Mathematical theory of financial markets

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1.1. Risk-free asset. Present value

The risk-free asset R_t is an asset which has not a financial risk. We can assume for convenience that $R_0 = 1$. It can be viewed as an amount in a bank. Assume that the risk-free rate of return is a constant r.

1. If the interest payment is once per year, then after the first year we have an amount

$$R_1 = R_0 \left(1 + r\right). \tag{1.1}$$

2. If the interest payments are twice per year, then

$$R_1 = R_0 \left(1 + \frac{r}{2}\right)^2. \tag{1.2}$$

3. If the interest payments are n per year, then

$$R_1 = R_0 \left(1 + \frac{r}{n}\right)^n. \tag{1.3}$$

4. If we have a continuous interest payment, $n \to \infty$, we derive

$$R_1 = R_0 e^r. \tag{1.4}$$

5. If the time period is not one, but t, then

$$R_t = R_0 e^{rt} \equiv e^{rt}. \tag{1.5}$$

We can easily check that (1.5) can be written in the dynamical form as

$$dR_t = rR_t dt. \tag{1.6}$$

If the risk free rate is not a constant, then equation (1.6) turns to

$$dR_t = r_t R_t dt. \tag{1.7}$$

and its solution is

$$R_t = e^{\int\limits_0^t r_u du}. \tag{1.8}$$

We shall use characterizations (1.5), (1.6), (1.7), and (1.8) of the risk-free asset. The risk-free rate is calculated mainly by the use of 13-week Treasury bills. Of course we may use different obligations – Britain, different European, or Japan.

Definition 1.1. *Let the current moment be t. The present value of some amount K, payable at a future moment $T > t$, is the amount at moment t, which the risk-free asset will turn to K at moment T. We shall denote it by $PV\,(K; t, T)$. Obviously, if the risk-free rate is a constant, then*

$$PV\,(K; t, T) = e^{-r(T-t)}K. \tag{1.9}$$

1.2. One-step binomial model

Let the initial asset value be S_0. We assume that we have two alternatives for the stock price after time T – $S_u = S_0 u$ with probability p or $S_d = S_0 d$ with probability $1 - p$, where $d < 1 < u$. This probability measure shall be denoted by P. It is known as a real-world measure. Something more, since S is a risky asset, a stronger condition holds

$$d < 1 < e^{rt} < u. \tag{1.10}$$

Definition 1.2. *The European call/put options give the holder the right to buy/sell the underlying asset S at the moment T for previously defined amount K. It is known as a strike price or simply the strike. Thus the values of the call and put options at the moment T are*

$$G\,(S_T) = (S_T - K)^+ = \max\,(S_T - K, 0) \ - \ \textit{call} \tag{1.11}$$

$$G\,(S_T) = (K - S_T)^+ = \max\,(K - S_T, 0) \ - \ \textit{put}. \tag{1.12}$$

We shall examine the problem for call option pricing. We shall investigate separately the cases $K \leq S_0 d$, $K \geq S_0 u$, and $S_0 d < K < S_0 u$. First, suppose that $K \leq S_0 d$. The scheme 1 describes the call option pricing. We can view the option as a portfolio which consists of one stock and \$$K$ debt. Using the present value formula (1.9), we conclude $C_0 = S_0 - e^{-rt}K$.

Figure 1: One-step binomial model – $K \leq S_0 d$.

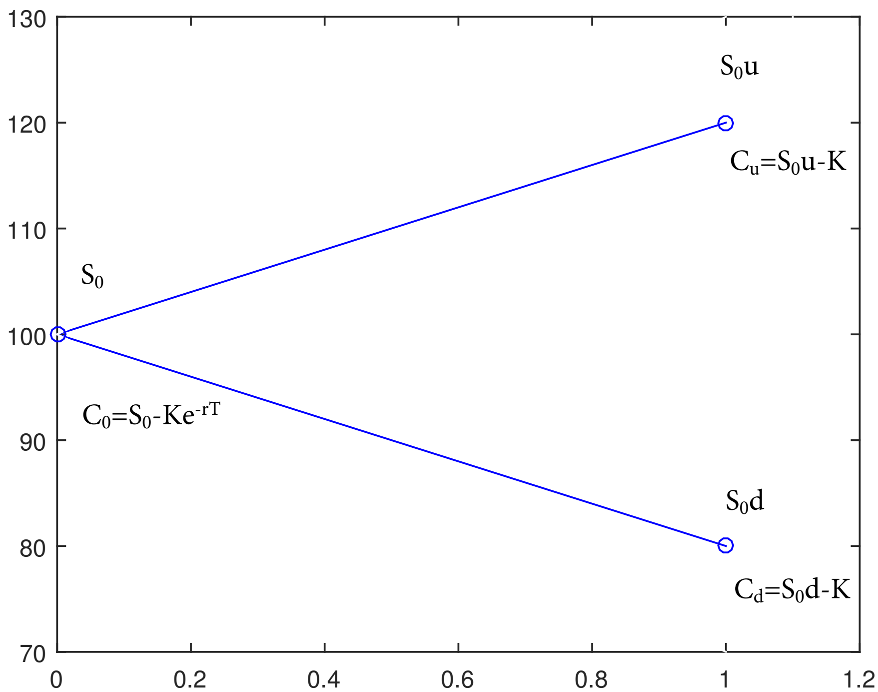
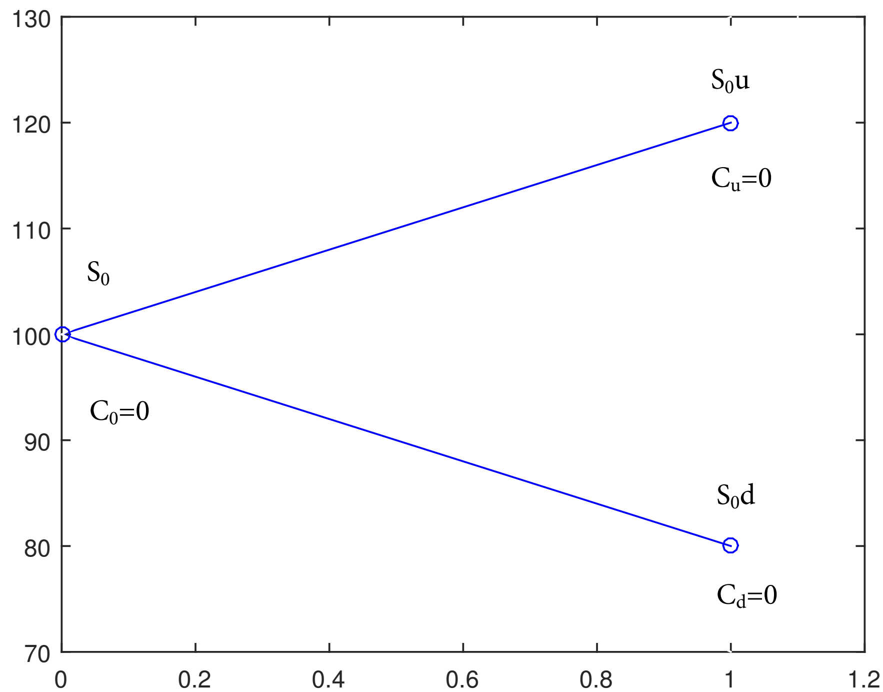


Figure 2: One-step binomial model – $K \geq S_0u$.



Second, if $K \geq S_0 u$, then the option price is zero, since the option pays nothing in both alternatives – see scheme 2.

Suppose now that $S_0 d < K < S_0 u$ – see scheme 3. Let us construct a portfolio Π which consists of x shares of the underlying asset and one short position of the option. We choose x in a way which makes the portfolio payments equal in both of down and up scenarios. This portfolio is risk free. If we have an up movement, the portfolio value is $\Pi_u = xS_0 u - (S_0 u - K) = (x-1) S_0 u + K$. The down movement leads to the portfolio value $\Pi_d = xS_0 d$. The requirement $\Pi_u = \Pi_d$ leads to

$$x = \frac{S_0 u - K}{S(u-d)}. \tag{1.13}$$

Therefore, the portfolio value at the maturity turns to

$$\Pi_T = xS_0 d = \frac{(S_0 u - K)\, d}{u-d}. \tag{1.14}$$

This portfolio has to provide the risk free rate of return:

$$\Pi_0 = e^{-rT} \frac{(S_0 u - K)\, d}{u-d}. \tag{1.15}$$

On the other hand, the initial portfolio value is

$$\Pi_0 = xS_0 - C_0 = \frac{S_0 u - K}{u-d} - C_0. \tag{1.16}$$

Combining equations (1.15) and (1.16) we conclude

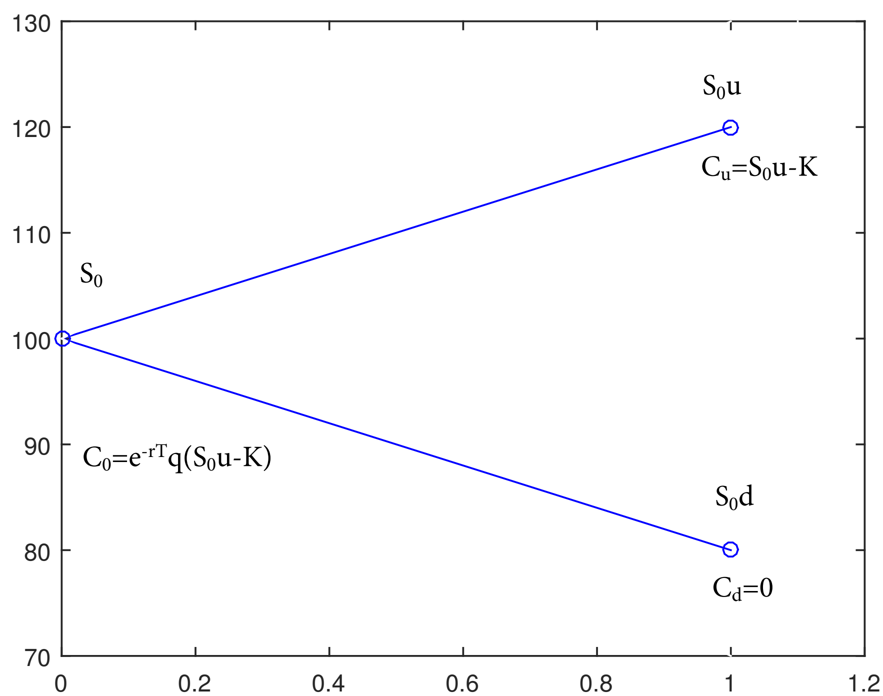
$$\begin{aligned} C_0 &= (S_0 u - K) \frac{1 - e^{-rT} d}{u-d} \\ &= e^{-rT} (S_0 u - K) \frac{e^{rT} - d}{u-d} \\ &= e^{-rT} (S_0 u - K)\, q, \end{aligned} \tag{1.17}$$

where

$$q = \frac{e^{rT} - d}{u-d}. \tag{1.18}$$

Note that $0 < q < 1$ due to inequalities (1.10).

Figure 3: One-step binomial model - $S_0d < K < S_0u$.



Let us examine now a general derivative which pays amount $G(S_T)$ at the maturity date T. Thus we have $G_u = G(S_0 u)$ and $G_d = G(S_0 d)$ – look at figure 4. The portfolio values turns $\Pi_u = xS_0 u - G_u$ and $\Pi_d = xS_0 d - G_d$, which we set to be equal. Hence, the value of x is

$$x = \frac{G_u - G_d}{(u-d)\, S_0} \tag{1.19}$$

and therefore the portfolio value at the maturity is

$$\Pi_T = \frac{G_u u - G_d d}{u - d}. \tag{1.20}$$

Hence, we can obtain the initial portfolio value discounting (1.20):

$$e^{-rT} \frac{G_u u - G_d d}{u - d} = xS_0 - G_0. \tag{1.21}$$

Using formulas (1.18) and (1.19) for q and x we derive

$$G_0 = e^{-rT} \left[q G_u + (1-q)\, G_d \right]. \tag{1.22}$$

We have mentioned above that $0 < q < 1$. We can view the pair $\{q, 1-q\}$ as a probability measure, which we shall denote by Q. Equation (1.22) means that the current derivative value can be obtained as the discounted expectation under the measure Q. This statement is obvious for the risk free asset since $R_u \equiv R_d = e^{rT}$. We shall check this statement for the underlying asset

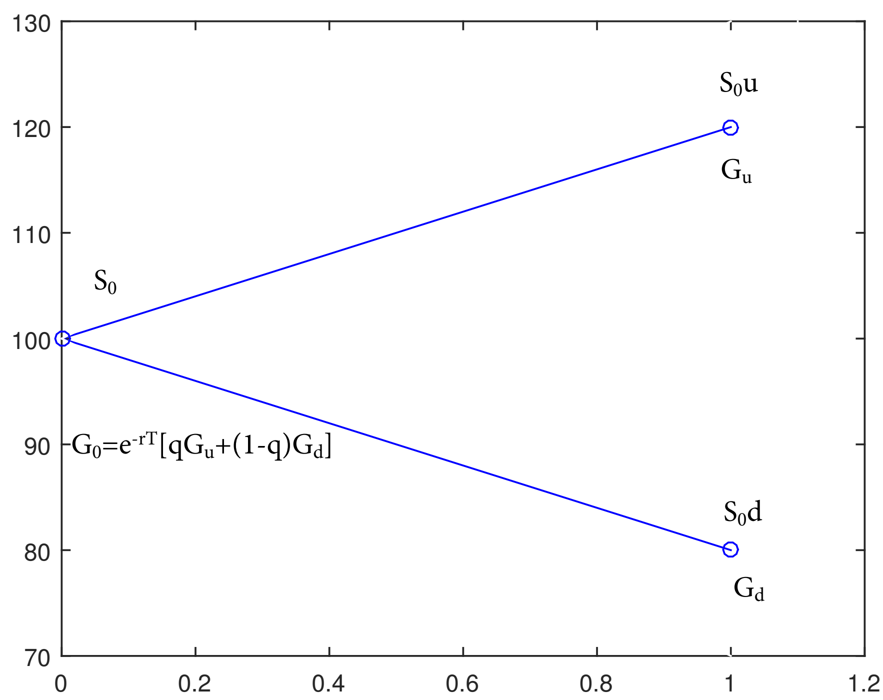
$$e^{-rT} \left[q S_0 u + (1-q)\, S_0 d \right] = e^{-rT} S_0 \left[q\,(u-d) + d \right] = e^{-rT} S_0 \left[\frac{e^{rT} - d}{u - d} (u-d) + d \right] = S_0. \tag{1.23}$$

We are ready to state our first fundamental result.

Definition 1.3. *The measure Q is known as a risk-neutral measure.*

Theorem 1.1. *Every asset can be priced as the discounted expectation w.r.t. the risk-neutral measure.*

Figure 4: One-step binomial model - general derivatives.



1.3. Two steps binomial model. American option pricing.

We consider now two steps binomial model – the corresponding tree is presented at figure 5. The pricing algorithm is as follows

$$\begin{aligned}
G_{uu} &= G\left(S_0 u^2\right)\\
G_{ud} &= G\left(S_0 ud\right)\\
G_{dd} &= G\left(S_0 d^2\right)\\
G_u &= e^{-rT/2}\left[qG_{uu} + (1-q)\,G_{ud}\right]\\
G_d &= e^{-rT/2}\left[qG_{ud} + (1-q)\,G_{dd}\right]\\
G &= e^{-rT/2}\left[qG_u + (1-q)\,G_d\right].
\end{aligned} \tag{1.24}$$

When we price European options we can calculate directly the expectation at the initial node. The probability of the node $\{uu\}$ is q^2, of the node $\{ud\}$ is $2q\,(1-q)$ (since this node can be reached by two ways), and the node $\{dd\}$ has the probability $(1-q)^2$. This way the pricing formula is

$$G = e^{-rT}\left[q^2 G_{uu} + 2q\,(1-q)\,G_{ud} + (1-q)^2\,G_{dd}\right]. \tag{1.25}$$

We can easily check that formulas (1.24) and (1.25) lead to one and the same result.

A simple modification of the algorithm (1.24) can be obtained for the American derivatives pricing problem. Note that an American derivative gives to its holders the right to exercise at every moment receiving amount $G\,(S_t)$ (a time dependence of the function $G\,(t, S_t)$ is admissible). Hence, we have to chose the larger result between immediate exercising or keeping the derivative. Therefore, the algorithm (1.24) turns to

$$\begin{aligned}
G_{uu} &= G\left(S_0 u^2\right)\\
G_{ud} &= G\left(S_0 ud\right)\\
G_{dd} &= G\left(S_0 d^2\right)\\
G_u &= \max\left\{G\left(S_u\right), e^{-rT/2}\left[qG_{uu} + (1-q)\,G_{ud}\right]\right\}\\
G_d &= \max\left\{G\left(S_d\right), e^{-rT/2}\left[qG_{ud} + (1-q)\,G_{dd}\right]\right\}\\
G &= \max\left\{G\left(S\right), e^{-rT/2}\left[qG_u + (1-q)\,G_d\right]\right\}.
\end{aligned} \tag{1.26}$$

Figure 5: Two-steps binomial tree.

