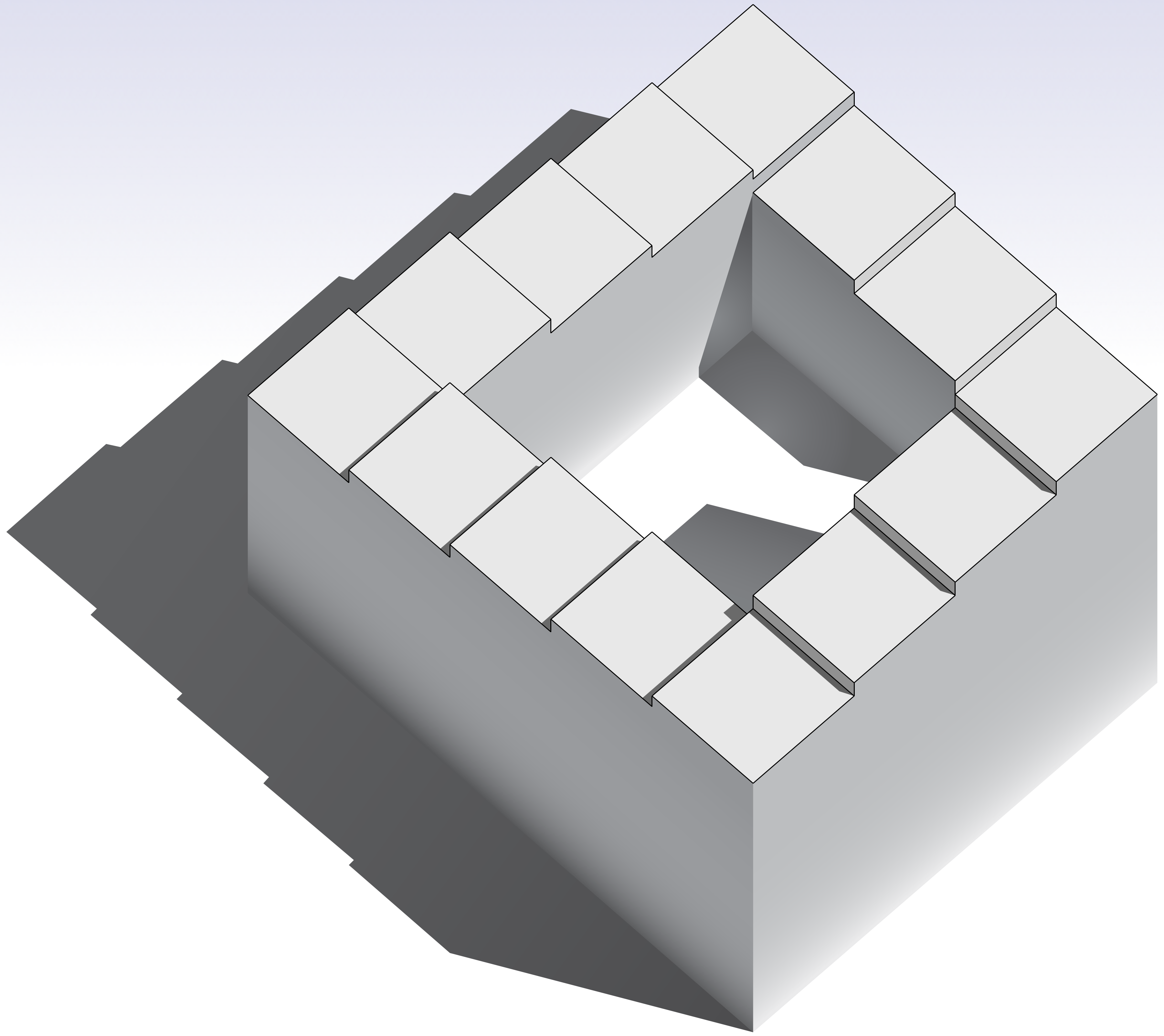
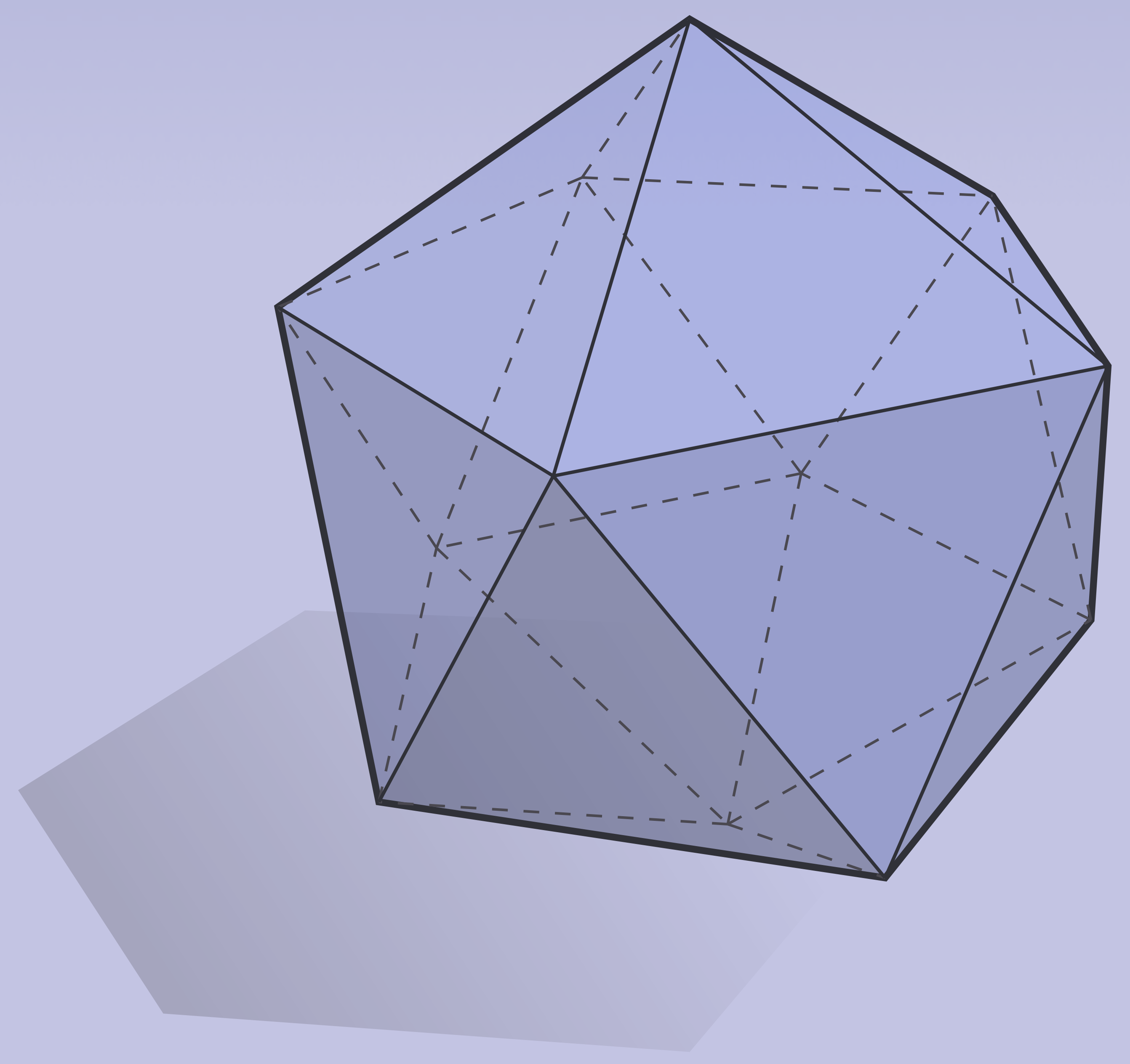


Discrete Differential Geometry: An Applied Introduction

Keenan Crane • CMU 15-458/858

LECTURE 10:

INTRODUCTION TO CURVES



DISCRETE DIFFERENTIAL GEOMETRY: AN APPLIED INTRODUCTION

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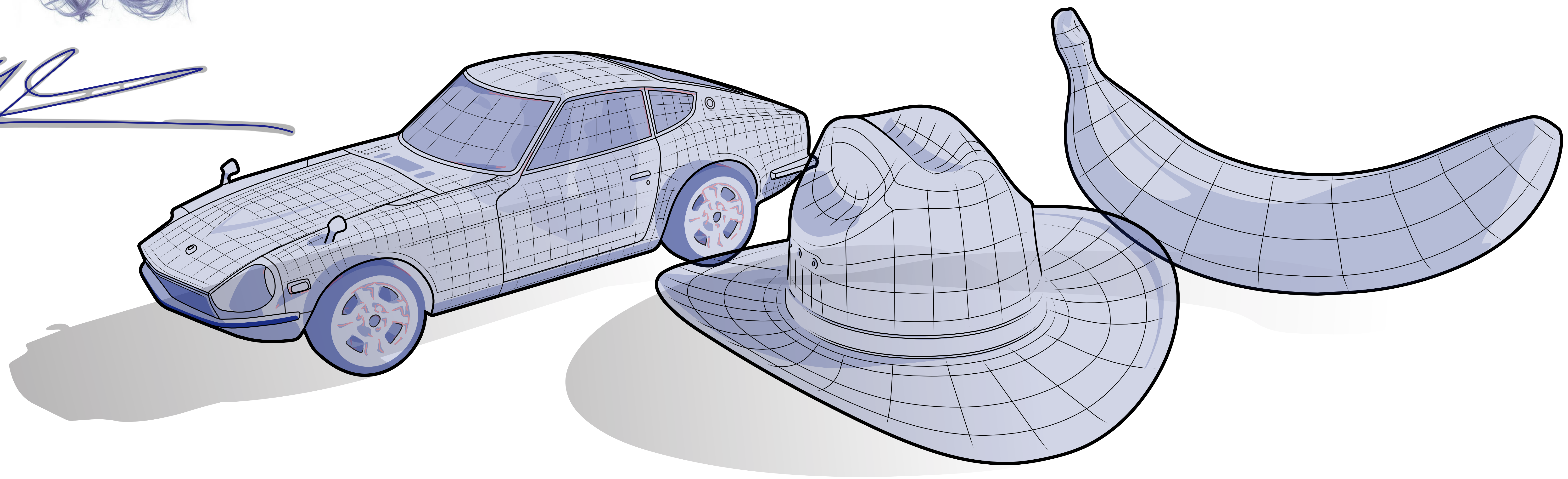
Curves & Surfaces

- Much of the geometry we encounter in life well-described by *curves* and *surfaces**

(Curves)

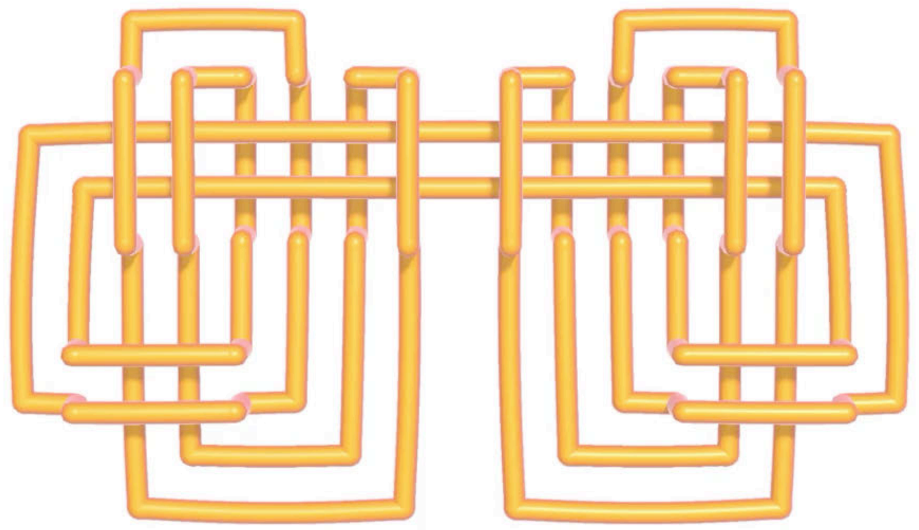
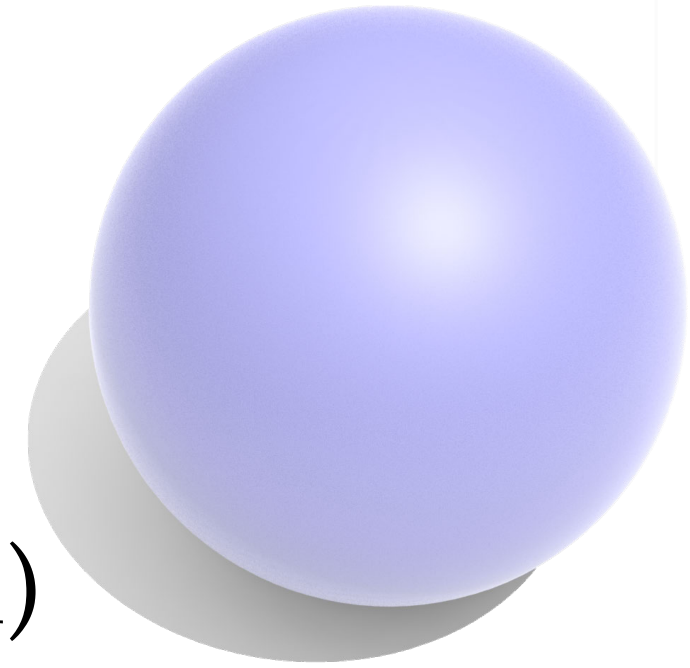
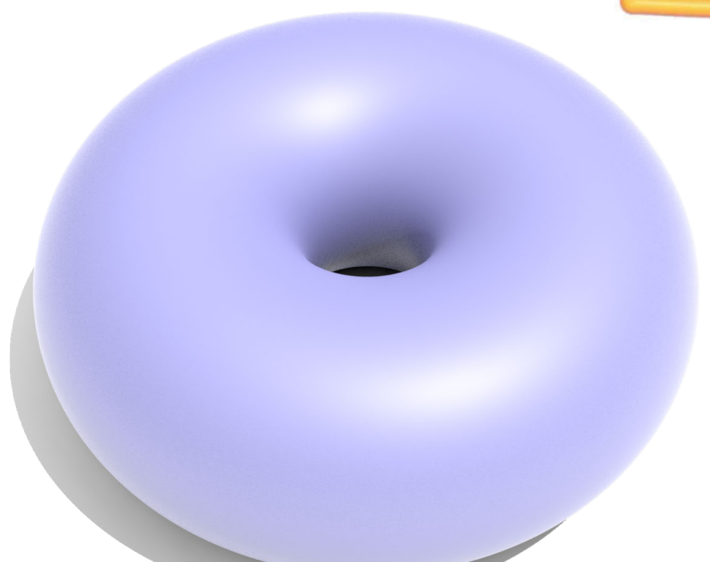
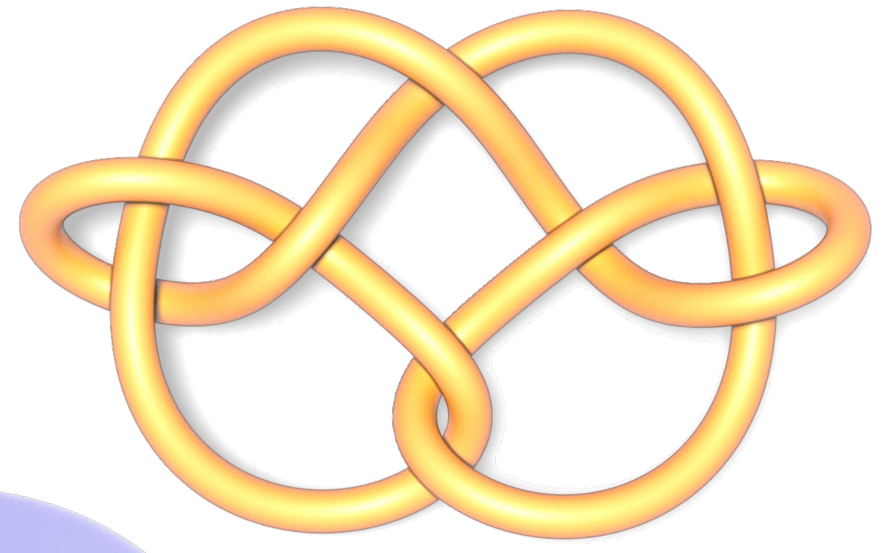
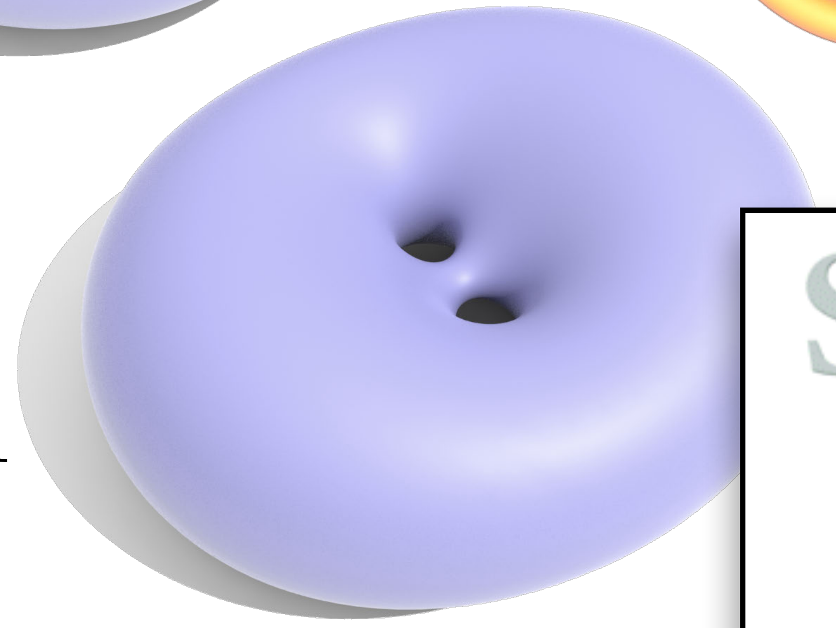


*Or solids… but the boundary of a solid is a surface!

(Surfaces)

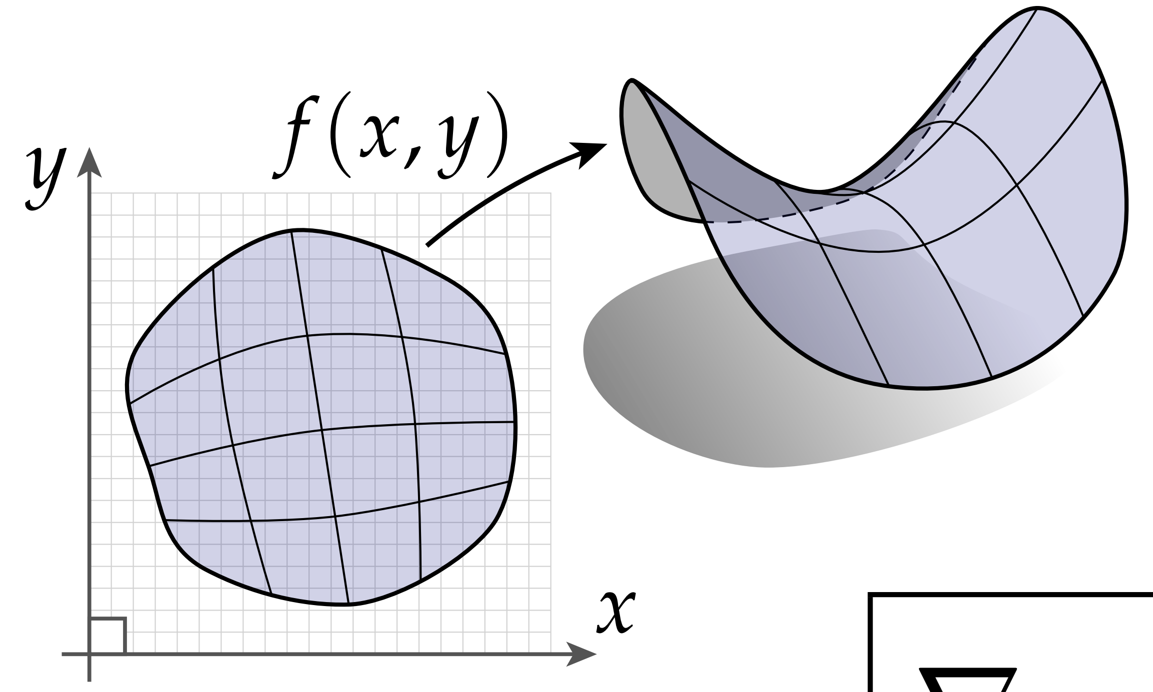
Much Ado About Manifolds

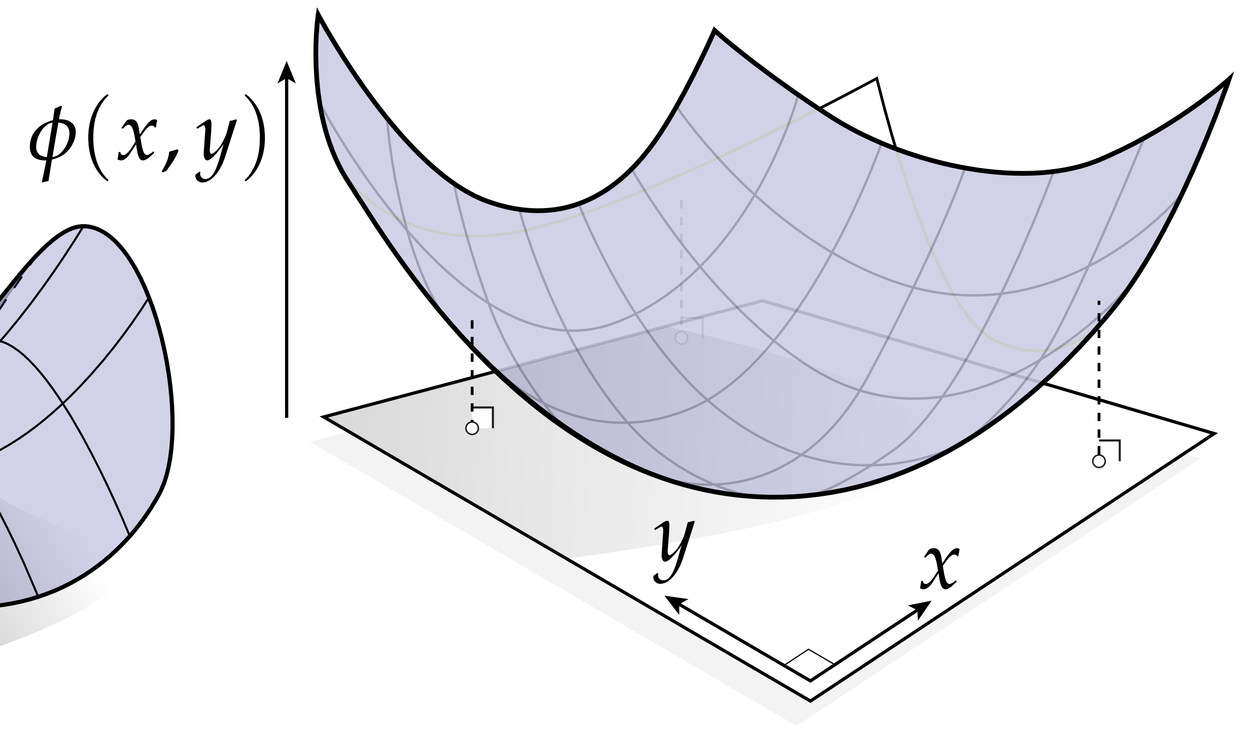
- In general, differential geometry studies n-dimensional manifolds; we'll focus on low dimensions: curves (n=1), surfaces (n=2), and volumes (n=3)
- Why? Geometry we encounter in everyday life/applications
- Low-dimensional manifolds are not "baby stuff!"
  - ***n*=1:** unknot recognition (open as of 2021)
  - ***n*=2:** Willmore conjecture (2012 for genus 1)
  - ***n*=3:** Geometrization conjecture (2003, $1 million)
- Serious intuition gained by studying low-dimensional manifolds
- Conversely, problems involving very high-dimensional manifolds (*e.g.*, data analysis/machine learning) involve less "deep" geometry than you might imagine!
  - *fiber bundles, Lie groups, curvature flows, spinors, symplectic structure, …*
- Moreover, curves and surfaces are <u>beautiful!</u> (And in some cases, high-dimensional manifolds have *less* interesting structure…)

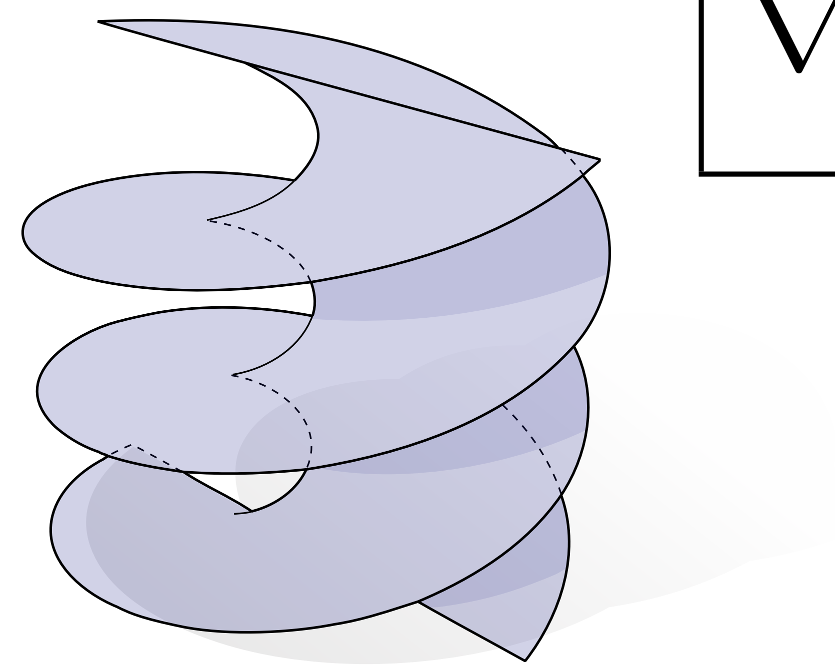

Smooth Descriptions of Curves & Surfaces

- Many ways to express the geometry of a curve or surface:
  - height function over tangent plane
  - local parameterization
  - Christoffel symbols — coordinates/indices
  - **differential forms** — “coordinate free”
  - moving frames — change in *adapted frame*
  - Riemann surfaces (*local*); Quaternionic functions (*global*)
- People can get religious about these different “dialects”... best to be multilingual!
- We'll dive deep into one description (**differential forms**) and touch on others





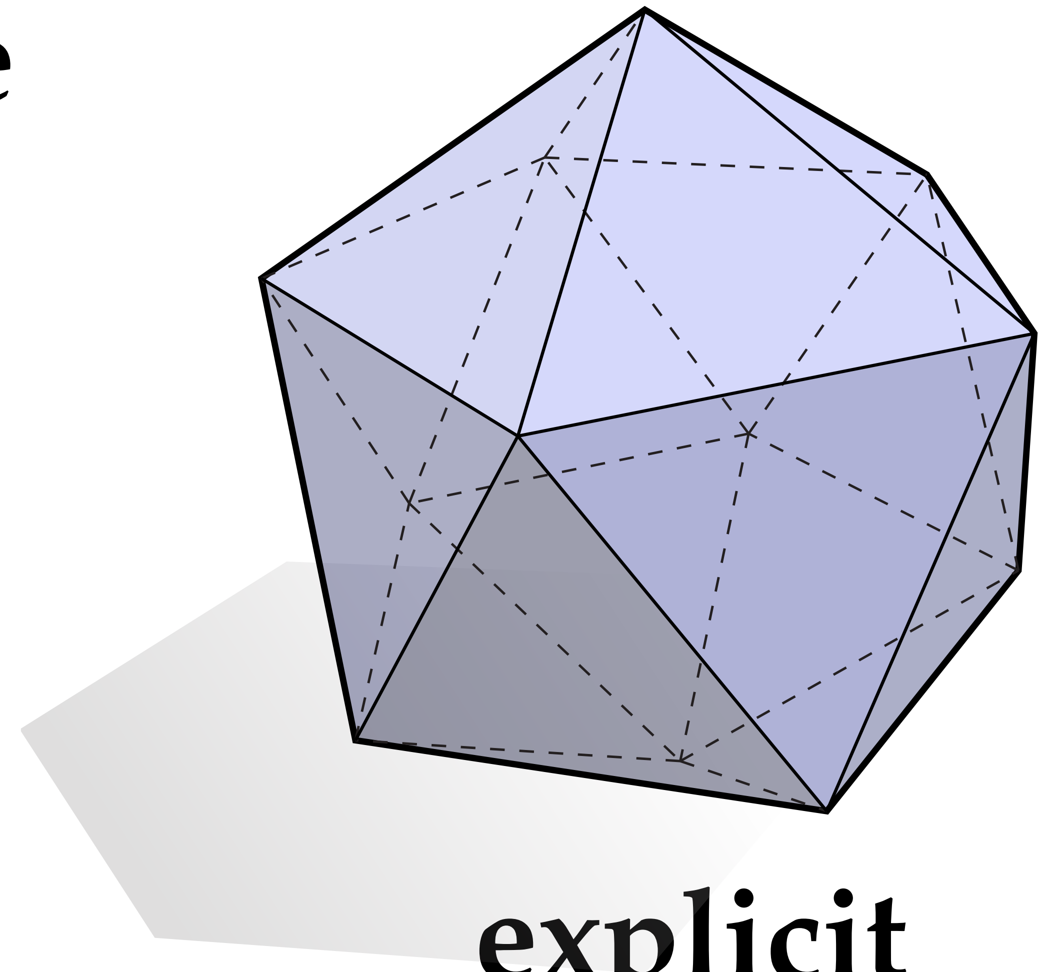
$$\boxed{\nabla_i e_j = \Gamma_{ij}^k e_k}$$



