PROBLEM SOLVING VIGNETTES

No. 14

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Playing with Probability

Probability problems can be easily stated, but their solutions may seem elusive or counterintuitive. Careful attention has to be placed on counting the number of outcomes for each event and recognizing if the outcomes we are counting are equally likely or not. As a simple example, when we roll two dice the possible sums range from 2 to 12, inclusive, but they are not all equally likely. We also have to understand when events are independent or not. For example, if I flip a coin then roll a die, the outcome of the die does not depend on the outcome of the flip because the outcomes are independent of each other. On the other hand, if I draw a card from a deck of cards and then draw a second card, these actions are dependent. That is, if I am interested in the second card being a heart, the probability depends on whether the first card was a heart or not. It seems our intuition about probability is sometimes flawed, probably because we overlook these subtle properties of probability.

We will begin with a problem from the first *Canadian Mathematical Gray Jay Competition* (CMGC). The CMGC is a new, multiple choice competition from the CMS for elementary school students . The first competition was written on Thursday October 8, 2020 by just under 2000 students world-wide.

> **6.** Alice and Bill play a game. They go to separate rooms, flip a coin and try to predict what the other person flipped. They win if at least one of them predicts correctly. They decide that Alice will always guess the same thing that she flips and Bill will always predict the opposite of what he flips. What percentage of the time should they win?
> **(A)** 0% **(B)** 25% **(C)** 50% **(D)** 75% **(E)** 100%

First, we should determine all possible outcomes to the process of Alice and Bill flipping their coins. We will use H and T for heads and tails, and we will list the results from Alice first them Bill, so TH means Alice flipped tails and Bill flipped heads. Thus all possible outcomes are $\{HH, HT, TH, TT\}$. We will assume that we have fair coins, so heads and tails are equally likely for each person. Also note that the result of Alice's flip will in no way influence Bill's flip, so we can see that our outcomes are all equally likely, with probability of $\frac{1}{4}$ each.

The key to solving this problem is deciding which of the events constitute a "win". To do that, let us look at the information in a bit more detail:

A's Flip	A's Guess	Correct?	B's Flip	B's Guess	Correct?	Win?
H	H	✓	H	T	✗	✓
H	H	✗	T	H	✓	✓
T	T	✗	H	T	✓	✓
T	T	✓	T	H	✗	✓

We see that Alice and Bill's strategy yields a win every time! It is interesting that we are dealing with a totally random process but we set up a situation where we can guarantee at least one person guesses correctly. Can you find a strategy that guarantees at least one of three people guesses correctly if A guesses B's who guesses C's who guesses A's?

Next, we will examine problem B2 from another CMS competition, the 2020 *Canadian Open Mathematics Challenge*.

> **B2.** Alice places a coin, heads up, on a table then turns off the light and leaves the room. Bill enters the room with 2 coins and flips them onto the table and leaves. Carl enters the room, in the dark, and removes a coin at random. Alice reenters the room, turns on the light and notices that both coins are heads. What is the probability that the coin Carl removed was also heads?

We know that before Carl went in there were three coins and after he left there were two coins, both heads. Since Carl could have removed either a head or a tail, before he went in there was either three heads (and he removed a head), or two heads and a tail (and he removed a tail). At this point, since there were two possible "starting" states, we may be fooled into thinking that our desired probability is $\frac{1}{2}$. Unfortunately, this is not the case since, counter-intuitively, the starting points are not equally likely in a couple of ways!

Looking closer, since Alice placed a coin heads up in the room, we already have a head (H_A, for Alice's head). When Bill goes in, he flips two coins yielding

$$H_{B_1}H_{B_2},\ H_{B_1}T_{B_2},\ T_{B_1}H_{B_2} \text{ or } T_{B_1}T_{B_2}$$

(where the subscripts B_1 and B_2 indicate Bill's first and second flip). At this point we see that it was impossible for Bill to have flipped two tails, because Carl could never have left behind two heads. Thus, when Carl enters the room there are three possible, equally likely, configurations:

$$H_AH_{B_1}H_{B_2},\ H_AH_{B_1}T_{B_2},\ \text{and } H_AT_{B_1}H_{B_2}.$$

If Carl went in and there were three heads, he could remove any of them and satisfy the conditions of the problem. However, if there were two heads and a tail, he could *only* remove the tail. Hence, as shown in the table below, the probability that Carl removed a head was $\frac{3}{5}$.