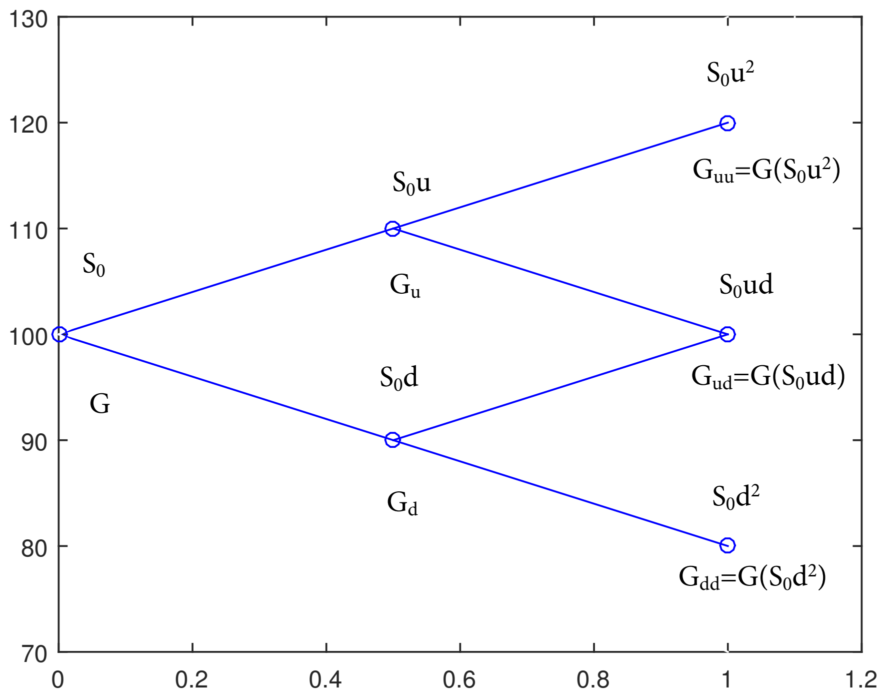
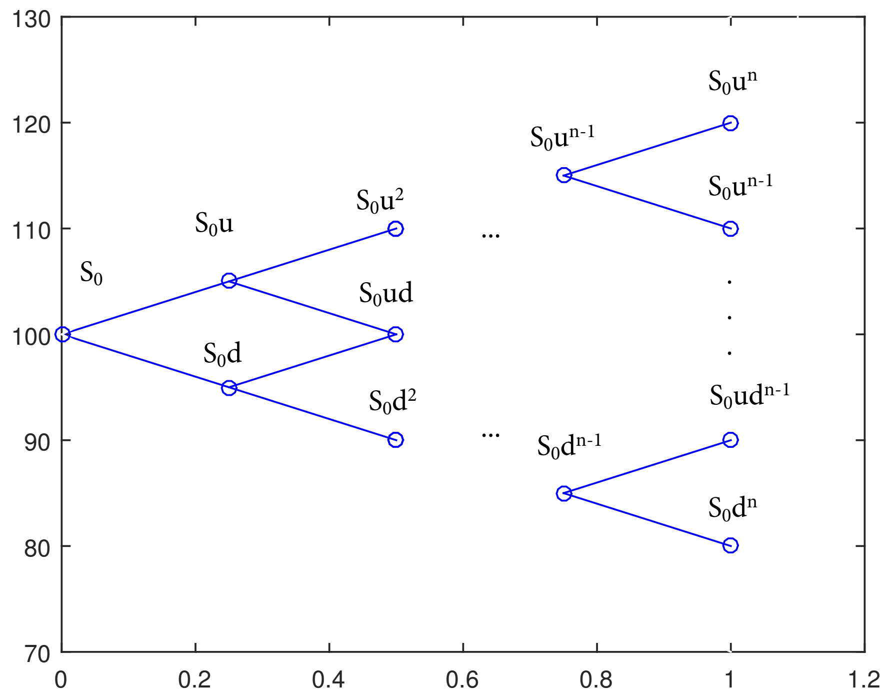


Figure 6: n-steps binomial tree.



1.4. Limiting case

Let us examine n-steps algorithm – figure 6 describes it.

We shall work w.r.t. the real-world measure P. The number of the up movements, we denote it by ξ, is binomial distributed, $\xi \sim Bi\left(n,p\right)$. Its expectation is $E = np$, whereas the variance is $D = np\left(1-p\right)$. The probability that we have k up movements and $n-k$ down is

$$P\left(\xi = k\right) = \left(\begin{array}{c} n \\ k \end{array}\right) p^{k}\left(1-p\right)^{n-k}. \tag{1.27}$$

The underlying asset price can be presented as

$$S_t = S_0 u^{\xi} d^{n-\xi} \tag{1.28}$$

and therefore the log-return is

$$\begin{aligned} \ln\left(\frac{S_T}{S_0}\right) &= \xi\ln\left(u\right) + \left(n-\xi\right)\ln\left(d\right) \\ &= \xi\ln\left(\frac{u}{d}\right) + n\ln\left(d\right). \end{aligned} \tag{1.29}$$

Since $\xi \sim Bi\left(n,p\right)$, we can calculate the expectation and the variance of the log-return as

$$\begin{aligned} E\left[\ln\left(\frac{S_T}{S_0}\right)\right] &= E\left[\xi\right]\ln\left(\frac{u}{d}\right) + n\ln\left(d\right) \\ &= np\ln\left(\frac{u}{d}\right) + n\ln\left(d\right) \\ &= n\left[p\ln\left(\frac{u}{d}\right) + \ln\left(d\right)\right] \\ &= n\left[p\ln\left(u\right) + \left(1-p\right)\ln\left(d\right)\right] \\ D\left[\ln\left(\frac{S_T}{S_0}\right)\right] &= \left[\ln\left(\frac{u}{d}\right)\right]^2 p\left(1-p\right)n. \end{aligned} \tag{1.30}$$

For different n's we shall vary p, u, and d – we denote them by p_n, u_n, and d_n– in a way in which the binomial tree converges for $n \to \infty$. We shall use the notations $x = \ln\left(u\right)$ and $y = \ln\left(u\right)$. A natural assumption is $du = 1$, or alternatively $y = -x$. Later we shall see what happens if we remove this restriction. Suppose that $p_n \to \overline{p}$ and $p_n = \overline{p} + \epsilon_n$ for $\epsilon_n \to 0$. Under these assumptions formulas (1.30) turn to

$$\begin{aligned} E_n &= n x_n \left(2\overline{p} - 1 + 2\epsilon_n\right) \\ D_n &= 4 n x_n^2 p_n \left(1 - p_n\right). \end{aligned} \tag{1.31}$$

From (E) we have $\overline{p} = 0.5$. We see from (D) that $x_n \sim \frac{A}{\sqrt{n}}$. Again (E) leads to $\epsilon_n \sim \frac{C}{\sqrt{n}}$. Finally we conclude

$$\begin{aligned} E_n &\to 2AC \\ D_n &\to A^2. \end{aligned} \tag{1.32}$$

Therefore, we can rewrite formula (1.29) as

$$\begin{aligned} \ln\left(\frac{S_T}{S_0}\right) &= 2x_n\xi - n x_n \\ &= A\frac{\xi_n - \frac{n}{2}}{\sqrt{\frac{n}{4}}} \\ &= A\frac{\xi_n - \frac{n}{2} - C\sqrt{n}}{\sqrt{\frac{n}{4}}} + 2AC \to AN\left(0,1\right) + 2AC \equiv N\left(2AC, A^2\right) \end{aligned} \tag{1.33}$$

We have used above the central limit theorem. Note that the distribution $N\left(2AC, A^2\right)$ confirms equations (1.32) . Also, we introduce a time dependence by the following notations $\overline{A} = A\sqrt{T}$ and $\overline{C} = C\sqrt{T}$. Therefore, $\ln\left(\frac{S_T}{S_0}\right) \sim N\left(2\overline{A}\overline{C}T, \overline{A}^2 T\right)$.

Let us remove the restriction $ud = 1$. The equations (1.31) turn to

$$\begin{aligned} E_n &= n\left[p_n\left(x_n - y_n\right) - y_n\right] \\ D_n &= \left(x_n - y_n\right)^2 p_n\left(1 - p_n\right). \end{aligned} \tag{1.34}$$

We see from (D) $x_n \sim \frac{A}{\sqrt{n}}$ and $y_n \sim \frac{B}{\sqrt{n}}$. Therefore first equation of (1.34) turns to

$$E_n = \sqrt{n}\left[p_n\left(A-B\right)+B\right] \tag{1.35}$$

and hence

$$\overline{p} = -\frac{B}{A-B}. \tag{1.36}$$

Note that $B<0$. Thus equation (1.35) turns to

$$E_n = \sqrt{n}\left[\left(\overline{p}+\epsilon_n\right)\left(A-B\right)+B\right] = \sqrt{n}\epsilon_n\left(A-B\right) \tag{1.37}$$

Hence $\epsilon_n \sim \frac{C}{\sqrt{n}}$. We conclude that formulas (1.34) converges to

$$\begin{aligned} E_n &= C\left(A-B\right) \\ D_n &= \left(A-B\right)^2\left(-\frac{B}{A-B}\right)\left(1+\frac{B}{A-B}\right) = -AB. \end{aligned} \tag{1.38}$$

Analogously to equation (1.33) we can see again the convergence to a normal distributed random variable $N\left(C\left(A-B\right),-AB\right)$.

$$\begin{aligned} \ln\left(\frac{S_T}{S_0}\right) &= \xi_n\frac{A-B}{\sqrt{n}}+\sqrt{n}B \\ &= \sqrt{-AB}\frac{\xi_n+n\frac{B}{A-B}-c\sqrt{n}}{\frac{\sqrt{-ABn}}{A-B}}+C\left(A-B\right) \\ &\to \sqrt{-AB}N\left(0,1\right)+C\left(A-B\right) \equiv N\left(C\left(A-B\right),-AB\right). \end{aligned} \tag{1.39}$$

2. Black-Scholes model, equation, and formulas

2.1. Stochastic differential equations driven by a Brownian motion

Let B_t be a Brownian motion under the filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$. Let us recall the definition.

Definition 2.1. *A stochastic process B_t is a Brownian motion if*

1. $B_0 = 0$.

2. *The process has independent increments, i.e. if* $t_1 < t_2 < ... < t_n$*, then the random variables* $B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}$ *are independent.*

3. *The increments are normally distributed,* $B_t - B_s \sim N(0, t-s)$.

4. *The process has continuous sample paths.*

Proposition 2.1 (Itô formula). *Let* μ_t *and* σ_t *be adapted processes and* X_t *be defined as*

$$dX_t = \mu_t dt + \sigma_t dB_t \tag{2.1}$$

or alternatively as

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s. \tag{2.2}$$

Let the function $f(t,x)$ *be* $C^{1,2}$ *and* $Y_t = f(t, X_t)$*. Then*

$$dY_t = \left(f_t(t, X_t) + \mu_t f_x(t, X_t) + \frac{\sigma_t^2}{2} f_{xx}(t, X_t) \right) dt + \sigma_t f_x(t, X_t) dB_t. \tag{2.3}$$

In the integral form this is

$$\begin{aligned} Y_t = f(0, X_0) + \int_0^t \left(f_t(s, X_s) + \mu_s f_x(s, X_t) + \frac{\sigma_s^2}{2} f_{xx}(s, X_s) \right) ds \\ + \int_0^t \sigma_s f_x(s, X_s) dB_s. \end{aligned} \tag{2.4}$$

Proof: We shall only sketch the proof. Let us expand the function $f(t,x)$ in Taylor series

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} dx^2 + ... \tag{2.5}$$

We substitute equation (2.1) in (2.5) and derive

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\left(\mu_t dt + \sigma_t dB_t\right) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left({\mu_t}^2\left(dt\right)^2 + 2\mu_t\sigma_t dt dB_t + {\sigma_t}^2\left(dB_t\right)^2\right) + ... \tag{2.6}$$

We have to observe that $(dt)^2 = 0$, $dtdB_t = 0$, and $(dB_t)^2 = dt$ and rearange to finish the proof. □

The difference between the Itô differential rule and the ordinary (non-stochastic) differentiation can be viewed in the following example.

Example 2.1. *Let $B(t)$ be a deterministic C^1 function. Let $\mu(t)$ and $\sigma(t)$ be deterministic functions too and let us consider the process $Y(t) = f(t, X(t))$, where*

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dB(t)\,. \tag{2.7}$$

Equation (2.7) is equivalent to

$$X'(t) = \mu(t) + \sigma(t)\,B'(t)\,. \tag{2.8}$$

We have for the derivative of the function $Y(t)$

$$Y'(t) = \frac{df(t, X(t))}{dt} = f_t(t, X(t)) + f_x(t, X(t))\,X'(t)\,. \tag{2.9}$$

After we substitute (2.8) in (2.9), we derive

$$\begin{aligned} Y'(t) &= f_t(t, X(t)) + f_x(t, X(t))\left(\mu(t) + \sigma(t)\,B'(t)\right) \\ &= \left(f_t(t, X(t)) + f_x(t, X(t))\,\mu(t)\right) + f_x(t, X(t))\,\sigma(t)\,B'(t)\,, \end{aligned} \tag{2.10}$$

which can be written as

$$dY(t) = \left(f_t(t, X(t)) + f_x(t, X(t))\,\mu(t)\right)dt + f_x(t, X(t))\,\sigma(t)\,dB(t)\,. \tag{2.11}$$

We can see that the difference between equations (2.3) and (2.11) is the presence of the term

$$\frac{\sigma_t^2}{2} f_{xx}(t, X_t)\,dt. \tag{2.12}$$

It appears from the Brownian motion property $dB_t dB_t = dt$.

Proposition 2.2. *Under suitable Lipschitz conditions for the functions $\mu(t,x)$ and $\sigma(t,x)$ the stochastic differential equation (SDE hereafter)*

$$\begin{aligned} dS_t &= \mu(t,S_t)\,dt + \sigma(t,S_t)\,dB_t \\ S_0 &= x \end{aligned} \tag{2.13}$$

has a unique solution.

Example 2.2 (Log-normal process, Geometric Brownian motion). *Let μ and $\sigma > 0$ be constants. Let us examine a SDE*

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dB_t \\ S_0 &= x. \end{aligned} \tag{2.14}$$

Let us examine the process $Y_t = \ln(S_t)$. The Itô formula (2.3) gives us

$$\begin{aligned} dY_t &= \left(\mu S_t \frac{1}{S_t} - \frac{\sigma^2 S_t^2}{2}\frac{1}{S_t^2}\right) dt + \sigma S_t \frac{1}{S_t} dB_t \\ &= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB_t, \end{aligned} \tag{2.15}$$

or

$$Y_t = \ln(x) + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t. \tag{2.16}$$

Therefore, the solution of the SDE (2.14) is

$$S_t = x e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t}. \tag{2.17}$$

2.2. Black-Scholes model. Deriving the Black-Scholes equation

Let the risk free rate be the constant r. Let we have the following three assets at a financial market

1. A risk-free asset (1.5), (1.6) –

$$dR_t = rR_t dt. \tag{2.18}$$

2. A risky asset with log-normal dynamics

$$dS_t = \mu S_t dt + \sigma S_t dB_t. \tag{2.19}$$

The drift coefficient μ represents the rate of return of the asset. The diffusion coefficient σ (also known as volatility) describes the risk of the asset.

3. A T-derivative Y_t. Its main property is that at the moment T it pays a previously defined amount $G(S_T)$.

Our aim is to derive the fair derivative price. Suppose that its price process Y_t can be presented as a function of the time and the current asset price

$$Y_t = g(t, S_t). \tag{2.20}$$

Obviously, we have a terminal condition

$$g(T, x) = G(x). \tag{2.21}$$

Let us use the Itô differential rule (2.3) to (2.20) to derive

$$\begin{aligned} dY_t = &\left(g_t(t, S_t) + \mu S_t g_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 g_{xx}(t, S_t) \right) dt \\ &+ \sigma S_t g_x(t, S_t)\, dB_t. \end{aligned} \tag{2.22}$$

We want to construct a portfolio Π_t between the risky asset S_t and the derivative Y_t without the risk part. Obviously, this portfolio consists of one derivative and $g_x(t, S_t)$ short positions of the risky asset. Thus the value of this portfolio is

$$\begin{aligned} d\Pi_t = &\left(g_t(t, S_t) + \mu S_t g_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 g_{xx}(t, S_t) - g_x(t, S_t)\, \mu S_t \right) dt \\ &+ \left(\sigma S_t g_x(t, S_t) - g_x(t, S_t)\, \sigma S \right) dB_t \\ = &\left(g_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 g_{xx}(t, S_t) \right) dt. \end{aligned} \tag{2.23}$$

Since the portfolio is risk-free, it has to satisfy

$$d\Pi_t = r\Pi_t dt = r\left(g\left(t,S_t\right) - g_x\left(t,S_t\right)S_t\right)dt. \tag{2.24}$$

Comparing (2.23) and (2.24) we see that

$$r\left(g\left(t,S_t\right) - g_x\left(t,S_t\right)S_t\right) = \left(g_t\left(t,S_t\right) + \frac{1}{2}\sigma^2 S_t^2 g_{xx}\left(t,S_t\right)\right). \tag{2.25}$$

Rearranging (2.25) we conclude that the function $g\left(t,x\right)$ satisfies the following terminal value problem (TVP, hereafter)

$$\begin{aligned} &g_t\left(t,x\right) + rxg_x\left(t,x\right) + \frac{1}{2}\sigma^2 x^2 g_{xx}\left(t,x\right) - rg\left(t,x\right) = 0 \\ &g\left(T,x\right) = G\left(x\right). \end{aligned} \tag{2.26}$$

This is the famous Black-Scholes equation.

Remark 2.1. *Note that the drift parameter μ disappears. In such a way the derivative price is independent of it.*

2.3. Solving the Black-Scholes equation

The Black-Scholes equation (2.26) is a heat type equation. Our aim is by several changes of variables to reach to the classical form of the heat equation.

