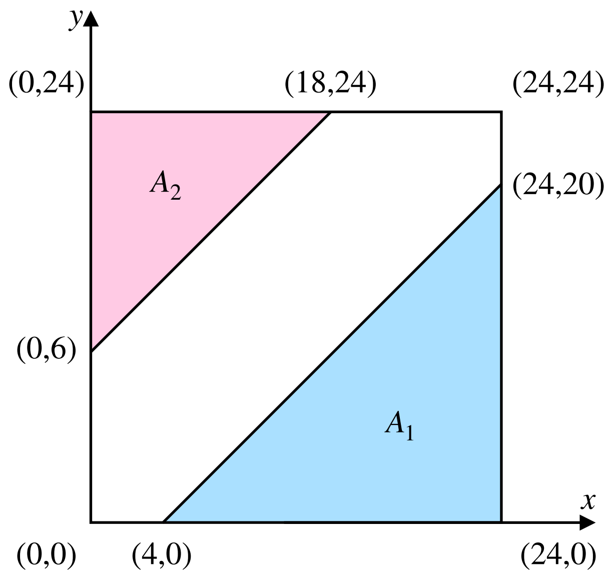
Задача 3. $\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x \le 24,\ 0 \le y \le 24\} = [0,24] \times [0,24]$, т.е.



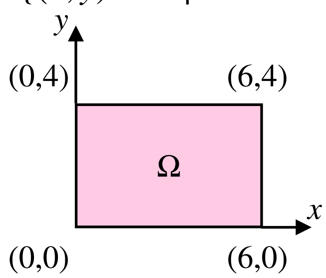
$\Omega = [0,24]^2$ – лвадрат със страна 24.

$A_1 \begin{cases} x - y \ge 4 \\ 0 \le y < x \le 24 \end{cases}, A_2 \begin{cases} y - x \ge 6 \\ 0 \le x < y \le 24 \end{cases}.$

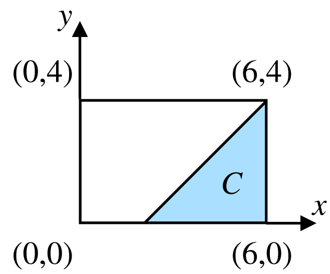
$$A = A_1 \cup A_2,\ \mathbb{P}(A) = \mathbb{P}(A_1 \cup A_2) = \frac{S_{A_1} + S_{A_2}}{S_\Omega} = \frac{\frac{20^2}{2} + \frac{18^2}{2}}{24^2}.$$

Задача 4. $\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x < 6,\ 0 \le y < 4\} = [0,6] \times [0,4]$, т.е. Ω е правоъгълник:

а) $C = \{(x, y) \in \Omega \,|\, 6 - x < 4 - y\} = \{(x, y) \in \Omega \,|\, x - y > 2\}$

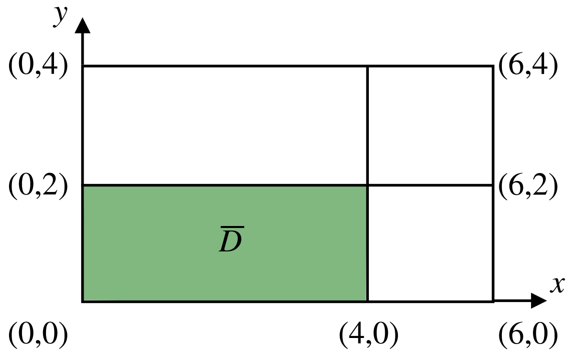


$$\mathbb{P}(C) = \frac{S_C}{S_\Omega} = \frac{\frac{4 \times 4}{2}}{6 \times 4} = \frac{1}{3};$$



б) $\overline{D} = \{(x, y) \in \Omega \,|\, 6 - x > 2,\ 4 - y > 2\} = \{(x, y) \in \Omega \,|\, x < 4,\ y < 2\}$

