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IMO Shortlist 2004

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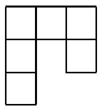
a certain move, his adversary is allowed to write $n+1$ or $2n$ (provided the number he writes does not exceed N). The player who writes N wins. We say that N is of type A or of type B according as A or B has a winning strategy.

(a) Determine whether $N = 2004$ is of type A or of type B.

(b) Find the least $N > 2004$ whose type is different from that of 2004.

13. **C6 (IRN)** For an $n \times n$ matrix A, let X_i be the set of entries in row i, and Y_j the set of entries in column j, $1 \le i, j \le n$. We say that A is *golden* if $X_1, \dots, X_n, Y_1, \dots, Y_n$ are distinct sets. Find the least integer n such that there exists a 2004×2004 golden matrix with entries in the set $\{1, 2, \dots, n\}$.

14. **C7 (EST)**[IMO3] Determine all $m \times n$ rectangles that can be covered with *hooks* made up of 6 unit squares, as in the figure:



Rotations and reflections of hooks are allowed. The rectangle must be covered without gaps and overlaps. No part of a hook may cover area outside the rectangle.

15. **C8 (POL)** For a finite graph G, let $f(G)$ be the number of triangles and $g(G)$ the number of tetrahedra formed by edges of G. Find the least constant c such that
$$g(G)^3 \le c \cdot f(G)^4 \ \text{ for every graph } G.$$

16. **G1 (ROM)**[IMO1] Let ABC be an acute-angled triangle with $AB \ne AC$. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the line segment BC.

17. **G2 (KAZ)** The circle Γ and the line ℓ do not intersect. Let AB be the diameter of Γ perpendicular to ℓ, with B closer to ℓ than A. An arbitrary point $C \ne A, B$ is chosen on Γ. The line AC intersects ℓ at D. The line DE is tangent to Γ at E, with B and E on the same side of AC. Let BE intersect ℓ at F, and let AF intersect Γ at $G \ne A$. Prove that the reflection of G in AB lies on the line CF.

18. **G3 (KOR)** Let O be the circumcenter of an acute-angled triangle ABC with $\angle B < \angle C$. The line AO meets the side BC at D. The circumcenters of the triangles ABD and ACD are E and F, respectively. Extend the sides BA and CA beyond A, and choose on the respective extension points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60°$.

13. Since X_i, Y_i, $i = 1, \dots, 2004$, are 4008 distinct subsets of the set $S_n = \{1, 2, \dots, n\}$, it follows that $2^n \geq 4008$, i.e. $n \geq 12$.
Suppose $n = 12$. Let $\mathcal{X} = \{X_1, \dots, X_{2004}\}$, $\mathcal{Y} = \{Y_1, \dots, Y_{2004}\}$, $\mathcal{A} = \mathcal{X} \cup \mathcal{Y}$. Exactly $2^{12} - 4008 = 88$ subsets of S_n do not occur in $\mathcal{A}$.
Since each row intersects each column, we have $X_i \cap Y_j \neq \emptyset$ for all i, j. Suppose $|X_i|, |Y_j| \leq 3$ for some indices i, j. Since then $|X_i \cup Y_j| \leq 5$, any of at least $2^7 > 88$ subsets of $S_n \setminus (X_i \cap Y_j)$ can occur in neither $\mathcal{X}$ nor $\mathcal{Y}$, which is impossible. Hence either in $\mathcal{X}$ or in $\mathcal{Y}$ all subsets are of size at least 4. Suppose w.l.o.g. that $k = |X_l| = \min_i |X_i| \geq 4$. There are

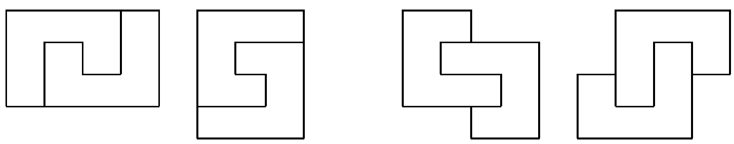
$$n_k = \binom{12-k}{0} + \binom{12-k}{1} + \cdots + \binom{12-k}{k-1}$$

subsets of $S \setminus X_l$ with fewer than k elements, and none of them can be either in $\mathcal{X}$ (because $|X_l|$ is minimal in $\mathcal{X}$) or in $\mathcal{Y}$. Hence we must have $n_k \leq 88$. Since $n_4 = 93$ and $n_5 = 99$, it follows that $k \geq 6$. But then none of the $\binom{12}{0} + \cdots + \binom{12}{5} = 1586$ subsets of S_n is in $\mathcal{X}$, hence at least $1586 - 88 = 1498$ of them are in $\mathcal{Y}$. The 1498 complements of these subsets also do not occur in $\mathcal{X}$, which adds to 3084 subsets of S_n not occurring in $\mathcal{X}$. This is clearly a contradiction.
Now we construct a golden matrix for $n = 13$. Let

$$A_1 = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A_m = \begin{bmatrix} A_{m-1} & A_{m-1} \\ A_{m-1} & B_{m-1} \end{bmatrix} \text{ for } m = 2, 3, \dots,$$

where B_{m-1} is the $2^{m-1} \times 2^{m-1}$ matrix with all entries equal to $m+2$. It can be easily proved by induction that each of the matrices A_m is golden. Moreover, every upper-left square submatrix of A_m of size greater than 2^{m-1} is also golden. Since $2^{10} < 2004 < 2^{11}$, we thus obtain a golden matrix of size 2004 with entries in S_{13}.