First, we have to move the boundary condition from $t = T$ to $t = 0$. We make this changing t to s by $s = T - t$. Thus equation (2.26) turns to the boundary value problem (BVP hereafter)

$$\begin{aligned} g_s\left(s,x\right) &= rxg_x\left(s,x\right) + \frac{1}{2}\sigma^2 x^2 g_{xx}\left(s,x\right) - rg\left(s,x\right) \\ g\left(0,x\right) &= G\left(x\right). \end{aligned} \tag{2.27}$$

The second change of variables is $y = \ln x$ ($x = e^y$) to remove x and x^2 in the coefficients before g_x and g_{xx}. Let

$$h\left(s,y\right) = g\left(s,e^y\right). \tag{2.28}$$

We have

$$\begin{aligned} g_x(s,x) &= \frac{\partial g(s,x)}{\partial x}\frac{\partial y}{\partial y} = \frac{\partial g(s,e^y)}{\partial y}\frac{\partial(\ln x)}{\partial x} \\ &= h_y(s,y)\frac{1}{x} = h_y(s,y)e^{-y} \end{aligned} \tag{2.29}$$

$$\begin{aligned} g_{xx}(s,x) &= \frac{\partial g_x(s,x)}{\partial x}\frac{\partial y}{\partial y} = \frac{1}{x}\frac{\partial(h_y(s,y)e^{-y})}{\partial y} \\ &= e^{-2y}(h_{yy}(s,y) - h_y(s,y)). \end{aligned} \tag{2.30}$$

Replacing (2.29) and (2.30) in equation (2.27) we derive

$$\begin{aligned} h_s(s,y) &= rxh_y(s,y)\frac{1}{x} + \frac{1}{2}\sigma^2x^2\frac{1}{x^2}(h_{yy}(s,y) - h_y(s,y)) - rh(s,y) \\ &= \left(r - \frac{1}{2}\sigma^2\right)h_y(s,y) + \frac{1}{2}\sigma^2h_{yy}(s,y) - rh(s,y). \end{aligned} \tag{2.31}$$

Thus equation (2.27) turns to the BVP

$$\begin{aligned} &h_s(s,y) = \left(r - \frac{1}{2}\sigma^2\right)h_y(s,y) + \frac{1}{2}\sigma^2h_{yy}(s,y) - rh(s,y) \\ &h(0,y) = G(e^y). \end{aligned} \tag{2.32}$$

Let

$$A = \frac{\sigma^2}{2} \tag{2.33}$$

$$B = r - \frac{\sigma^2}{2} \tag{2.34}$$

$$C = -r. \tag{2.35}$$

We shall show now how to solve an equation of the form

$$\begin{aligned} &h_s(s,y) = Ah_{yy}(s,y) + Bh_y(s,y) + Ch(s,y). \\ &h(0,y) = G(e^y). \end{aligned} \tag{2.36}$$

Note that $A > 0$.

We make the following change of variables to eliminate the terms h_y and h from (2.36)

$$f(s,y) = e^{(\alpha y+\beta s)} h(s,y) \tag{2.37}$$

$$h(s,y) = e^{-(\alpha y+\beta s)} f(s,y). \tag{2.38}$$

Therefore

$$\begin{aligned} h_s(s,y) &= -\beta e^{(\alpha y+\beta s)} f(s,y) + e^{-(\alpha y+\beta s)} f_s(s,y) \\ &= e^{-(\alpha y+\beta s)} \left(-\beta f(s,y) + f_s(s,y)\right) \end{aligned} \tag{2.39}$$

$$\begin{aligned} h_y(s,y) &= -\alpha e^{(\alpha y+\beta s)} f(s,y) + e^{-(\alpha y+\beta s)} f_y(s,y) \\ &= e^{-(\alpha y+\beta s)} \left(-\alpha f(s,y) + f_y(s,y)\right) \end{aligned} \tag{2.40}$$

$$\begin{aligned} h_{yy}(s,y) &= -\alpha e^{-(\alpha y+\beta s)} \left(-\alpha f(s,y) + f_y(s,y)\right) \\ &\quad + e^{-(\alpha y+\beta s)} \left(-\alpha f_y(s,y) + f_{yy}(s,y)\right) \\ &= e^{-(\alpha y+\beta s)} \left(\alpha^2 f(s,y) - 2\alpha f_y(s,y) + f_{yy}(s,y)\right). \end{aligned} \tag{2.41}$$

After replacing (2.39), (2.40), and (2.41) in equation (2.32) we derive

$$\begin{aligned} &-\beta e^{(\alpha y+\beta s)} f(s,y) + e^{-(\alpha y+\beta s)} f_s(s,y) \\ &= B e^{-(\alpha y+\beta s)} \left(-\alpha f(s,y) + f_y(s,y)\right) \\ &+ A e^{-(\alpha y+\beta s)} \left(\alpha^2 f(s,y) - 2\alpha f_y(s,y) + f_{yy}(s,y)\right) + C e^{-(\alpha y+\beta s)} f(s,y), \end{aligned} \tag{2.42}$$

which is equivalent to

$$f_s(s,y) = A f_{yy}(s,y) + (B - 2\alpha A) f_y(s,y) + \left(\beta - B\alpha + A\alpha^2 + C\right) f(s,y). \tag{2.43}$$

We choose α and β in such a way the terms f and f_y to vanish. Hence

$$\alpha = \frac{B}{2A} \tag{2.44}$$

$$\beta = \frac{B^2}{4A} - C. \tag{2.45}$$

Hence equation (2.32) turns to the heat equation

$$\begin{aligned} f_s\left(s, y\right) &= A f_{yy}\left(s, y\right) \\ f\left(0, y\right) &= e^{\alpha y} G\left(e^y\right). \end{aligned} \tag{2.46}$$

Let us recall that its fundamental solution is

$$\Phi\left(s, y\right) = \frac{1}{\sqrt{4\pi A s}} e^{-\frac{y^2}{4As}}. \tag{2.47}$$

Remark 2.2. *Recall that a fundamental solution means the solution of the equation with a δ-boundary function.*

Therefore the solution of equation (2.46) is given as the convolution

$$\begin{aligned} f\left(s, y\right) &= \int\limits_{-\infty}^{\infty} G\left(e^z\right) \Phi\left(s, y - z\right) dz \\ &= \frac{1}{\sqrt{4\pi A s}} \int\limits_{-\infty}^{\infty} e^{-\frac{(y-z)^2}{4As}} e^{\frac{B}{2A} z} G\left(e^z\right) dz. \end{aligned} \tag{2.48}$$

Let us consider

$$
\begin{aligned}
&\exp\left(-\frac{(y-z)^2}{4As}+\frac{B}{2A}z\right)=\exp\left(-\frac{y^2-2yz+z^2-2Bsz}{4As}\right)\\
&=\exp\left(-\frac{y^2-2z\left(y+Bs\right)+z^2\pm\left(y+Bs\right)^2z^2}{4As}\right)\\
&=\exp\left(-\frac{\left(z-y-Bs\right)^2}{4As}\right)\exp\left(\frac{\left(y+Bs\right)^2-y^2}{4As}\right)\\
&=\exp\left(-\frac{\left(z-y-Bs\right)^2}{4As}\right)\exp\left(\frac{B^2s^2+2yBs}{4As}\right)\\
&=\exp\left(-\frac{\left(z-y-Bs\right)^2}{4As}\right)\exp\left(\frac{yB}{2A}\right)\exp\left(\frac{B^2s}{4A}\right)\\
&=\exp\left(-\frac{\left(z-y-Bs\right)^2}{4As}\right)\exp\left(\alpha y\right)\exp\left(\left(\beta+C\right)s\right).
\end{aligned}
\tag{2.49}
$$

Thus equation (2.48) turns to

$$
f\left(s,y\right)=\frac{e^{\alpha y+\beta s}e^{Cs}}{\sqrt{4\pi As}}\int\limits_{-\infty}^{\infty}e^{-\frac{(z-y-Bs)^2}{4As}}G\left(e^z\right)dz. \tag{2.50}
$$

Replacing A, B, and C by (2.33), (2.34), and (2.35) we derive

$$
\begin{aligned}
h\left(s,y\right)&=\frac{e^{Cs}}{\sqrt{4\pi As}}\int\limits_{-\infty}^{\infty}e^{-\frac{(z-y-Bs)^2}{4As}}G\left(e^z\right)dz\\
&=\frac{e^{-rs}}{\sigma\sqrt{2\pi s}}\int\limits_{-\infty}^{\infty}e^{-\frac{\left(z-y-\left(r-\frac{\sigma^2}{2}\right)s\right)^2}{2\sigma^2 s}}G\left(e^z\right)dz.
\end{aligned}
\tag{2.51}
$$

Let us turn to call option pricing and hence we have to use the form (1.11) for the boundary condition

$$
\begin{aligned}
h(s,y) &= \frac{e^{-rs}}{\sigma\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{-\frac{\left(z-y-\left(r-\frac{\sigma^2}{2}\right)s\right)^2}{2\sigma^2 s}} (e^z-K)^+ dz. \\
&= \frac{e^{-rs}}{\sigma\sqrt{2\pi s}} \int_{\ln K}^{\infty} e^{-\frac{\left(z-y-\left(r-\frac{\sigma^2}{2}\right)s\right)^2}{2\sigma^2 s}} (e^z-K) dz.
\end{aligned} \tag{2.52}
$$

Now we change the variable in the integral

$$
\begin{aligned}
u &= \frac{z-y-\left(r-\frac{\sigma^2}{2}\right)s}{\sigma\sqrt{s}} \\
z &= u\sigma\sqrt{s}+y+\left(r-\frac{\sigma^2}{2}\right)s.
\end{aligned} \tag{2.53}
$$

Therefore equation (2.52) turns to

$$
\begin{aligned}
h(s,y) &= \frac{e^{-rs}\sigma\sqrt{s}}{\sigma\sqrt{2\pi s}} \int_{\frac{\ln K-y-\left(r-\frac{\sigma^2}{2}\right)s}{\sigma\sqrt{s}}}^{\infty} e^{-\frac{u^2}{2}} \left(e^{u\sigma\sqrt{s}+y+\left(r-\frac{\sigma^2}{2}\right)s}-K\right)du \\
&= x\frac{e^{-rs}}{\sqrt{2\pi}} \int_{\frac{\ln\frac{K}{x}-\left(r-\frac{\sigma^2}{2}\right)s}{\sigma\sqrt{s}}}^{\infty} e^{-\frac{u^2}{2}} e^{u\sigma\sqrt{s}+y+\left(r-\frac{\sigma^2}{2}\right)s}du - K\frac{e^{-rs}}{\sqrt{2\pi}} \int_{\frac{\ln\frac{K}{x}-\left(r-\frac{\sigma^2}{2}\right)s}{\sigma\sqrt{s}}}^{\infty} e^{-\frac{u^2}{2}}du.
\end{aligned} \tag{2.54}
$$

We have used above that $y=\ln x$. The second integral in (2.54) is equal to

$$
-Ke^{-r(T-t)}N(d_2) \tag{2.55}
$$

for

$$
d_2 = \frac{\ln\frac{x}{K}+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{2.56}
$$

We use above the notation $N(\cdot)$ for the cumulative density function (CDF) of the standard normal (Gaussian) distribution.

Let us turn to the first integral in equation (2.54). We have

$$x\frac{e^{-rs}}{\sqrt{2\pi}}\int\limits_{\frac{\ln\frac{K}{x}-\left(r-\frac{\sigma^2}{2}\right)s}{\sigma\sqrt{s}}}^{\infty} e^{-\frac{u^2}{2}}e^{u\sigma\sqrt{s}+y+\left(r-\frac{\sigma^2}{2}\right)s}du = x\frac{1}{\sqrt{2\pi}}\int\limits_{-\frac{\ln\frac{x}{K}+\left(r-\frac{\sigma^2}{2}\right)s}{\sigma\sqrt{s}}}^{\infty} e^{-\frac{u^2-2u\sigma\sqrt{s}+\sigma^2 s}{2}}du$$
$$= x\frac{1}{\sqrt{2\pi}}\int\limits_{-\frac{\ln\frac{x}{K}+\left(r-\frac{\sigma^2}{2}\right)s}{\sigma\sqrt{s}}}^{\infty} e^{-\frac{\left(u-\sigma\sqrt{s}\right)^2}{2}}du. \tag{2.57}$$

After we change the variable

$$v = u - \sigma\sqrt{s}, \tag{2.58}$$

we conclude that the first integral in equation (2.54) is equal to

$$\frac{x}{\sqrt{2\pi}}\int\limits_{-\frac{\ln\frac{x}{K}+\left(r+\frac{\sigma^2}{2}\right)s}{\sigma\sqrt{s}}}^{\infty} e^{-\frac{v^2}{2}}du = xN(d_1) \tag{2.59}$$

where

$$d_1 = \frac{\ln\frac{x}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \equiv d_2 + \sigma\sqrt{T-t}. \tag{2.60}$$

We shall prove now the following interesting proposition

Proposition 2.3 (Put-Call parity). *Let the prices of put and call options at moment t be P_t and C_t, respectively. Then*

$$C_t = P_t + S_t - e^{-r(T-t)}K. \tag{2.61}$$

Proof: Let us examine a portfolio Π, which consists of one call option and minus one (a short position) put option. Thus the portfolio value at the maturity date T is

$$\Pi_T = \max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K. \tag{2.62}$$

Therefore the portfolio value at moment t has to be

$$C_t - P_t \equiv \Pi_t = S_t - PV(K) = S_t - e^{-r(T-t)}K. \tag{2.63}$$

We use above formula (1.9) for the present value. We have only to rearrange.

□

Now we can formulate the following theorem for the price of European options in the Black-Scholes model

Theorem 2.1. *The prices of European call and put options written on the asset with dynamics* (2.19) *is*

$$C_t = S_t N(d_1) - Ke^{-r(T-t)} N(d_2) \tag{2.64}$$

$$P_t = Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1) \tag{2.65}$$

where d_1 and d_2 are given by (2.60) *and* (2.56)*, respectively*

Proof: We have to derive the put option price. We shall use the put-call parity – proposition 2.3. We have

$$\begin{aligned} P_t &= C_t - S_t + e^{-r(T-t)}K \\ &= S_t N(d_1) - Ke^{-r(T-t)} N(d_2) - S_t + e^{-r(T-t)}K \\ &= e^{-r(T-t)}K(1 - N(d_2)) - (1 - N(d_1)) S_t \\ &= e^{-r(T-t)}KN(-d_2) - N(-d_1) S_t. \end{aligned} \tag{2.66}$$

□

3. Risk premium. Risk neutral measure. A second approach to Black-Scholes model

3.1. Risk premium

Definition 3.1. 1. **Risk premium** *is the difference between the total return of a risky asset and the risk-free asset.*

2. **Elasticity of the risk premium** *is the change of the return w.r.t. the change of the risk.*

Let us examine again the Black-Scholes model. We have that the risk premium of the risky asset is

$$\mu S_t - rS_t = (\mu - r) S_t \tag{3.1}$$

and thus the corresponding elasticity turns to

$$l_{asset} = \frac{(\mu - r) S_t}{\sigma S_t} = \frac{\mu - r}{\sigma}. \tag{3.2}$$

Remind that the derivative price is presented as $Y_t = g(t, S_t)$ where the function $g(t, x)$ solves the Black-Scholes equation (2.26). Therefore, using the Itô differential rule (2.3) we derive

$$\begin{aligned} dY_t &= \left(g_t(t, S_t) + \mu S_t g_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 g_{xx}(t, S_t) \right) dt \\ &\quad + \sigma S_t g_x(t, S_t) \, dB_t \\ &= ((\mu - r) S_t g_x(t, S_t) + rY_t) \, dt + \sigma S_t g_x(t, S_t) \, dB_t \end{aligned} \tag{3.3}$$

This leads to derivative elasticity

$$l_{derivative} = \frac{(\mu - r) S_t g_x(t, S_t) + rY_t - rY_t}{\sigma S_t g_x(t, S_t)} = \frac{\mu - r}{\sigma}. \tag{3.4}$$

We can see that both elasticity coincide.

3.2. Risk neutral measure

Definition 3.2. *The term arbitrage means the opportunity for a risk-free profit above the profit which is given by the risk-free asset. We assume that the market is arbitrage free.*

Later we shall formalize this definition. Note that we derive the Black-Scholes equation using namely this assumption.

Definition 3.3. *The process X_t is martingale (w.r.t. the filtration $\mathcal{F}_t$) if*

$$E[X_t | \mathcal{F}_s] = X_s \tag{3.5}$$

for every $s < t$.

Proposition 3.1. *The Brownian motion is a martingale. Something more, if α_t is an adapted process, then*

$$\int_0^t \alpha_s dB_s \tag{3.6}$$

is a martingale.

Definition 3.4. *Let the measures P and Q be equivalent (i.e. $P(A) = 0$ iff $Q(A) = 0$). Then there exists a random variable ξ, such that*

$$E^Q[\eta] = E^P[\xi\eta]. \tag{3.7}$$

Something more, if $\mathcal{G}$ is a sub-sigma algebra of $\mathcal{F}$, then

$$E^Q[\eta|\,\mathcal{G}] = \frac{E^P[\eta\xi|\,\mathcal{G}]}{E^P[\xi|\,\mathcal{G}]}. \tag{3.8}$$

The random variable ξ is known as a Radon-Nikodym derivative and it is denoted by

$$\xi = \frac{dQ}{dP}. \tag{3.9}$$

Since the probability space is filtered we shall use the following notation too

$$\left.\frac{dQ}{dP}\right|_t = E^P[\xi|\,\mathcal{F}_t]. \tag{3.10}$$

Defined in a such way, the process (3.10) is a martingale with expectation 1.

