Задача (НСВ). Нека случайните величини $X_1,\ X_2 \in \mathcal{U}nif(0,1)$ са независими. Да се намери разпределението на случайната величина Y, където $Y = 2X_1 - 3X_2$.

Решение:

Знаем, че $f_{X_1}(x) = f_{X_2}(x) = \dfrac{1}{r-l} = \dfrac{1}{1-0} = 1$, за $x \in (l, r) = (0,1)$.

$$f_{X_1,X_2}(x_1, x_2) \overset{X_1 \perp\!\!\!\perp X_2}{=} f_{X_1}(x_1) f_{X_2}(x_2) = 1 \times 1 \times \mathbb{1}_{\{x_1, x_2 \in [0,1]\}}.$$

Първи подход с **якобиан**:

$$\begin{cases} Y_1 = 2X_1 - 3X_2 \\ Y_2 = X_2 \end{cases} \Rightarrow \begin{cases} X_1 = \frac{1}{2}(Y_1 + 3X_2) \\ X_2 = Y_2 \end{cases} \Rightarrow \begin{cases} X_1 = \frac{1}{2}(Y_1 + 3Y_2) \\ X_2 = Y_2 \end{cases}$$

$$\text{abs}\left(|J|\right) = \text{abs}\left(\begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{vmatrix} \right) = \text{abs}\left(\begin{vmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 1 \end{vmatrix} \right) = \frac{1}{2}$$

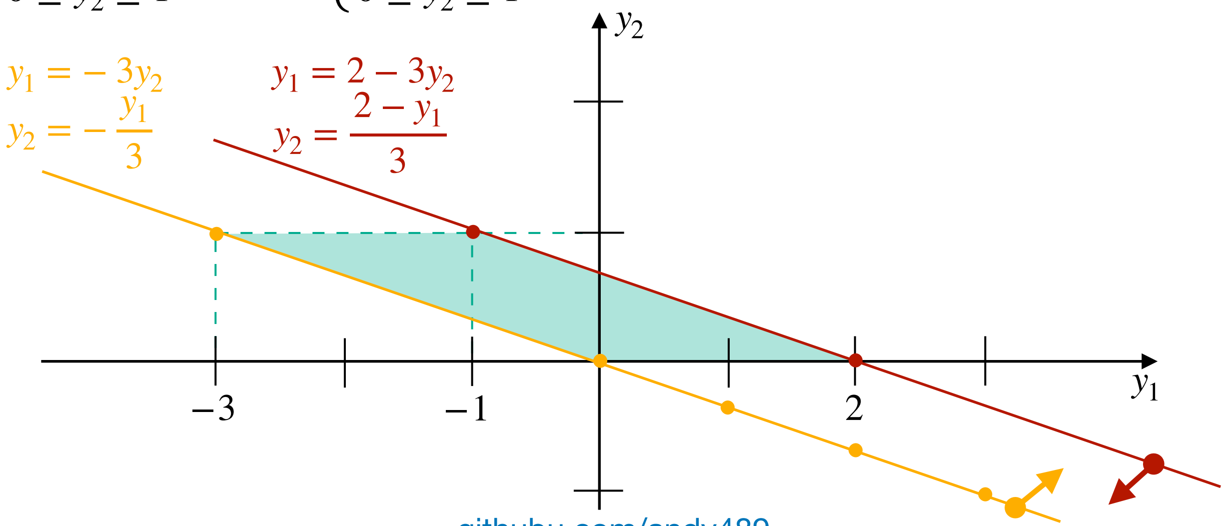
$$\Rightarrow f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\left(\frac{1}{2}(y_1 + 3y_2),\ y_2\right) \times \text{abs}\left(|J|\right) =$$

$$= \frac{1}{2} f_{X_1,X_2}\left(\frac{1}{2}(y_1 + 3y_2),\ y_2\right) =$$

$$= \frac{1}{2} \times \mathbb{1}_{\left\{ \frac{y_1 + 3y_2}{2} \in [0,1] \text{ и } y_2 \in [0,1] \right\}}$$

За да имаме ненулева съвместна плътност за (Y_1, Y_2) трябва да са изпълнени неравенствата от индикаторната функция по-горе:

$$\begin{cases} 0 \le y_1 + 3y_2 \le 2 \\ 0 \le y_2 \le 1 \end{cases} \Leftrightarrow \begin{cases} -3y_2 \le y_1 \le 2 - 3y_2 \\ 0 \le y_2 \le 1 \end{cases}$$



$$f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{Y_1,Y_2(y_1,y_2)}\,\mathrm{d}\,y$$