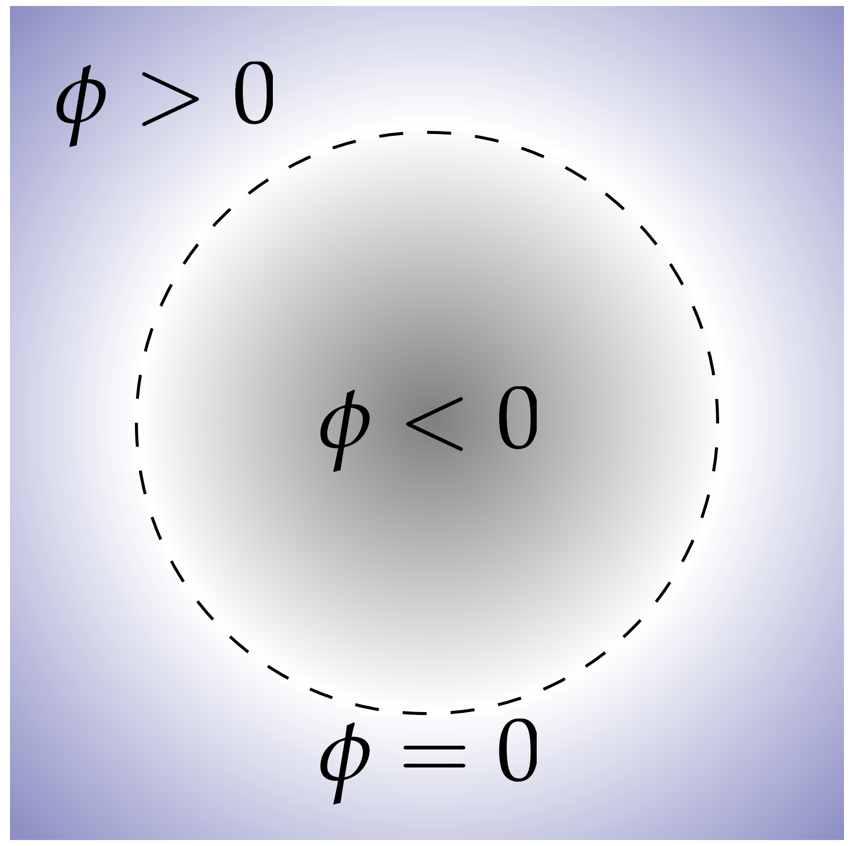
$$\boxed{\begin{aligned} d\sigma &= \omega \wedge \sigma \\ d\omega &= \omega \wedge \omega \end{aligned}}$$

Discrete Descriptions of Curves & Surfaces

- Also *many* discrete representations of geometry
- For instance:
  - **implicit** — *e.g.*, zero set of scalar function on a grid
    - good for changing topology
    - expensive to store/adaptivity is hard
    - hard to solve sophisticated equations *on* surface
  - **explicit** — *e.g.*, polygonal surface mesh
    - changing topology is harder
    - cheaper to store / adaptivity is easier
    - more mature tools for equations *on* surfaces
- Don't be "religious"—use the right tool for the job!



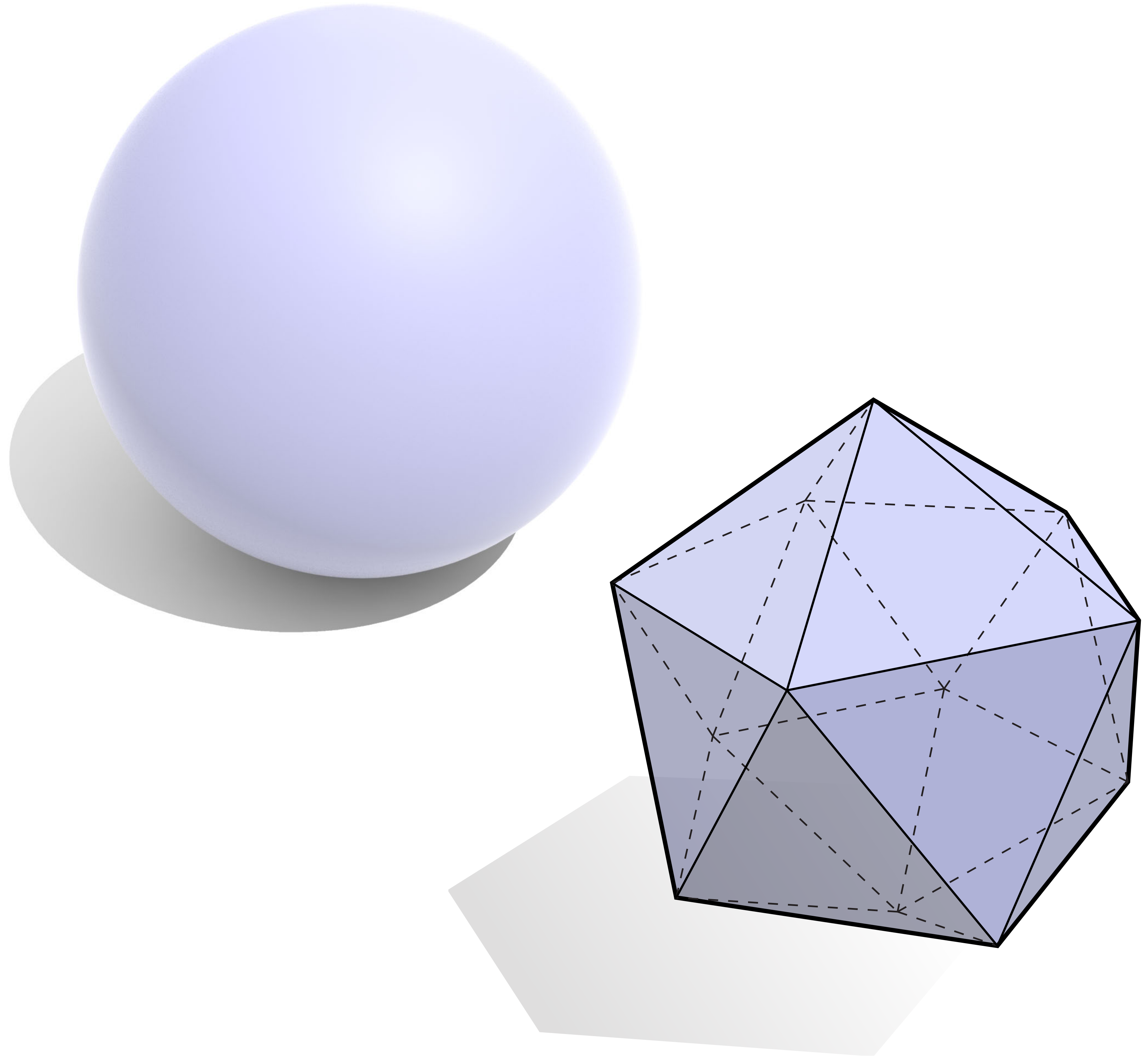
explicit



implicit

Curves & Surfaces—Overview

- **Goal:** understand curves & surfaces from complementary smooth and discrete points of view.
- <u>**Smooth setting:**</u>
  - express geometry via differential forms
  - will first need to think about *vector-valued* forms
- <u>**Discrete setting:**</u>
  - use explicit mesh as domain
  - express geometry via discrete differential forms

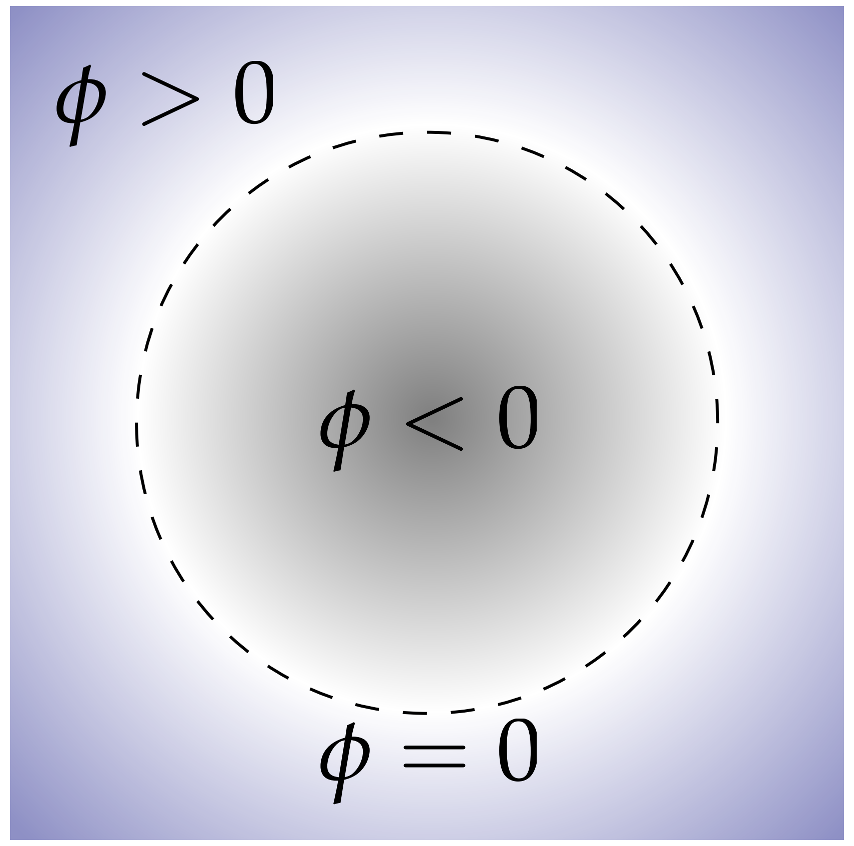


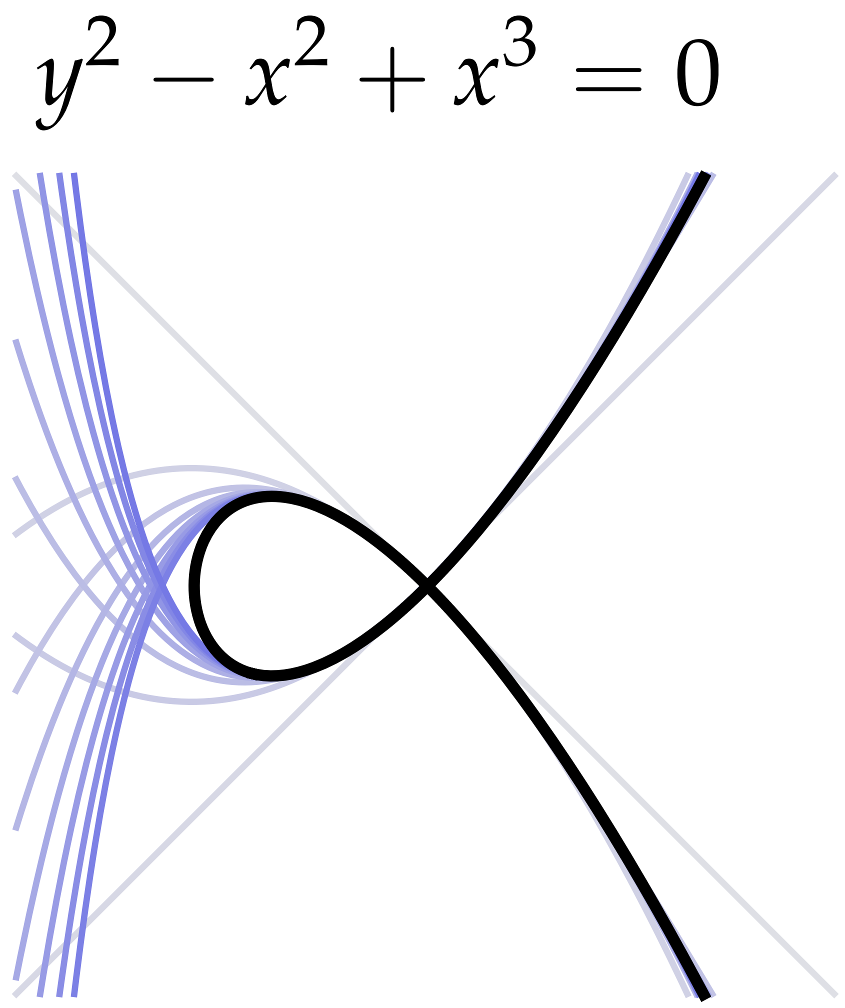
- **Payoff:** will become very easy to switch back & forth between smooth setting (*scribbling in a notebook*) and discrete setting (*running algorithms on data*)

Planar Curves

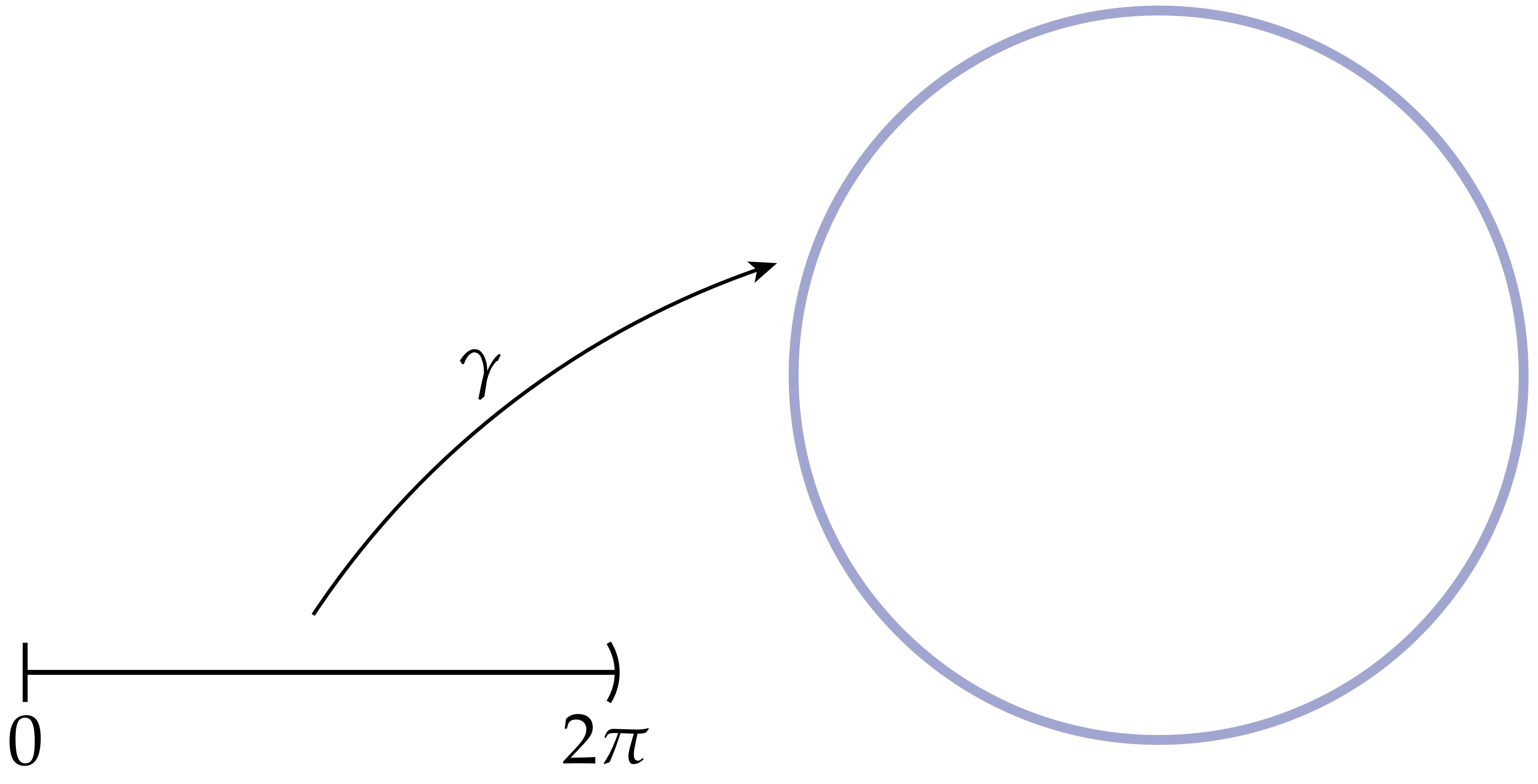
Planar Curves — Overview

- How can we describe curves in the plane?
- One way: <u>subset</u> of plane (*e.g.*, *"algebraic curve"*)
  - points where a function $\phi(x,y)$ vanishes
  - *e.g.*, circle is points where $x^2 + y^2 - 1 = 0$
  - becomes tricky when curve crosses itself…





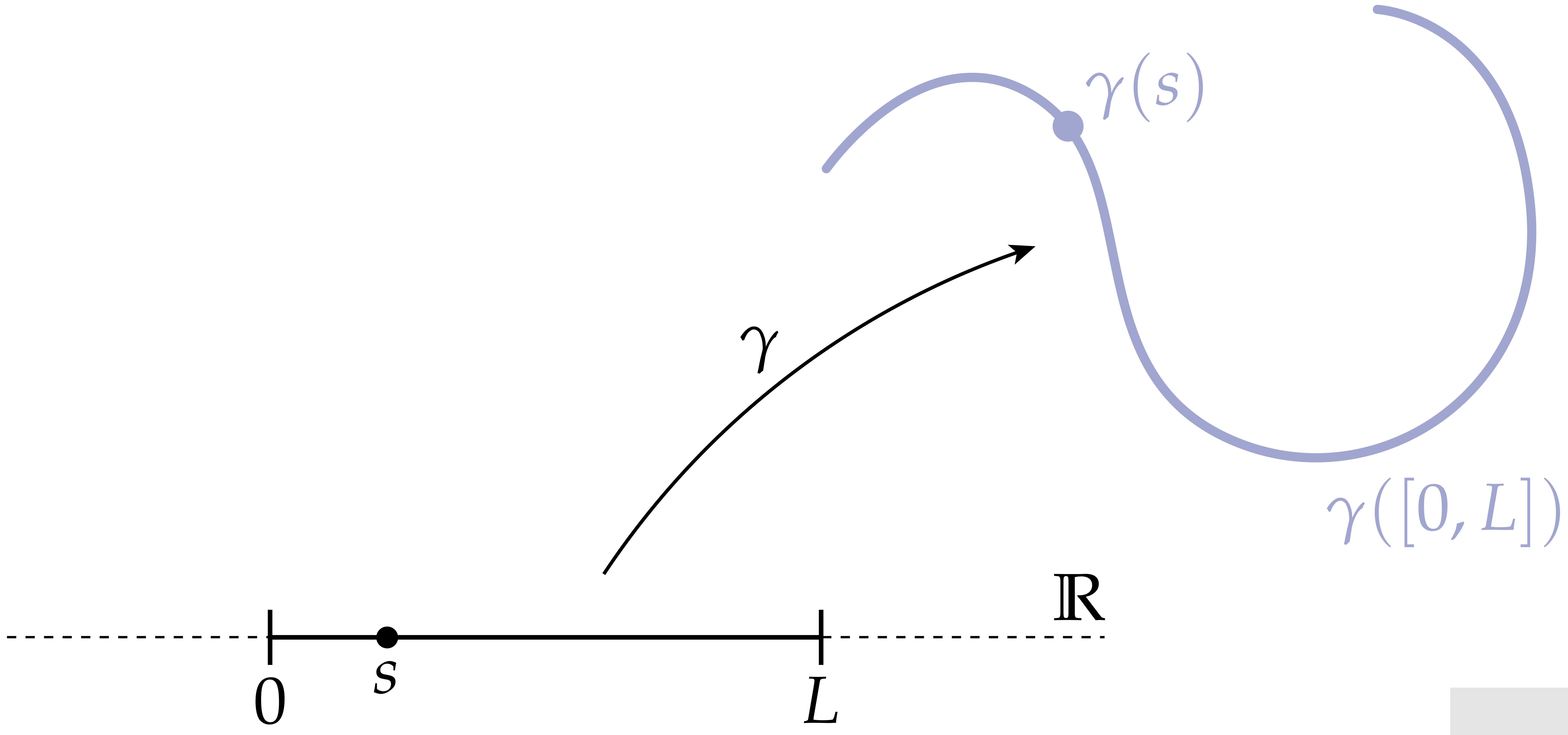
- Curve as <u>map</u> to plane (*"parameterized curve"*)
  - *e.g.*, circle is $\gamma(\theta) = (\cos\theta, \sin\theta)$
  - easy to talk about crossings (*immersed vs. embedded*)
  - can also encode velocity (*e.g.*, physical trajectory)



- Parameterized curves will have natural discrete analog (polygonal curves)

Parameterized Plane Curve

A **parameterized plane curve** is a map* taking each point in an interval $[0,L]$ of the real line to some point in the plane $\mathbb{R}^2$:



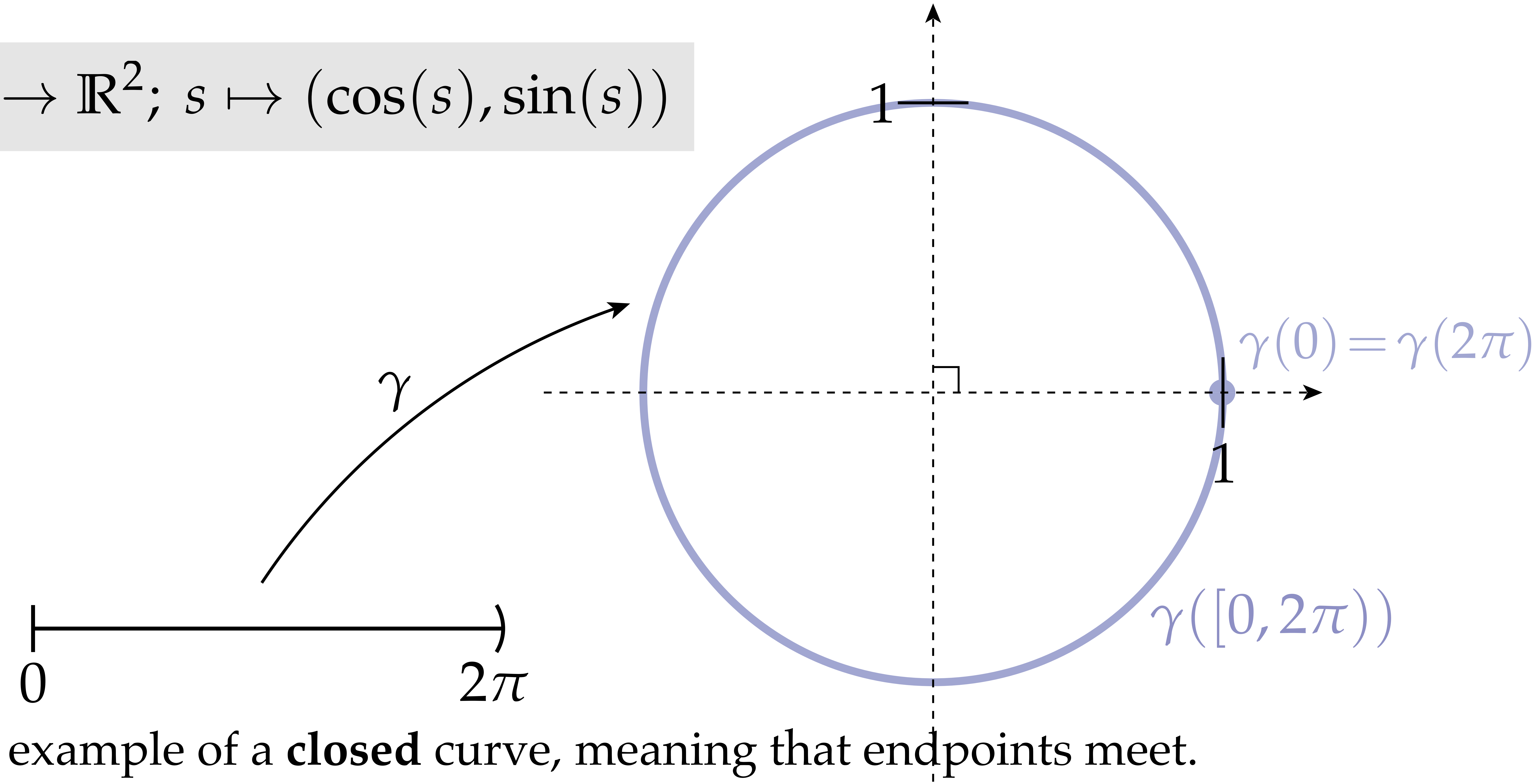
*Continuous, differentiable, smooth…

$$\gamma : [0, L] \to \mathbb{R}^2$$

Curves in the Plane—Example

As an example, we can express a circle as a parameterized curve γ:

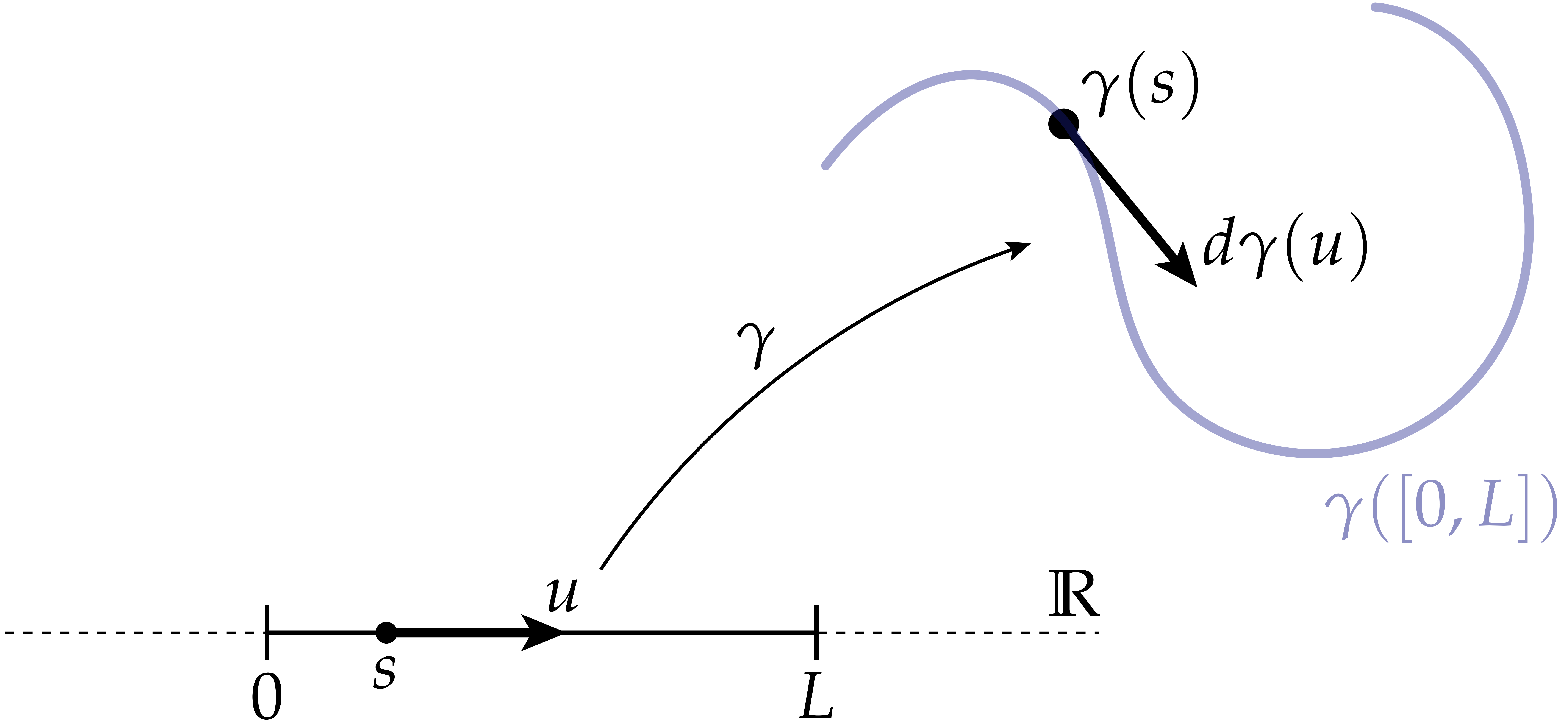
$$\gamma : [0, 2\pi) \to \mathbb{R}^2;\ s \mapsto (\cos(s), \sin(s))$$



The circle is an example of a **closed** curve, meaning that endpoints meet.