Theorem 3.1 (Girsanov theorem). *Let θ be a stochastic process which satisfies the Novikov's condition*

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)\right] < \infty. \tag{3.11}$$

Let us define a probability measure Q by the Radon-Nikodym derivative

$$\left.\frac{dQ}{dP}\right|_t = L_t, \tag{3.12}$$

where the process L_t is the solution of the SDE

$$dL_t = -\theta_t L_t dB_t^P. \tag{3.13}$$

Note that proposition 3.1 gives us that the process L_t is a P-martingale. Then the process

$$dB_t^Q = \theta_t dt + dB_t^P \tag{3.14}$$

is a Brownian motion w.r.t. the measure Q.

Remark 3.1. *Note that*

$$L_t = e^{-\frac{\theta^2}{2} - \theta B_t^P}. \tag{3.15}$$

Let us note that changing the probability measure, we can change the drift of the processes.

Let us turn back to the elasticity – we have proved that it is one and the same for the risky asset and for the derivative. Let us use it to change the measure –

$$\theta = \frac{\mu - r}{\sigma}. \tag{3.16}$$

Thus we define the Q-Brownian motion as in equation (3.14). Therefore the asset and derivative dynamics (2.19) and (3.3) turn to

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dB_t \\ &= \mu S_t dt + \sigma S_t \left(-\theta dt + dB_t^Q \right) \\ &= \mu S_t dt - \sigma \frac{\mu - r}{\sigma} S_t dt + \sigma S_t dB_t^Q \\ &= r S_t dt + \sigma S_t dB_t^Q. \end{aligned} \tag{3.17}$$

$$\begin{aligned} dY_t &= \left((\mu - r) S_t g_x (t, S_t) + r Y_t \right) dt + \sigma S_t g_x (t, S_t) \left(-\theta dt + dB_t^Q \right) \\ &= \left((\mu - r) S_t g_x (t, S_t) + r Y_t \right) dt - \sigma \frac{\mu - r}{\sigma} S_t g_x (t, S_t) dt \\ &+ \sigma S_t g_x (t, S_t) dB_t^Q \\ &= r Y_t dt + \sigma S_t g_x (t, S_t) dB_t^Q. \end{aligned} \tag{3.18}$$

Remark 3.2. *Note hat when we change measures the diffusion keeps its form.*

Thus the assets are

1. The risk-free asset

$$dR_t = rR_t dt. \tag{3.19}$$

2. The risky asset

$$dS_t = rS_t dt + \sigma S_t dB_t^Q. \tag{3.20}$$

3. The derivative

$$dY_t = rY_t dt + \sigma S_t g_x\left(t, S_t\right) dB_t^Q. \tag{3.21}$$

We can see that all assets have one and the same rate of return w.r.t. the measure Q. That is namely the reason the measure Q to be called *risk-neutral measure*.

Let us see what is the dynamics of the discounted asset processes.

1. Obviously, we have for the risk-free asset

$$e^{-rt} R_t = 1. \tag{3.22}$$

2. The discounted risky asset

$$\begin{aligned} d\left(e^{-rt} S_t\right) &= \left(-re^{-rt} S_t + rS_t e^{-rt}\right) dt + \sigma S_t e^{-rt} dB_t^Q \\ &= \sigma S_t e^{-rt} dB_t^Q. \end{aligned} \tag{3.23}$$

3. The discounted derivative

$$\begin{aligned} d\left(e^{-rt} Y_t\right) &= \left(-re^{-rt} Y_t + rY_t e^{-rt}\right) dt + \sigma S_t g_x\left(t, S_t\right) e^{-rt} dB_t^Q \\ &= \sigma S_t g_x\left(t, S_t\right) e^{-rt} dB_t^Q. \end{aligned} \tag{3.24}$$

We can see from proposition 3.1, that all of the discounted price processes are martingales. Thus we can define the risk neutral measure as

Definition 3.5. *The risk neutral measure is a measure (if it exists) under which all discounted assets are martingales.*

We can formulate now the following hypothesis, which we shall prove later.

Hypothesis 3.1. *If there exists a risk neutral measure, then the market is arbitrage free.*

3.3. Option pricing. Black-Scholes formulas

Suppose that S_t is the price process of the underlying asset – not necessary log-normal. Let the measure Q be risk-neutral. This means that the discounted option price process is a martingale. We shall examine a call option. We have

$$e^{-rt}Y_t = E^Q\left[e^{-rT}Y_T \middle| \mathcal{F}_t\right] = E^Q\left[e^{-rT}(S_T - K)^+ \middle| \mathcal{F}_t\right]. \tag{3.25}$$

Therefore

$$\begin{aligned} Y_t &= e^{-r(T-t)}E^Q\left[(S_T-K)^+ \middle| \mathcal{F}_t\right] \\ &= e^{-r(T-t)}E^Q\left[(S_T-K)\, I_{\{S_T>K\}} \middle| \mathcal{F}_t\right] \\ &= e^{-r(T-t)}E^Q\left[S_T I_{\{S_T>K\}} \middle| \mathcal{F}_t\right] - Ke^{-r(T-t)}Q\left(S_T > K \middle| \mathcal{F}_t\right). \end{aligned} \tag{3.26}$$

We shall use another change of measures for the first term of equation (3.26). Let the measure R be define by the Radon-Nikodym derivative – see definition 3.4 –

$$\left.\frac{dR}{dQ}\right|_t = e^{-rt}\frac{S_t}{S_0}. \tag{3.27}$$

Note that $e^{-rt}\frac{S_t}{S_0}$ is a martingale with expectation 1. Equation (3.8) gives us

$$
\begin{aligned}
&e^{-r(T-t)}E^Q\left[S_T I_{\{S_T>K\}} \middle| \mathcal{F}_t\right] = e^{rt}S_0E^Q\left[e^{-rT}\frac{S_T}{S_0}I_{\{S_T>K\}} \middle| \mathcal{F}_t\right] \\
&= e^{rt}S_0E^Q\left[e^{-rT}\frac{S_T}{S_0} \middle| \mathcal{F}_t\right] E^R\left[I_{\{S_T>K\}} \middle| \mathcal{F}_t\right] \\
&= e^{rt}S_0e^{-rt}\frac{S_t}{S_0}R\left(S_T > K \middle| \mathcal{F}_t\right) \\
&= S_tR\left(S_T > K \middle| \mathcal{F}_t\right).
\end{aligned} \tag{3.28}
$$

We have proved the following theorem

Theorem 3.2. *If Q is a risk neutral measure and the Radon-Nikodym derivative between measures Q and R is given by equation (3.27), then the call option price is*

$$
Y_t = S_tR\left(S_T > K \middle| \mathcal{F}_t\right) - Ke^{-r(T-t)}Q\left(S_T > K \middle| \mathcal{F}_t\right). \tag{3.29}
$$

Now we turn to the Black-Scholes model supposing that the asset price is described by the log-normal process (2.19). We first examine the second term in equation (3.29). Q-dynamics of the asset process is given by equation (3.20) and it leads to

$$
\begin{aligned}
S_T &= S_0e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma B_T^Q} \\
&= S_0e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma B_t}e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma\left(B_T^Q-B_t^Q\right)} \\
&= S_te^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma \widetilde{B}_{T-t}^Q}
\end{aligned} \tag{3.30}
$$

We have use above that for fixed s, the process $\widetilde{B}_t = B_{t+s} - B_s$ is again Brownian motion. Thus the second term of equation (3.29) turns to

$$
\begin{aligned}
Q\left(S_T > K \middle| \mathcal{F}_t\right) &= Q\left(S_t e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma \widetilde{B}^Q_{T-t}} > K \middle| \mathcal{F}_t\right) \\
&= Q\left(x e^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma \widetilde{B}^Q_{T-t}} > K\right)\Bigg|_{x=S_t} \\
&= Q\left(\widetilde{B}^Q_{T-t} > \frac{\ln\frac{K}{x} - \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma}\right)\Bigg|_{x=S_t} \\
&= N\left(\frac{\ln\frac{S_t}{K} + \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= N\left(d_2\right)
\end{aligned}
\tag{3.31}
$$

for

$$
d_2 = \frac{\ln\frac{S_t}{K} + \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{3.32}
$$

Let us turn back to the first term of equation (3.29). Since the Radon-Nikodym derivative between measures Q and R is given by equations (3.27), we have

$$
L_t = \frac{dR}{dQ}\bigg|_t = \frac{e^{-rt}S_t}{S_0} = e^{-\frac{\sigma^2}{2}+\sigma B^Q_t}, \tag{3.33}
$$

which of course is the solution of the SDE

$$
dL_t = \sigma L_t dB^Q_t. \tag{3.34}
$$

Thus the R-Brownian motion is defined as

$$
dB^R_t = -\sigma dt + dB^Q_t. \tag{3.35}
$$

Therefore the R-dynamics of the underlying asset is

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma S_t dB^Q_t \\
&= rS_t dt + \sigma S_t\left(\sigma dt + dB^R_t\right) \\
&= \left(r+\sigma^2\right) S_t dt + \sigma S_t dB^R_t.
\end{aligned}
\tag{3.36}
$$

Analogously to (3.30) we derive

$$S_T = S_t e^{\left(r+\frac{\sigma^2}{2}\right)(T-t)+\sigma \widetilde{B}^R_{T-t}}. \tag{3.37}$$

Similarly to (3.31) we find

$$R\left(S_T > K\right) = N\left(d_1\right), \tag{3.38}$$

where

$$d_1 = d_2 + \sigma\sqrt{T-t}. \tag{3.39}$$

Finally, combining theorem 3.2 with equations (3.31), (3.32), (3.38), and (3.39) we derive again the Black-Scholes call option pricing formula (2.64).

3.4. A stochastic point of view of the Black-Scholes equation

We shall denote by S^x_t the solution of some SDE with initial condition $S_0 = x$. Let us define the infinitesimal generator of this process as

Definition 3.6. *The infinitesimal generator of the process S_t is defined as the operator $\mathcal{A}$ on the set of $C^{1,2}$-functions*

$$\mathcal{A}f\left(x\right) = \lim_{\varepsilon\to 0} \frac{E\left(f\left(S^x_\varepsilon\right)\right) - f\left(x\right)}{\varepsilon}. \tag{3.40}$$

Let the process S_t is the solution of SDE (2.13). We can be prove the following proposition.

Proposition 3.2. *The infinitesimal generator of the process (2.13) is the second order differential operator*

$$\mathcal{A}f\left(x\right) = \mu\left(x\right) f'\left(x\right) + \frac{\sigma^2\left(x\right)}{2} f''\left(x\right). \tag{3.41}$$

Proof: Using the Itô formula (2.3) we derive

$$\begin{aligned}\mathcal{A}f\left(x\right) &= \lim_{\varepsilon\to 0} \frac{E\left(f\left(S^x_\varepsilon\right)\right) - f\left(x\right)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{E\left(\int\limits_0^\varepsilon \mu\left(S^x_u\right) f'\left(S^x_u\right) + \frac{\sigma^2\left(S^x_u\right)}{2} f''\left(S^x_u\right) du + \int\limits_0^\varepsilon \mu\left(S^x_u\right) f'\left(S^x_u\right) dB_u\right)}{\varepsilon}\end{aligned} \tag{3.42}$$

The expectation of the stochastic integral is zero since it is a martingale. Changing the order of the expectation and the integration we derive

$$\begin{aligned}\mathcal{A}f(x) &= \lim_{\varepsilon\to 0}\frac{E\left(\int\limits_0^{\varepsilon}\mu\left(S_u^x\right)f'\left(S_u^x\right)+\frac{\sigma^2\left(S_u^x\right)}{2}f''\left(S_u^x\right)du+\right)}{\varepsilon}\\ &= E\left(\mu\left(S_0^x\right)f'\left(S_0^x\right)+\frac{\sigma^2\left(S_0^x\right)}{2}f''\left(S_0^x\right)\right)\\ &= \mu(x)f'(x)+\frac{\sigma^2(x)}{2}f''(x).\end{aligned} \tag{3.43}$$

□

We have the following Markovian property

Proposition 3.3. *We have for $t<T$*

$$E\left[f\left(S_T\right)\middle|\mathcal{F}_t\right]=E\left[f\left(S_{T-t}^y\right)\right]\Big|_{y=S_t^x} \tag{3.44}$$

Proposition 3.4 (Kolmogorov backward equation). *The deferential equation*

$$\begin{aligned}&f_t(t,x)+\mathcal{A}f(t,x)=0\\ &f(0,x)=F(x)\end{aligned} \tag{3.45}$$

has a unique solution which is given by

$$f(t,x)=E\left[F\left(S_t^x\right)\right]. \tag{3.46}$$

The Feynman-Kac extension of the equation (3.45) is

$$\begin{aligned}&f_t(t,x)=\mathcal{A}f(t,x)-r(t)f(t,x)\\ &f(0,x)=F(x)\end{aligned} \tag{3.47}$$

and its solution is

$$f(t,x)=E\left[e^{\int\limits_0^t r\left(S_u^x\right)du}F\left(t,S_t^x\right)\right]. \tag{3.48}$$

We shall prove the following proposition

Proposition 3.5. *The following three statements are equivalent.*

1. *The process $f(t+a, S_t^x)$ is a martingale for every a and x.*
2. *The expectation $E[f(t+a, S_t^x)]$ is time independent.*
3. *The function $f(t,x)$ solves the differential equation*

$$f_t(t,x) + \mathcal{A}f(t,x) = 0. \tag{3.49}$$

Proof: For simplicity, we shall present first the proof when the function $f(\cdot)$ is time independent. The general proof can be found below.

1. We shall prove that 1. is equivalent to 2. If $f(S_t^x)$ is a martingale, then its expectation is time independent, and therefore

$$E[f(S_t^x)] = f(x). \tag{3.50}$$

   Suppose now that (3.50) is true. We use Markovian property 3.3 to derive

$$\begin{aligned} E[f(S_T^x)|\,\mathcal{F}_t] &= E\left[f\left(S_{T-t}^y\right)\right]\big|_{y=S_t^x} \\ &= f(y)|_{y=S_t^x} = f(S_t^x). \end{aligned} \tag{3.51}$$

2. Second, we shall prove that 2. is equivalent to 3. Definition of the infinitesimal generator 3.6 shows that 3. follows 2. Now suppose that 3. is true. Let us define the function $g(t,x)$ as

$$g(t,x) = E[f(S_t^x)]. \tag{3.52}$$

   We shall prove that its t-derivative is zero:

$$
\begin{aligned}
g_t(t,x) &= \lim_{\varepsilon\to 0} \frac{E\left[f\left(S^x_{t+\varepsilon}\right)\right] - E\left[f\left(S^x_t\right)\right]}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{E\left[E\left[f\left(S^x_{t+\varepsilon}\right)\middle|\mathcal{F}_t\right]\right] - E\left[f\left(S^x_t\right)\right]}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{E\left[E\left[f\left(S^y_\varepsilon\right)\right]\big|_{y=S^x_t}\right] - E\left[f\left(S^x_t\right)\right]}{\varepsilon} \\
&= E\left[\lim_{\varepsilon\to 0} \frac{E\left[f\left(S^y_\varepsilon\right)\right] - f(y)}{\varepsilon}\bigg|_{y=S^x_t}\right] \\
&= E\left[f_t(t,y) + \mathcal{A}f(t,y)\big|_{y=S^x_t}\right] = 0.
\end{aligned} \tag{3.53}
$$

□

Proof:

1. First, we shall prove that 2) follows 1). Since $f\left(t+a, S^x_t\right)$ is a martingale, its expectation does not depend on time.

2. Suppose that 2) is true. Then the Markov property of S_t leads to

$$
\begin{aligned}
E\left[f\left(T+a, S^x_T\right)\middle|\mathcal{F}_t\right] &= E\left[f\left(T+a, S^y_{T-t}\right)\right]\big|_{y=S^x_t} \\
&= E\left[f\left(T-t+t+a, S^y_{T-t}\right)\right]\big|_{y=S^x_t} \\
&= f(t+a,y)\big|_{y=S^x_t} = f\left(t+a, S^x_t\right).
\end{aligned} \tag{3.54}
$$

   2). Thus we have proved that 1) is equivalent to 2).

3. Suppose again that 2) is true. We have from the definition of the infinitesimal generator (assuming $a=0$)

$$
\begin{aligned}
0 &= \lim_{t\to 0} \frac{E\left[f\left(t, S^x_t\right)\right] - f(t,x) \pm E\left[f\left(0, S^x_t\right)\right]}{t} \\
&= f_t(t,x) + \mathcal{A}f(t,x).
\end{aligned} \tag{3.55}
$$

4. Suppose that 3) is true. Let us define the function $f^a(t,x)$ as

$$f^a(t,x) = f(t+a,x). \tag{3.56}$$

Therefore

$$0 = f_t^a(t,x) + \mathcal{A} f^a(t,x). \tag{3.57}$$

Let for fixed a, the function $g(t,x)$ be defined as

$$g(t,x) = E\left[f^a\left(t, S_t^x\right)\right]. \tag{3.58}$$

We have for its t-derivative

$$\begin{aligned}
g_t(t,x) &= \lim_{\varepsilon\to 0} \frac{E\left[f^a\left(t+\varepsilon, S_{t+\varepsilon}^x\right)\right] - E\left[f^a\left(t, S_t^x\right)\right]}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{E\left[E\left[f^a\left(t+\varepsilon, S_{t+\varepsilon}^x\right)\middle|\mathcal{F}_t\right]\right] - E\left[f^a\left(t, S_t^x\right)\right]}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{E\left[E\left[f^a\left(t+\varepsilon, S_\varepsilon^y\right)\right]\big|_{y=S_t^x}\right] - E\left[f^a\left(t, S_t^x\right)\right]}{\varepsilon} \\
&= E\left[\lim_{\varepsilon\to 0} \frac{E\left[f^a\left(t+\varepsilon, S_\varepsilon^y\right)\right] - f^a(t,y)}{\varepsilon}\Bigg|_{y=S_t^x}\right] \\
&= E\left[f_t^a(t,y) + A f^a(t,y)\big|_{y=S_t^x}\right] = 0
\end{aligned} \tag{3.59}$$

Therefore the function (3.58) is time independent. This finishes the proof.