I сл.) $y_1 \in [-3,-1]$:

$$f_{Y_1}(y_1) = \int_{-\frac{y_1}{3}}^{1} \frac{1}{2}\,\mathrm{d}\,y = \frac{1}{2}y\Big|_{-\frac{y_1}{3}}^{1} = \frac{1}{2}\left(1+\frac{y_1}{3}\right) \Rightarrow$$

$$F_{Y_1}(y_1) = \mathbb{P}(Y_1 \le y_1) = \mathbb{P}(Y_1 \le -1) = \int_{-3}^{y_1} \frac{1}{2}\left(1+\frac{y}{3}\right)\mathrm{d}\,y =$$

$$= \frac{1}{2}\left(y_1 + 3 + \frac{y_1^2-9}{6}\right) = \frac{y_1^2+6y_1+9}{12}.$$

II сл.) $y_1 \in (-1,0]$:

$$f_{Y_1}(y_1) = \int_{-\frac{y_1}{3}}^{\frac{2-y_1}{3}} \frac{1}{2}\,\mathrm{d}\,y_2 = \frac{1}{2}\left(\frac{2-y_1}{3}+\frac{y_1}{3}\right) = \frac{1}{2}\times\frac{2}{3} = \frac{1}{3} \Rightarrow$$

$$F_{Y_1}(y_1) = \mathbb{P}(Y_1 \le y_1) =$$

$$= \mathbb{P}(Y_1 \le -1) + \mathbb{P}(-1 \le Y_1 \le y_1) = \frac{1-6+9}{12} + \int_{-1}^{y_1}\frac{1}{3}\,\mathrm{d}\,y =$$

$$= \frac{1}{3}+\frac{1}{3}\left(y_1+1\right) = \frac{y_1+2}{3}.$$

III сл.) $y_1 \in (0,2]$:

$$f_{Y_1}(y_1) = \int_{0}^{\frac{2-y_1}{3}} \frac{1}{2}\,\mathrm{d}\,y = \frac{2-y_1}{6} \Rightarrow$$

$$F_{Y_1}(y_1) = F_{Y_1}(0) + \int_0^{y_1}\frac{2-y}{6}\,\mathrm{d}\,y = \frac{2}{3}+\frac{y_1}{3}-\frac{y_1^2}{12} = \frac{-y_1^2+4y_1+8}{12}.$$

Окончательно:

$$F_{2X_1-3X_2}(t) = \begin{cases} 0, & t < -3 \\ \frac{(t+3)^2}{12}, & t \in [-3,-1] \\ \frac{t+2}{3}, & t \in t(-1,0] \\ \frac{-t^2+4t+8}{12}, & t \in (0,2] \\ 1, & t > 2 \end{cases}$$

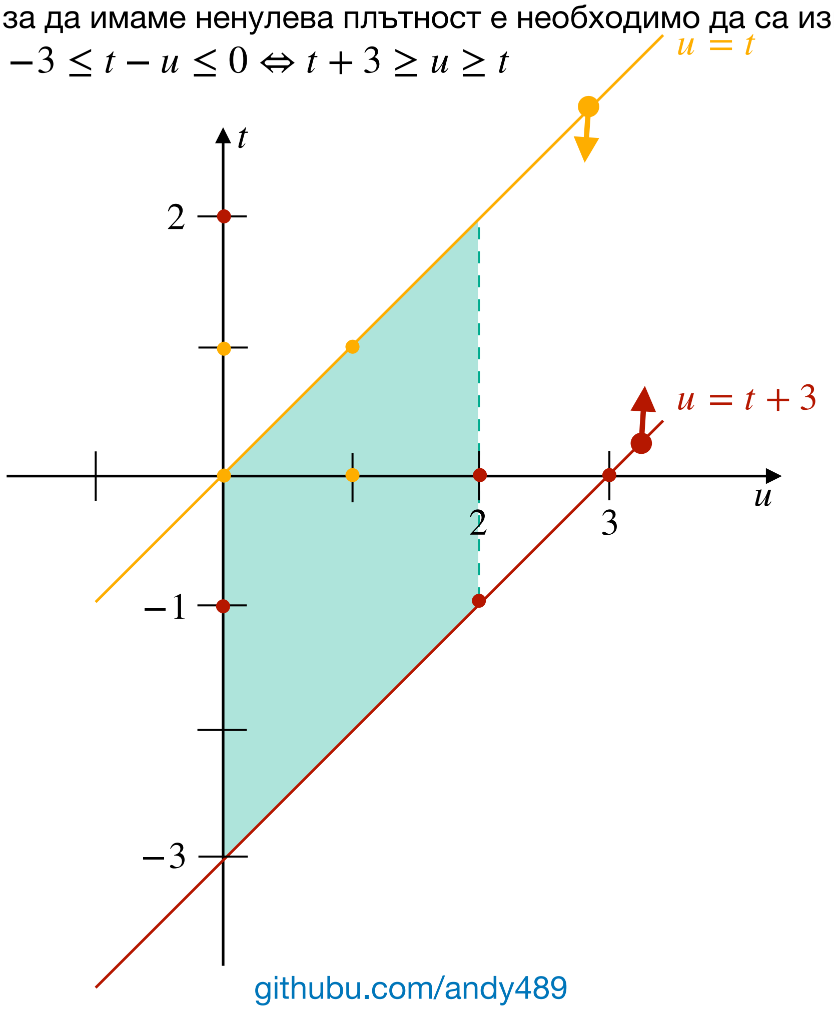
Втори подход с **конволюция**:

$$\begin{cases} f_{2X_1}(x_1) = \frac{1}{2}\mathbb{1}_{\{x_1\in[0,2]\}} \\ f_{-3X_2}(x_2) = \frac{1}{3}\mathbb{1}_{\{x_2\in[-3,0]\}} \end{cases} \Rightarrow$$

$$f_{2X_1-3X_2}(t) = \int_{\mathbb{R}} f_{2X_1}(u) \times f_{-3X_2}(t-u)\mathrm{d}\,u =$$

$$= \int_{\mathbb{R}} \frac{1}{6}\mathbb{1}_{\{u\in[0,2],\ t-u\in[-3,0]\}}\,\mathrm{d}\,u$$

Следователно, за да имаме ненулева плътност е необходимо да са изпълнени неравенствата: $-3 \le t-u \le 0 \Leftrightarrow t+3 \ge u \ge t$



I сл.) $t \in [-3, -1]$:

$$f_{2X_1-3X_2}(t) = \int_0^{t+3} \frac{1}{6}\,\mathrm{d}\,u = \frac{1}{6}(t+3)$$

$$F_{2X_1-3X_2}(t) = \int_{-3}^{t} \frac{1}{6}(z+3)\mathrm{d}\,t = \frac{1}{6}\frac{(z+3)^2}{2}\Bigg|_{-3}^{t} =$$

$$= \frac{1}{12}t^2 + \frac{1}{6}t + \frac{3}{4} = \frac{(t+3)^2}{12}.$$

II сл.) $t \in (-1,0]$:

$$f_{2X_1-3X_2}(t) = \int_0^{2} \frac{1}{6}\,\mathrm{d}\,u = \frac{1}{3}$$

$$F_{2X_1-3X_2}(t) = \mathbb{P}(2X_1 - 3X_2 < t) = \mathbb{P}(2X_1 - 3X_2 \leq -1) + \mathbb{P}(-1 < 2X_1 - 3X_2 < t) =$$

$$= F_{2X_1-3X_2}(-1) + \int_{-1}^{t} \frac{1}{3}\,\mathrm{d}\,u = \frac{(-1+3)^2}{12} + \frac{t+1}{3} =$$

$$= \frac{t+2}{3}.$$

III сл.) $t \in (0,2]$:

$$f_{2X_1-3X_2}(t) = \int_t^{2} \frac{1}{6}\,\mathrm{d}\,u = \frac{2-t}{6}$$

$$F_{2X_1-3X_2}(t) = \mathbb{P}(2X_1 - 3X_2 \leq 0) + \mathbb{P}(0 < 2X_1 - 3X_2 \leq t) =$$

$$= F_{Y_1}(0) + \int_0^{t} \frac{2-t}{6}\,\mathrm{d}\,t = \frac{2}{3} + \frac{t}{3} - \frac{t^2}{12} = \frac{-t^2 + 4t + 8}{12}.$$

И по двата метода получаваме един и същ отговор, което е индикатор за коректност на решението.