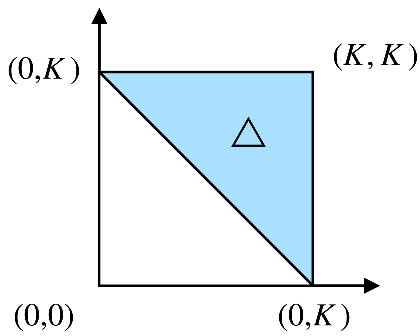
$$\mathbb{P}(D) = 1 - \mathbb{P}(\overline{D}) = 1 - \frac{S_{\overline{D}}}{S_\Omega} = 1 - \frac{2 \times 4}{6 \times 4} = 1 - \frac{1}{3} = \frac{2}{3}.$$



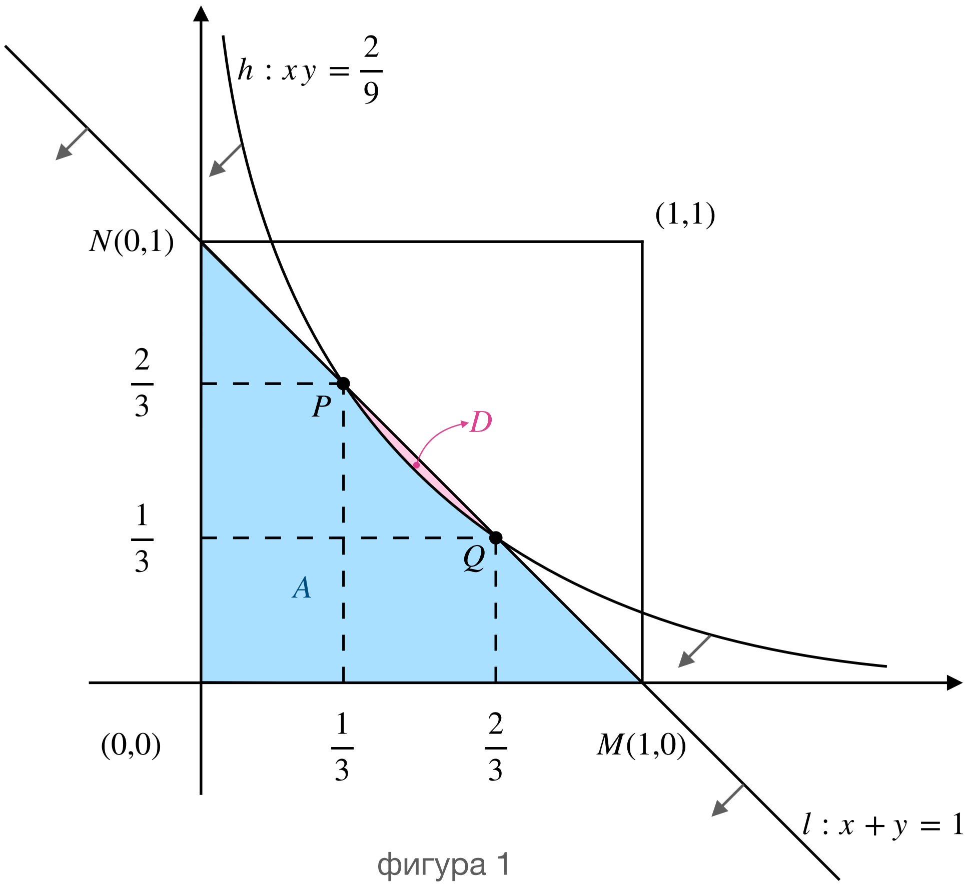
Задача 5. $\Omega = \{(x,y) \in \mathbb{R}^2 \,|\, 0 < x, y < K\} = (0,K) \times (0,K)$

$\triangle = \{(x,y) \in \Omega \,|\, x + y > K\}$

$\mathbb{P}(\triangle) = \dfrac{S_\triangle}{S_\Omega} = \dfrac{K^2/2}{K^2} = \dfrac{1}{2}.$



Задача 5. (Упражнение 5) – Чертеж и Решение



фигура 1

$\Omega = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq x, y \leq 1\} = [0,1]^2$ – единичният квадрат на фигура 1.