□

Let us turn to the Black-Scholes model. We already proved that under the risk neutral measure, the asset price dynamics is given by equation (3.20). Also if the derivative price can be presented as $Y_t = g(t, S_t)$, then $e^{-rt} g(t, S_t)$ is a martingale. Therefore we can use the third statement of proposition 3.5 to conclude

$$-re^{-rt}g(t,x)+e^{-rt}g_t(t,x)+rxe^{-rt}g_x(t,x)+\frac{\sigma^2}{2}x^2e^{-rt}g_{xx}(t,x)=0, \quad (3.60)$$

which is exactly the Black-Scholes equation.

Let us look at the Kolmogorov backward equation (3.45). Since $e^{-rt}Y_t$ is a martingale, we have

$$\begin{aligned} Y_t &= e^{-r(T-t)}E\left[G(S_T)\mid \mathcal{F}_t\right] \\ &= e^{-r(T-t)}E\left[G\big(S^y_{T-t}\big)_t\right]\Big|_{y=S_t} \\ &= h(T-t,S_t). \end{aligned} \quad (3.61)$$

where the function $h(u,y)$ is defined as

$$h(u,y)=e^{-ru}E\left[G\left(S^y_u\right)\right] \quad (3.62)$$

and it solves the Feynman-Kac extension (3.47) of the Kolmogorov backward equation.

3.5. A general scheme for pricing derivatives

1. First, we have to recognize the risk-neutral measure Q – we have to find the asset parameters under the measure Q. We use proposition 3.5 for the function $f(t,x)=e^{-rt}x$. Statement 3. says

   $$0=f_t(t,x)+\mathcal{A}f(t,x)=-re^{-rt}x+e^{-rt}\mathcal{A}f_1(x), \quad (3.63)$$

   where we denote above by f_1 the function $f_1(x)=x$. Equation (3.63) is equivalent to

   $$\mathcal{A}f_1(x)-rx=0. \quad (3.64)$$

2. Once we derive the risk-neutral measure we can price a derivative with a terminal value $Y_T=G(S_T)$ as

   $$Y_t=e^{-r(T-t)}E^Q\left[Y_T\mid \mathcal{F}_t\right]\equiv e^{-r(T-t)}E^Q\left[G(S_T)\mid \mathcal{F}_t\right]. \quad (3.65)$$

   If we have to price an option we can use theorem 3.2.

Example 3.1. *We shall show now how the Black-Scholes model looks in this scheme*

1. *Suppose that the risk-neutral dynamics of the asset is*

$$dS_t = \mu^Q S_t dt + \sigma S_t dB_t^Q. \tag{3.66}$$

   Equation (3.64) is equivalent to

$$\mu^Q x - rx = 0 \tag{3.67}$$

   which of course means $\mu^Q = r$.

2. *Since in the Black-Scholes model we price an option, we can use theorem 3.2. We have shown above that it leads to the Black-Scholes formula.*

4. General market model

Let the market consists of $n+1$ assets S_t^0, S_t^1,..., S_t^n which are right continuous with left limits (RCLL, hereafter). Usually, the asset S_t^0 is assumed to be the risk-free asset (2.18), but this is not necessarily. We required only the strict positiveness $S_t^0 > 0$ for all $t > 0$.

4.1. Self-financing strategies

Let $\theta = \{\theta_t^0, \theta_t^1, ..., \theta_t^n\}$ be a portfolio between the assets.

Definition 4.1. *We shall call the strategy θ self-financing if buying of some new asset is financed bu selling of another.*

The previous definition means that for the self-financing portfolios there is not external transactions. Let us examine a time grid $0 = t_0 < t_1 < ... < t_m = t$. The portfolio value at moment t_h is

$$\theta_{t_h} S_{t_h} = \sum_{i=0}^{n} \theta_{t_h}^i S_{t_h}^i. \tag{4.1}$$

Thus the income value of the portfolio at moment t_{h+1} is

$$\theta_{t_h} S_{t_{h+1}} = \sum_{i=0}^{n} \theta^i_{t_h} S^i_{t_{h+1}}. \tag{4.2}$$

The self-financing condition 4.1 allows to rearrange the portfolio keeping its value. Therefore

$$\theta_{t_h} S_{t_{h+1}} = \theta_{t_{h+1}} S_{t_{h+1}}. \tag{4.3}$$

We sum equations (4.3) for $h = 0, ..., m-1$ to derive

$$\sum_{h=0}^{m-1} \theta_{t_h} S_{t_{h+1}} = \sum_{h=0}^{m-1} \theta_{t_{h+1}} S_{t_{h+1}}. \tag{4.4}$$

After rearranging we deduce

$$\begin{aligned} \theta_t S_t \equiv \theta_{t_m} S_{t_m} &= \theta_{t_0} S_{t_1} + \sum_{h=1}^{m-1} \theta_{t_h} \left(S_{t_{h+1}} - S_{t_h}\right) \pm \theta_{t_0} S_{t_0} \\ &= \theta_{t_0} S_{t_0} + \sum_{h=0}^{m-1} \theta_{t_h} \left(S_{t_{h+1}} - S_{t_h}\right). \end{aligned} \tag{4.5}$$

We take the length of the time grid to tends to zero to establish the following proposition

Proposition 4.1. *If the strategy θ_t is self-financing, then*

$$\theta_t S_t = \theta_{t_0} S_{t_0} + \int_0^t \theta_u dS_u. \tag{4.6}$$

In dynamic form equation (4.6) looks as

$$d\left(\theta_t S_t\right) = \theta_t dS_t. \tag{4.7}$$

Remark 4.1. *Note that in the sum in equation (4.5), the value of θ is taken in the left interval end – θ_{t_h}. This allows us to take the limit $\Delta t \to 0$.*

Remark 4.2. *Note that defined in this way the self-financing strategies are left-continuous.*

Corollary 4.1. *If the assets are martingales, then the self-financing portfolio is martingale too.*

Let us define the following sets of assets.

Definition 4.2. *1. For a fixed moment t, t-asset is an asset which has value φ – $\mathcal{F}_t$-measurable random variable, i.e. its value is known in moment t. We shall denote by $L(t,P)$ the set of all such assets. Note that we mark the dependence from the probability measure P.*

2. Let us denote by $L_+(t,P)$ the following subset

$$L_+(t,P) = \{\varphi : \varphi \in L(t,P), P(\varphi \geq 0) = 1\}. \tag{4.8}$$

The inequality $P(\varphi \geq 0) = 1$ means that the asset has a non-negative value at moment t.

3. Let us denote by $L_{++}(t,P)$ the subset

$$L_{++}(t,P) = \{\varphi : \varphi \in L_+(t,P), P(\varphi > 0) > 0\}. \tag{4.9}$$

This means that φ is non-negative and there is a positive probability to be positive.

Now we are ready to define the meaning of *arbitrage*.

Definition 4.3. *θ is an arbitrage strategy if*

1. It is self-financing.

2.

$$\theta_{t_0} S_{t_0} = 0. \tag{4.10}$$

3.

$$\theta_t S_t \in L_{++}(t,P) \tag{4.11}$$

for some $t > 0$.

The meaning of this definition is that the self-financing portfolio θ starts from zero and in some moment t it can have positive value without possibility to be negative.

Corollary 4.2. *If θ_t is an arbitrage strategy for a moment τ, then*

$$E\left[\theta_\tau S_\tau\right] > 0. \tag{4.12}$$

Definition 4.4. *The market is arbitrage-free if there is not arbitrage strategies.*

4.3. Deflator

A deflator Y_t is a RCLL strictly positive process. It can be viewed as units in which we measured the value of different assets. In such a way we can define the deflated prices as

Definition 4.5. *The deflated processes are defined as*

$$S_t^{0,Y} = \frac{S_t^0}{Y_t}, S_t^{1,Y} = \frac{S_t^1}{Y_t}, ..., S_t^{n,Y} = \frac{S_t^n}{Y_t}. \tag{4.13}$$

We shall use also the notation

$$S_t^Y = \frac{S_t}{Y_t}. \tag{4.14}$$

Note that the division is by components.

Proposition 4.2 (Itô product formula). *Let X_t and Y_t be two processes. Then*

$$d\left(X_t Y_t\right) = X_t dY_t + Y_t dX_t + d\left[X, Y\right]\big|_t. \tag{4.15}$$

The term $\left[X, Y\right]\big|_t$ is called quadratic covariation and it is defined as the limit

$$\left[X, Y\right]\big|_t = \lim_{\Delta t \to 0} \sum_i \left(X_{t_{i+1}} - X_{t_i}\right)\left(Y_{t_{i+1}} - Y_{t_i}\right). \tag{4.16}$$

Remark 4.3. *We have for a Brownian motion*

$$d\left[B, B\right]\big|_t = dt. \tag{4.17}$$

Proposition 4.3. *If the strategy is self-financing w.r.t. the assets S_t^0, S_t^1,..., S_t^n, then it is self-financing w.r.t. the assets $S_t^{0,Y}$, $S_t^{1,Y}$,..., $S_t^{n,Y}$ too.*

Proof: The Itô product formula gives us

$$d\left(\theta_t S_t \frac{1}{Y_t}\right) = \theta_t S_t d\frac{1}{Y_t} + \frac{1}{Y_t} d\left(\theta_t S_t\right) + \left.[\theta S, Y]\right|_t. \tag{4.18}$$

Since θ is self-financing, we derive for the second term in equation (4.18)

$$d\left(\theta_t S_t\right) = \theta_t dS_t. \tag{4.19}$$

Since the self-financing strategy is left continuous, we derive for the third term

$$\left.[\theta S, Y]\right|_t = \theta_t \left.[S, Y]\right|_t. \tag{4.20}$$

Combining (4.18), (4.19), and (4.20) we conclude

$$\begin{aligned} d\left(\theta_t S_t \frac{1}{Y_t}\right) &= \theta_t S_t d\frac{1}{Y_t} + \frac{1}{Y_t}\theta_t dS_t + \theta_t \left.[S, Y]\right|_t \\ &= \theta_t \left(S_t d\frac{1}{Y_t} + \frac{1}{Y_t} dS_t + \left.[S, Y]\right|_t\right) \\ &= \theta_t d\left(S_t \frac{1}{Y_t}\right) \equiv \theta_t dS_t^Y, \end{aligned} \tag{4.21}$$

which means that the strategy θ is self-financing w.r.t. the assets $S_t^{0,Y}$, $S_t^{1,Y}$,..., $S_t^{n,Y}$. □

Corollary 4.3. *Let Y_t be a deflator. Then*

1. $$\varphi \in L_+(t, P) \Leftrightarrow \frac{\varphi}{Y_t} \in L_+(t, P) \tag{4.22}$$

2. $$\varphi \in L_{++}(t, P) \Leftrightarrow \frac{\varphi}{Y_t} \in L_{++}(t, P) \tag{4.23}$$

3. *The market with assets S_t^0, S_t^1,..., S_t^n is arbitrage free if and only if the market which consists of the deflated processes $S_t^{0,Y}$, $S_t^{1,Y}$,..., $S_t^{n,Y}$ is arbitrage free.*

Definition 4.6. *A deflator is called a martingale deflator if the deflated assets $S_t^{0,Y}$, $S_t^{1,Y}$,..., $S_t^{n,Y}$ are martingales. (w.r.t. the real-world measure P)*

Theorem 4.1. *If there exists a martingale deflator, then the market is arbitrage free.*

Proof: Let Y_t be a martingale deflator and suppose that θ_t is an arbitrage self-financing strategy for a moment τ. The self-financing condition gives us

$$\theta_\tau S_\tau^Y = \theta_0 S_0^Y + \int_0^\tau \theta_u dS_u^Y. \tag{4.24}$$

Since S_t^Y is a martingale, the integral $\int_0^t \theta_u dS_u^Y$ is a martingale too and therefore its expectation is zero. Taking expectation in equation (4.24) and using corollary 4.2 we derive

$$0 < E\left[\theta_\tau S_\tau^Y\right] = E\left[\theta_0 S_0^Y + \int_0^\tau \theta_u dS_u^Y\right] = E\left[\theta_0 S_0^Y\right] = 0. \tag{4.25}$$

The contradiction proves the theorem. □

4.4. Martingale measure

Definition 4.7. *A measure Q is equivalent to the measure P when $Q(A) = 0 \Leftrightarrow P(A) = 0$. We shall denote $Q \sim P$.*

Corollary 4.4. *Let Q be an equivalent to P measure. Then*

1. $$\varphi \in L_+(t, P) \Leftrightarrow \varphi \in L_+(t, Q) \tag{4.26}$$

2. $$\varphi \in L_{++}(t, P) \Leftrightarrow \varphi \in L_{++}(t, Q) \tag{4.27}$$

3. *The market model is arbitrage free w.r.t. the measure P if and only if it is arbitrage free w.r.t. the measure Q too.*

Definition 4.8. *A measure Q is called a martingale measure if there exists a deflator Y_t such that*

1. *$Q \sim P$.*
2. *The deflated processes $S_t^{0,Y} = \frac{S_t^0}{Y_t}, S_t^{1,Y} = \frac{S_t^1}{Y_t}, ..., S_t^{n,Y} = \frac{S_t^n}{Y_t}$ are Q-martingales.*

Definition 4.9. *If the deflator is the risk-free asset R_t, then the measure is called risk-neutral.*

Theorem 4.2. *If the market admits a martingale measure, then it is arbitrage free.*

Proof: Suppose that Q is a martingale measure w.r.t the deflator Y. Suppose that there exists an arbitrage and the arbitrage strategy is θ for a moment τ. Similarly to theorem 4.2 we derive

$$0 = \theta_0 S_0^Y = E^Q \left[\theta_\tau S_\tau^Y - \int\limits_0^\tau \theta_u dS_u^Y \right] = E^Q \left[\theta_\tau S_\tau^Y \right] > 0. \tag{4.28}$$

The contradiction proves the theorem. □

Remark 4.4. *Often the deflator is assumed to be the asset S_0. If the deflator is a process, which is not an asset, we can add it in the model without adding an arbitrage.*

4.5. The inverse theorem

Now we shall prove the inverse theorem, i.e. that under some additional conditions the lapse of arbitrage leads to the existence of a martingale measure. Suppose that the market is arbitrage-free.

Definition 4.10.

1. *Let we denote by Θ the set of all self-financing strategies.*
2. *Let M_t be the set all $\mathcal{F}_t$-measurable random variables, which value can be reached by some self-financing strategy:*

$$M_t = \{\theta_t S_t, \ \theta_t \in \Theta\}. \tag{4.29}$$

Lemma 4.1. *The set M_t is linear.*

Proof: Let $\varphi_1, \varphi_2 \in M_t$ and they are reached by the strategies θ^1 and θ^2. Hence

$$
\begin{aligned}
\varphi_1 &= \theta_t^1 S_t = \theta_0^1 S_0 + \int_0^t \theta_u^1 dS_u \\
\varphi_2 &= \theta_t^2 S_t = \theta_0^2 S_0 + \int_0^t \theta_u^2 dS_u.
\end{aligned}
\tag{4.30}
$$

We use that the integral is a linear operator and deduce that for all constants α_1 and α_2

$$
\begin{aligned}
\alpha_1\varphi_1 + \alpha_2\varphi_2 &= \alpha_1\theta_t^1 S_t + \alpha_2\theta_t^2 S_t \\
&= \alpha_1\left(\theta_0^1 S_0 + \int_0^t \theta_u^1 dS_u\right) + \alpha_2\left(\theta_0^2 S_0 + \int_0^t \theta_u^2 dS_u\right) \\
&= \left(\alpha_1\theta_0^1 + \alpha_2\theta_0^2\right) S_0 + \int_0^t \left(\alpha_1\theta_u^1 + \alpha_2\theta_u^2\right) dS_u.
\end{aligned}
\tag{4.31}
$$

Therefore $(\alpha_1\varphi_1 + \alpha_2\varphi_2) \in M_t$. □

Definition 4.11 (Initial investment function). *We shall define the following initial investment function $\psi(\cdot,\cdot)$*

$$
M_t \xrightarrow{\psi} \mathbb{R} \tag{4.32}
$$

as follows. Let $\varphi \in M_t$ and it is reached by the self-financing strategy θ. Thus the function $\psi(\cdot,\cdot)$ is defined as

$$
\psi(t,\varphi) = \theta_0 S_0. \tag{4.33}
$$

Proposition 4.4. *The function $\psi(\cdot,\cdot)$ is well defined.*

Proof: Suppose that there exist two self-financing strategies α and β such that

$$\begin{aligned} \alpha_t S_t &= \beta_t S_t \\ \alpha_0 S_0 &> \beta_0 S_0. \end{aligned} \tag{4.34}$$

Let us examine the following strategy γ

$$\gamma_t = \left\{ \frac{(\alpha_0 - \beta_0)\, S_0}{S_0^0}, \underbrace{0, ..., 0}_{n} \right\} + (\beta_t - \alpha_t)\,. \tag{4.35}$$

We have

$$\gamma_0 S_0 = \frac{(\alpha_0 - \beta_0)\, S_0}{S_0^0} S_0^0 + (\beta_0 - \alpha_0)\, S_0 = 0. \tag{4.36}$$

Lemma 4.1 says that the M_t is linear and therefore $\gamma \in M_t$. Thus we have

$$\begin{aligned} E\left[\gamma_t S_t\right] &= E\left[\frac{(\alpha_0 - \beta_0)\, S_0}{S_0^0} S_t^0 + (\alpha_t - \beta_t)\, S_t\right] \\ &= E\left[\frac{(\alpha_0 - \beta_0)\, S_0}{S_0^0} S_t^0\right] > 0. \end{aligned} \tag{4.37}$$

Therefore γ is an arbitrage strategy. The contradiction proves that the function $\psi(\cdot, \cdot)$ is well defined. □

Lemma 4.2. *The function $\psi(\cdot, \cdot)$ is linear.*

Proof: The proof is based on the linearity of the integration. □

Lemma 4.3. *For a fixed t the function $\psi(t, \cdot)$ is strictly increasing.*

Proof: Suppose that there exist two self-financing strategies α and β such that

$$\begin{aligned} &\alpha_0 S_0 \geq \beta_0 S_0 \\ &P\left(\alpha_t S_t < \beta_t S_t\right) = 1 \end{aligned} \tag{4.38}$$

We examine a strategy γ defined in equation (4.35) and prove in the same way as in proposition 4.4 that it is an arbitrage strategy. $\square$

We shall use the following proposition

Proposition 4.5. *Let $\Psi(\cdot)$ be linear continuous function $L(t,P) \overset{\Psi}{\to} \mathbb{R}$. Then there exists an unique random variable $\xi \in L(t,P)$ such that*

$$\Psi(\alpha) = E[\xi\alpha] \ \ \forall \alpha \in L(t,P). \tag{4.39}$$

Now we can prove the inverse martingale theorem.