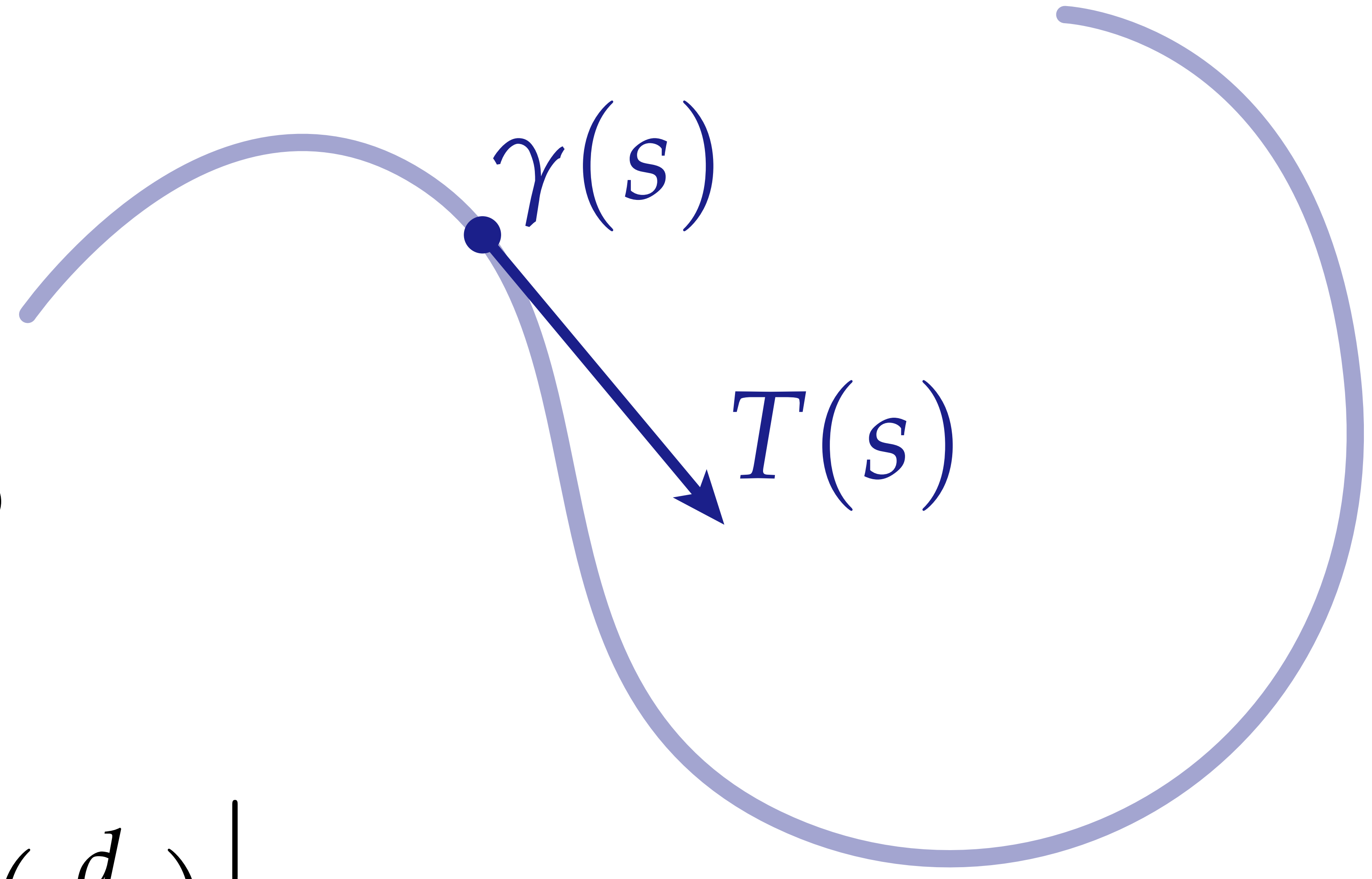
Differential of a Curve

- If we think of a parameterized curve as an $\mathbb{R}^2$-valued 0-form on an interval of the real line, the *differential* (or exterior derivative) says how vectors on the domain get mapped/"stretched" into the plane:



Tangent of a Curve

- Informally, a vector is *tangent* to a curve if it "just barely grazes" the curve.
- More formally, the **unit tangent** (or just **tangent**) of a parameterized curve is the map obtained by normalizing its first derivative*:



$$T(s) := \tfrac{d}{ds}\gamma(s) \Big/ \left|\tfrac{d}{ds}\gamma(s)\right| = d\gamma(\tfrac{d}{ds}) \Big/ \left|d\gamma(\tfrac{d}{ds})\right|$$

- If the derivative already has unit length, then we say the curve is **arc-length parameterized** and can write the tangent as just

$$T(s) := \tfrac{d}{ds}\gamma(s) = d\gamma(\tfrac{d}{ds})$$

*Assuming curve never slows to a stop, *i.e.*, assuming it's "regular"

Tangent of a Curve — Example

Let's compute the unit tangent of a circle:

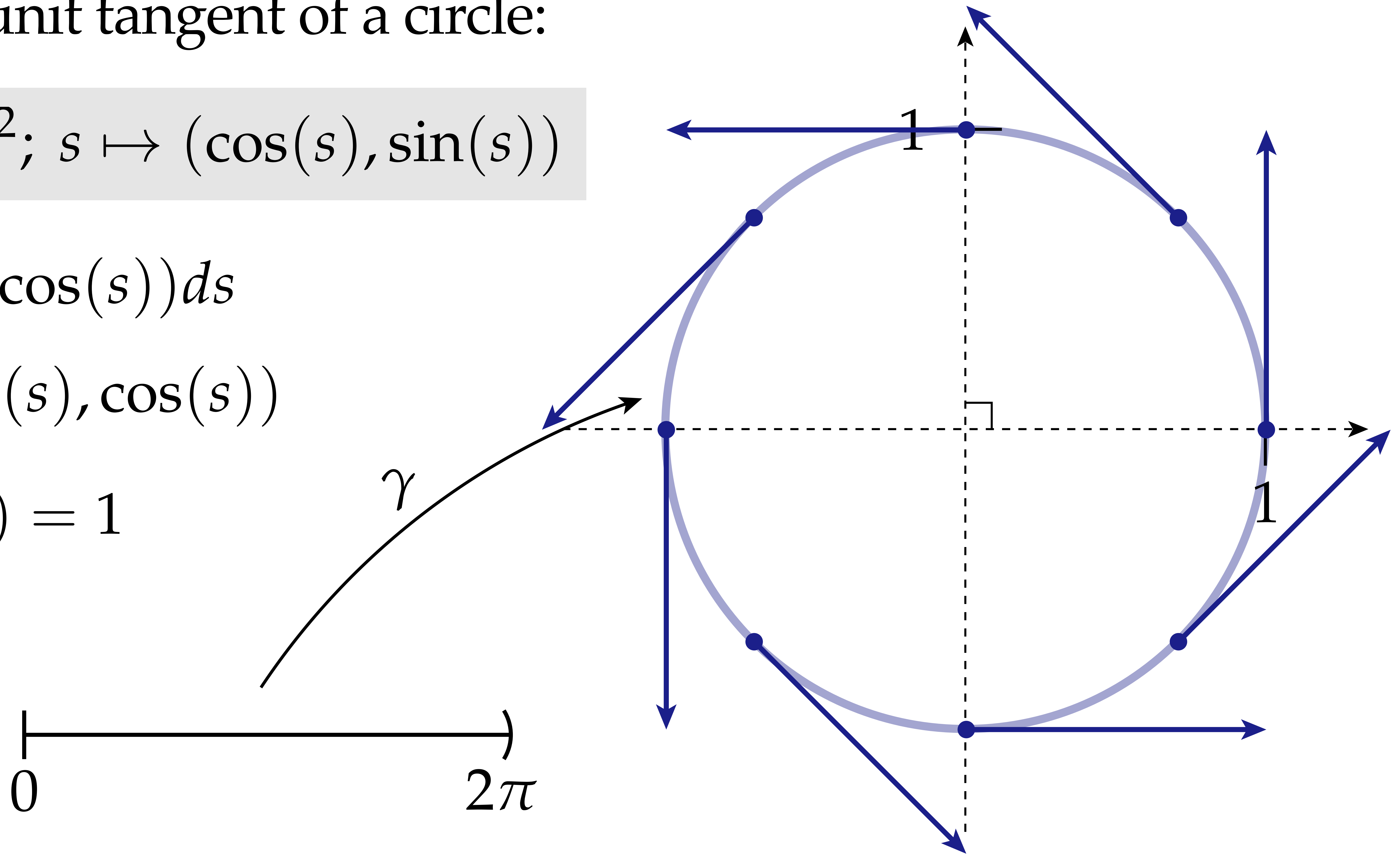
$$\gamma : [0, 2\pi) \to \mathbb{R}^2;\ s \mapsto (\cos(s), \sin(s))$$

$$d\gamma = (-\sin(s), \cos(s))ds$$

$$d\gamma(\tfrac{\partial}{\partial s}) = (-\sin(s), \cos(s))$$

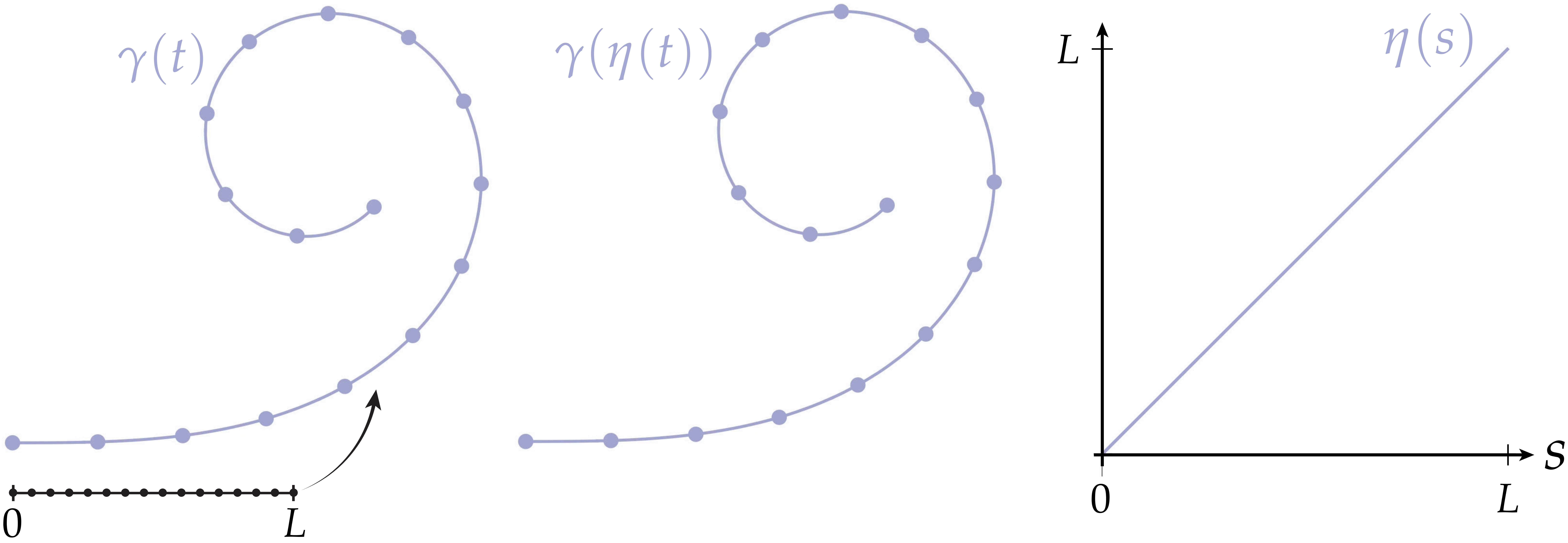
$$\cos^2(s) + \sin^2(s) = 1$$

$$\Rightarrow T = d\gamma(\tfrac{d}{ds})$$



Reparameterization of a Curve

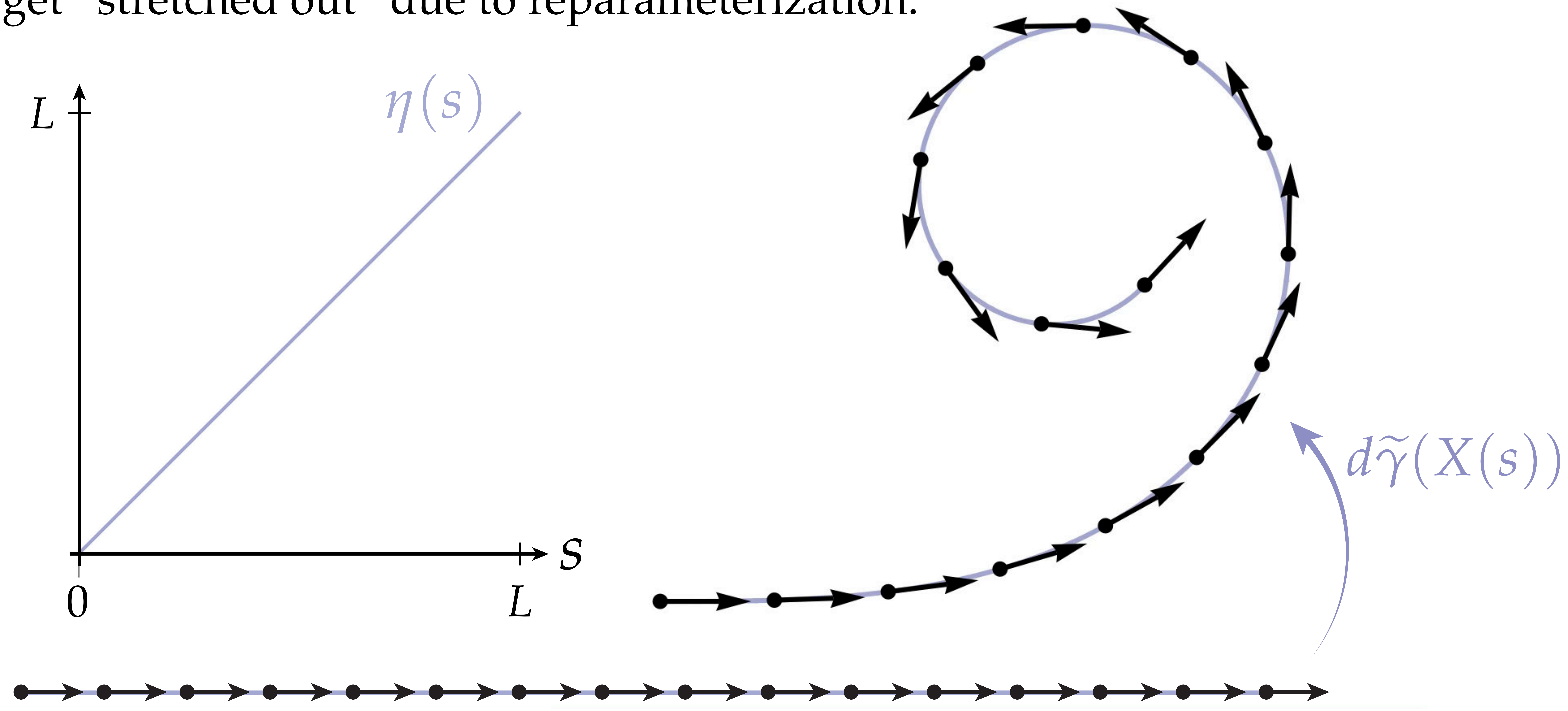
Can *reparameterize* a curve $\gamma : [0, L] \to \mathbb{R}^2$ by first applying a bijection $\eta : [0, L] \to [0, L]$ to obtain a new parameterized curve $\widetilde{\gamma}(t) := \gamma(\eta(t))$



Key idea: new curve looks identical (same image), even though the map describing it changes.

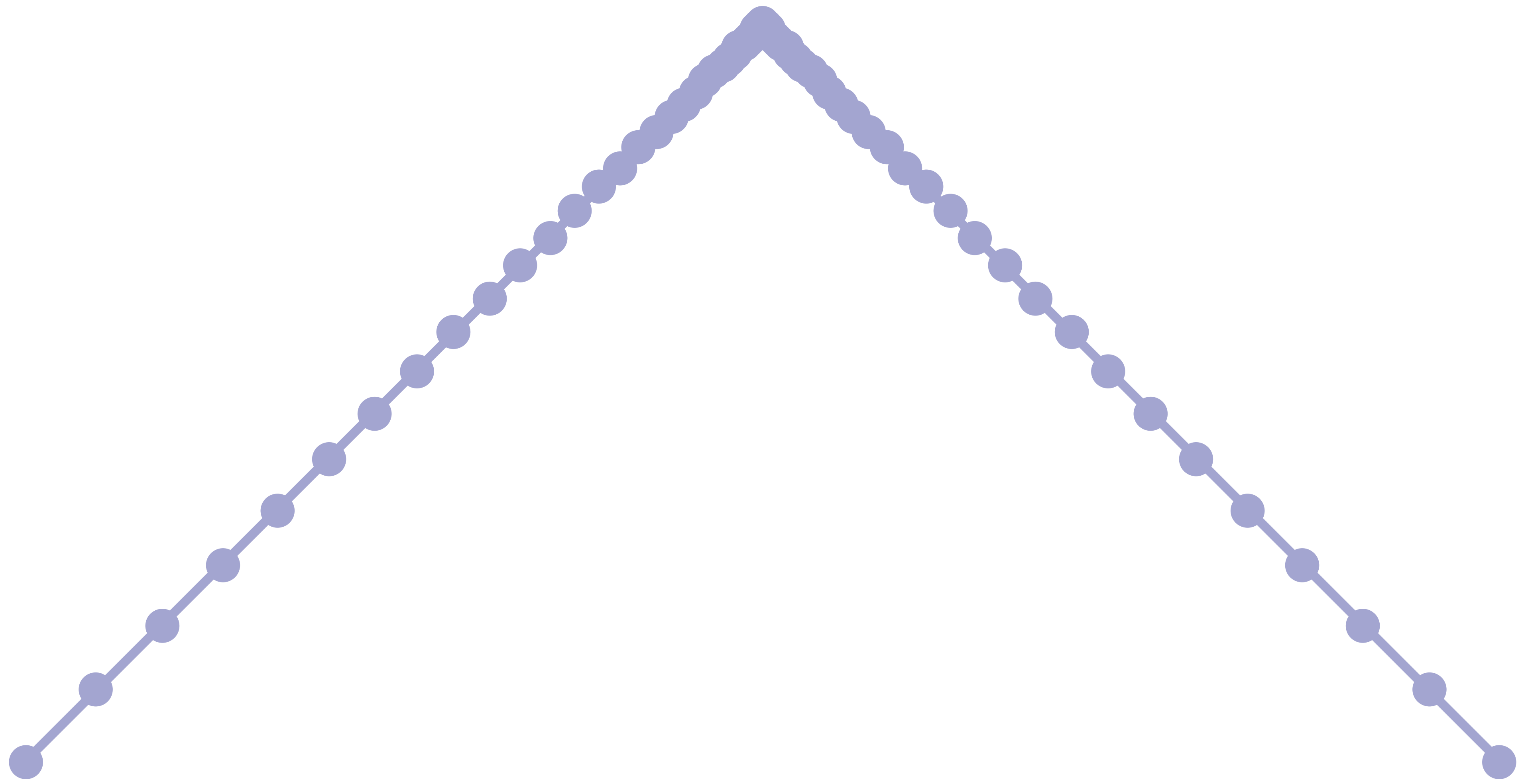
Differential & Reparameterization

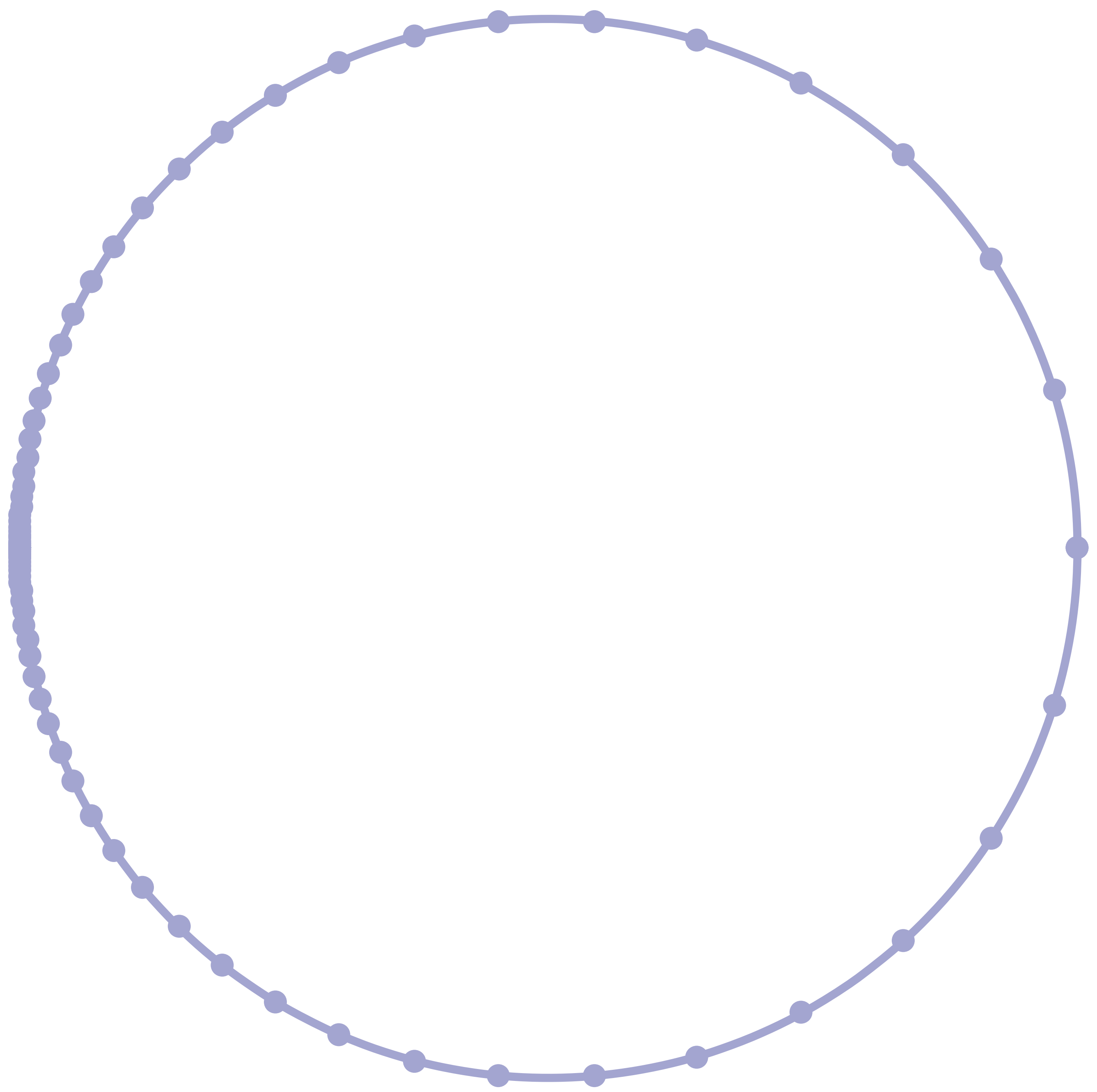
For a curve $\widetilde{\gamma}(t) := \gamma(\eta(t))$, the differential $d\widetilde{\gamma} = d\gamma \circ d\eta$ captures how vectors get "stretched out" due to reparameterization:



Regular Curve / Immersion

- A parameterized curve is *regular* (or *immersed*) if the differential is nonzero everywhere, *i.e.*, if the curve "never slows to zero"
- Without this condition, a parameterized curve may look non-smooth but actually be differentiable everywhere, or look smooth but fail to have well-defined tangents.



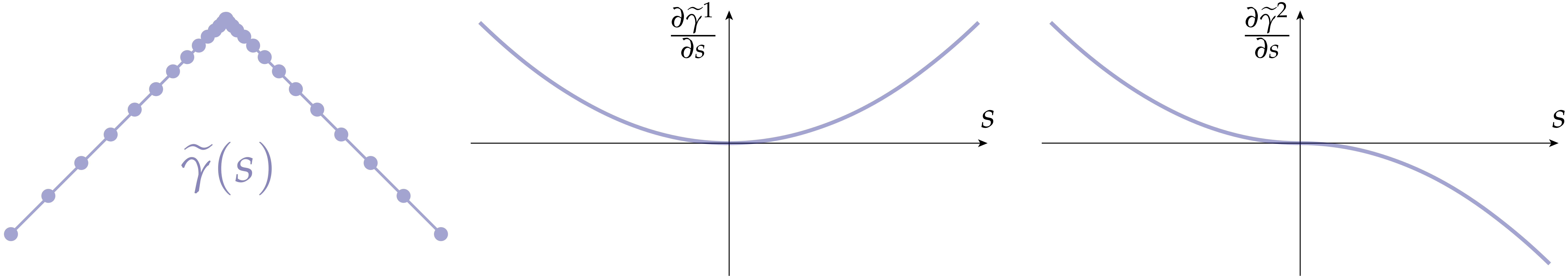


Irregular Curve—Example

- Consider the reparameterization of a piecewise linear curve:

$$\eta(s) := s^3 \qquad \gamma(s) := \begin{cases} (s,s) & s \leq 0 \\ (s,-s) & s > 0 \end{cases} \qquad \tilde{\gamma}(s) = \begin{cases} (s^3,s^3) & s \leq 0, \\ (s^3,-s^3) & s > 0 \end{cases}$$

- Even though the reparameterized curve has a continuous first derivative, this derivative goes to zero at $s = 0$:

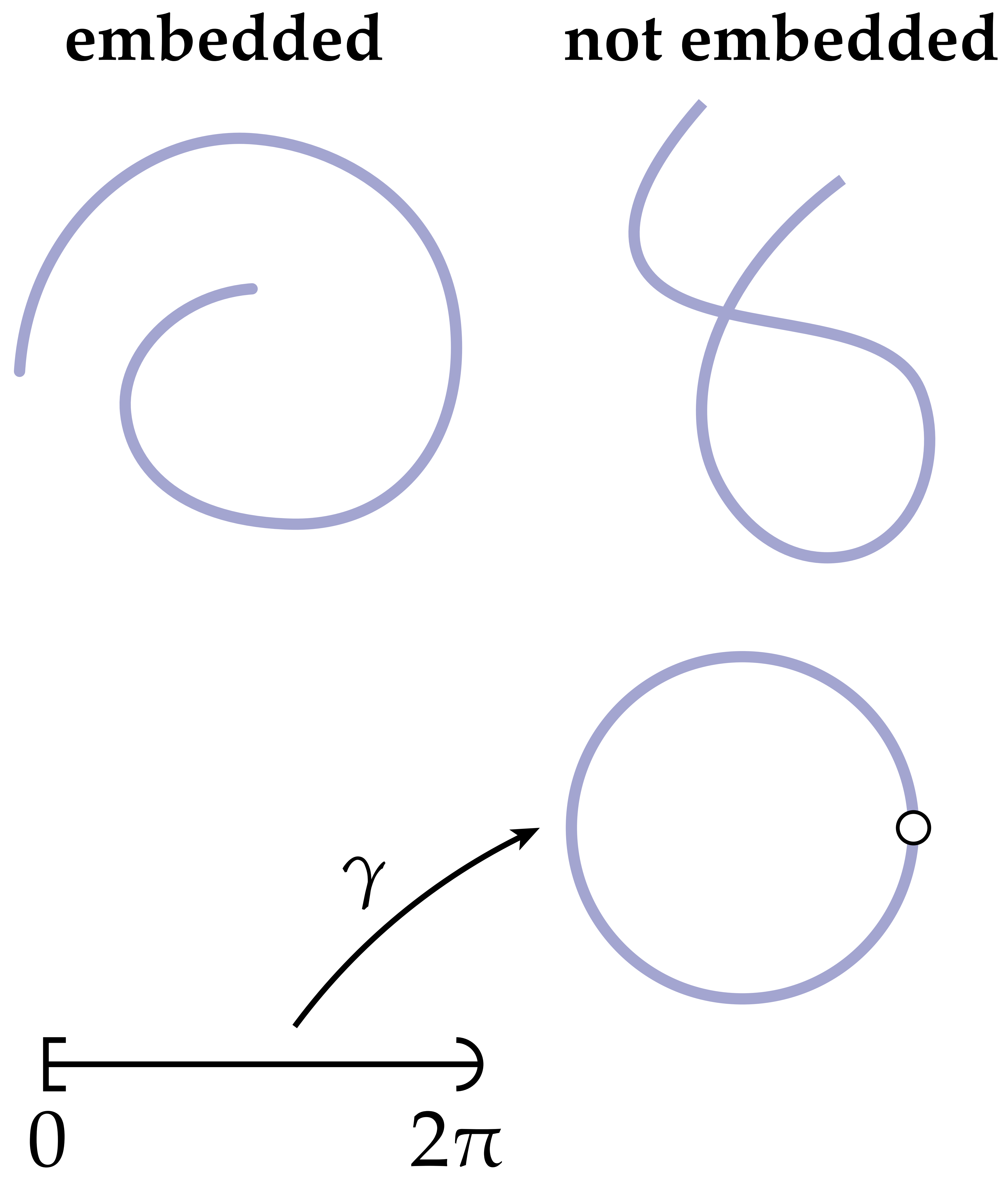


- Hence, (still) can't define tangent at s=0 (or normal, curvature, …)

Key idea: Even a differentiable curve may not be "nice."