Нека $A = \{(x,y) \in \Omega \,|\, x + y < 1 \wedge xy < \frac{2}{9}\}$. Търсим вероятността на A : $\mathbb{P}(A)$.

Нека $l : x + y = 1$ и $h : xy = \frac{2}{9}$. l – права, h – клон на хипербола (от чертежа на фиг. 1).

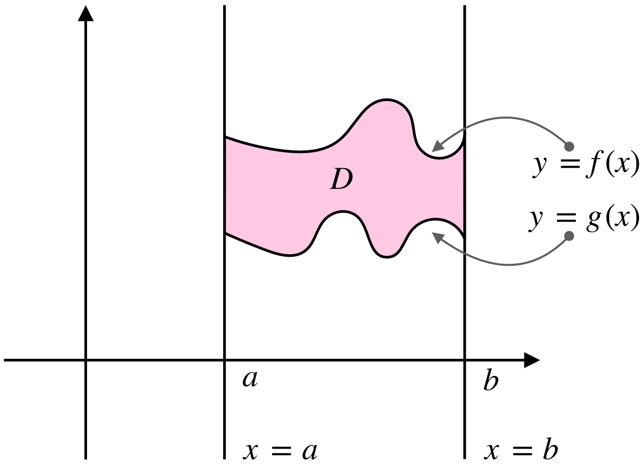
Ще намерим пресечните точки на $h \cap l$: $\begin{cases} x + y = 1, \\ xy = \frac{2}{9}, \\ x \geq 0, \\ y \geq 0. \end{cases}$

Графично A е синята част от чертежа, а D е розовата част.

$\mathbb{P}(A) = \dfrac{\mu(A)}{\mu(\Omega)} = \dfrac{S_A}{S_\Omega} = S_A$. За да пресметнем S_A – ще приложим следната Лема 1: Нека $D := \{(x, y) \in \Omega \,|\, xy \geq \dfrac{2}{9},\ x + y \leq 1\}$, т.е. D е фигурата заградена от правата $l = x + y$ и хиперболата $h : xy = \dfrac{2}{9}$.

<u>Лема 1</u>: Нека f, g са интегруеми функции в $[a, b]$, дефинирани в $[a, b]$ и приемащи реални стойности (накратко $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ са интегруеми в $[a, b]$ функции). Нека още $g(x) \leq f(x)$, $\forall x \in [a, b]$. Тогава лицето на фигурата D (чертежа на фиг. 2), заградена от кривите $x = a$, $x = b$, $y = f(x)$, $y = g(x)$ се дава с формулата

$S_D = \displaystyle\int_a^b \big(f(x) - g(x)\big)\, \mathrm{d}\, x.$



Прилагаме Лема 1 за пресмятане на S_D:

$S_D = \displaystyle\int_{\frac{1}{3}}^{\frac{2}{3}} \left[(1 - x) - \frac{2}{9}x\right] \mathrm{d}\, x = \int_{\frac{1}{3}}^{\frac{2}{3}} \left(1 - x - \frac{2}{9x}\right)$. Тук $y_1 = 1 - x$ и $y_2 = \dfrac{2}{9x}$.

$$\mathbb{P}(A) = S_A = S_{\triangle OMN} - S_D = \frac{1}{2} - \int_{\frac{1}{3}}^{\frac{2}{3}} \left(1 - x - \frac{2}{9x}\right) \mathrm{d}\, x = \frac{1}{2} - \left(x - \frac{x^2}{2} - \frac{2 \ln x}{9}\right)\Bigg|_{\frac{1}{3}}^{\frac{2}{3}} =$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{4}{18} + \frac{2}{9} \times (\ln 2 - \ln 3) + \frac{1}{3} - \frac{1}{18} - \frac{2}{9} \times (\ln 1 - \ln 3) = \frac{1}{2} - \frac{1}{3} + \frac{3}{18} + \frac{2}{9} \ln 2 =$$

$$= \frac{9 - 6 + 3}{18} + \frac{2}{9} \times \ln 2 \approx \frac{1}{3} + \frac{2}{9} \times 0.69314718056 \approx 0.48736604012 \approx 0.487 \qquad \square$$