Theorem 4.3. *If the function $\psi(T,\cdot)$ is continuous and the set M_T is closed, then there exists a martingale measure w.r.t. the deflator S_t^0.*

Proof: Corollary 4.3 gives us the right to work directly with the deflated market. We know from proposition 4.5 that there exists a random variable ξ such that equation (4.39) is satisfied for every α. We have $\frac{S_t^0}{Y_t} \equiv 1$ and therefore

$$E^P[\xi] = E^P\left[\xi S_T^{0,Y}\right] = \psi\left(T, S_T^{0,Y}\right) = S_0^{0,Y} = 1. \tag{4.40}$$

Since the function $\psi(T,\cdot)$ is strictly increasing (lemma 4.3), the random variable ξ is strictly positive. Hence we can use it for a Radon-Nikodym derivative. So we define the measure Q as

$$\frac{dQ}{dP} = \xi. \tag{4.41}$$

Obviously $S_t^{0,Y} \equiv 1$ is a martingale. We have to prove that $S_t^{i,Y}$ is a martingale for $i = 1,...,n$. Let τ be an arbitrary moment $\tau \le T$. Let us define the strategy θ as

1. $\theta^j \equiv 0, \ j \neq i, \ j \neq 0.$

2.

$$\begin{aligned} \theta_t^i &= 1, t \le \tau \\ \theta_t^i &= 0, t > \tau. \end{aligned} \tag{4.42}$$

3.

$$\begin{aligned} \theta_t^0 &= 0, t \le \tau \\ \theta_t^0 &= S_\tau^{i,Y}, t > \tau. \end{aligned} \tag{4.43}$$

This strategy is a unit of the i-th asset till moment τ which is converted to $S^{0,Y}$ after that. Obviously θ is self-financing with terminal value

$$\theta_T S_T^Y = S_\tau^{i,Y}. \tag{4.44}$$

Its initial investment is

$$S_0^{i,Y} = \psi\left(T, \theta_T S_T^Y\right) = \psi\left(T, S_\tau^{i,Y}\right) = E^P\left[\xi S_\tau^{i,Y}\right] = E^Q\left[S_\tau^{i,Y}\right]. \tag{4.45}$$

Therefore the deflated asset process is a martingale, since its expectation is time independent. □

4.6. Relation between martingale measures and martingale deflators

Proposition 4.6. *The existence of a martingale measure w.r.t the deflator Y_t is equivalent to the existence of a martingale deflator Z_t*

$$Z_t = Y_t E^P\left[\left.\frac{dP}{dQ}\right| \mathcal{F}_t\right] = \frac{Y_t}{E^P\left[\left.\frac{dQ}{dP}\right| \mathcal{F}_t\right]} = \frac{Y_t}{\xi_t}, \tag{4.46}$$

where

$$\xi_t = E^P\left[\left.\frac{dQ}{dP}\right| \mathcal{F}_t\right]. \tag{4.47}$$

Proof: Suppose that there exists a martingale measure Q. Let $t < u$. We have

$$E^P\left[\left.\frac{S_u}{Z_u}\right| \mathcal{F}_t\right] = E^P\left[\left.\xi_u \frac{S_u}{Y_u}\right| \mathcal{F}_t\right] = \xi_t E^Q\left[\left. S_u^Y \right| \mathcal{F}_t\right] = \xi_t S_t^Y = \frac{S_t}{Z_t} \tag{4.48}$$

and therefore the delator Z_t is martingale.

Inverse – let Z_t be a martingale deflator. Therefore

$$S_t^Y = \frac{S_t^Z}{\xi_t} = \frac{1}{\xi_t} E^P \left[S_u^Z \middle| \mathcal{F}_t \right] = \frac{1}{\xi_t} E^P \left[\xi_u S_u^Y \middle| \mathcal{F}_t \right] = E^Q \left[S_u^Y \middle| \mathcal{F}_t \right] \tag{4.49}$$

which means that the measure Q is a martingale measure. $\square$

Example 4.1 (Black-Scholes model). *In this case the deflator is*

$$Y_t \equiv R_t = e^{rt}. \tag{4.50}$$

We prove in subsection 3.2 that the risk-neutral measure has Radon-Nikodym derivative

$$\left. \frac{dQ}{dP} \right|_t = L_t, \tag{4.51}$$

where L_t is the solution of the SDE

$$dL_t = -\theta L_t dB_t^P \tag{4.52}$$

for

$$\theta = \frac{\mu - r}{\sigma}. \tag{4.53}$$

The solution of equation (4.52) is

$$L_t = e^{-\frac{\theta^2}{2}t - \theta B_t^P}. \tag{4.54}$$

In this way we define the deflator (4.46) as

$$Z_t = \frac{Y_t}{\xi_t} = e^{rt} e^{\frac{\theta^2}{2}t + \theta B_t^P}. \tag{4.55}$$

We have to prove that S_t^Z is a P-martingale. We have

$$
\begin{aligned}
S_t^Z &\equiv \frac{S_t}{Z} \\
&= e^{-rt} e^{-\frac{\theta^2}{2}t-\theta B_t^P} S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma B_t^P} \\
&= S_0 e^{(\mu-r)t} e^{-\frac{\theta^2+\sigma^2\pm 2\sigma\theta}{2}t+(\sigma-\theta)B_t^P} \\
&= S_0 e^{(\mu-r)t} e^{-\sigma\theta t} e^{-\frac{(\sigma-\theta)^2}{2}t+(\sigma-\theta)B_t^P} \\
&= S_0 e^{(\mu-r)t} e^{-\sigma\frac{\mu-r}{\sigma}t} e^{-\frac{(\sigma-\theta)^2}{2}t+(\sigma-\theta)B_t^P} \\
&= S_0 e^{-\frac{(\sigma-\theta)^2}{2}t+(\sigma-\theta)B_t^P},
\end{aligned}
\tag{4.56}
$$

which is a martingale.

5. Lévy processes

Definition 5.1. *A stochastic process Y_t is called a Lévy process if*

1. $Y_0 = 0$.
2. *The process has independent increments, i.e. if $t_1 < t_2 < ... < t_n$, then the random variables $Y_{t_1}, Y_{t_2} - Y_{t_1}, ..., Y_{t_n} - Y_{t_{n-1}}$ are independent.*
3. *The increments are stationary, i.e $Y_{t+s} - Y_s$ is independent of s.*
4. *Stochastic continuity, i.e. for $t \geq 0$ and $\varepsilon > 0$:* $\lim_{s\to t} P\left[|Y_s - Y_t| > \varepsilon\right] = 0$.

We can see that the difference with the Brownian motion is that we do not impose a normality of the increments. It can be proven that a Lévy process has a RCLL version.

Proposition 5.1. *Suppose that $E\left[Y_t\right] < \infty$. Then we have*

1. $$E\left[Y_t\right] = tE\left[Y_1\right]. \tag{5.1}$$
2. *The process*
$$Y_t - tE\left[Y_1\right] \tag{5.2}$$
is a martingale.

Proof: We shall prove the first statement.

1. Let $t = 2$. Then

$$E[Y_t] = E[(Y_2 - Y_1) + Y_1] = E[Y_2 - Y_1] + E[Y_1] = 2E[Y_1]. \quad (5.3)$$

2. If $t = n$, then

$$E[Y_t] = E\left[\sum_{i=1}^{t} Y_i - Y_{i-1}\right] = \sum_{i=1}^{t} E[Y_i - Y_{i-1}] = tE[Y_1]. \quad (5.4)$$

We can conclude that equation (5.1) is true. The proof of the second statement is a consequence from theorem 3.5. □

Definition 5.2. *If the Lévy process has a finite expectation, then the term $tE[Y_1]$ is its compensator; $Y_t - tE[Y_1]$ is the compensated process. It is a martingale.*

5.1. Poisson process

A random variable is exponentially distributed with intensity λ if its density is

$$p(x) = \lambda e^{-\lambda x}, \quad x \geq 0. \quad (5.5)$$

Its cumulative density function is

$$\begin{aligned} F(x) \equiv P(\xi < x) &= \int_0^x \lambda e^{-\lambda u} du \\ &= -\int_0^x e^{-\lambda u} d(-\lambda u) = -\int_0^{-\lambda x} e^v dv = \int_{-\lambda x}^0 e^v dv \\ &= 1 - e^{-\lambda x}. \end{aligned} \quad (5.6)$$

Proposition 5.2 (Memorylessness). *The exponential distribution is memorylessness, in sense*

$$P(\xi > t + s | \, \xi > t) = P(\xi > s). \tag{5.7}$$

Proof: We have

$$\begin{aligned} P(\xi > t + s | \, \xi > t) &= \frac{P(\xi > t + s, \xi > t)}{P(\xi > t)} \\ = \frac{P(\xi > t + s)}{P(\xi > t)} &= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = P(\xi > s). \end{aligned} \tag{5.8}$$

□

Proposition 5.3. *The exponential distribution is the unique continuous memorylessness distribution.*[1]

Proof: Let $G(x) = P(\xi > x)$. As we can see in equation (5.8)

$$G(t + s) = G(t) G(s). \tag{5.9}$$

1. If $s = t$,

$$G(2t) = (G(t))^2. \tag{5.10}$$

2. If $s = 2t$,

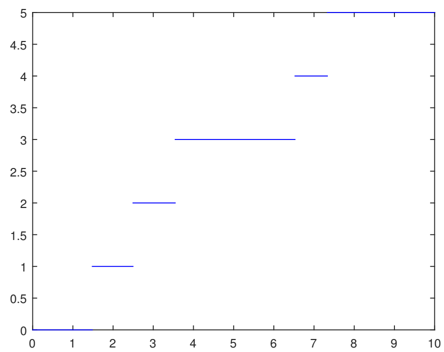
$$G(3t) = G(t) G(2t) = (G(t))^3. \tag{5.11}$$

3. More generally,

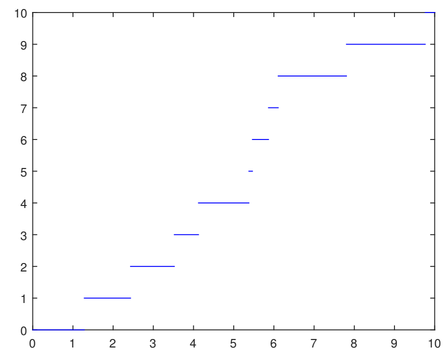
$$G(x) = (G(1))^x. \tag{5.12}$$

[1]Similarly, the geometric distribution is the only discrete distribution with the property memorylessness.

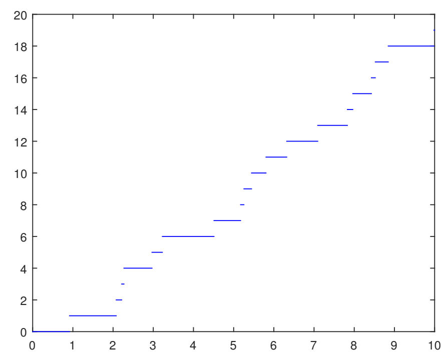
Figure 7: Poisson process simulations



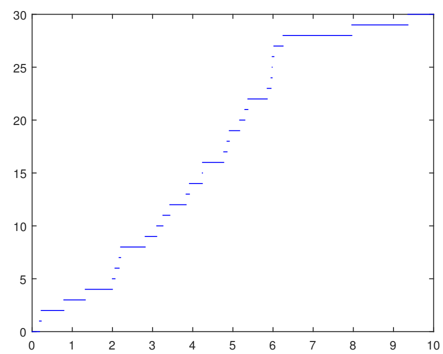
(a) Intensity $\lambda = 0.5$



(b) Intensity $\lambda = 1$



(c) Intensity $\lambda = 2$



(a) Intensity $\lambda = 3$

We can rewrite equation (5.12) as

$$G(x) = (G(1))^x = e^{\ln(G(1))x} = e^{-\lambda x}, \tag{5.13}$$

where $\lambda = -\ln(G(1))$. Note that $\lambda > 0$ since $G(x)$ is a probability. $\square$

Definition 5.3 (Poisson process). *The Poisson process with intensity λ is a step process with jumps with size* 1*. The jump moments are independent exponentially distributed with rate λ. We shall denote it by N_t.*

The process can be viewed as follows. Let τ_1, τ_2,...,τ_n,... be a sequence of independent exponentially distributed random variables. The first jump happens in moment τ_1, the second jump occurs τ_2 time after the first jump, i.e in $\tau_1 + \tau_2$ moment. Similarly, the n-th jump happens τ_n time after the $n-1$-th jump, which means that it happens in moment $\tau_1+\tau_2+...+\tau_n$. The memorylessness guarantees the independence of the increments. The process stationarity is due to the fact that the jumps are of size one. Note that since the exponential distribution is the only one which exhibits memorylessness, all Lévy pure jump processes are driven by the Poisson process in some sense. Some MATLAB simulations of the Poisson processes sample paths are presented at figure 7. Varing the jump intensity, we can see that the jumps happen more often for larger value of the intensity λ.

We shall prove the following proposition for the distribution of a Poisson process.

Proposition 5.4. *The probabilities of a Poisson process are given by*

$$P\left(N_t = n\right) = \frac{\left(\lambda t\right)^n}{n!} e^{-\lambda t}. \tag{5.14}$$

Proof: We shall use a mathematical induction. First, the event $N_t = 0$ is equivalent to $\tau_1 > t$ for the exponentially distributed random variable τ_1. Therefore

$$P\left(N_t = 0\right) = P\left(\tau_1 > t\right) = e^{-\lambda t}, \tag{5.15}$$

i.e. equation (5.14) is true for $n = 0$. Suppose now that equation (5.14) is true for some $n = k$. Thus the probability $N_t = k+1$ can be decomposed to the probability $\tau_{k+1} = du$ together with $N_{t-u} = k$. Using the memorylessness we derive

$$
\begin{aligned}
P\left(N_t = k+1\right) &= \int_0^t P\left(N_{t-u} = u\right) dP\left(\tau_{k+1} \in du\right) \\
&= \int_0^t \frac{\left(\lambda\left(t-u\right)\right)^k}{k!} e^{-\lambda(t-u)} \lambda e^{-\lambda u} du \\
&= \frac{\lambda^{k+1} e^{-\lambda t}}{k!} \int_0^t \left(t-u\right)^k du \\
&= \frac{\lambda^{k+1} e^{-\lambda t}}{k!} \int_0^t v^k dv \\
&= \frac{\lambda^{k+1} e^{-\lambda t}}{k!} \left(\left. \frac{v^{k+1}}{k+1} \right|_0^t \right) \\
&= \frac{\lambda^{k+1} e^{-\lambda t} t^{k+1}}{(k+1)!}.
\end{aligned}
\tag{5.16}
$$

□

The infinitesimal generator of a Poisson process is presented in the following proposition.

Proposition 5.5. *The infinitesimal generator of a Poisson process is*

$$
\mathcal{A} f\left(z\right) = \lambda\left(f\left(z+1\right) - f\left(z\right)\right). \tag{5.17}
$$

Proof: We have

$$
E\left[f\left(Y_t^z\right)\right] = \sum_{n=0}^{\infty} f\left(n+z\right) \frac{\left(\lambda t\right)^n}{n!} e^{-\lambda t} \equiv \sum_{n=0}^{\infty} f^n\left(t, z\right). \tag{5.18}
$$

Let us take the t-derivative at the point $t = 0$.

1. If $n = 0$, we derive

$$
f_t^0\left(t, z\right) = -\lambda f\left(z\right) e^{-\lambda t} \tag{5.19}
$$

and therefore

$$f_t^0(0,z) = -\lambda f(z)\,. \tag{5.20}$$

2. If we have $n=1$, then

$$\begin{aligned} f_t^1(t,z) &= \frac{\partial\left(f(1+z)\,\lambda t e^{-\lambda t}\right)}{\partial t} \\ &= \lambda f(1+z)\left(e^{-\lambda t} - t\lambda e^{-\lambda t}\right) \end{aligned} \tag{5.21}$$

and therefore

$$f_t^1(0,z) = \lambda f(1+z)\,. \tag{5.22}$$

3. Obviously, the terms for $n \geq 2$ vanishes, because there exists a multiplication by t.

Finally, we derive

$$\begin{aligned} (\mathcal{A}f)(z) &= \lim_{t\to 0} \frac{E\left[f\left(Y_t^z\right)\right] - f(z)}{t} \\ &= \lambda\left(f(z+1) - f(z)\right). \end{aligned} \tag{5.23}$$

□

Remark 5.1. *We can see that the generator is the difference of the function values after and before a jump, multiplied by the intensity.*

Corollary 5.1. *The process*

$$\overline{Y}_t = Y_t - \lambda t \tag{5.24}$$

is a martingale.