Embedded Curve

- Roughly speaking, an *embedded* curve does not cross itself
- More precisely, a curve is embedded if it is a continuous and bijective map from its domain to its image, and the inverse map is also continuous
- **Q:** What's an example of a continuous injective curve that is not embedded?
- **A:** A half-open interval mapped to a circle (inverse is not continuous)



Normal of a Curve

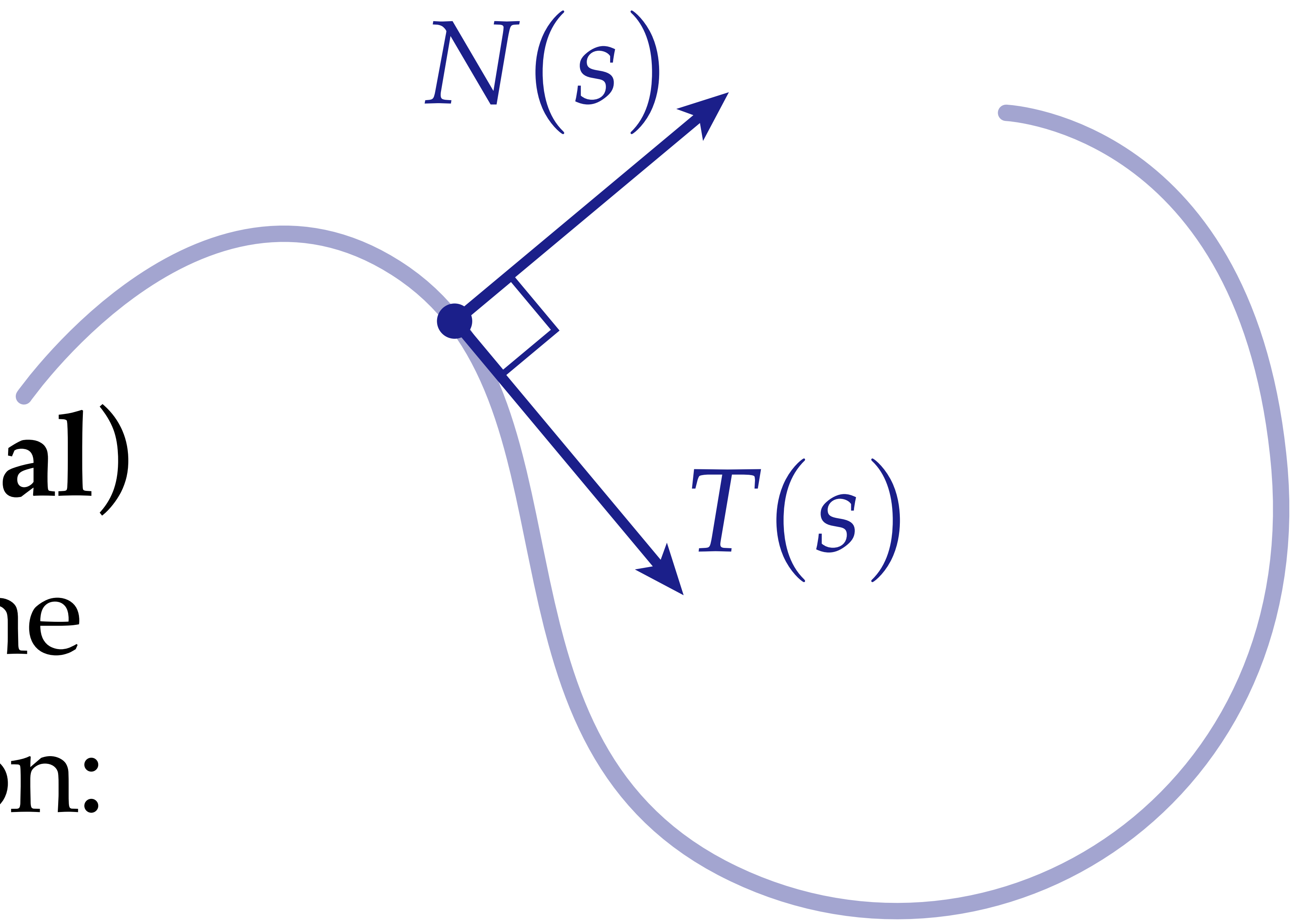
- Informally, a vector is *normal* to a curve if it "sticks straight out" of the curve.

- More formally, the **unit normal** (or just **normal**) can be expressed as a quarter-rotation $\mathcal{J}$ of the unit tangent in the counter-clockwise direction:

$$N(s) := \mathcal{J}T(s)$$

- In coordinates (*x*,*y*), a quarter-turn can be achieved by* simply exchanging *x* and *y*, and then negating *y*:

$$(x, y) \overset{\mathcal{J}}{\mapsto} (-y, x)$$



*Why does this work?

Normal of a Curve—Example

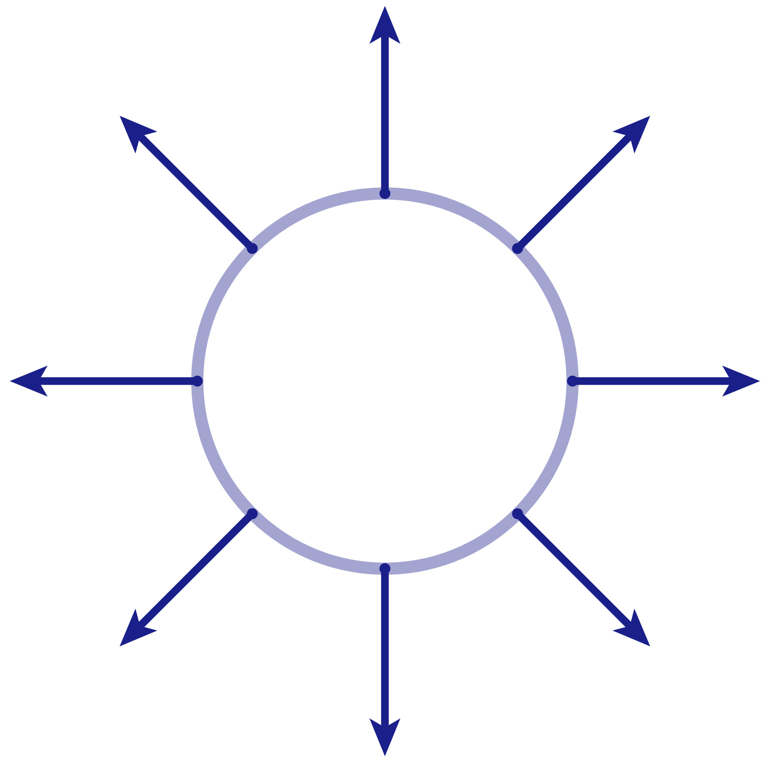
Let's compute the unit normal of a circle:

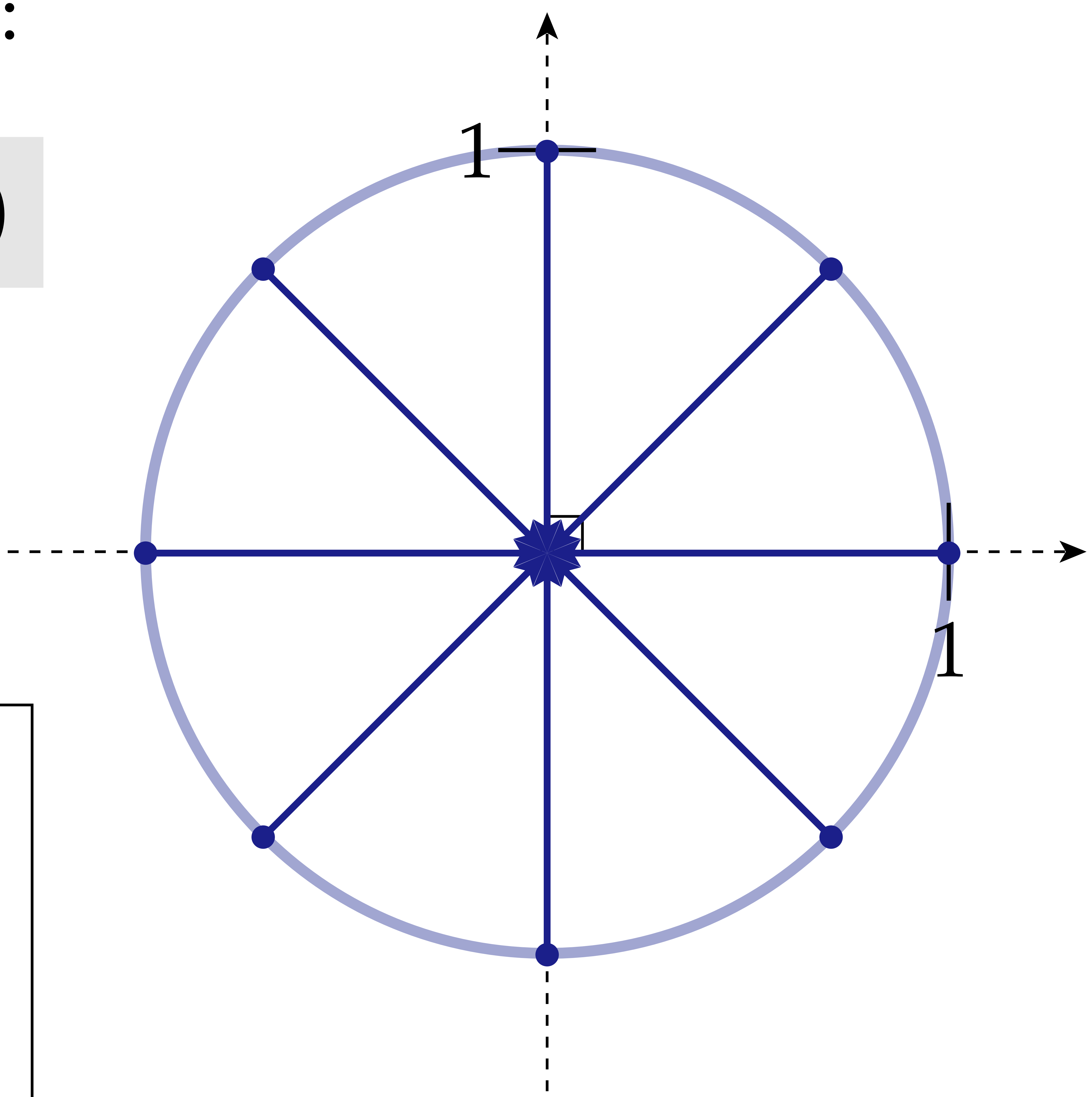
$$\gamma : [0, 2\pi) \to \mathbb{R}^2;\ s \mapsto (\cos(s), \sin(s))$$

$$T(s) = (-\sin(s), \cos(s))$$

$$N(s) = \mathcal{J}T(s) = (-\cos(s), -\sin(s))$$

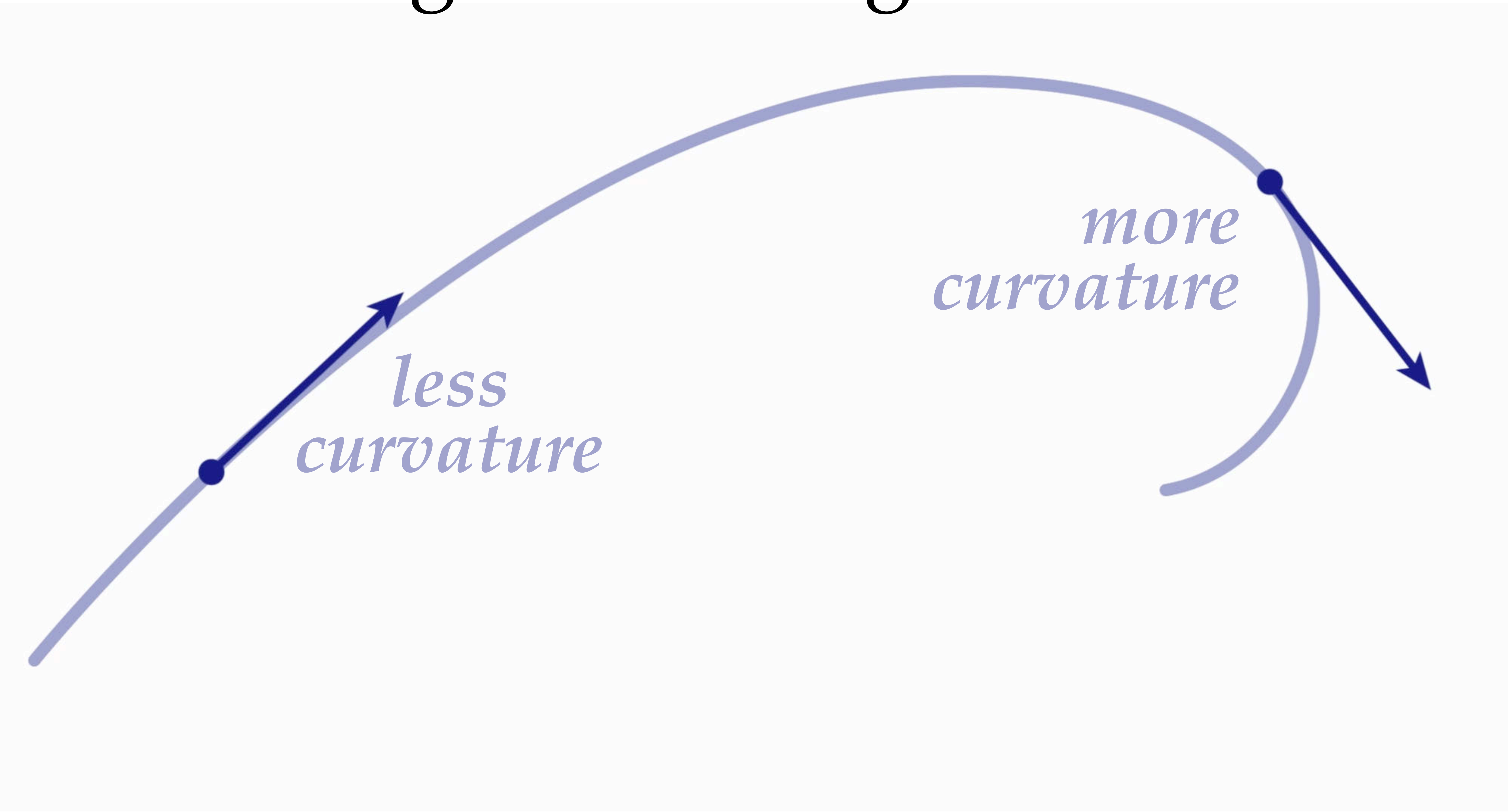
Note: could also adopt the convention $N = -\mathcal{J}T$.
(Just remain consistent!)





Curvature of a Plane Curve

- Informally, curvature describes "how much a curve bends"
- More formally, the **curvature** of an arc-length parameterized plane curve can be expressed as the rate of change in the tangent*



Curvature of a Plane Curve

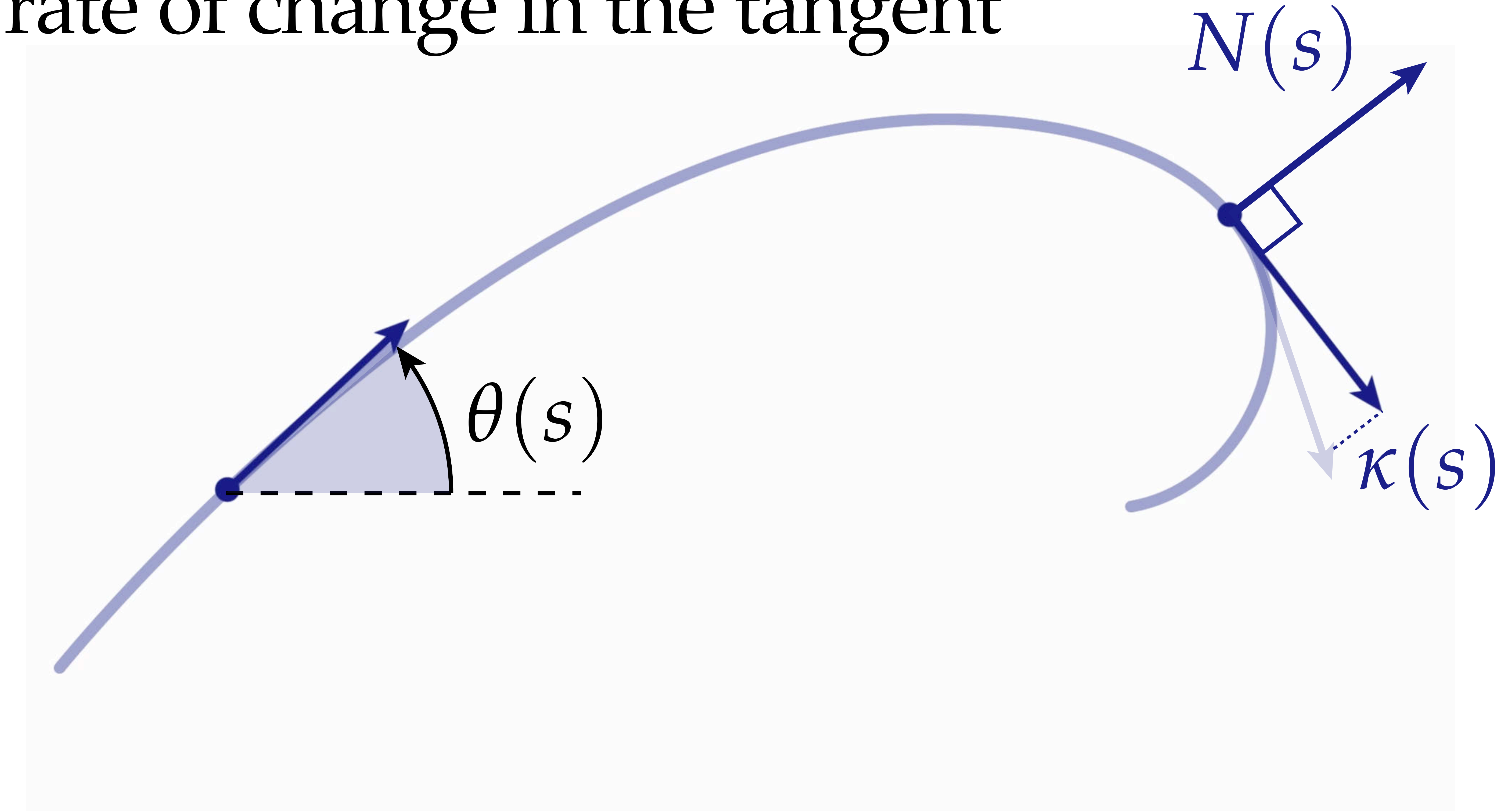
- Informally, curvature describes "how much a curve bends"
- More formally, the **curvature** of an arc-length parameterized plane curve can be expressed as the rate of change in the tangent*

$$\kappa(s) := \langle N(s), \tfrac{d}{ds}T(s)\rangle$$

$$= \langle N(s), \tfrac{d^2}{ds^2}\gamma(s)\rangle$$

Equivalently:

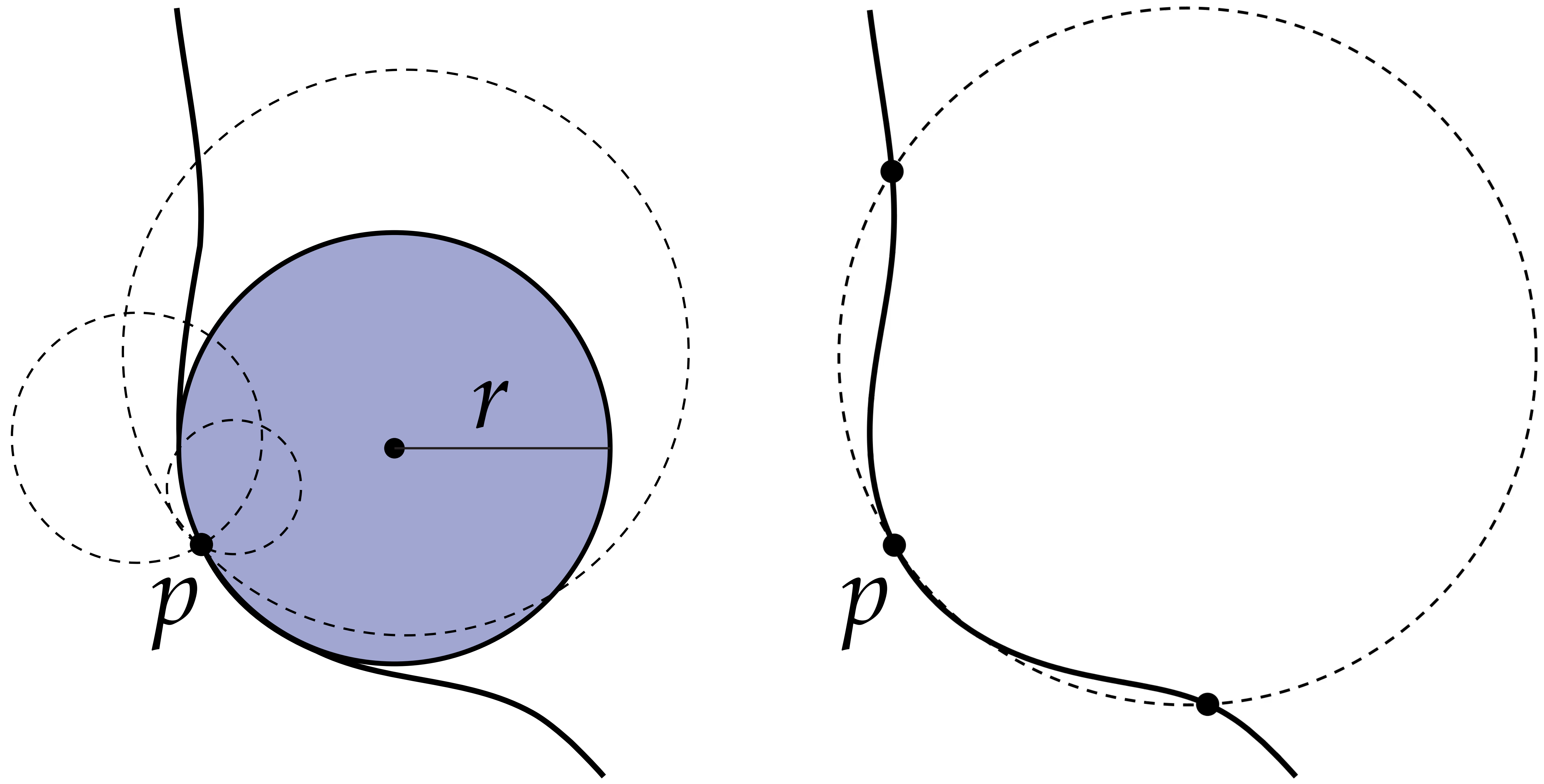
$$\kappa(s) = \frac{d}{ds}\theta(s)$$



*Here, angle brackets denote the usual dot product: $\langle (a,b),(x,y)\rangle := ax + by$

Osculating Circle

- Can also define curvature via the **osculating circle**, which is (roughly speaking) the circle that best approximates a curve at a given point p

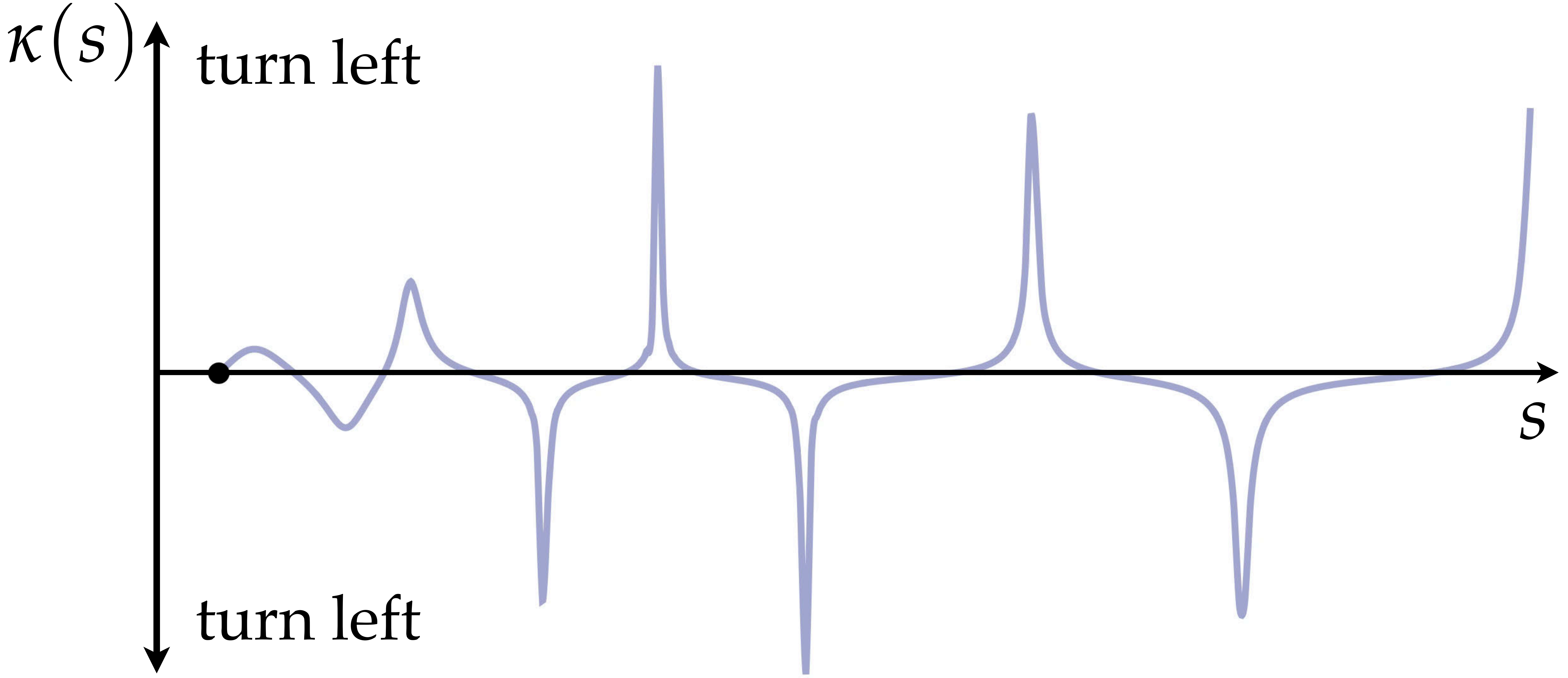


- More precisely, if we consider a circle passing through p and two equidistant neighbors to the "left" and "right" (resp.), the osculating circle is the limiting circle as these neighbors approach p.
- The curvature is then the reciprocal of the radius: $\kappa(p) = \dfrac{1}{r(p)}$

Fundamental Theorem of Plane Curves

Theorem. Up to rigid motions, an arc-length parameterized plane curve is uniquely determined by its curvature.

