Non-symmetric Ruin

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1 Introduction

In the file Martingales we gave outline about martingales. Let us recall the non-symmetric problem. We have a random walk based on the sequence of identically distributed, independent random variables $(\xi_i)_{i\geq 1}$, where $\xi_i = 1$ with probability p and $\xi_i = -1$ with probability q. The random walk is formally defined as

$$S_0 = 0; \quad S_n = \sum_{j=1}^{n} \xi_j$$

and describes the process of net win or loss of a game where at each step a player wins or loses one unit. S_n the net win or loss of the first n bets.

2 Problem

Given that we have 59 units of money we decide that we shall play until we lose all our capital or achieve a net win of 71 units. Mathematically, we stop the game at the random time

$$\tau = \min\{k \geq 0 : S_k = 71 \quad or \quad S_k = -59\}$$

Find $P(S_\tau = 71) =?$ and $E[\tau] =?$

3 Solution

When $p \neq 1/2$ then the process S_n is not a martingale but $Y_n = S_n - nE[\xi_1]$ is. However, if apply the Optional Sampling Theorem, that is $E[Y_\tau] = E[Y_0] = 0$, we arrive at

$$0 = E[S_\tau] - E[\tau]E[\xi_1].$$

We know that S_τ takes only two values and therefore

$$0 = E[S_\tau] - E[\tau]E[\xi_1] = 71P(S_\tau = 71) - 59(1 - P(S_\tau = 71)) - E[\tau](2p - 1), \quad (1)$$

because $E[\xi_1] = 2p - 1$. **Unfortunately, this equation has two unknowns!**

It turns out that useful **different** martingales can be constructed from S_n.

3.1 Exponential martingales

It can be checked that the process $Z_n = e^{\lambda S_n}(E[e^{\lambda \xi_1}])^{-n}, Z_0 = 1$, is a martingale for any real λ. However,

$$E[e^{\lambda \xi_1}] = pe^{\lambda} + qe^{-\lambda}.$$

We set $f(\lambda) = \ln(pe^{\lambda} + qe^{-\lambda})$.

Therefore,

$$Z_n = e^{\lambda S_n - nf(\lambda)}.$$

If we directly apply the Optional Sampling Theorem then, that is $E[Z_\tau] = E[Z_0] = 1$ we arrive at

$$1 = E[e^{\lambda S_\tau - \tau f(\lambda)}]. \tag{2}$$

The problem is that the term $\tau f(\lambda)$ is again intractable? Is it ?

3.2 Choosing a special exponential martingale

If we choose $\lambda \neq 0$ such that $f(\lambda) = 0$ the annoying term will be eliminated. But when $p \neq 1$ the equation

$$f(\lambda) = \ln(pe^{\lambda} + qe^{-\lambda}) = 0 \iff pe^{\lambda} + qe^{-\lambda} = 1$$

can be solved with $\lambda^* = \ln(q/p)$. Then

$$Z_n = e^{\lambda^* S_n - nf(\lambda^*)} = e^{\lambda^* S_n}, Z_0 = 1$$

is a martingale and from equation (2) we have that

$$1 = E[e^{\lambda^* S_\tau}].$$

However,

$$E[e^{\lambda^* S_\tau}] = P(S_\tau = 71)e^{71\lambda^*} + (1 - P(S_\tau = 71))e^{-59\lambda^*},$$

because S_τ can take only two possible values, i.e. 71 if we complete the game with net profit or -59 if we lose our capital. Then

$$1 = P(S_\tau = 71)e^{71\lambda^*} + (1 - P(S_\tau = 71))e^{-59\lambda^*}$$

gives

$$P(S_\tau = 71) = \frac{1 - e^{-59\lambda^*}}{e^{71\lambda^*} - e^{-59\lambda^*}}.$$

And the difficult problem is therefore solved.

3.3 Finding $E[\tau]$

Now when we know $P(S_\tau = 71)$ we can go back to the now tractable equation (1)

$$71P(S_\tau = 71) - 59(1 - P(S_\tau = 71)) - E[\tau](2p - 1) = 0$$

and evaluate $E[\tau]$.

4 Concluding remarks

The problem is significantly harder than the case $p = 1/2$. The useful notion is the exponential martingale which as an object has very far-reaching applications/implications in the theory of probability. Suffices to mention the martingale measures. However, as the example shows, these can be used for specific computations related to exit problems for stochastic processes.