Start	Remove	Left	Conditions?
$H_A H_{B_1} H_{B_2}$	H_A	$H_{B_1} H_{B_2}$	✓
$H_A H_{B_1} H_{B_2}$	H_{B_1}	$H_A H_{B_2}$	✓
$H_A H_{B_1} H_{B_2}$	H_{B_2}	$H_A H_{B_1}$	✓
$H_A H_{B_1} T_{B_2}$	H_A	$H_{B_1} T_{B_2}$	✗
$H_A H_{B_1} T_{B_2}$	H_{B_1}	$H_A T_{B_2}$	✗
$H_A H_{B_1} T_{B_2}$	T_{B_2}	$H_A H_{B_1}$	✓
$H_A T_{B_1} H_{B_2}$	H_A	$T_{B_1} H_{B_2}$	✗
$H_A T_{B_1} H_{B_2}$	T_{B_1}	$H_A H_{B_2}$	✓
$H_A T_{B_1} H_{B_2}$	H_{B_2}	$H_A T_{B_1}$	✗

The next few problems will come from the problem solving course I took with Professor Honsberger. Problems from the three assignments and problems 1 through 25 have been featured in earlier columns. Below is the next set of problems from the course.

#26. A normal die bearing the numbers $1, 2, 3, 4, 5, 6$ on its faces is thrown repeatedly until the running total first exceeds 12. What is the most likely total that will be obtained?

#27. Find all natural numbers, not ending in zero, which have the property that if the final digit is deleted, the integer obtained divides into the original.

#28. Let n denote an odd natural number greater than one. Let A denote an $n \times n$ symmetric matrix such that each row and each column consists of some permutation of the numbers $1, 2, 3, \ldots, n$. Show that each of the numbers $1, 2, 3, \ldots, n$ must occur in the main diagonal of A.

#29. A,B, and C are to fight a 3-cornered duel. All of them know that A's chance of hitting his target is 0.3, that C's chance is 0.5, and that B never misses. They are to fire at their choice of target in succession $A, B, C, A, B, \ldots$ etc. until only one man is left unhit (once a man is hit, he drops out of the duel). What is the best strategy for A?

#30. What are the final two digits of $\left(\cdots\left(7^7\right)^7\cdots\right)^7$, containing 1001 7's?

We will consider problem 26 next. To get a feel for what is happening, you may want to do an experiment. I rolled a die a number of times to perform the process four times and came up with the following:

Rolls	Total
$3, 4, 2, 1, 4$	14
$2, 3, 4, 1, 1, 3$	14
$5, 5, 4$	14
$4, 5, 6$	15

The first thing that we should notice is even though a single roll of the die does not have any effect on future rolls of the die, since our outcomes must sum to a number greater than 12, they will have different probabilities. The first entry took 5 rolls, the second 6 and the last two took 3. The probability of rolling the last two results are the same, but different from the first two which are also different from each other.

Other strange things happen as well. For the last roll $4, 5, 6$, we could have also gotten $4, 6, 5$, or any other permutation of those three numbers and we would have gotten the same result. This is useful for counting our results. On the other hand, looking at the first roll $3, 4, 2, 1, 4$, we could have also gotten $4, 1, 2, 4, 3$ for the same total of 14, but $4, 2, 4, 3, 1$ would not occur! Since $4 + 2 + 4 + 3 = 13$, we would have stopped before we reached 14. So, some of the strategies that we might have considered have to be discarded. Instead, let us look at how things could end. We will use the notation $(11, 3)$ to represent any roll whose last sum, before being greater than 12, was 11 and the last roll is 3. Hence, the second roll $2, 3, 4, 1, 1, 3$ would fall under this category. Listing all possible outcomes and their final result we get:

Final Sum

13	14	15	16	17	18
$(7, 6)$					
$(8, 5)$	$(8, 6)$				
$(9, 4)$	$(9, 5)$	$(9, 6)$			
$(10, 3)$	$(10, 4)$	$(10, 5)$	$(10, 6)$		
$(11, 2)$	$(11, 3)$	$(11, 4)$	$(11, 5)$	$(11, 6)$	
$(12, 1)$	$(12, 2)$	$(12, 3)$	$(12, 4)$	$(12, 5)$	$(12, 6)$

Looking at a total of 15 as an example we see that all the events that give us our desired sum; $(9, 6)$, $(10, 5)$, $(11, 4)$, and $(12, 3)$; all have different probabilities. On the bright side, all events that start with the same number; such as $(9, 4)$, $(9, 5)$, and $(9, 6)$; have the same probability. Hence we can easily see that

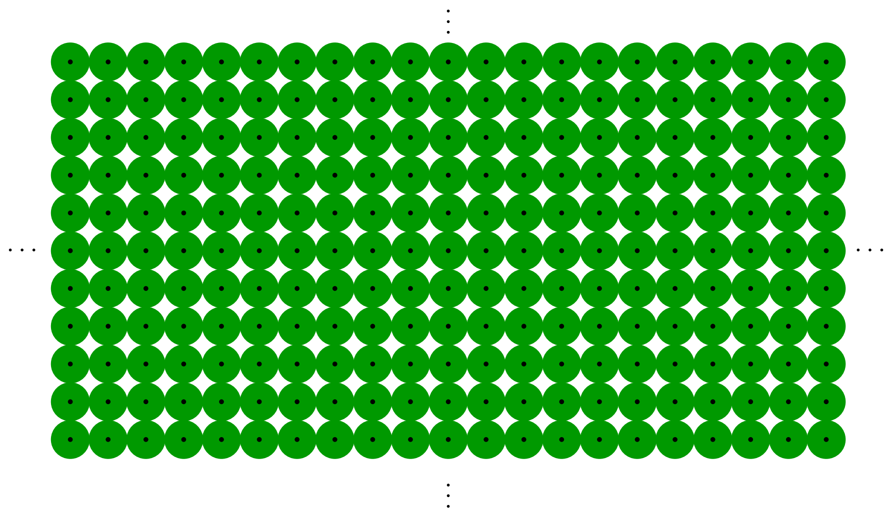
$$P(18) < P(17) < P(16) < P(15) < P(14) < P(13)$$