Intuition: curvature tells us how to "steer" as we move at <u>unit</u> speed.



$\gamma(s)$

Recovering a Curve from Curvature

Q: Given only the curvature function, how can we recover the curve?

A: Just "invert" the two relationships $\frac{d}{ds}\theta = \kappa$, $\frac{d}{ds}\gamma = T$

First integrate curvature to get angle: $\theta(s) := \int_0^s \kappa(t)\, dt$

Then evaluate unit tangents: $T(s) = (\cos(s), \sin(s))$

Finally, integrate tangents to get curve: $\gamma(s) := \int_0^s T(t)\, dt$

Q: What determines the rigid motion?

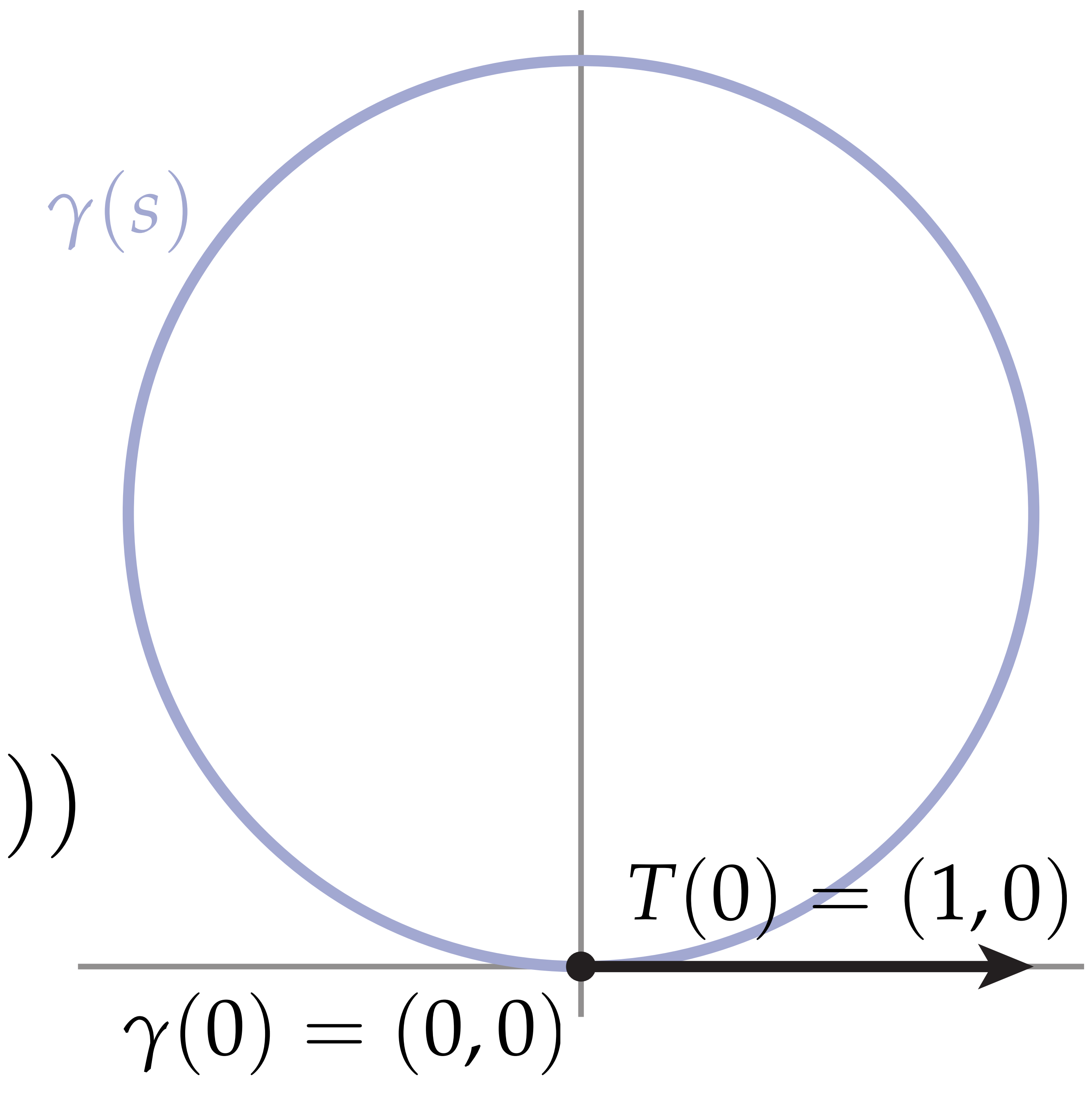
Recovering a Curve from Curvature—Example

- Suppose we have a curvature function $\kappa(s) = 1$

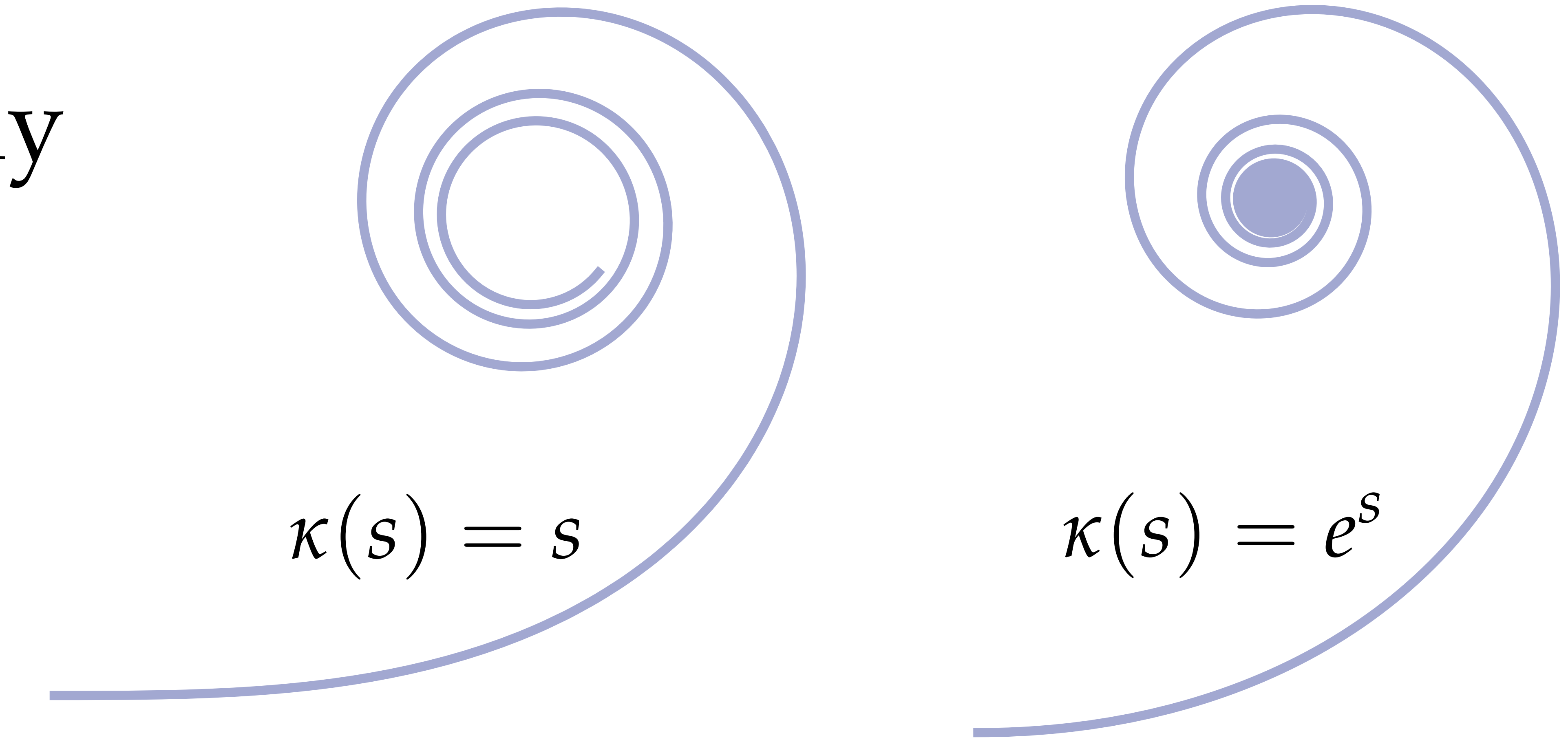
$$\theta(s) = \int_0^s 1\,ds = s$$

$$T(s) = (\cos(\theta(s)), \sin(\theta(s))) = (\cos(s), \sin(s))$$

$$\gamma(s) = \int_0^s T(t)\,dt = \int_0^s (\cos(t), \sin(t))\,dt = (\sin(s), 1 - \cos(s))$$

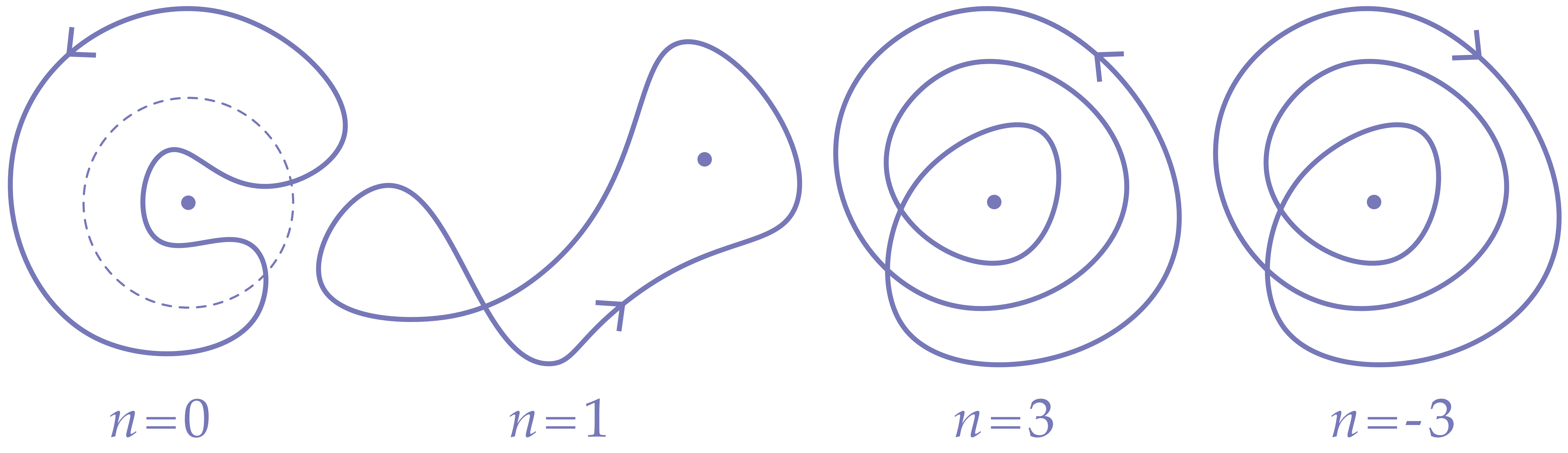


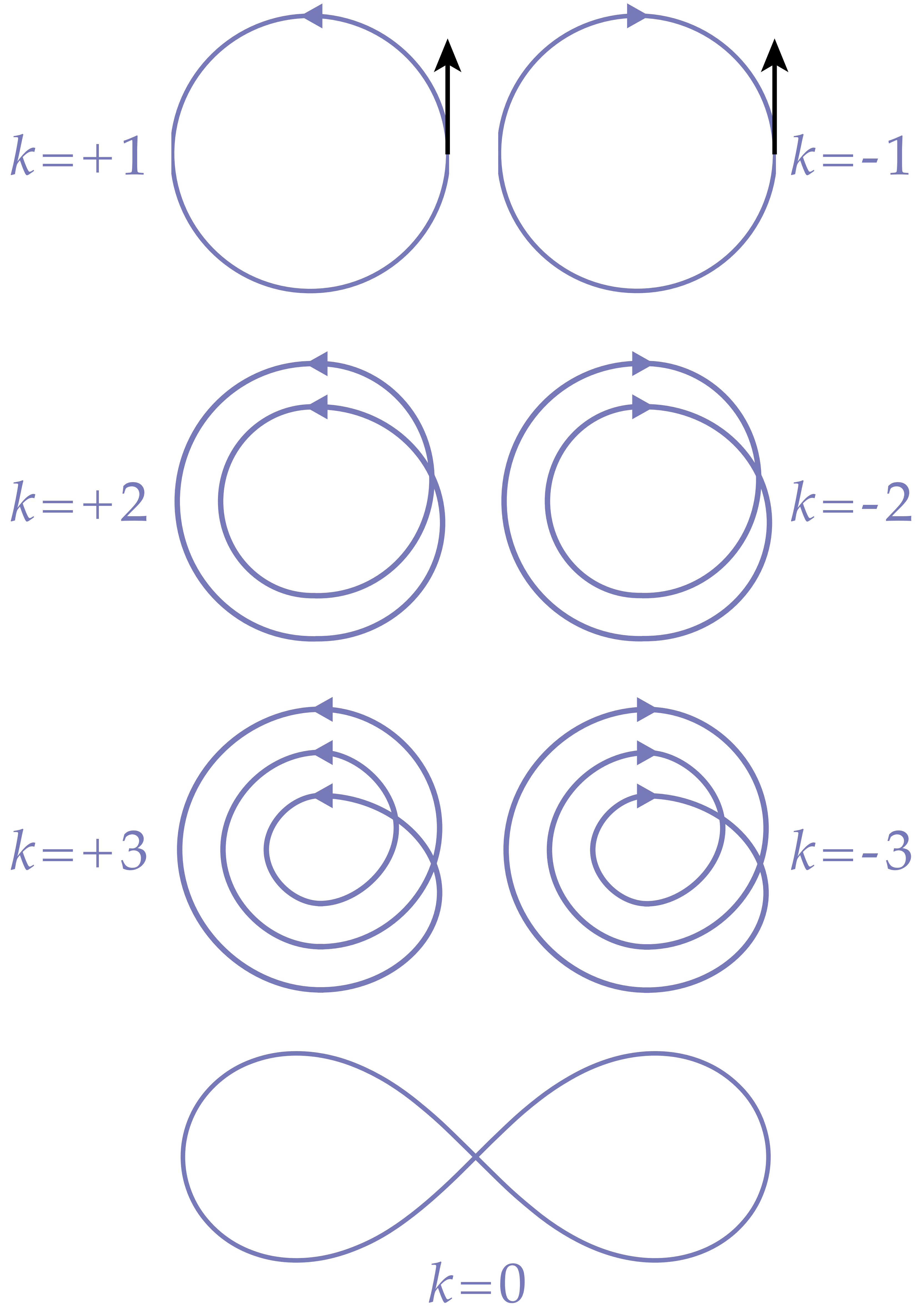
- Is that what you expected?
- In general, hard to integrate curvature analytically
- But *very* easy in discrete case! (Next lecture)



Turning and Winding Numbers

- For a closed regular curve in the plane:
  - **turning number** k is the number of counter-clockwise turns made by the tangent*
  - **winding number** n is the number of times the curve "goes around" a particular point p
    - more precisely: total *signed* length of the projection of curve onto unit circle around p





*__Food for thought__: k is the *total curvature* over 2π. Why?

Tangent vs. Winding Number

- These two concepts sound similar—how exactly are they related / different?

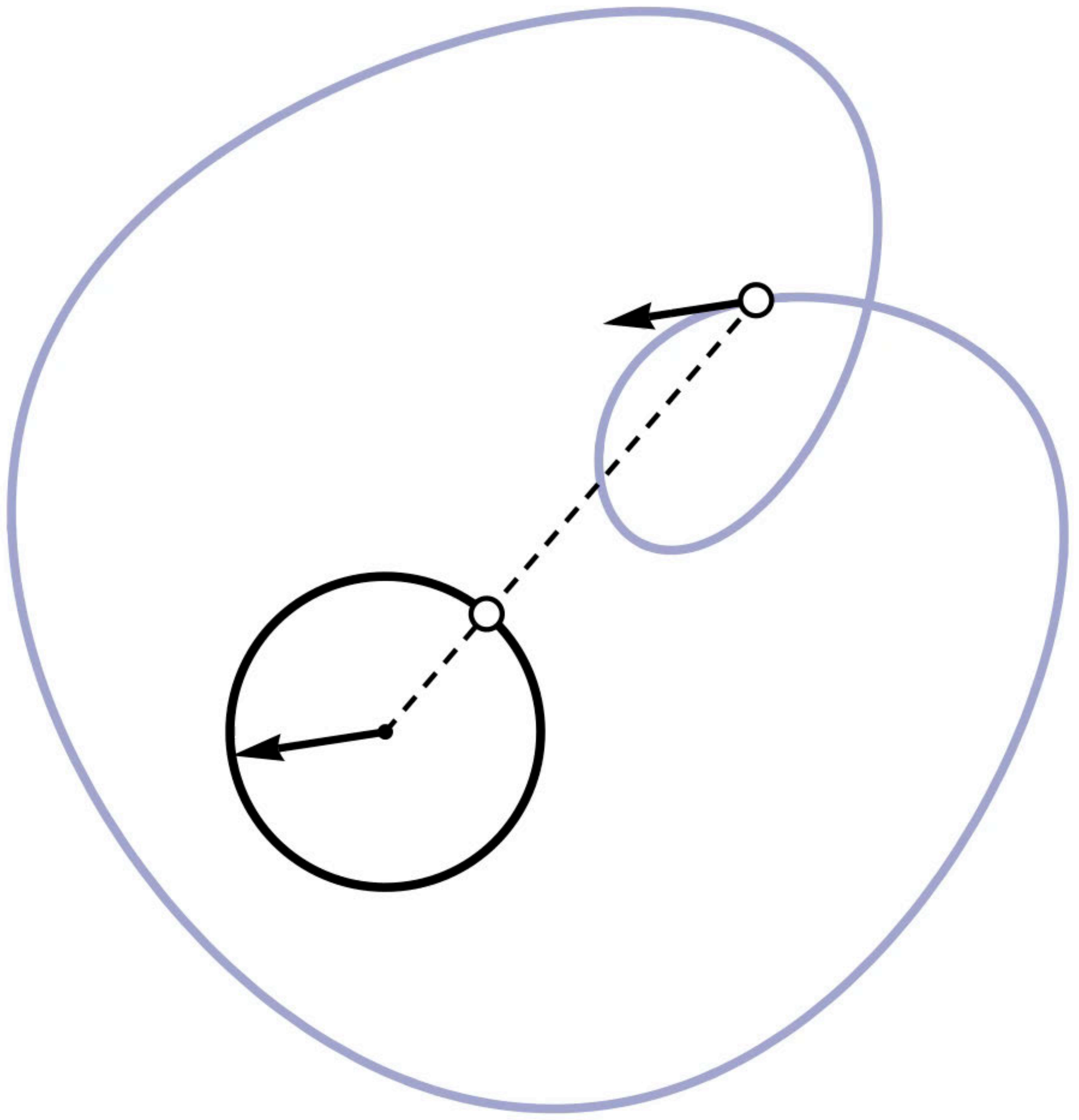
– **Turning number.** Can think of the tangent T as a map to the unit circle. The turning number k is the *topological degree* of this map, *i.e.*, the number of times it covers the circle as we go once around the curve.

– **Winding number.** Consider the map $\hat{\gamma}_p(s)$ obtained by projecting the curve onto the circle around p. The winding number n is the topological degree of this map.

- No reason these numbers have to be the same!
- However, the turning number is the winding number *of the tangent map* around the origin

$$T(s) := \frac{\gamma'(s)}{|\gamma'(s)|} \qquad k = \text{degree}(T(s))$$

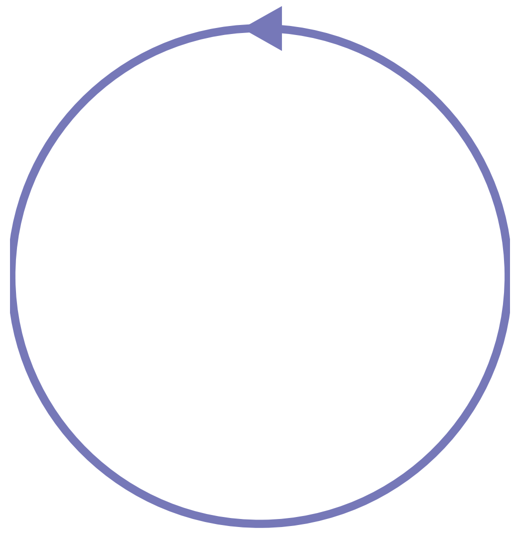
$$\hat{\gamma}_p(s) := \frac{\gamma(s) - p}{|\gamma(s) - p|} \qquad n = \text{degree}(\hat{\gamma}_p(s))$$



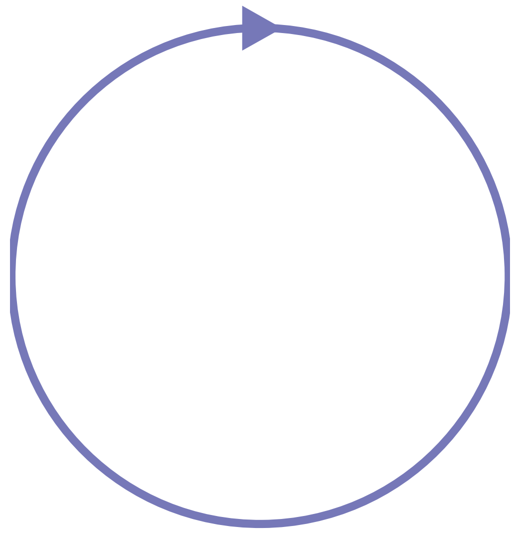
Whitney-Graustein Theorem

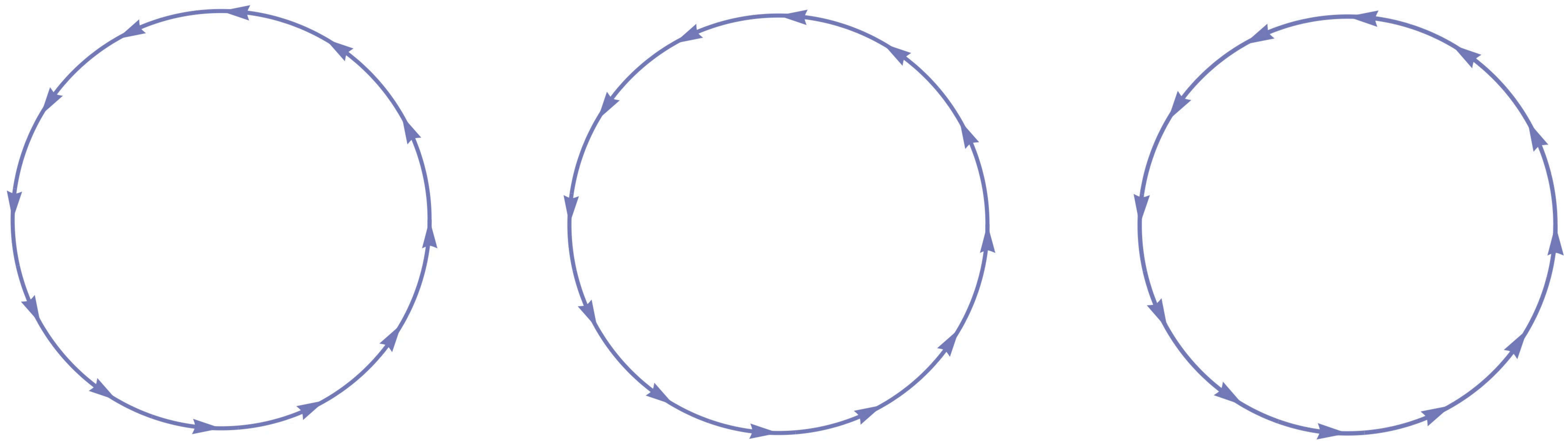
- Two curves are related by *regular homotopy* if one can continuously "deform" one into the other while remaining regular (immersed).
- **Theorem (Whitney-Graustein).** Two curves have the same turning number k if and only if they are regularly homotopic.
- **Corollary.** There is no way to turn the circle "inside-out", *i.e.*, no regular homotopy between circles with turning number $k = \pm 1$.

