зад.1 $X_i \sim N(163; 5{,}6^2)$

1. $\dfrac{X_1+\dots+X_{20}}{20} \sim N\left(163, \dfrac{5{,}6^2}{20}\right)$

Забележете, че тук няма нужда от ЦГТ, т.к. работим с $N(\mu,\sigma^2)$!

$$\mathbb{P}\left(N\left(163, \frac{5\cdot 6^2}{20}\right) > 165\right) = \mathbb{P}\left(N(0,1) > \frac{2\sqrt{20}}{5{,}6}\right)$$

$$\approx 1 - \Phi(1.60) \approx \underline{5.48\,\%}$$

2. $\mathbb{P}(X_i < 180) = \mathbb{P}\left(N(0,1) < \dfrac{17}{5{,}6}\right)$

$\approx \Phi(3{,}04) \approx 0{,}9988$

$\mathbb{P}(\text{поне } 1 \text{ над } 180) = 1 - \mathbb{P}(\text{всички под } 180)$

$\approx 1 - 0{,}9988^{20} \approx \underline{2{,}37\,\%}.$

зад.2 $\displaystyle\int_0^\infty \int_0^y c e^{-y}\, dx\, dy = 1 \implies$

$$1 = \int_0^\infty y e^{-y}\, dy = c\left[-e^{-y}(y+1)\right]_0^\infty = c, \text{ т.е.}$$

$\underline{c = 1}.$

За $x > 0$: $f_X(x) = \displaystyle\int_x^\infty e^{-y}\, dy = e^{-x}$ и

значи $X \sim Exp(1)$, откъдето $\mathbb{E}X = 1$, $DX = 1$.
Може и директно: $\mathbb{E}X = \int_0^\infty x e^{-x} dx = 1$
$\mathbb{E}X^2 = \int_0^\infty x^2 e^{-x} dx = 2$, $DX = 1$

Аналогично, за $y > 0$: $f_Y(y) = \int_0^y e^{-y} dx = y e^{-y}$
$\mathbb{E}Y = \int_0^\infty y^2 e^{-y} dy = 2$, $\mathbb{E}Y^2 = \int_0^\infty y^3 e^{-y} dy = 6$
и значи $DY = 2$

$$\mathbb{E}XY = \int_0^\infty \int_0^y xy e^{-y} \, dx \, dy$$

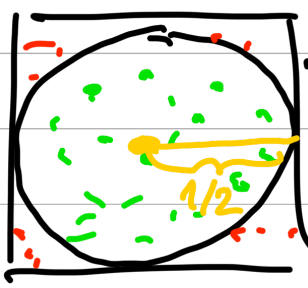
$$= \int_0^\infty y e^{-y} \cdot \frac{y^2}{2} dy = \frac{1}{2} \int_0^\infty y^3 e^{-y} dy$$

$$= 3.$$

Следователно, $Cor(X,Y) = \frac{3 - 1 \cdot 2}{\sqrt{1 \cdot 2}} = \frac{\sqrt{2}}{2}$.

2. $f_{X|Y}(x|y=1) = \frac{e^{-1}}{e^{-1}} = 1$ за $x \in (0,1)$
и $\mathbb{E}(X|Y=1) = \int_0^1 x \cdot 1 \, dx = \frac{1}{2}$.

зад. 3



Нека $X_i = \begin{cases} 1, & \text{ако точка } i \text{ е в кръга} \\ 0, & \text{ако не е.} \end{cases}$

$$\mathbb{P}(X_i = 1) = \frac{\pi \cdot \left(\frac{1}{2}\right)^2}{1} = \frac{\pi}{4}$$

Ако симулираме n точки, то частта, попаднали в кръг е $\frac{X_1+\dots+X_n}{n}$.

$X_i \sim Ber\left(\frac{\pi}{4}\right)$, $\mathbb{E}X_i = \frac{\pi}{4} < \infty$ и от

ЗГЧ: $\frac{X_1+\dots+X_n}{n} \xrightarrow[n\to\infty]{п.с.} \frac{\pi}{4}$, т.е. възможна оценка за π е:

$$\hat{\pi}_n = \frac{4(X_1+\dots+X_n)}{n}$$

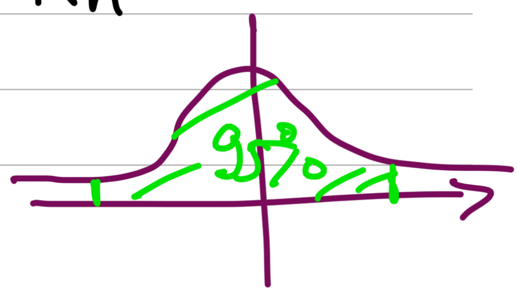
Търсим n, така че

$$\mathbb{P}\left(-0.001 \le \hat{\pi}_n - \pi \le 0{,}001\right) = 95\%$$

Т.к. $DX_1 < \infty$, от ЦГТ: $\frac{\frac{X_1+\dots+X_n}{n} - \frac{\pi}{4}}{\sigma/\sqrt{n}} \to N(0,1)$, където $\sigma^2 = \frac{\pi}{4}\left(1-\frac{\pi}{4}\right)$