Proof: Let us examine the function

$$f(t,x) = x - \lambda t. \tag{5.25}$$

We have

$$f_t + \mathcal{A}f = -\lambda + \lambda\left[(x+1-\lambda t) - (x-\lambda t)\right] = 0. \tag{5.26}$$

Theorem (3.5) gives us that $f(t, Y_t)$ is a martingale. □

Remark 5.2. *This propositions says that the compensator of the Poisson process is λt.*

5.2. Fourier transform/ Characteristic function

The Fourier transform of a function $f(x)$ is defined as the function $\hat{f}(u)$:

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{ixu} f(x)\,dx \equiv \int_{-\infty}^{\infty} (\cos(xu) + i\sin(xu)) f(x)\,dx. \tag{5.27}$$

The inverse transform appears as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixu} \hat{f}(u)\,dx. \tag{5.28}$$

Analogously, the Fourier transform of a random variable ξ or its characteristic function is defined as

$$\Psi_\xi(u) = E\left[e^{iu\xi}\right] = \int_{-\infty}^{\infty} e^{ixu} p_\xi(x)\,dx. \tag{5.29}$$

Hence, the characteristic function of a random variable is just the Fourier transform of its density.

Example 5.1 (Normal distribution). *Let ξ be a normal distributed random variable $N(\mu,\sigma)$. Then its Fourier transform is*

$$
\begin{aligned}
\Psi_\xi(u) &= \frac{1}{\sqrt{2\pi\sigma}} \int\limits_{-\infty}^{\infty} e^{ixu} \exp\left(-\frac{(x-\mu)^2}{2\sigma}\right) dx \\
&= \frac{1}{\sqrt{2\pi\sigma}} \int\limits_{-\infty}^{\infty} \exp\left(-\frac{x^2 - 2x(\mu + iu\sigma) + \mu^2 \pm (\mu + iu\sigma)^2}{2\sigma}\right) dx \\
&= \frac{\exp\left(iu\mu - \frac{u^2\sigma}{2}\right)}{\sqrt{2\pi\sigma}} \int\limits_{-\infty}^{\infty} \exp\left(-\frac{(x - (\mu + iu\sigma))^2}{2\sigma}\right) dx \\
&= \exp\left(iu\mu - \frac{u^2\sigma}{2}\right).
\end{aligned}
\tag{5.30}
$$

Example 5.2 (Exponential distribution). *Let ξ be an exponential distributed random variable with intensity λ. Then its Fourier transform is*

$$
\begin{aligned}
\Psi_\xi(u) &= \lambda \int\limits_{-\infty}^{\infty} e^{ixu} e^{-\lambda x} dx \\
&= \lambda \int\limits_{-\infty}^{\infty} e^{ix(u+i\lambda)} dx \\
&= \lambda \frac{-i(u+i\lambda)}{-i(u+i\lambda)} \int\limits_{-\infty}^{\infty} e^{ix(u+i\lambda)} dx \\
&= \frac{\lambda}{\lambda - iu}.
\end{aligned}
\tag{5.31}
$$

Example 5.3 (Poisson distribution). *The characteristic function of the Poisson process distribution is*

$$
\begin{aligned}
\Psi_{\xi}(u) &= E\left[e^{iu\xi}\right] \\
&= \sum_{n=0}^{\infty} e^{inu} \frac{(\lambda t)^{n}}{n!} e^{-\lambda t} \\
&= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\left(e^{iu} \lambda t\right)^{n}}{n!} \\
&= e^{-\lambda t} \exp\left(\lambda t e^{iu}\right) \\
&= \exp\left(t\lambda\left(e^{iu}-1\right)\right).
\end{aligned}
\tag{5.32}
$$

The last is true because the Taylor's expansion of the exponent is

$$
e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!}. \tag{5.33}
$$

Let us differentiate in the following equation

$$
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixy} \hat{f}(y)\, dy \tag{5.34}
$$

to derive

$$
\begin{aligned}
f'(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixy}(-iy) \hat{f}(y)\, dy \\
f''(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixy}(-iy)^{2} \hat{f}(y)\, dy
\end{aligned}
\tag{5.35}
$$

and so on. Hence, if $\alpha(y)$ is a polynomial with constant coefficients, then

$$
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixy} \alpha(-iy) \hat{f}(y)\, dy \tag{5.36}
$$

is a differential operator completely defined by the polynomial $\alpha(y)$. Analogously we can state the following definition

Definition 5.4. *Let $\alpha(x,y)$ be a function. Then*

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-ixy}\alpha(x,-iy)\,\hat{f}(y)\,dy \tag{5.37}$$

is called pseudo-differential operator with symbol $\alpha(x,y)$.

Let the random variables ξ_1, ξ_2,...,ξ_n be independent. Then the characteristic function of their sum is

$$\begin{aligned}\Psi_{\xi_1+\ldots+\xi_n}(u) &= E\left[e^{iu(\xi_1+\ldots+\xi_n)}\right] \\ &= E\left[e^{iu\xi_1}e^{iu\xi_2}\ldots e^{iu\xi_1}\right] \\ &= E\left[e^{iu\xi_1}\right]E\left[e^{iu\xi_2}\right]E\left[e^{iu\xi_1}\right] \\ &= \prod_{i=1}^{n}\Psi_{\xi_i}(u).\end{aligned} \tag{5.38}$$

We can see that the characteristic function of the sum of independent random variables is the product of their characteristic functions. Let now Y_t be a Lévy process. Let t be a natural number. Then we can rewrite the process as

$$Y_t = \sum_{i=1}^{t}(Y_i - Y_{i-1}). \tag{5.39}$$

Hence, using equation (5.38) we can conclude that

$$\Psi_{Y_t}(x) = \prod_{j=1}^{t}\Psi_{Y_j-Y_{j-1}}(x) = (\Psi_{Y_1}(x))^t \tag{5.40}$$

since the random variables $Y_j - Y_{j-1}$ are identical distributed.

Definition 5.5. *Let us define the Lévy symbol of the process as*

$$\psi(x) = \ln\Psi_{Y_1}(x). \tag{5.41}$$

We can formulate the following proposition

Proposition 5.6. *The characteristic function of a Lévy process is*

$$\Psi_{Y_t}(x) = e^{t\psi(x)}. \tag{5.42}$$

Proof: The proof is based on equations (5.40) and (5.41). $\square$

Proposition 5.7. *The infinitesimal generator of a Lévy process is a pseudo-differential operator and its symbol is the Lévy symbol of the process, $\psi(-iv)$.*

Proof: We have

$$\begin{aligned} f(t,z) &\equiv E\left[f\left(Y_t^z\right)\right] \\ &= \int_{-\infty}^{\infty} f(y+z)\, p_t(y)\, dy \\ &= \int_{-\infty}^{\infty} f(y+z)\, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixy} e^{t\psi(x)} dx dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-ixy} f(y+z) dy \right) e^{t\psi(x)} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{iv(y+z)} f(y+z) dy \right) e^{-ivz} e^{t\psi(-v)} dv \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivz} e^{t\psi(-v)} \hat{f}(v)\, dv. \end{aligned} \tag{5.43}$$

Thus the generator turns to

$$\begin{aligned} g_t(0,z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivz} \psi(-v)\, e^{t\psi(-v)} \hat{f}(v)\, dv \Bigg|_{t=0} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivz} \psi(-i(-iv))\, e^{t\psi(-v)} \hat{f}(v)\, dv. \end{aligned} \tag{5.44}$$

$\square$

5.3. Compound Poisson process

Let ζ_1, ζ_2,... be independent identical distributed random variables with distribution $\kappa\left(dx\right)$. Let us examine the process

$$Y_t = \sum_{i=0}^{N_t} \zeta_i. \tag{5.45}$$

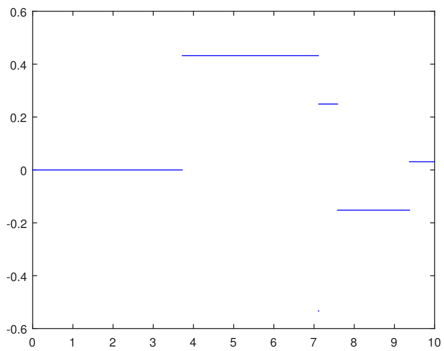
This process is named compound Poisson process. It jumps like the ordinary Poisson process, but its jump sizes are stochastic. Some simulations of the compound Poisson sample paths are presented at figures 8 and 9. At the first one jumps are normally distributed with parameters $N\left(0,1\right)$ whereas at the second one we use parameters $N\left(1,1\right)$. We can see that at figure 8 the downward and upward jumps are approximately equal, whereas at the second figure the upward jumps are significantly more. The fact that the jump sizes are independent and equal distributed guarantees that the compound Poisson process has independent and equal distributed increments. The form of its characteristic function is given in the following proposition.

Proposition 5.8. *The characteristic function of a compound Poisson process is*

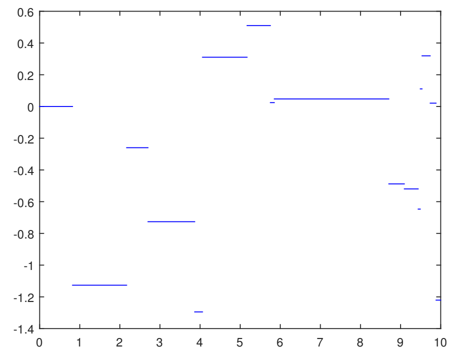
$$\Psi_{Y_t}\left(z\right) = \exp\left(\lambda t\left(\int\limits_{-\infty}^{\infty}\left(e^{iuz}-1\right)\kappa\left(du\right)\right)\right). \tag{5.46}$$

Proof: As we mentioned above, the characteristic function of the sum of independent identical distributed random variables is the product of their characteristic functions. Thus we obtain

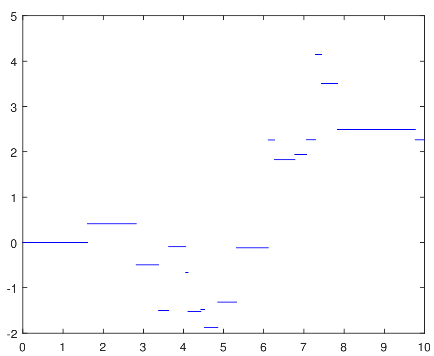
Figure 8:   Compound Poisson process simulations



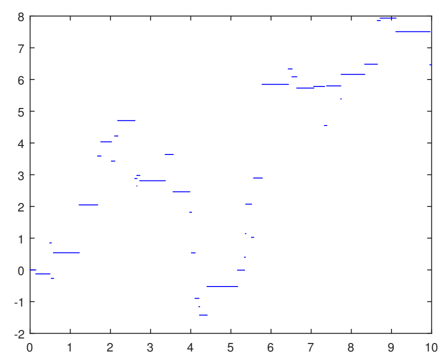
(a) $\lambda = 0.5$, $\mu = 0$, $\sigma = 1$



(b) $\lambda = 1$, $\mu = 0$, $\sigma = 1$

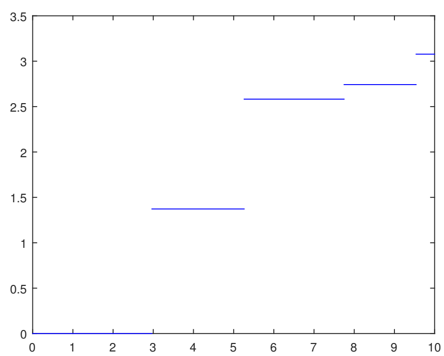


(c) $\lambda = 2$, $\mu = 0$, $\sigma = 1$

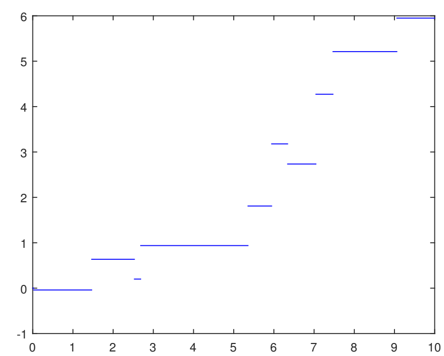


(a) $\lambda = 3$, $\mu = 0$, $\sigma = 1$

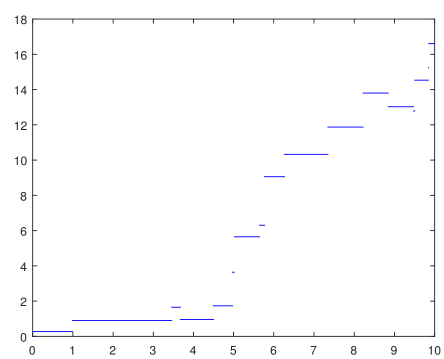
Figure 9: Compound Poisson process simulations



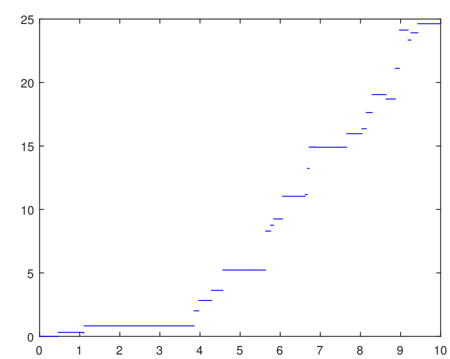
(a) $\lambda = 0.5$, $\mu = 1$, $\sigma = 1$



(b) $\lambda = 1$, $\mu = 1$, $\sigma = 1$



(c) $\lambda = 2$, $\mu = 1$, $\sigma = 1$



(a) $\lambda = 3$, $\mu = 1$, $\sigma = 1$

$$
\begin{aligned}
E\left[e^{izY_t}\right] &= \sum_{n=0}^{\infty} E\left[\exp\left(iz\sum_{i=0}^{n}\zeta_i\right)\right] P\left(N_t = n\right) \\
&= \sum_{n=0}^{\infty}\left(\prod_{i=1}^{n}\Psi_{\zeta_i}\left(z\right)\right) P\left(N_t = n\right) \\
&= \sum_{n=0}^{\infty}\frac{\left(\Psi_{\zeta}\left(z\right)\right)^n e^{-\lambda t}(\lambda t)^n}{n!} \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{\left(\lambda t\Psi_{\zeta}\left(z\right)\right)^n}{n!} \\
&= e^{-\lambda t}e^{\lambda t\Psi_{\zeta}(z)} \\
&= \exp\left(\lambda t\left(\Psi_{\zeta}\left(z\right) - 1\right)\right) \\
&= \exp\left(\lambda t\left(\int_{-\infty}^{\infty}\left(e^{iuz} - 1\right)\kappa\left(du\right)\right)\right).
\end{aligned}
\tag{5.47}
$$

□

Immediately, we conclude that the Lévy symbol of a compound Poisson process is

$$
\psi_{Y_t}\left(z\right) = \lambda\left(\int_{-\infty}^{\infty}\left(e^{iuz} - 1\right)\kappa\left(du\right)\right). \tag{5.48}
$$

The next step is to derive its infinitesimal generator.

Proposition 5.9. *The generator of a compound Poisson process is*

$$
\left(\mathcal{A}f\right)\left(z\right) = \lambda\int_{-\infty}^{\infty}\left(f\left(u+z\right) - f\left(z\right)\right)\kappa\left(du\right). \tag{5.49}
$$

Proof: We have

$$
\begin{aligned}
g(t,z) &= E\left[f\left(Y_t^z\right)\right] \\
&= \int_{-\infty}^{\infty} f(y+z)\, p_t(y)\, dy \\
&= \int_{-\infty}^{\infty} f(y+z) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixy} \Psi_{Y_t}(x)\, dxdy \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ixy} e^{t\psi(x)} f(y+z)\, dxdy.
\end{aligned} \tag{5.50}
$$

Therefore the generator, which is the t-derivative of the function $g(t,z)$ taken in $t=0$, is

$$
\begin{aligned}
(\mathcal{A}f)(z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ixy} \psi(x)\, f(y+z)\, dxdy \\
&= \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ixy} f(y+z) \int_{-\infty}^{\infty} \left(e^{iux} - 1\right) \kappa(du) dxdy \\
&= \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ixy} f(y+z) \left(e^{iux} - 1\right) \kappa(du) dxdy.
\end{aligned} \tag{5.51}
$$

We shall divide the integral into two parts. We have

$$
\begin{aligned}
I_1 &= \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ixy} f(y+z)\, e^{iux} \kappa(du) dx dy \\
&= \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ix(y-u)} f(y+z)\, \kappa(du) dx dy \\
&= \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-ix(y-u)} dx \right) f(y+z)\, \kappa(du) dy \\
&= \lambda \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(y-u)\, f(y+z)\, \kappa(du) dy \\
&= \lambda \int_{-\infty}^{\infty} f(u+z)\, \kappa(du).
\end{aligned} \tag{5.52}
$$

The second integral turns to

$$
\begin{aligned}
I_2 &= \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ixy} f(y+z)\, \kappa(du) dx dy \\
&= \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-ixy} dx \right) f(y+z)\, \kappa(du) dy \\
&= \lambda \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(y)\, f(y+z)\, \kappa(du) dy \\
&= \lambda \int_{-\infty}^{\infty} f(z)\, \kappa(du).
\end{aligned} \tag{5.53}
$$

Finally, we conclude

$$(\mathcal{A}f)(z) = I_1 - I_2$$
$$= \lambda \int_{-\infty}^{\infty} (f(u+z) - f(z))\, \kappa(du). \tag{5.54}$$

□

5.4. Lévy processes: general case

Let Y_t be a Lévy process. Let us define the associated Poisson random measure over $\mathbb{R}\setminus\{0\}$ as

$$N(t, A) = \{\#s : 0 < s < t, \Delta Y_s\}. \tag{5.55}$$

This means the following. Let A be a subset of the real numbers, which does not contain the zero with some its region. Hereafter we shall use only such sets. Then $N(t, A)$ is the number of jumps with size in the set A. Let us fix a set A. We can see that $N(t, A)$ is a Lévy process. Also it is a pure jump process with jump sizes one. Hence it is a Poisson process.