$k=+1$

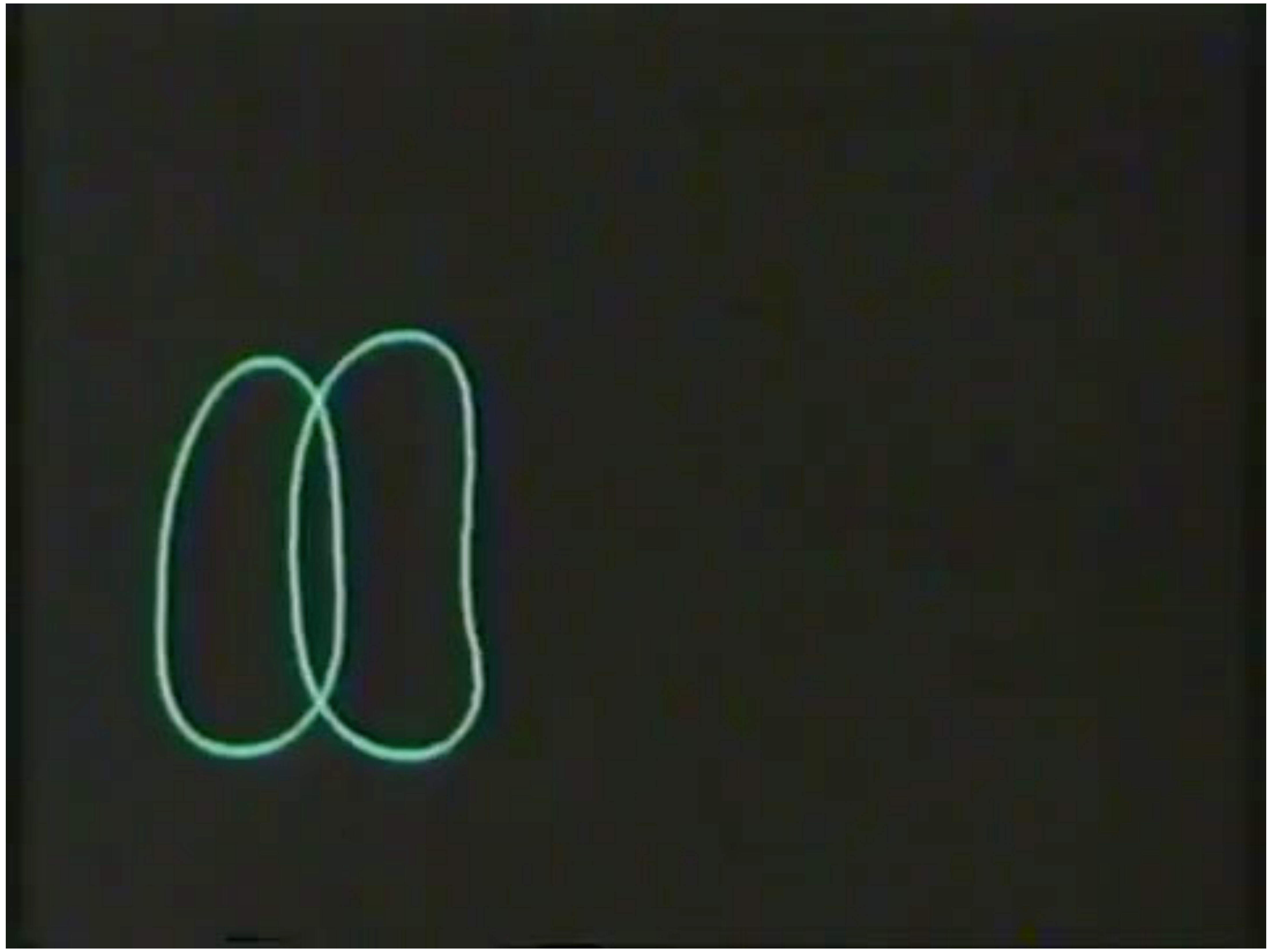


$k=-1$





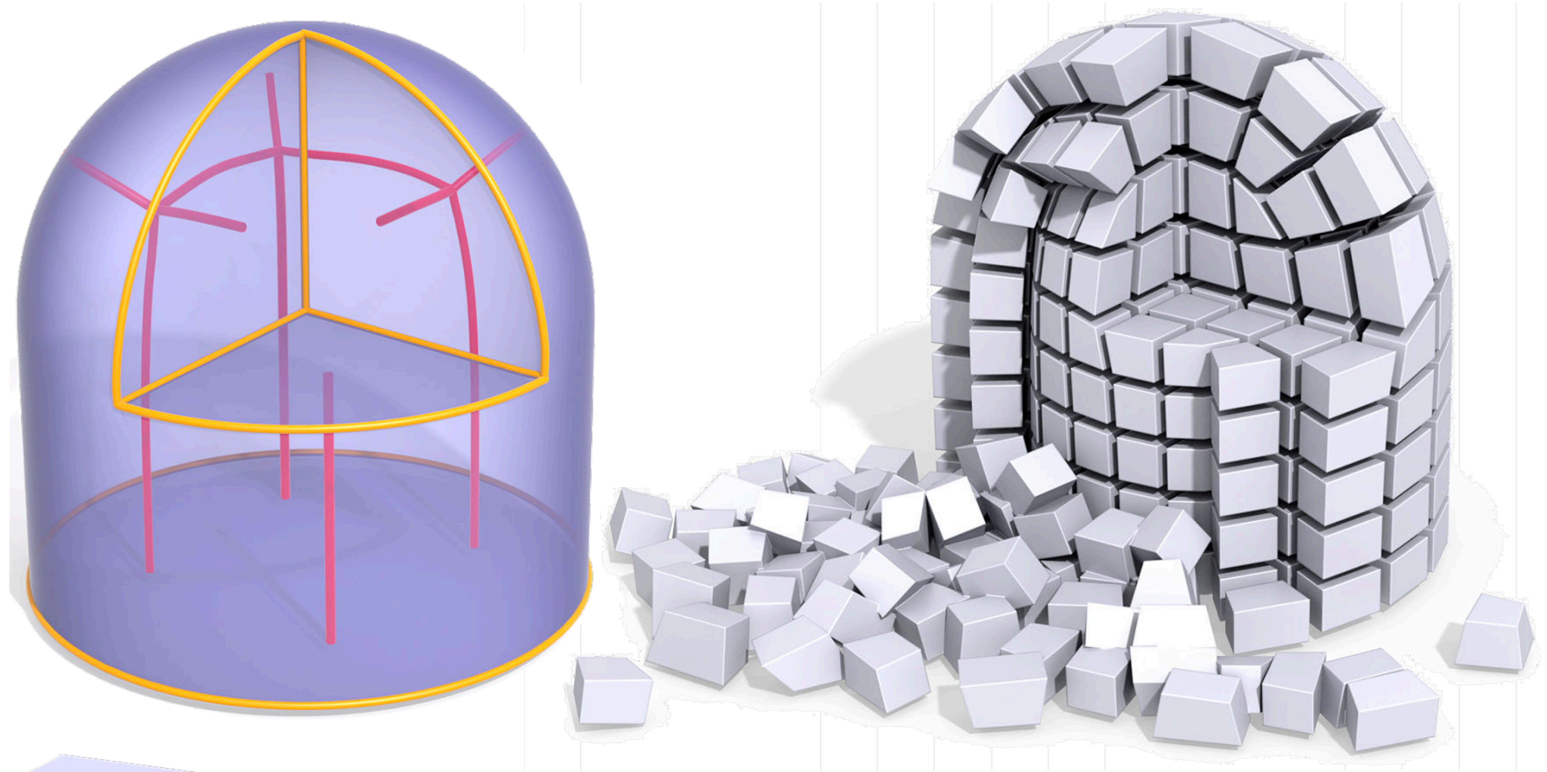
Whitney-Graustein — Visual Proof

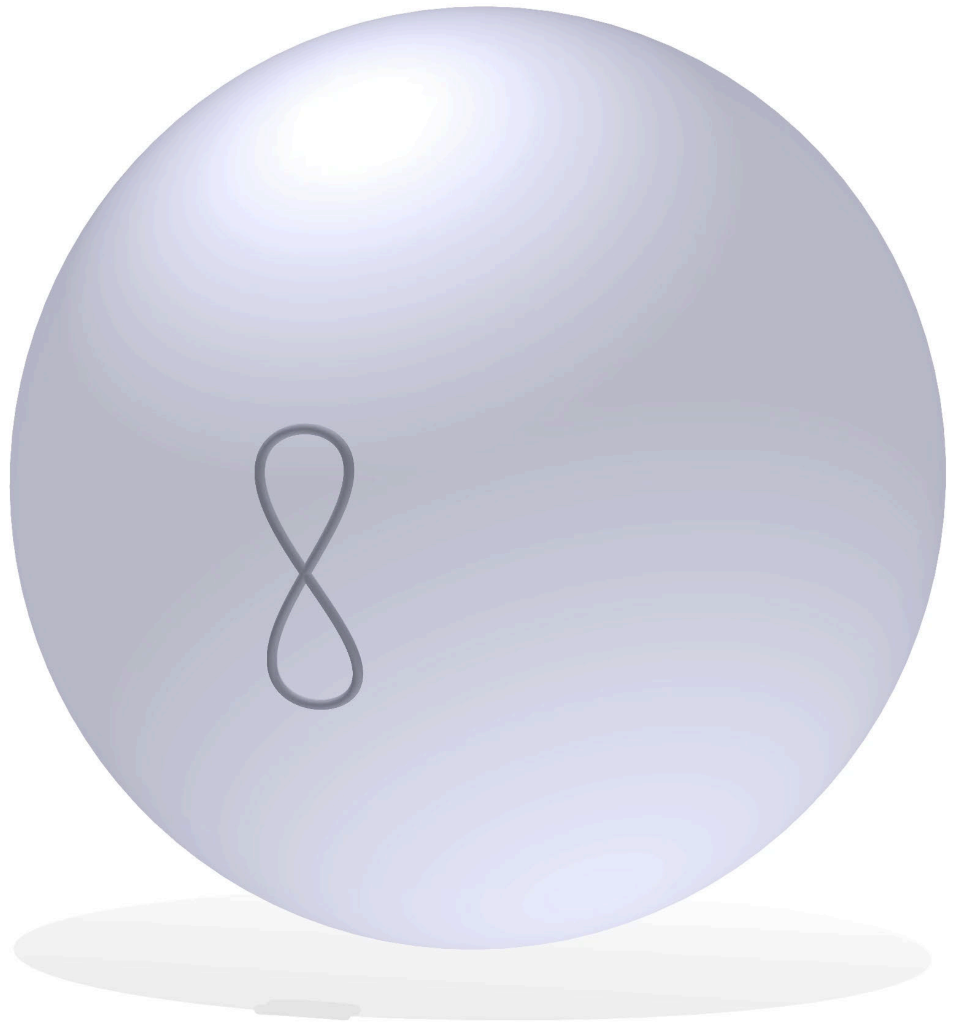


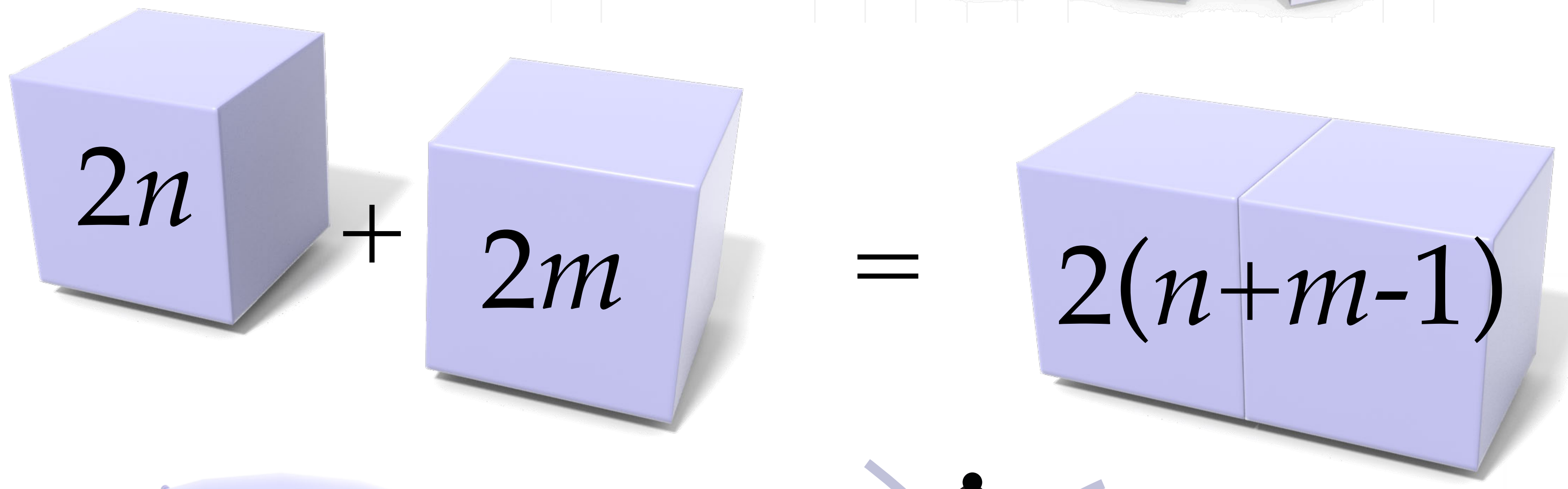
Movie: "Regular Homotopies in the Plane" (Nelson Max, 1972)

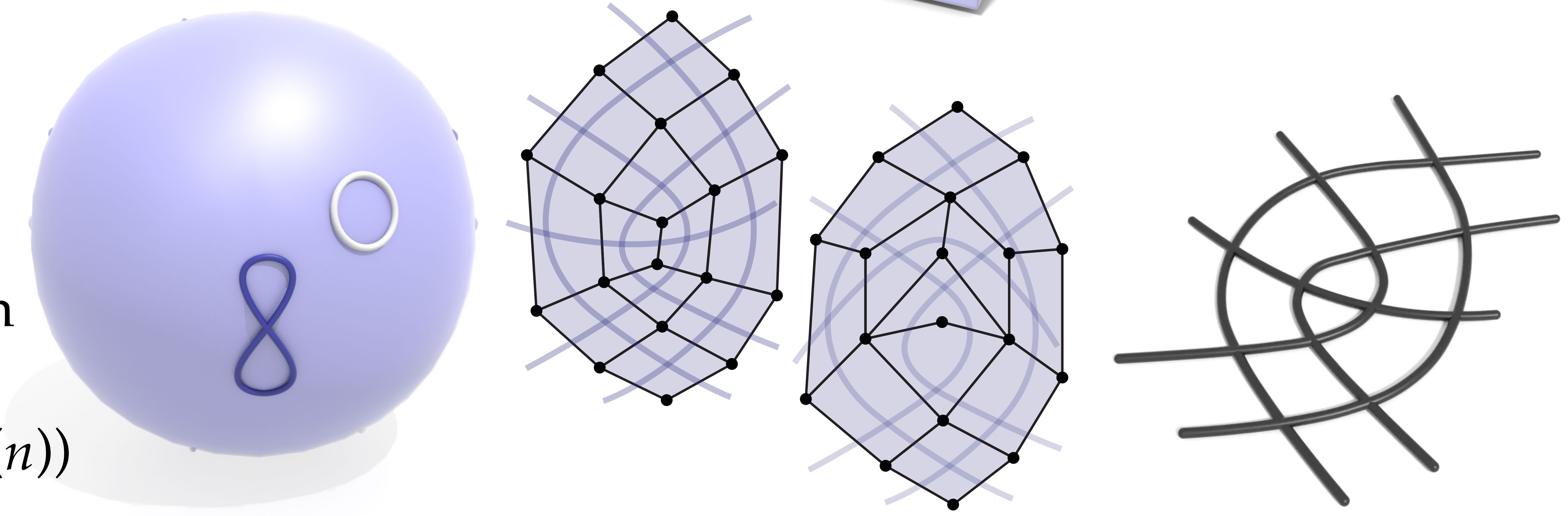
Application: Hexahedral Meshing

- **Hexahedral meshing:** break shape up into "little cubes" for simulation.
- **Theorem (Thurston-Mitchell).** A genus-0 quadrilateral mesh is the boundary of a hexahedral mesh if and only if it has an <u>even number of faces</u>.
- *Proof sketch*:
  - every curve on sphere is regularly homotopic to either circle or figure-8
  - deform dual graph into collection of disjoint circles and figure-8s
  - interpolate between dual graph and disjoint curves by shrinking sphere
  - sweeps out surfaces dual to a hex mesh (after adding additional widgets)
- Yields $O(n^2)$ hexahedra; can do better ($O(n)$)
- (Erickson) Can generalize to any genus





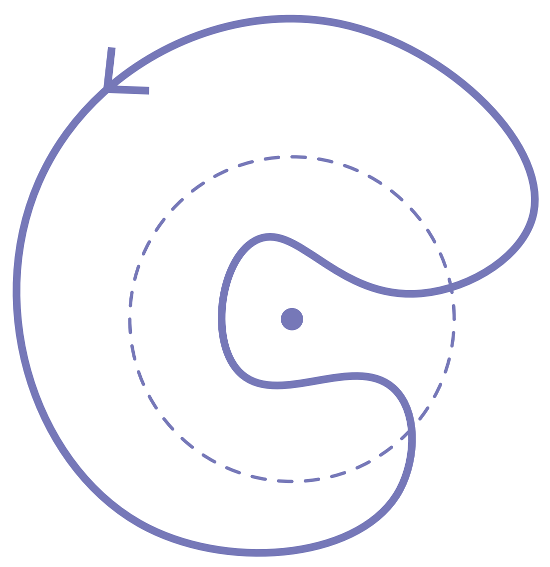


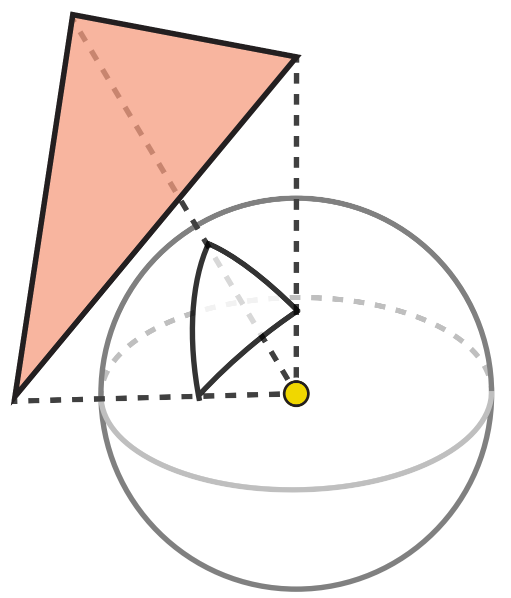


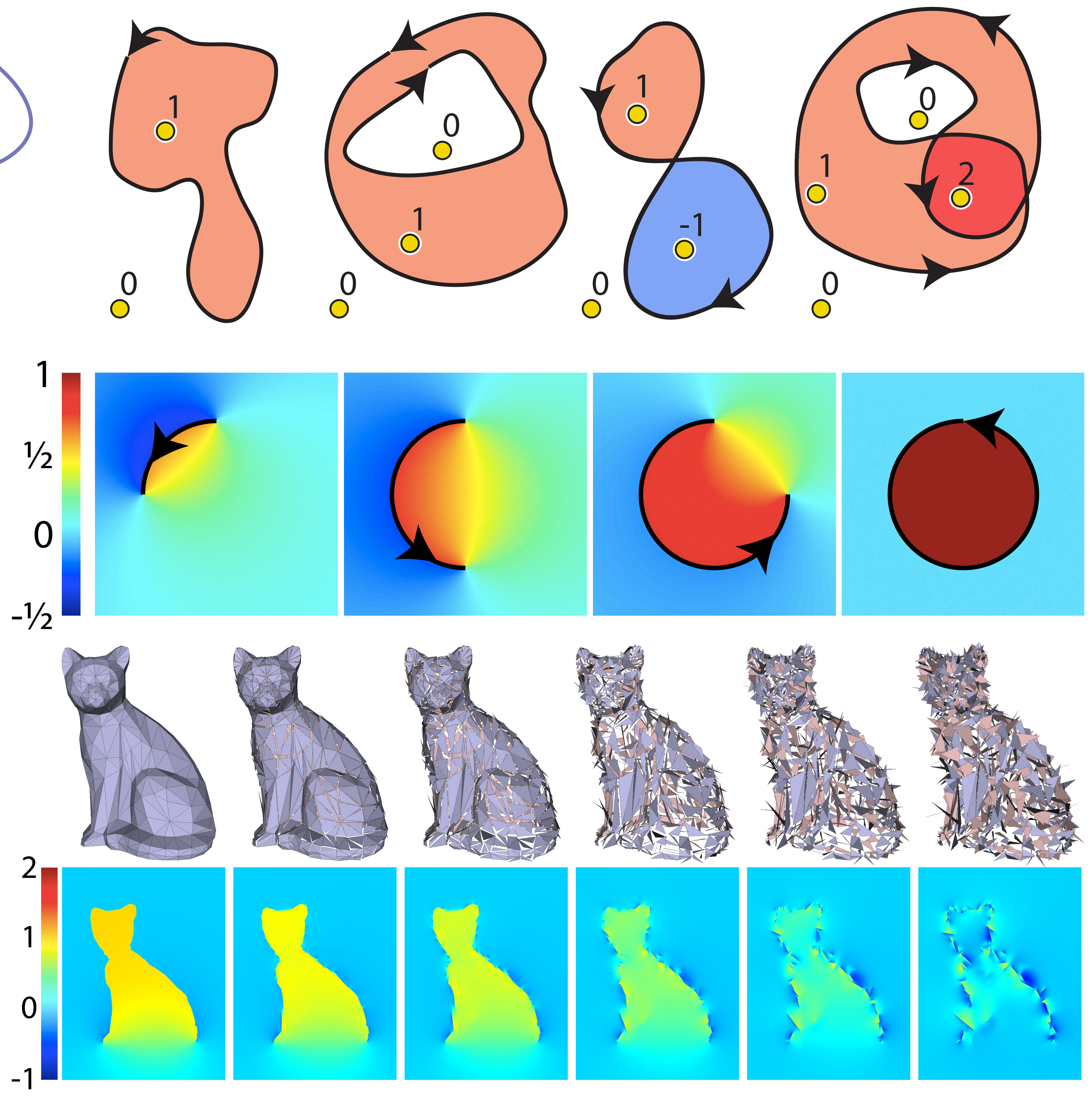
see: Erickson, *"Regular Homotopy and Hexahedral Meshing"*

Application: Generalized Winding Numbers

- Recall: *winding number* counts number of times a closed curve wraps around a given point—which points are inside/outside?
- Real data not always perfectly closed (noise, holes…). Still want to know in/out!
- Can still sum up signed projected lengths (or areas) to get *fractional* winding number
- Gives good sense of which points are likely "inside"/"outside" (threshold)
- Useful for many practical tasks: extracting watertight mesh, tetrahedral meshing, constructive solid geometry (booleans), …







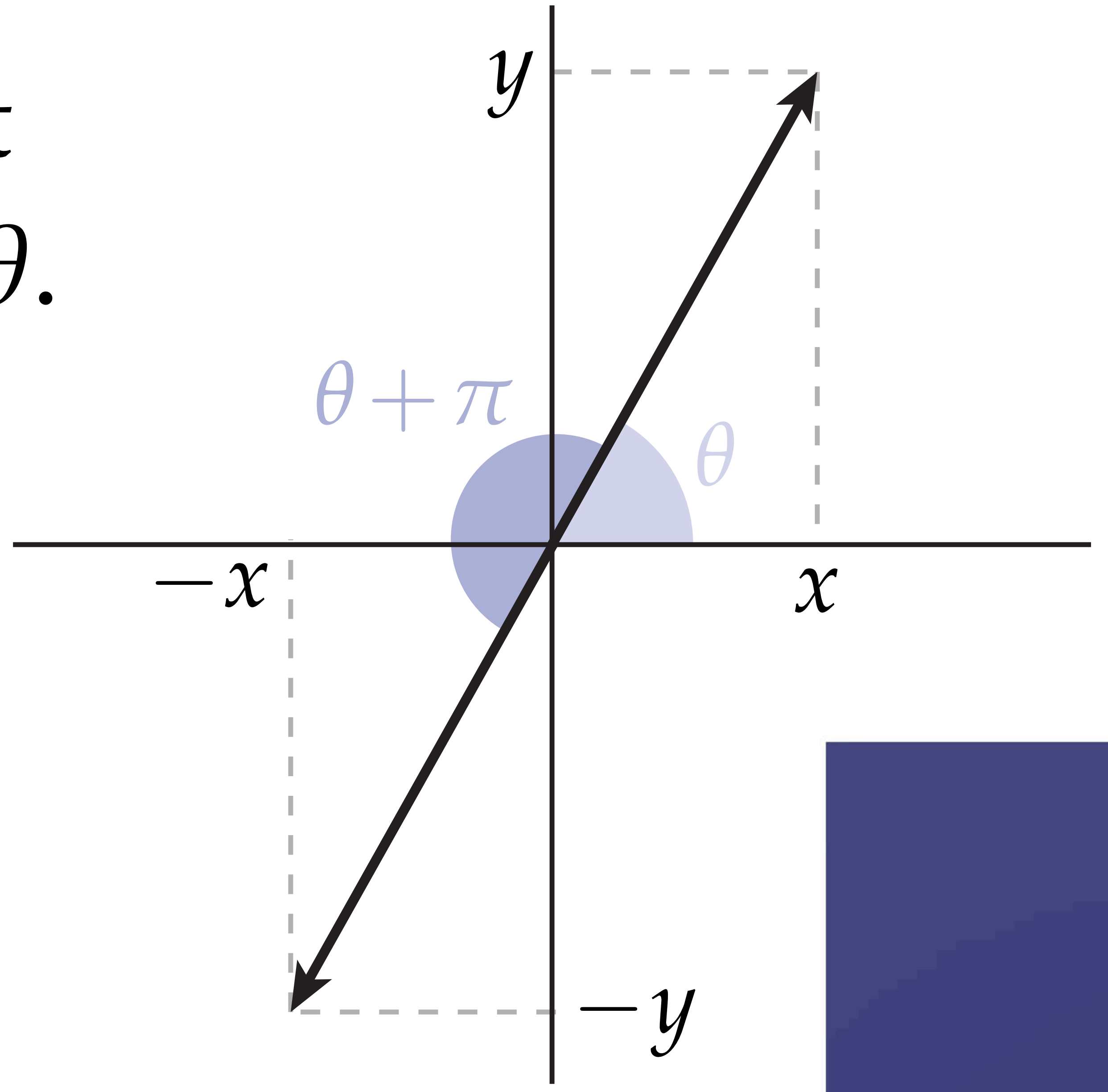
Jacobson et al, *"Robust Inside-Outside Segmentation using Generalized Winding Numbers"* (2013)

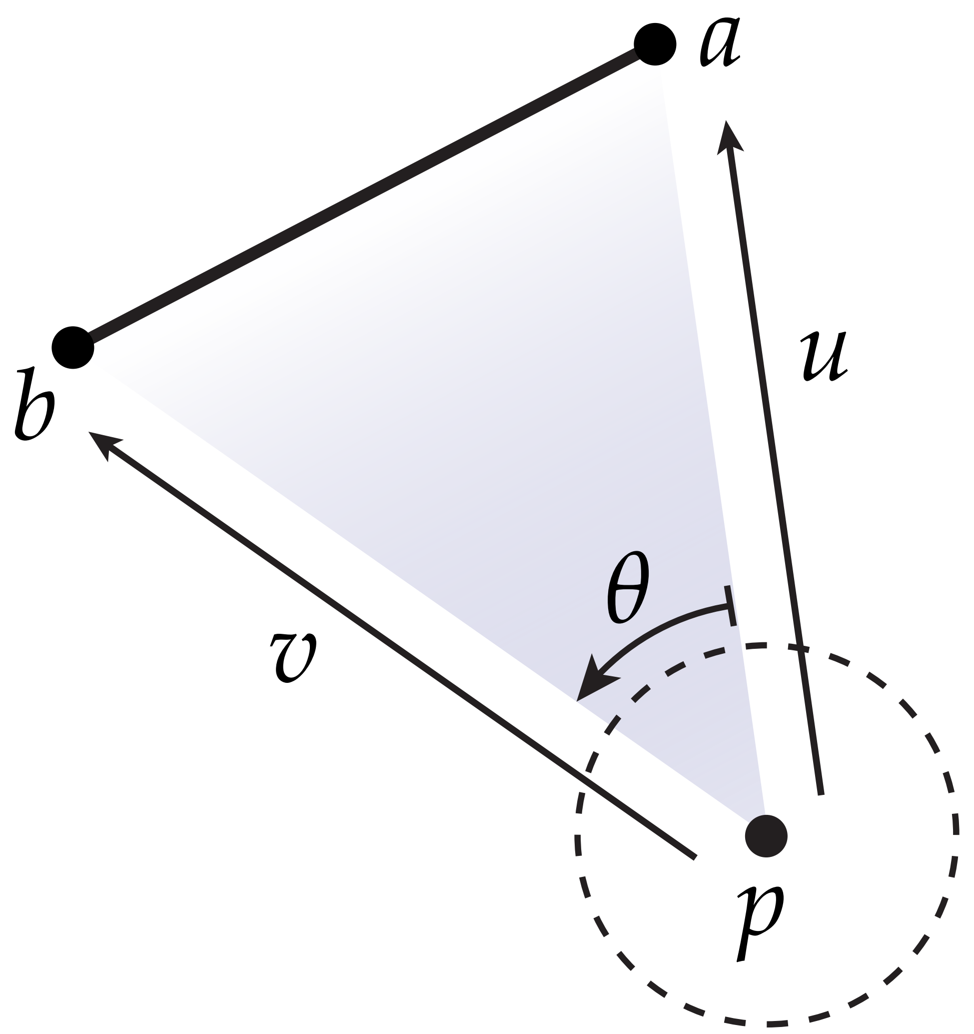
Computing Generalized Winding Numbers (2D)

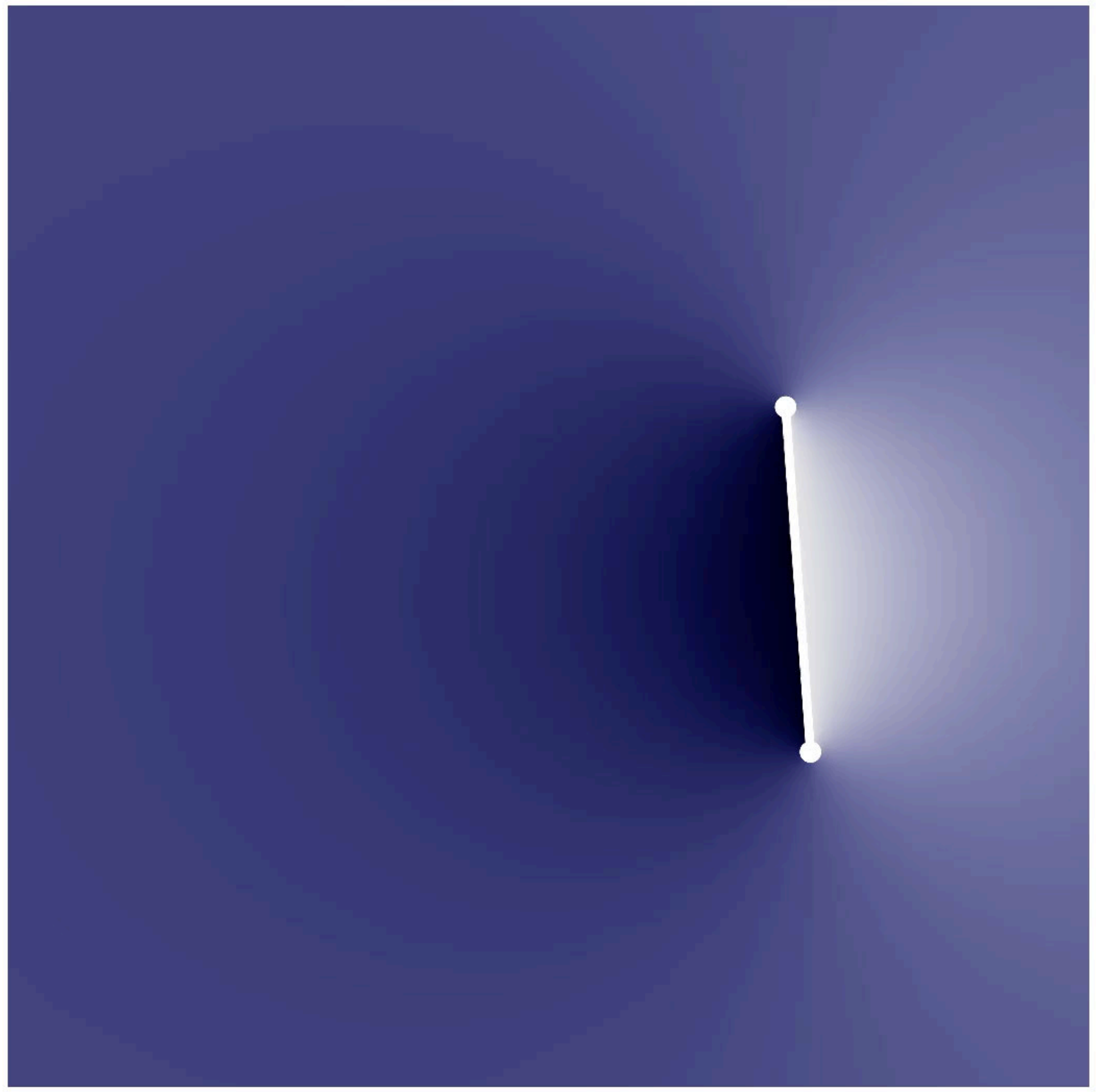
- For an edge (a,b), projected length onto unit circle around a point p is just signed angle θ.
- Recall: angle of a vector (x,y) is $\arctan(y/x)$
  - …but $(-x,-y)$ will yield the same angle!
  - need to use two-argument $\operatorname{atan2}(y,x)$
- In particular, to get signed angle θ:

$$u = a - p \qquad\qquad v = b - p$$

$$\theta = \operatorname{atan2}(\underbrace{u \times v}_{u_1v_2 - u_2v_1}, \overbrace{u \cdot v}^{u_1v_1 + u_2v_2})$$