14. Suppose that an $m \times n$ rectangle can be covered by "hooks". For any hook H there is a unique hook K that covers its " inside" square. Then also H covers the inside square of K, so the set of hooks can be partitioned into pairs of type $\{H, K\}$, each of which forms one of the following two figures consisting of 12 squares:



A_1 B_1 A_2 B_2

Thus the $m \times n$ rectangle is covered by these tiles. It immediately follows that $12 \mid mn$.
Suppose one of m, n is divisible by 4. Let w.l.o.g. $4 \mid m$. If $3 \mid n$, one can easily cover the rectangle by 3×4 rectangles and therefore by hooks. Also,

if $12 \mid m$ and $n \notin \{1,2,5\}$, then there exist $k, l \in \mathbb{N}_0$ such that $n = 3k+4l$, and thus the rectangle $m \times n$ can be partitioned into 3×12 and 4×12 rectangles all of which can be covered by hooks. If $12 \mid m$ and $n = 1, 2$, or 5, then it is easy to see that covering by hooks is not possible.

Now suppose that $4 \nmid m$ and $4 \nmid n$. Then m, n are even and the number of tiles is odd. Assume that the total number of tiles of types A_1 and B_1 is odd (otherwise the total number of tiles of types A_2 and B_2 is odd, which is analogous). If we color in black all columns whose indices are divisible by 4, we see that each tile of type A_1 or B_1 covers three black squares, which yields an odd number in total. Hence the total number of black squares covered by the tiles of types A_2 and B_2 must be odd. This is impossible, since each such tile covers two or four black squares.

15. Denote by $V_1, \dots, V_n$ the vertices of a graph G and by E the set of its edges. For each $i = 1, \dots, n$, let A_i be the set of vertices connected to V_i by an edge, G_i the subgraph of G whose set of vertices is A_i, and E_i the set of edges of G_i. Also, let v_i, e_i, and $t_i = f(G_i)$ be the numbers of vertices, edges, and triangles in G_i respectively.

    The numbers of tetrahedra and triangles one of whose vertices is V_i are respectively equal to t_i and e_i. Hence

$$\sum_{i=1}^{n} v_i = 2|E|, \qquad \sum_{i=1}^{n} e_i = 3f(G) \quad \text{and} \quad \sum_{i=1}^{n} t_i = 4g(G).$$

    Since $e_i \le v_i(v_i-1)/2 \le v_i^2/2$ and $e_i \le |E|$, we obtain $e_i^2 \le v_i^2|E|/2$, i.e., $e_i \le v_i\sqrt{|E|/2}$. Summing over all i yields $3f(G) \le 2|E|\sqrt{|E|/2}$, or equivalently $f(G)^2 \le 2|E|^3/9$. Since this relation holds for each graph G_i, it follows that

$$t_i = f(G_i) = f(G_i)^{1/3} f(G_i)^{2/3} \le \left(\frac{2}{9}\right)^{1/3} f(G)^{1/3} e_i.$$

    Summing the last inequality for $i = 1, \dots, n$ gives us

$$4g(G) \le 3\left(\frac{2}{9}\right)^{1/3} f(G)^{1/3} \cdot f(G), \qquad \text{i.e.} \qquad g(G)^3 \le \frac{3}{32} f(G)^4.$$

    The constant $c = 3/32$ is the best possible. Indeed, in a complete graph C_n it holds that $g(K_n)^3/f(K_n)^4 = \binom{n}{4}^3\binom{n}{3}^{-4} \to \frac{3}{32}$ as $n \to \infty$.

    Remark. Let N_k be the number of complete k-subgraphs in a finite graph G. Continuing inductively, one can prove that $N_{k+1}^k \le \frac{k!}{(k+1)^k} N_k^{k+1}$.

16. Note that $\triangle ANM \sim \triangle ABC$ and consequently $AM \ne AN$. Since $OM = ON$, it follows that OR is a perpendicular bisector of MN. Thus, R is the common point of the median of MN and the bisector of $\angle MAN$. Then it follows from a well-known fact that R lies on the circumcircle of $\triangle AMN$.