$$95\% = \mathbb{P}\left(\frac{-\frac{0{,}001}{4}}{\sigma/\sqrt{n}} \le \frac{\frac{X_1+\dots+X_n}{n} - \frac{\pi}{4}}{\sigma/\sqrt{n}} \le \frac{\frac{0{,}001}{4}}{\sigma\sqrt{n}}\right)$$

$$= 1 - 2\Phi\left(-\frac{0{,}001}{4\sigma/\sqrt{n}}\right)$$



и получаваме $\Phi\left(-\frac{0{,}001}{4\sigma/\sqrt{n}}\right) = 0{,}025$

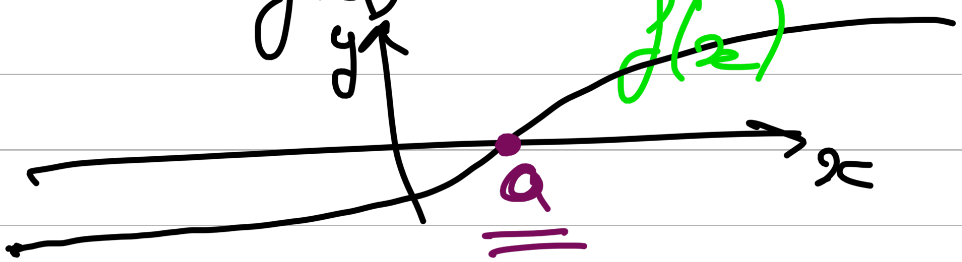
От таблицата: $\frac{0{,}001}{4\sigma/\sqrt{n}} \approx 1{,}96$

$\Rightarrow \quad n \approx \left(\frac{1{,}96 \cdot 4\sigma}{0.001}\right)^2$

≈ 3219

зад. 4 $f_X(x) = 1_{\{x\in[0,1]\}}$; $f_Y(y) = e^{-y} 1_{\{y>0\}}$

1. Да вярно е. Формално, $\mathbb{P}(X \le a) - \mathbb{P}(Y \ge a) =: f(a)$ е непрекъсната, $f(-\infty) = -1$, $f(\infty) = 1$ и по теоремата на Болцано-Вайерщрас (за средната ст-т), с-бува a за което $f(a) = 0$



2. $\begin{cases} Z = \frac{X}{Y} \\ W = Y \end{cases} \Rightarrow \begin{cases} X = ZW \\ Y = W \end{cases}$ и $|J| = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix} = w$

$$f_{Z,W}(z,w) = 1_{\{zw \in [0,1], w>0\}} e^{-w} \cdot w$$

$$= 1_{\{z>0, \frac{1}{z} > w > 0\}} e^{-w} \cdot w$$

и значи за $z>0$: $f_Z(z) = \int_0^{z^{-1}} e^{-w} \cdot w \, dw$

$$= \underline{1 - e^{-1/z}\left(\frac{1}{z}+1\right)}$$ и 0 иначе.

$\begin{cases} Z = \frac{Y}{X} \\ W = X \end{cases} \Rightarrow \begin{cases} X = W \\ Y = ZW \end{cases}$ и $|J| = \begin{vmatrix} 0 & 1 \\ w & z \end{vmatrix} = -w$.

Следователно $f_{Z,W}(z,w) = 1_{\{w \in [0,1], z>0\}} e^{-zw} \cdot w$

и в такъв случай, за $z>0$

$$f_Z(z) = \int_0^1 w e^{-zw} \, dw = \frac{1}{z^2} \int_0^z t e^{-t} \, dt$$

$$= \underline{\frac{1}{z^2}\left(1 - e^{-z}(z+1)\right)}$$

и 0 за $z \le 0$.

* За бонуса: Ако $Z = \frac{X}{Y}$, $W = \frac{Y}{X}$,

то $f_Z(z) \underset{\infty}{\sim} 1 - \left(1 - \frac{1}{z}\right)\left(\frac{1}{z} + 1\right) = \frac{1}{z^2}$

и следователно $\mathbb{E}Z = \int_0^\infty z f_Z(z)\, dz = \infty$

т.к. $\int_0^\infty \frac{1}{x}\, dx = \infty$.

За W: $f_W(x) \underset{\infty}{\sim} \frac{1}{x^2}$ и аналогично

$$\mathbb{E}W = \int_0^\infty x f_W(x)\, dx = \infty.$$