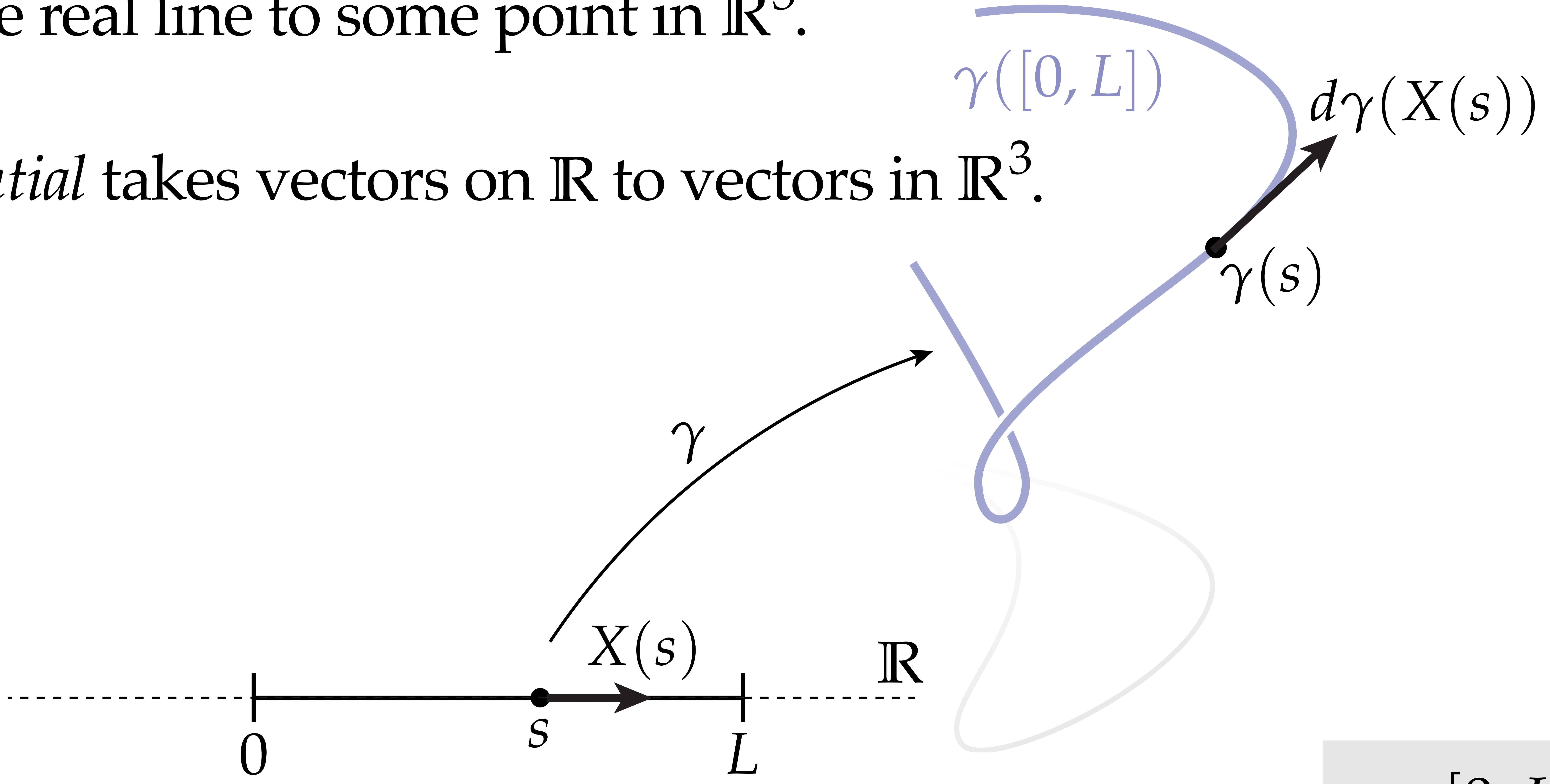


Space Curves

Parameterized Space Curve

A **parameterized space curve** is a map* taking each point in an interval $[0,L]$ of the real line to some point in $\mathbb{R}^3$.

Its *differential* takes vectors on $\mathbb{R}$ to vectors in $\mathbb{R}^3$.



*Continuous, differentiable, smooth…

$$\gamma : [0,L] \to \mathbb{R}^3$$

Pushforward of Vectors on a Space Curve

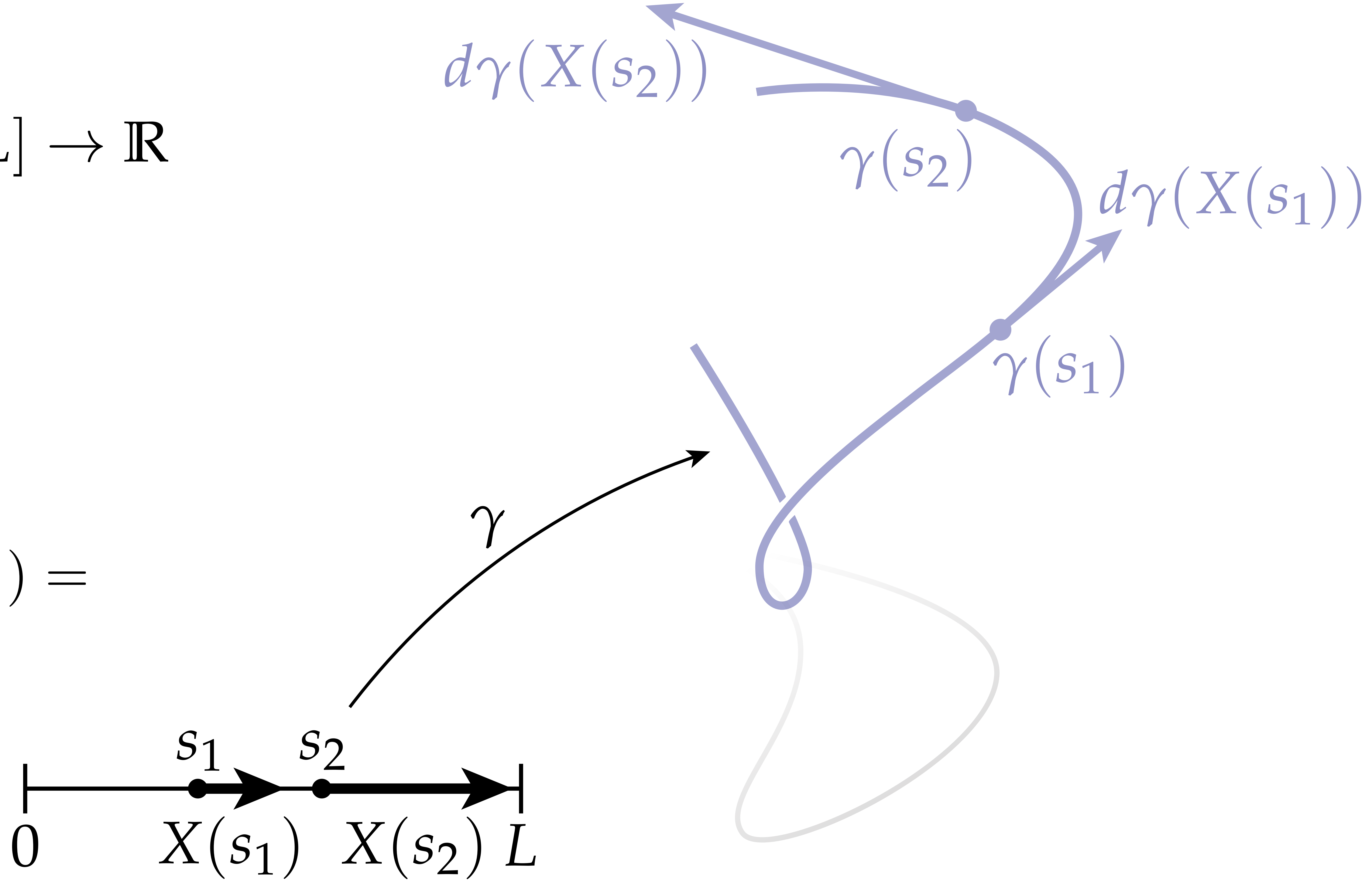
More explicitly, suppose we want to map a vector field X on $\mathbb{R}$ to the "stretched out" vector field on Υ.

$$\gamma := (x, y, z), \;\; x, y, z : [0, L] \to \mathbb{R}$$

$$X := a\tfrac{\partial}{\partial s}, \;\; a : [0, L] \to \mathbb{R}$$

$$d\gamma = (\tfrac{\partial x}{\partial s}, \tfrac{\partial y}{\partial s}, \tfrac{\partial z}{\partial s})ds$$

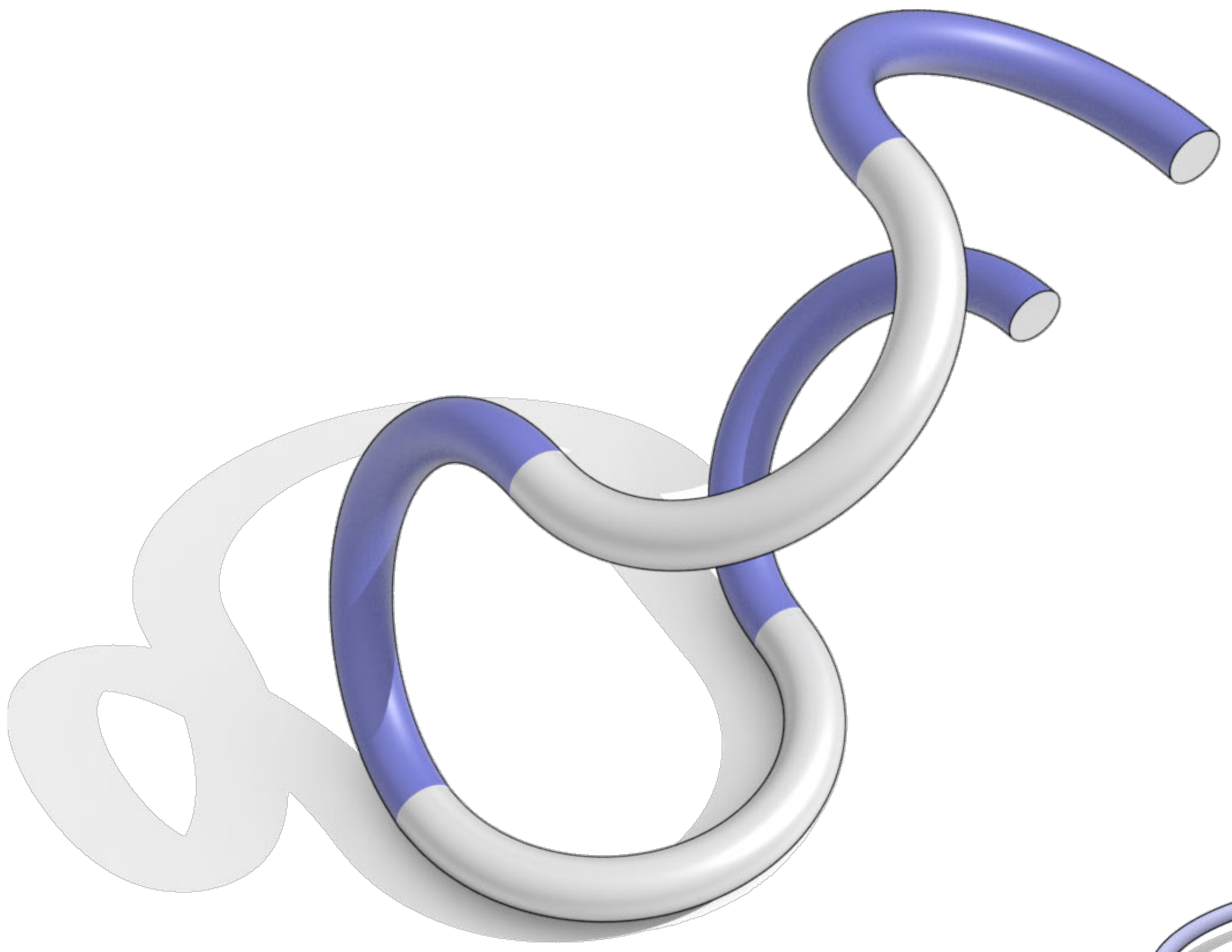
$$d\gamma(X) = (\tfrac{\partial x}{\partial s}, \tfrac{\partial y}{\partial s}, \tfrac{\partial x}{\partial z})ds(a\tfrac{\partial}{\partial s}) = a(\tfrac{\partial x}{\partial s}, \tfrac{\partial y}{\partial s}, \tfrac{\partial z}{\partial s})$$



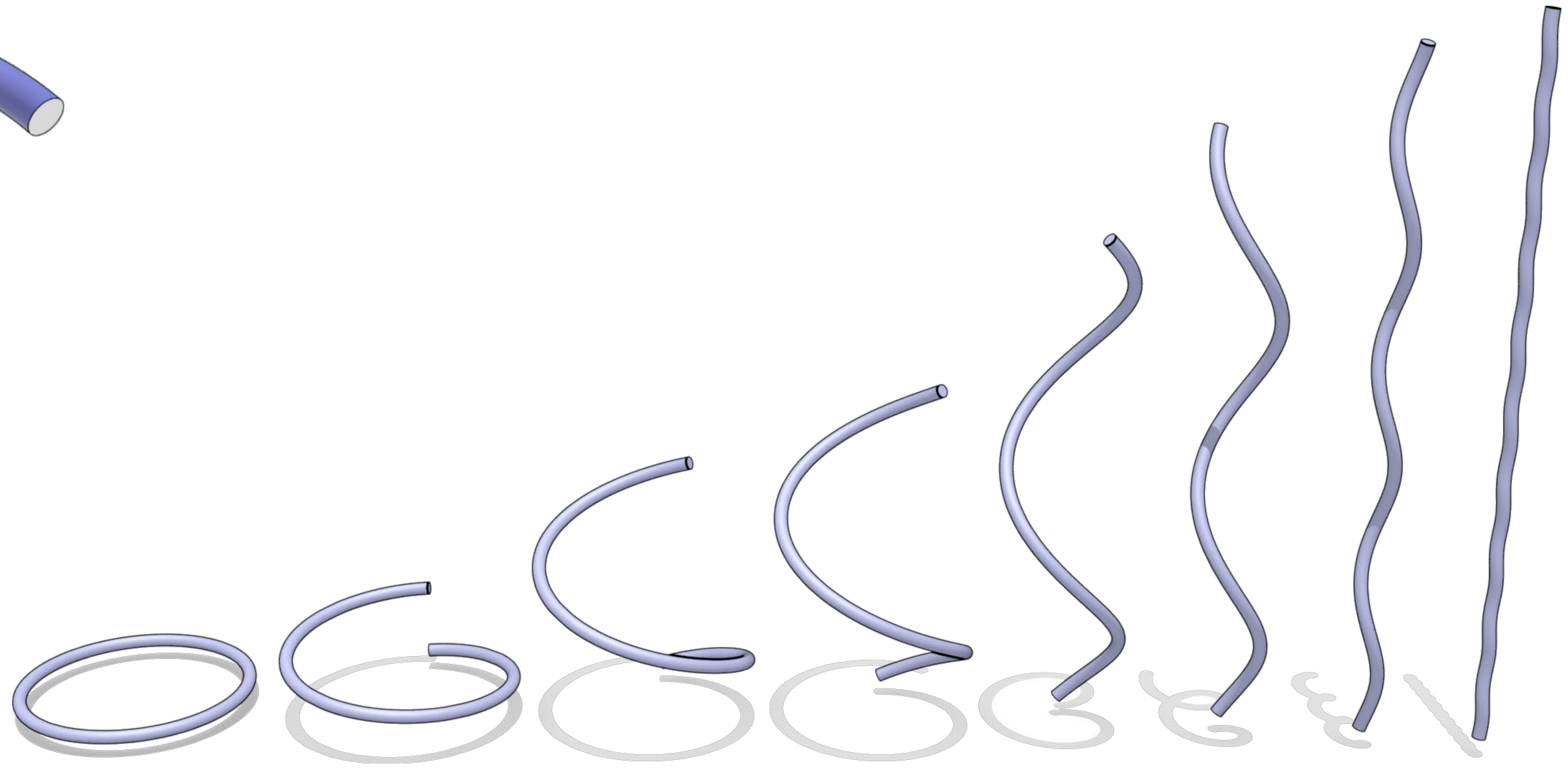
Analogy: Jacobian.

Curvature and Torsion of a Space Curve

- For a plane curve, *curvature* captured the notion of "bending"
- For a space curve we also have *torsion*, which captures "twisting"



Intuition: torsion is "out of plane bending"



increasing torsion

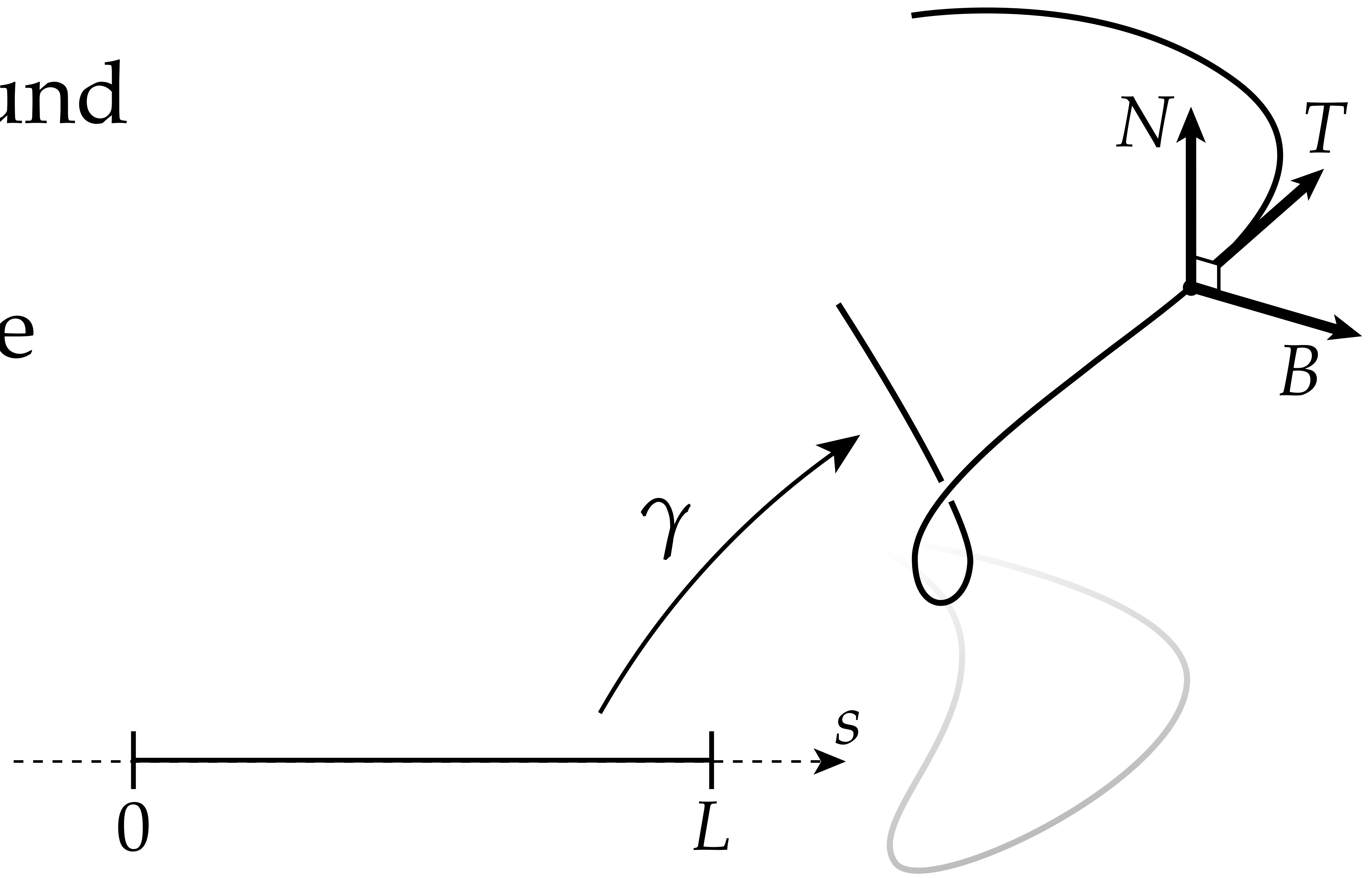
Frenet Frame

- Each point of a space curve γ has a natural coordinate frame called the *Frenet frame*, which depends only on the local geometry

- As in the plane, the tangent T is found by differentiating the curve, and differentiating the tangent yields the curvature times the normal N

- The binormal B then completes an orthonormal basis with T and N

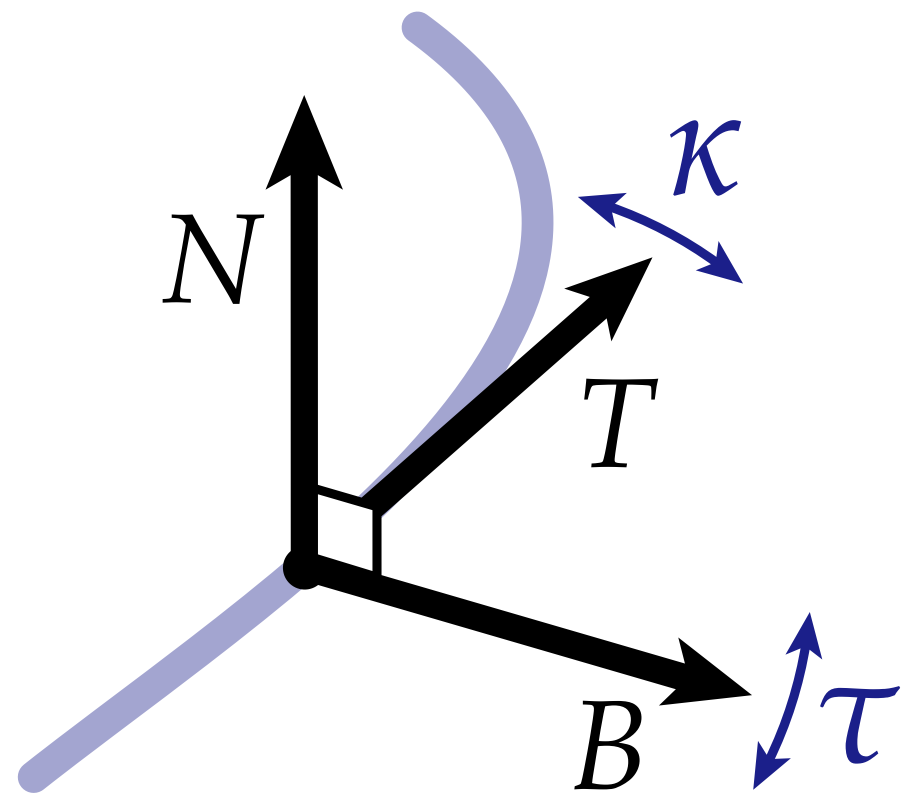
$$T(s) := \tfrac{d}{ds}\gamma(s)$$
$$N(s) := \tfrac{d}{ds}T/|\tfrac{d}{ds}T|$$
$$B(s) := T(s) \times N(s)$$



Frenet-Serret Equation

- Curvature κ and torsion τ can be defined in terms of the change in the Frenet frame as we move along the curve:

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}$$



- Most importantly, change in the tangent describes bending (*curvature*); change in binormal describes twisting (*torsion*)

$$\begin{aligned} \kappa &= -\langle N, \tfrac{d}{ds}T\rangle \\ \tau &= \langle N, \tfrac{d}{ds}B\rangle \end{aligned}$$

Example—Helix

Let's compute the Frenet frame, curvature, and torsion for a *helix**

$$\gamma(s) := (a\cos(s), a\sin(s), bs)$$

$$\frac{d}{ds}\gamma(s) = (-a\sin(s), a\cos(s), b)$$

$$\left|\frac{d}{ds}\gamma\right| = \sqrt{a^2+b^2} = 1$$

$$\Rightarrow T(s) = \frac{d}{ds}\gamma(s)$$

$$\frac{d}{ds}T(s) = -a(\cos(s), \sin(s), 0)$$

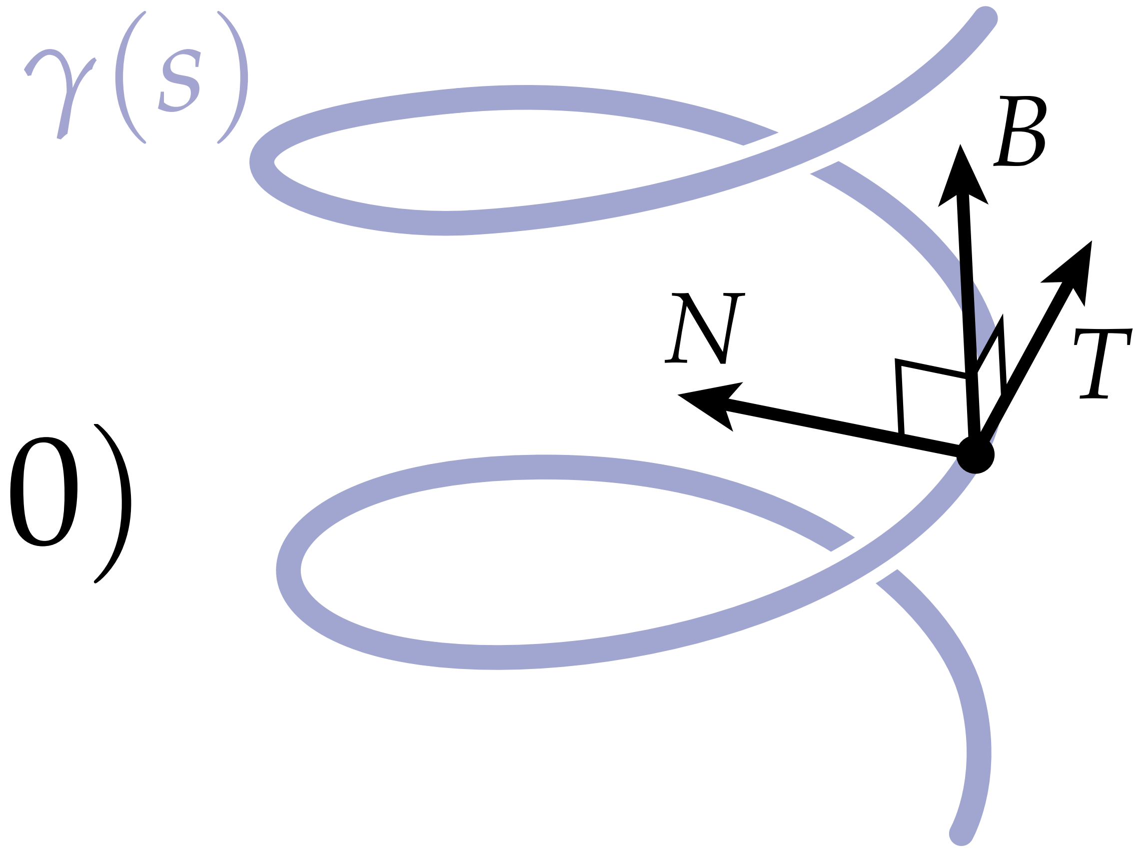
$$\Rightarrow \kappa(s) = -a,\ N(s) = (\cos(s), \sin(s), 0)$$

