiplied by weight ((322-13sqrt(70))/900)	0.0248771			
u=	-0.906179846				
	l1, l2, l3	0.8374613	0.2124932	-0.0499545	
	x(u), y(u), z(u)	0.9999276	0.0118255	0	
	derivative (l1, l2, l3)	-13.428028	17.30676	-3.8787318	
	(dx/du), (dy/du), (dz/du)	-0.0100486	1.0026154	0	
	sqrt((dx/du)^2+(dy/du)^2+(dz/du)^2)	0.1049989			
	multiplied by weight ((322-13sqrt(70))/900)	0.0248771			
sum =	0.209439092				

Appendix E - Repeated Calculations for a unit circle



Appendix F - GIF image of the knots

Trefoil Knot:
https://drive.google.com/file/d/0B86VGubhb9kwaENWYjhETEtzemM/view?usp=sharing
Torus Knot:
https://drive.google.com/file/d/0B86VGubhb9kwX0FBdV8yVHpyQnc/view?usp=sharing