so 13 is the most likely sum. In this case we were able to answer the question, without actually having to calculate the probabilities involved.

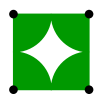
Next on the list is question 4 from the third assignment from professor Honsberger's class, featured in an earlier column [2017 : 43(10), p. 441-443].

> **4.** A circle of radius $\frac{1}{2}$ is tossed at random onto a coordinate plane. What is the probability that it covers a lattice point?

This is a geometric probability problem, where we are looking at areas rather than counting cases, since there are infinitely many! If we focus on a particular lattice point, we see that it will be covered by the circle if and only if the centre of the circle is a distance of at most $\frac{1}{2}$ away. Thus, the centre of the circle must land within a circle of radius $\frac{1}{2}$ centred on a lattice point to contain the lattice point. If we colour the possible locations of the centre of the circle green, the coordinate plane will look something like the diagram below.



Because of the symmetry, we can focus on a unit square formed by four lattice points, shown in the figure below.



Thus, for whichever unit square the centre lands in, if it lands in the green area the circle will contain a lattice point, otherwise it won't. Hence

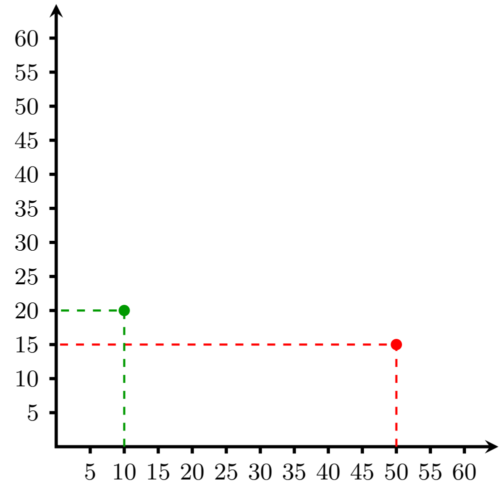
$$P = \frac{\text{Green area}}{\text{Total area}} = \frac{\pi\left(\frac{1}{2}\right)^2}{1} = \frac{\pi}{4}$$

or about 78.5%.

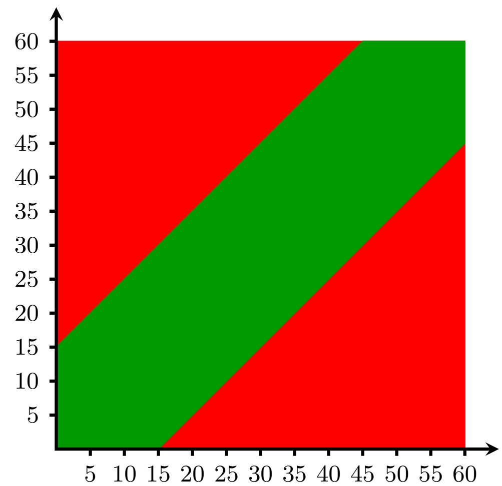
In many cases where you are dealing with continuous data, a geometric argument will work. Consider the following problem from the second assignment [2017: 43(8) 344-346].

> **3.** Two people agree to meet for lunch at their favourite restaurant, Each agrees to wait 15 minutes for the other, after which time he will leave. If each chooses his time of arrival at random between noon and 1 o'clock, what is the probability of a meeting taking place ?

Again, there are infinitely many possibilities to consider. Let's look at a graph where we indicate the number of minutes after noon that each person arrives on the axes. We can then colour the points in the plane either red (no meeting) or green (they meet). For example, if the first person arrives at 12 : 10 and the second at 12 : 20, they would meet, indicated by the green point at $(10, 20)$. On the other hand, if the first person arrives at 12 : 50 and the second at 12 : 15, they would not meet, indicated by a red point at $(50, 15)$.



Colouring the rest of the grid accordingly, we can find the desired probability as the ratio of the green area to the total area which is $\frac{7}{16}$.



You may want to consider a similar problem with three friends and calculate the probability that they all meet.

Probability problems can be interesting, because their solutions can require tools from different areas of mathematics. Enjoy the rest of the problems from Professor Honsberger's class. We will revisit probability in a future column.