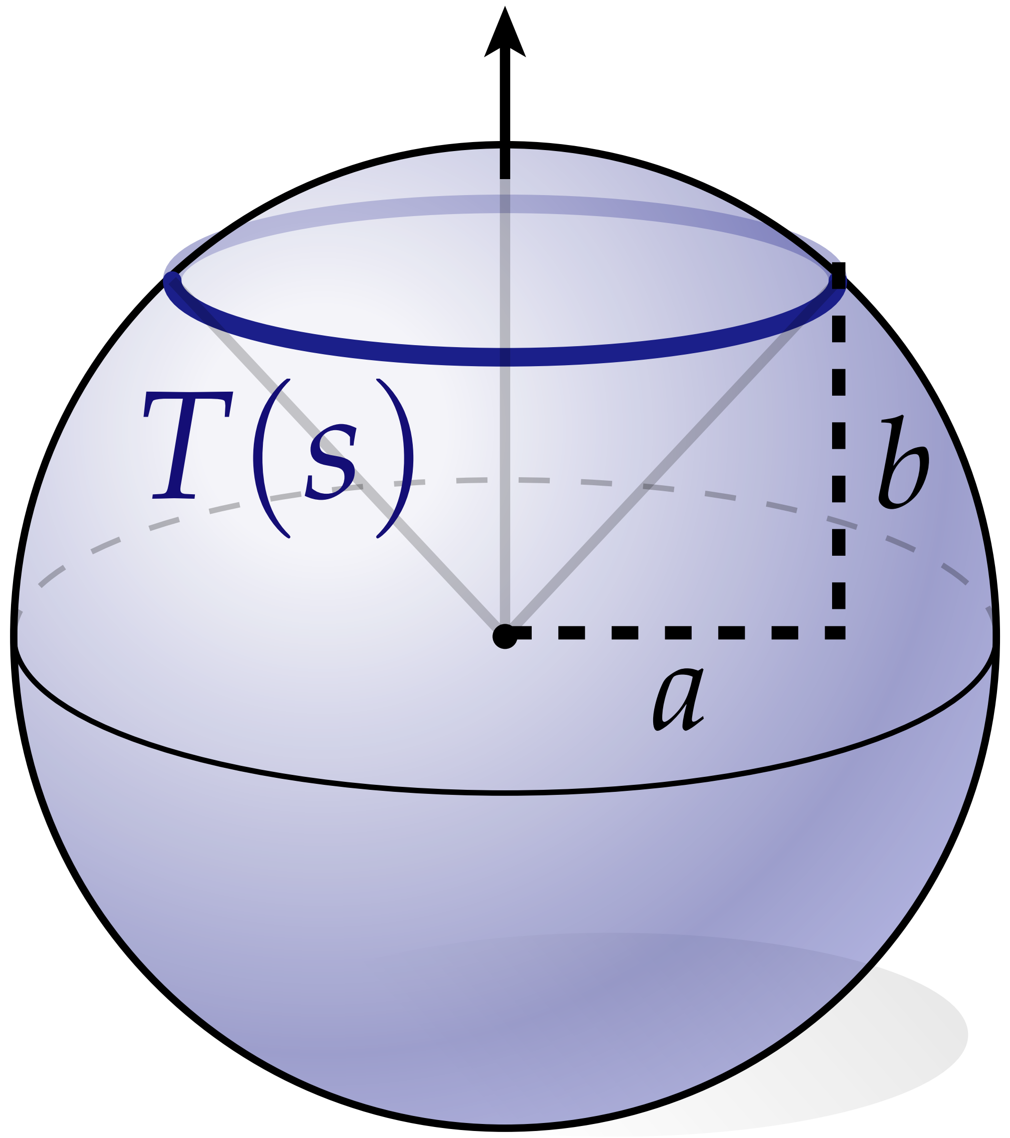
$$B(s) = T(s) \times N(s) =$$

$$(-b\sin(s), b\cos(s), -a)$$

$$\frac{d}{ds}B(s) = -b(\cos(s), \sin(s), 0)$$

$$\Rightarrow \tau(s) = -b$$



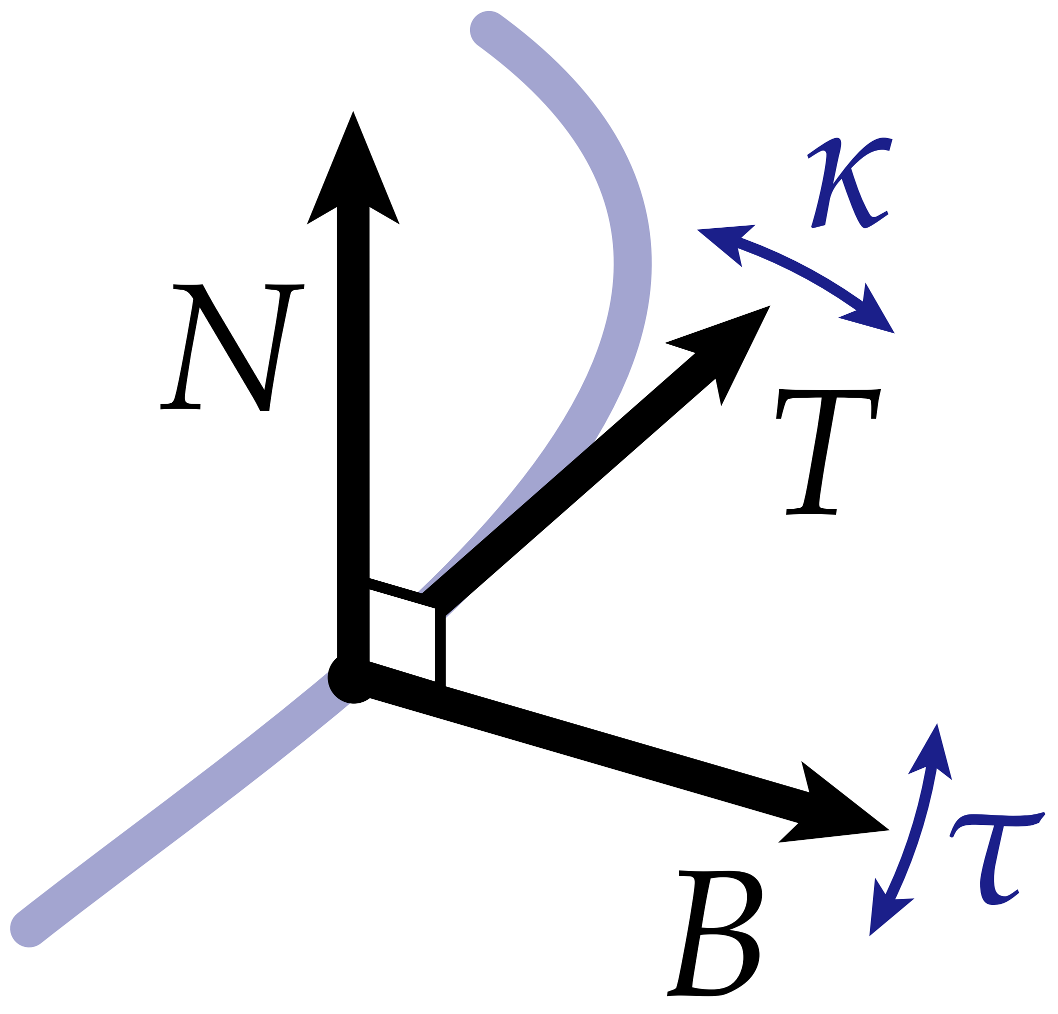


*For simplicity, let's pick a,b such that $a^2 + b^2 = 1$.

Fundamental Theorem of Space Curves

- The *fundamental theorem of space curves* tells us we can also go the other way: given the curvature and torsion of an arc-length parameterized space curve, we can recover the curve itself
- In 2D we just had to integrate a single ODE; here we integrate a system of three ODEs—namely, Frenet-Serret!

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}$$



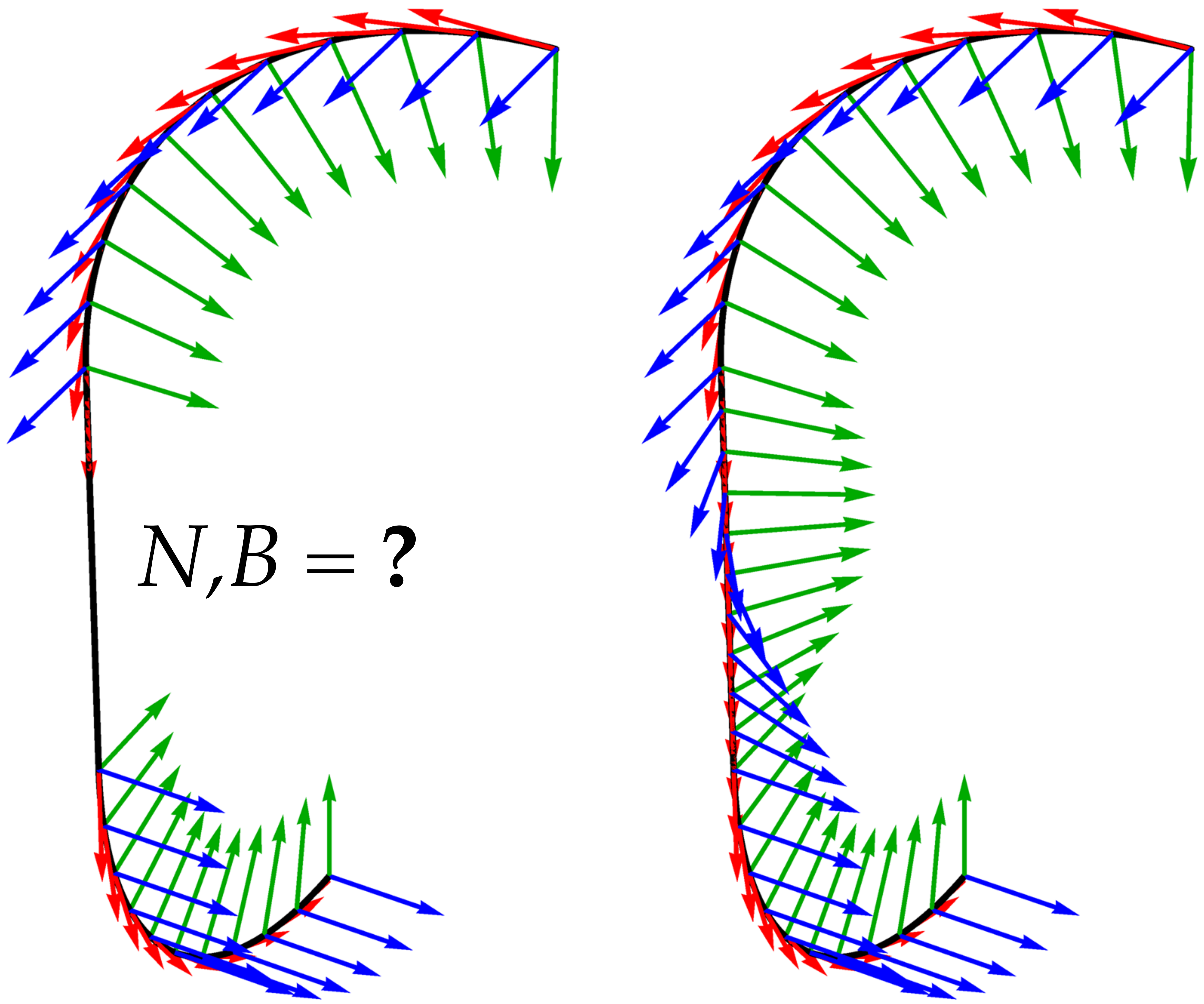
Intuition: curvature *and torsion* tell us how to "steer", while traveling at unit speed.

Adapted Frames on Curves

Q: If our curve has a straight piece, is the Frenet frame well-defined?

A: No, we don't have a clear normal/binormal (since, *e.g.*, $dT/ds = 0$)

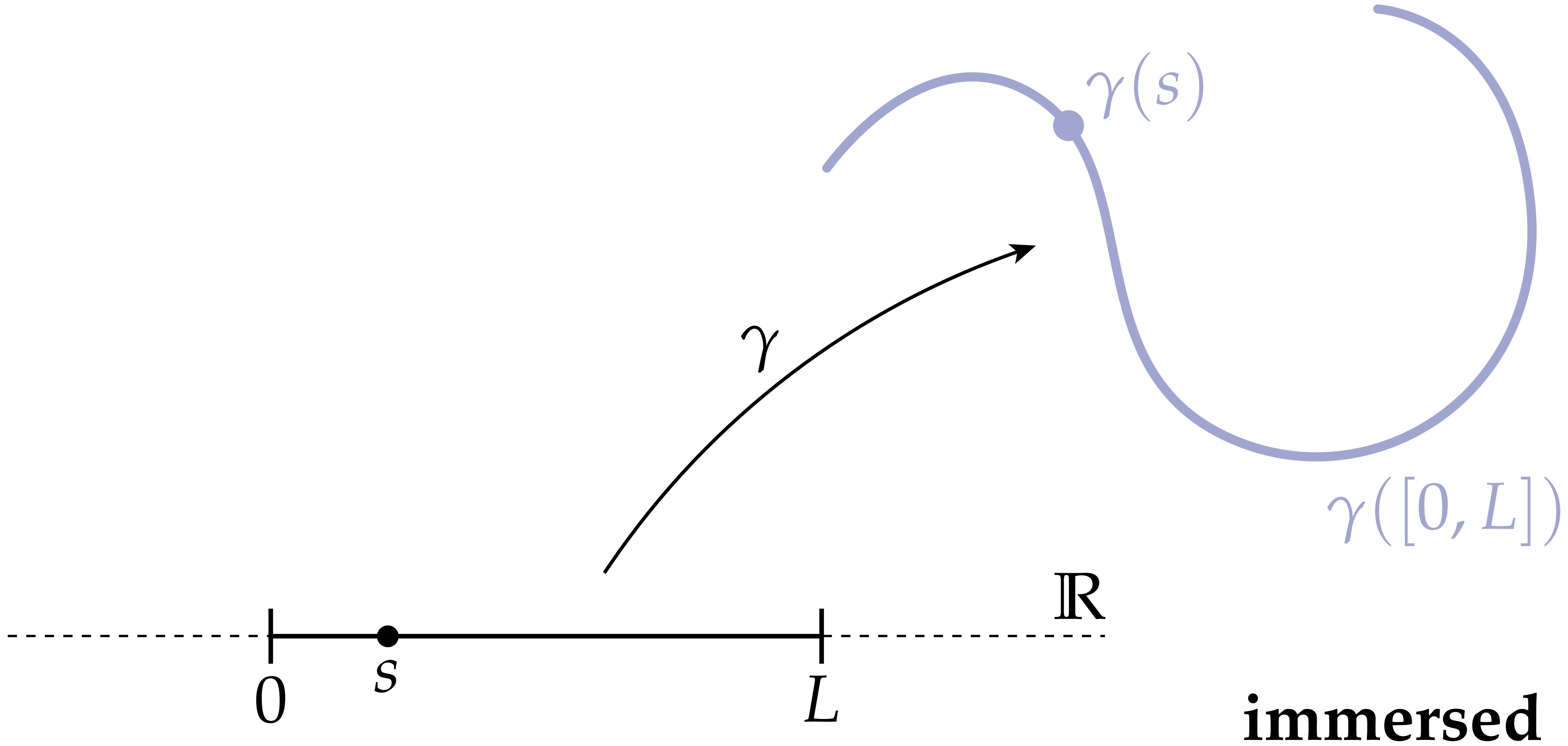
- However, there are many ways to choose an *adapted frame*
  - any orthonormal frame including T
- *E.g.*, *least-twisting* frame (Bishop)
  - unlike Frenet, *global* rather than *local*
- First example of *moving frames*, which provides a whole other language for surfaces…

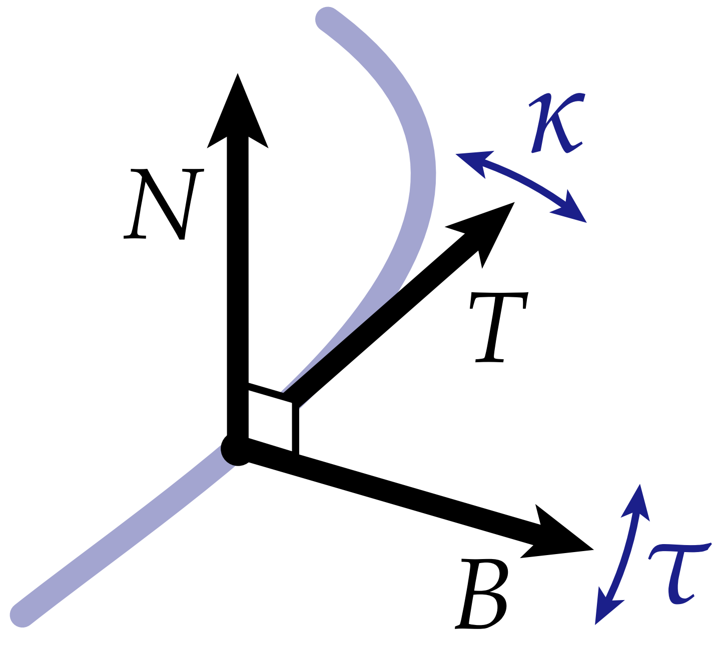


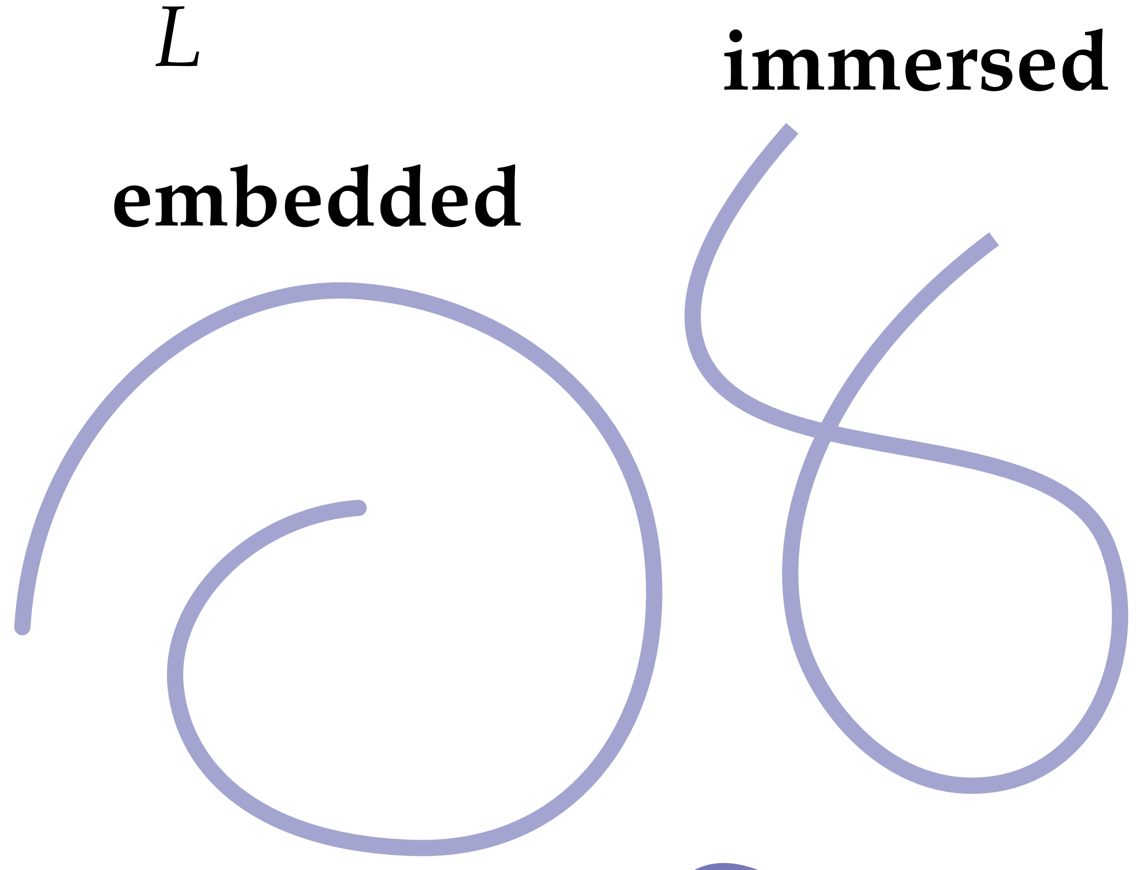
Summary

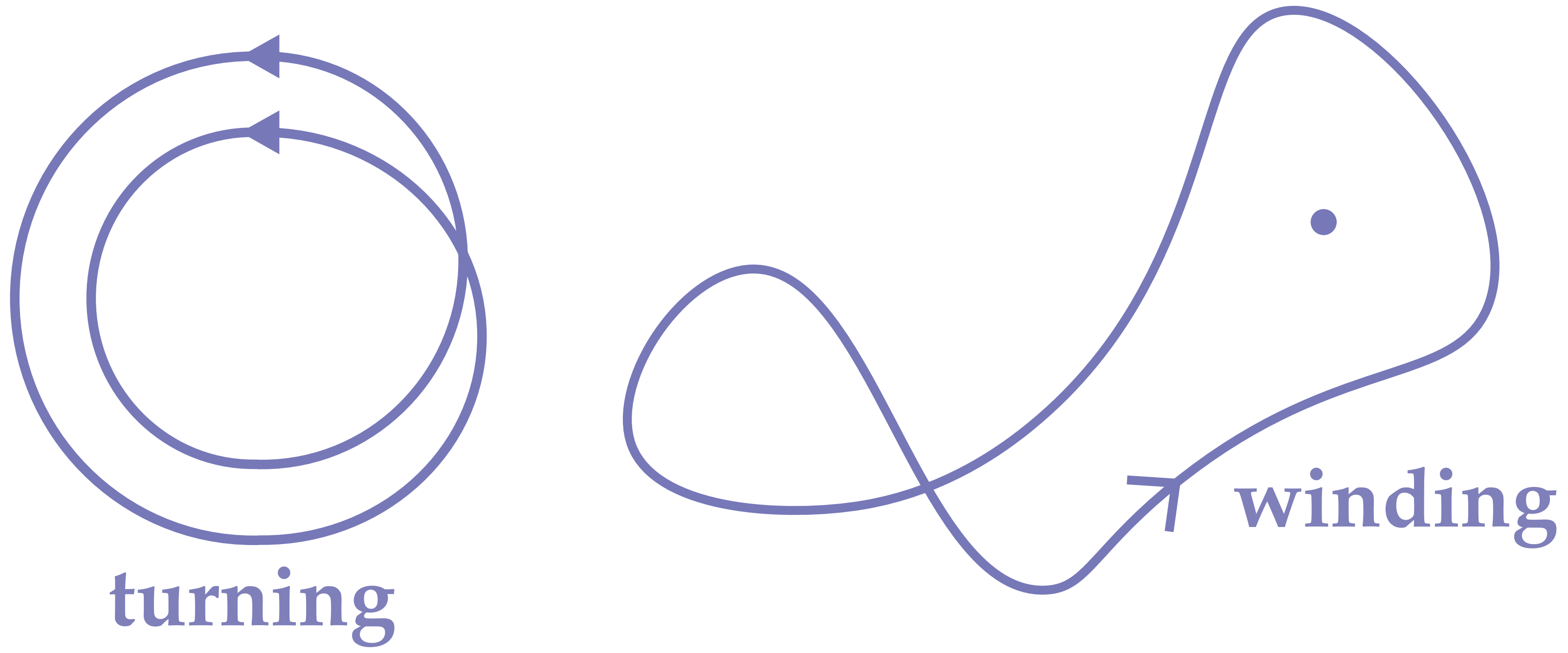
Curves—Summary

- Many ways to describe curves (*e.g.*, algebraic)
- Common approach in differential geometry: parameterized curves
  - Why? Can express important <u>local</u> quantities via derivatives of parameterization
  - tangent, normal, binormal (*Frenet frame*)
  - curvature, torsion
- Embedded vs. immersed / regular
- *Turning number*—degree of tangent map
- *Winding number*—degree of map around point
- **Fundamental theorem:** recover from curvatures

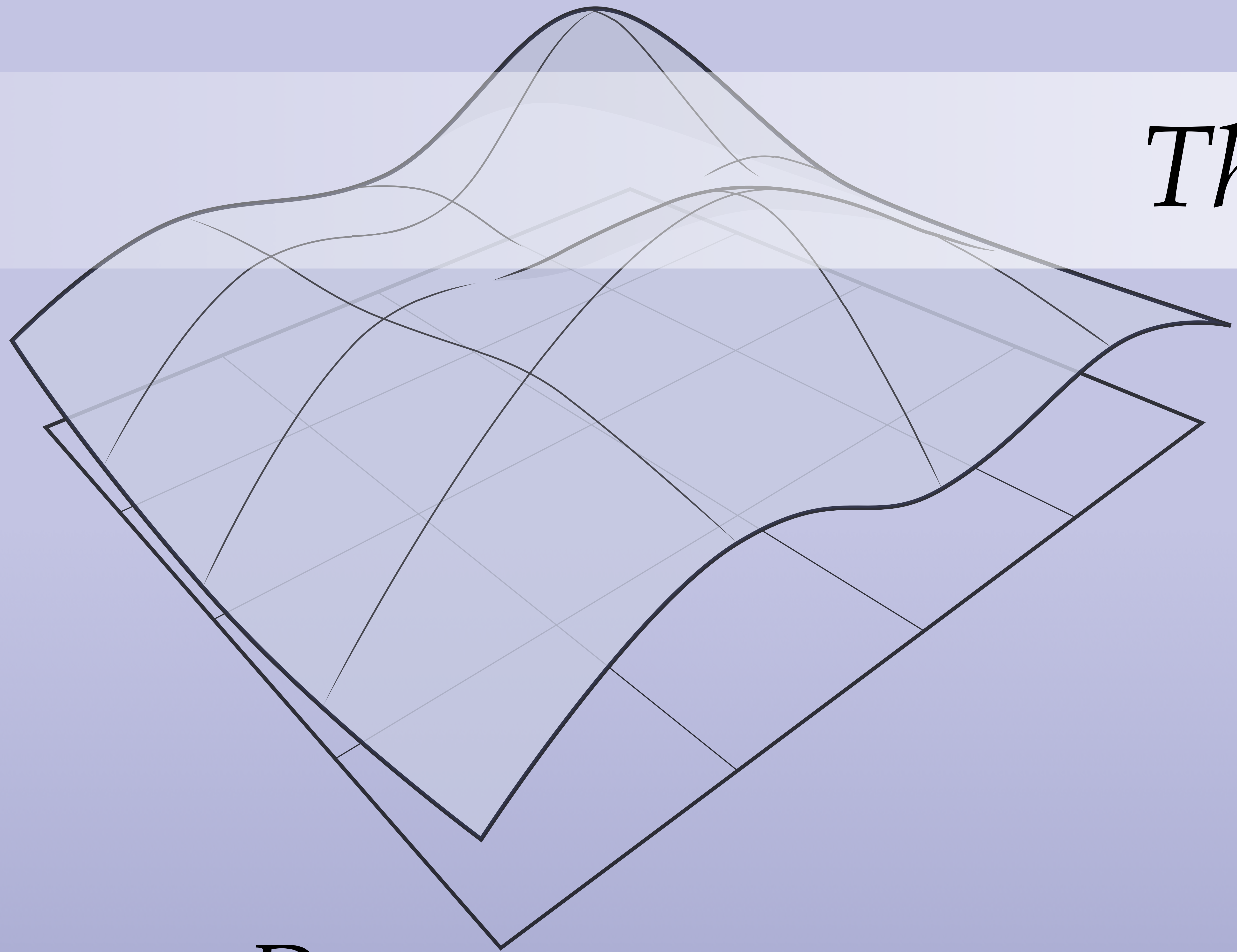








Thanks!



DISCRETE DIFFERENTIAL GEOMETRY:
AN APPLIED INTRODUCTION

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