Let us define the measure $\nu(A)$ as

$$\nu(A) = E[N(1, A)] = E[\{\#s : 0 < s < 1, \Delta Y_s \in A\}], \tag{5.56}$$

i.e. as the expected number of jumps for a unit time with size in the set A. This measure is known as a Lévy measure.

Example 5.4. *The Lévy measure of a Poisson process with intensity λ is $\nu(dx) = \lambda\delta_1(x)$. Note that*

$$\nu(A) = \int_A \nu(dx). \tag{5.57}$$

Hence, if $\{1\} \in A$, then $\nu(A) = \lambda$ and $\nu(A) = 0$, otherwise.

Example 5.5. *The Lévy measure of a compound Poisson process with intensity λ and probability density function of the jump sizes $p(x)$ is $\nu(dx) = \lambda p(dx)$. Hence*

$$\nu(A) = \int_A \lambda p(dx) = \lambda P(\xi \in A). \tag{5.58}$$

Proposition 5.1 gives that the compensator of $N(t,A)$ is $t\nu(A)$. Hence, the compensated process

$$\tilde{N}(t,A) = N(t,A) - t\nu(A) \tag{5.59}$$

is a martingale. The following proposition holds.

Proposition 5.10. *Let the sets A_1 and A_2 be disjoint. Then the processes $N(t,A_1)$ and $N(t,A_2)$ are independent.*

Proof: Since the sets A_1 and A_2 are disjoint, the number of jumps in the set A_1 is independent of the number of jumps in the set A_2. □

Definition 5.6. *Let $f(x)$ be a real valued function. The integral w.r.t. the Poisson random measure is defined as*

$$\int_A f(x)\, N(t,dx) = \sum_{x\in A} f(x)\, N(t,\{x\}). \tag{5.60}$$

We have the following proposition.

Proposition 5.11. *The integral (5.60) is a Compound Poisson process with associated measure $\nu_{f,A}(B)$, where*

$$\nu_{f,A}(B) = \nu\left(A \cap f^{-1}(B)\right). \tag{5.61}$$

Something more

$$E\left[\int_A f(x)\, N(t,dx)\right] = t\int_A f(x)\, \nu(dx). \tag{5.62}$$

Proof: Suppose that the function $f(\cdot)$ is a step function

$$f(x) = \sum_i c_i I_{x\in A_i}. \tag{5.63}$$

Therefore

$$
\begin{aligned}
E\left[\exp\left(iu\int_A f(x)\,N(t,dx)\right)\right] &= E\left[\exp\left(iu\sum_j c_j N(t,A_j)\right)\right] \\
&= \prod_j E\left[e^{iuc_j N(t,A_j)}\right] \\
&= \prod_j \exp\left(t\nu(A_j)\left(e^{iuc_j}-1\right)\right) \\
&= \exp\left(t\sum_j \left(e^{iuc_j}-1\right)\nu(A_j)\right) \qquad (5.64)\\
&= \exp\left(t\int_A \left(e^{iuf(x)}-1\right)\nu(dx)\right) \\
&= \exp\left(t\int_{-\infty}^{\infty} \left(e^{iux}-1\right)\nu_{f,A}(dx)\right).
\end{aligned}
$$

We can recognize the characteristic function of a compound Poisson process. We finish the proof taking the limit in the step function. The second statement is true since

$$
E[\xi] = -i\left.\frac{d\Psi_\xi(u)}{d\xi}\right|_{u=0}. \qquad (5.65)
$$

□

Definition 5.7. *Let*

$$
E[N(1,A)] = \int_A f(x)\,\nu(dx) < \infty. \qquad (5.66)
$$

Then the compensated Poisson integral is defined as

$$
\int_A f(x)\,\tilde{N}(t,dx) = \int_A f(x)\,N(t,dx) - t\int_A f(x)\,\nu(dx). \qquad (5.67)
$$

Note that proposition proposition 5.1 shows that the compensated Poisson integral is a martingale.

We have the following immediate corollary for its characteristic function.

Corollary 5.2. *The characteristic function of the compensated Poisson integral is*

$$E\left[\exp\left(iu\int_A f(x)\,\tilde{N}(t,dx)\right)\right] = \exp\left(t\int_{-\infty}^{\infty}\left(e^{iux}-1-iux\right)\nu_{f,A}(dx)\right). \tag{5.68}$$

Let us examine the process

$$Z_t = Y_t - \int_{x\geq 1} xN(t,dx). \tag{5.69}$$

Since the big jumps are truncated, the process Z_1 has finite first moment and it is

$$\mu = E\left[Y_1 - \int_{x\geq 1} xN(1,dx)\right]. \tag{5.70}$$

Thus proposition 5.1 gives us that the process

$$Z_t - \mu t \tag{5.71}$$

is a martingale. Finally, let us define the process X_t as

$$\begin{aligned} X_t = Y_t - \mu t - \int_{x\geq 1} xN(t,dx) - \int_{x<1} x\tilde{N}(t,dx) \\ = (Z_t - \mu t) - \int_{x<1} x\tilde{N}(t,dx). \end{aligned} \tag{5.72}$$

Remark 5.3. *Let us examine the small jumps*

$$\int\limits_{x<1} x\tilde{N}\,(t,dx). \tag{5.73}$$

Since the number of small jumps can be infinitely large, we can not directly examine the integral

$$\int\limits_{x<1} xN\,(t,dx), \tag{5.74}$$

because it may not converges. However, for every $0<\epsilon<1$ *the integral*

$$\int\limits_{\varepsilon<x<1} xN\,(t,dx) \tag{5.75}$$

is well define. Something more, the compensated integral

$$\int\limits_{\varepsilon<x<1} x\tilde{N}\,(t,dx) \tag{5.76}$$

is a martingale. Now we can take a limit and define the integral (5.73) *as*

$$\int\limits_{x<1} x\tilde{N}\,(t,dx) = \lim_{\varepsilon\to 0}\int\limits_{\varepsilon<x<1} x\tilde{N}\,(t,dx). \tag{5.77}$$

We can see that

1. X_t is a martingale.
2. Its expectation is zero.
3. Using the characteristic function we can derive that for some constant σ

$$D\,(X_t) = \sigma^2 t. \tag{5.78}$$

4. X_t has continuous paths

All these shows that X_t is a Brownian motion. Hence we can conclude

Theorem 5.1. *Every Lévy process can be decomposed to a drift, a multiplied by a constant Brownian motion, a pure jump martingale, and a compound Poisson process with possibly infinite expectation. All terms are independent.*

Corollary 5.3. *The characteristic function of a Lévy process is*

$$\Psi_{Y_t}(u) = \exp\left(t\left(i\mu u - \frac{\sigma^2}{2}u^2 + \int_{-\infty}^{\infty}\left(e^{iyu} - 1 - I_{(-1,1)}(y)\, iuy\right)\nu(dy)\right)\right) \tag{5.79}$$

and its Lévy symbol is

$$\psi(u) = i\mu u - \frac{\sigma^2}{2}u^2 + \int_{-\infty}^{\infty}\left(e^{iyu} - 1 - I_{(-1,1)}(y)\, iuy\right)\nu(dy). \tag{5.80}$$

Corollary 5.4. *The infinitesimal generator of a Lévy process is*

$$(\mathcal{A}f)(z) = \mu f'(z) + \frac{\sigma^2}{2}f'(z) + \int_{-\infty}^{\infty}\left(f(y+z) - f(z) - f'(z)\, I_{(-1,1)}(y)\, y\right)\nu(dy). \tag{5.81}$$

Proof: In equation (5.51) is proven that

$$(\mathcal{A}f)(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ixy}\psi(x)\, f(y+z)\, dx\nu(dy). \tag{5.82}$$

Therefore the generator is a pseudo-differential operator with symbol $\psi(x)$. Therefore using decomposition (theorem 5.1) we derive equation (5.81) $\square$

5.5. The inverse formulation

Let μ and $\sigma > 0$ be constants and $\nu(A)$ be a measure such that

$$\int_{-\infty}^{\infty}\min\left(y^2, 1\right)\nu(dy) < \infty. \tag{5.83}$$

Then there exist a Lévy process with decomposition as in theorem 5.1. We shall call

$$\{\mu, \sigma, \nu(\cdot)\} \tag{5.84}$$

the Lévy triple of the process.

If the measure $\nu(A)$ is finite, i.e.

$$\nu(\mathbb{R}) < \infty, \tag{5.85}$$

then we can set $\lambda = \nu(A)$ and $f(\cdot) = \frac{\nu(\cdot)}{\lambda}$. We can see that $f(\cdot)$ is a probability density and therefore the corresponding Lévy process is compound Poisson.

The following proposition gives the behavior of the moments of a Lévy process

Proposition 5.12. *1. If* $\int\limits_{|x|>1} |x|^n \nu(dx) < \infty$*, then* $E[|Y_t|^n] < \infty$*.*

2. If $\int\limits_{|x|>1} e^{ux} \nu(dx) < \infty$*, then* $E\left[e^{uY_t}\right] < \infty$*. Something more*

$$E\left[e^{uY_t}\right] = e^{t\psi(-iu)}. \tag{5.86}$$

5.6. Some classes of Lévy processes

We present two type of the so called infinite activity Lévy processes – stable and tempered stable processes. Their sample paths look as figure 10. We can see that the process moves only by jumps most of which are with very small sizes.

5.6.1. Stable processes

The Lévy measure of an α-stable process is given by

$$\nu(x) = \frac{c_1}{(-x)^{1+\alpha}} I_{x<0} + \frac{c_2}{x^{1+\alpha}} I_{x>0}. \tag{5.87}$$

Its behavior can be seen at figures 11c and 11d. We can see immediately that they do not have variance, and if $\alpha < 1$ their expectation is infinity too. The corresponding Lévy symbol is

Figure 10: Infinite activity sample paths

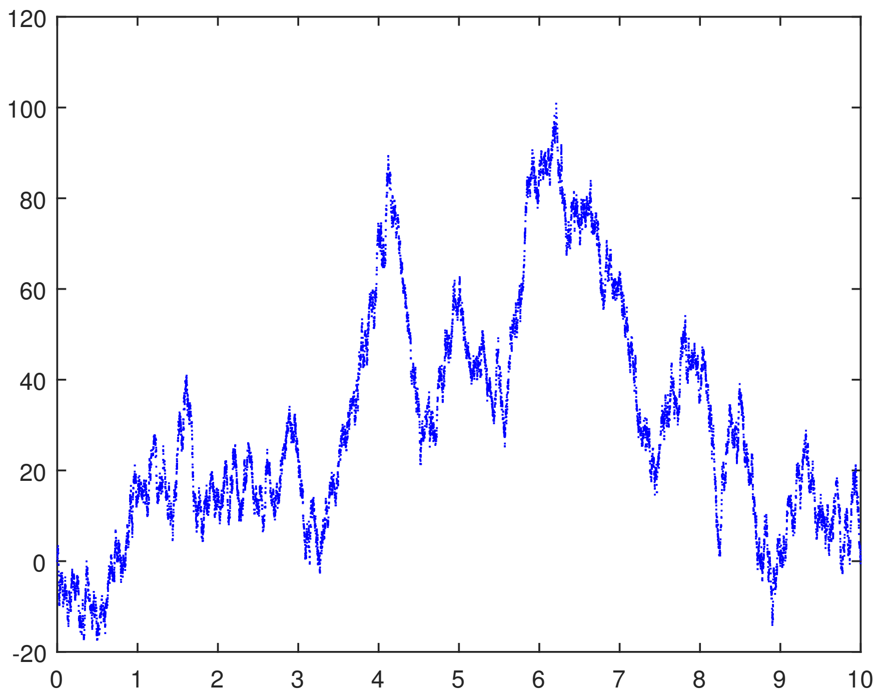
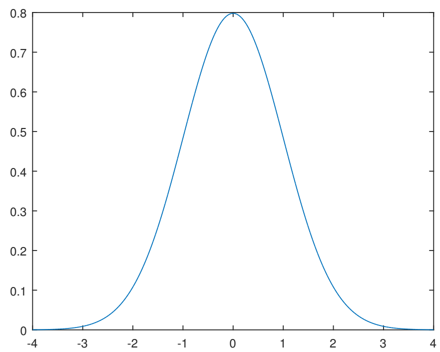
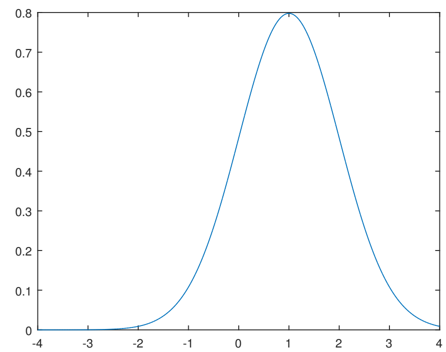


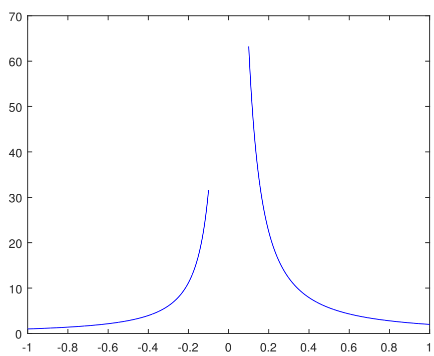
Figure 11: Lévy measures



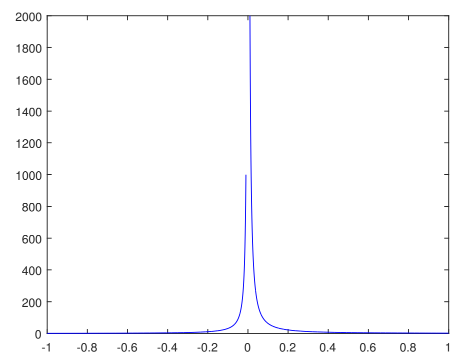
(a) Compound Poisson measure $\lambda = 2$, $\mu = 0$, $\sigma = 1$



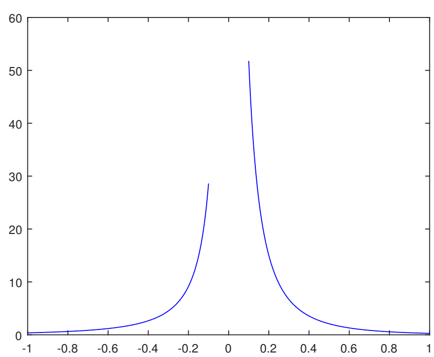
(b) Compound Poisson measure $\lambda = 2$, $\mu = 1$, $\sigma = 1$



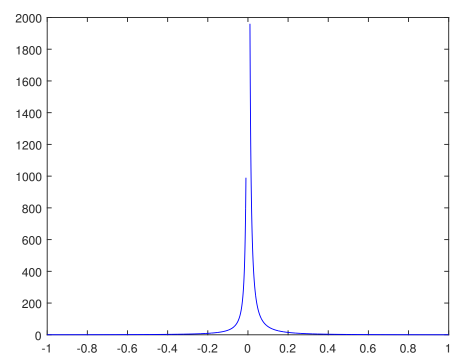
(c) Stable Lévy measure



(d) Stable Lévy measure



(e) Tempered stable Lévy measure



(f) Tempered stable Lévy measure

$$
\begin{aligned}
\psi(z) &= -\sigma^\alpha |z|^\alpha \left(1 - i\beta \operatorname{sgn}(z) \tan\left(\frac{\pi\alpha}{2}\right)\right) + i\mu z, \quad \alpha \neq 1 \\
\psi(z) &= -\sigma |z| \left(1 + i\beta \frac{2}{\pi} \operatorname{sgn}(z) \log(|z|)\right) + i\mu z, \quad \alpha = 1.
\end{aligned}
\tag{5.88}
$$

5.6.2. Tempered stable processes

Their Lévy measure is of the form

$$
\nu(x) = \frac{c_1}{(-x)^{1+\alpha_1}} \exp(\lambda_1 x) I_{x<0} + \frac{c_2}{x^{1+\alpha_2}} \exp(-\lambda_2 x) I_{x>0}. \tag{5.89}
$$

Its behavior can be seen at figures 11e and 11f. We can see that the big jumps in the stable processes are exponentially truncated. This means that the tempered stable processes have moments. The Lévy symbol is given by

$$
\begin{aligned}
\psi_1(u) &= \begin{cases} \Gamma(-\alpha_1) \lambda_1^{\alpha_1} c_1 \left(\left(1 + \frac{iu}{\lambda_1}\right)^{\alpha_1} - 1 - \frac{iu\alpha_1}{\lambda_1}\right), \alpha_1 \neq 1 \\ -iuc_1 + c_1(\lambda_1 + iu) \ln\left(1 + \frac{iu}{\lambda_1}\right), \alpha_1 = 1 \end{cases} \\
\psi_2(u) &= \begin{cases} \Gamma(-\alpha_2) \lambda_2^{\alpha_2} c_2 \left(\left(1 - \frac{iu}{\lambda_2}\right)^{\alpha_2} - 1 + \frac{iu\alpha_2}{\lambda_2}\right), \alpha_2 \neq 1 \\ iuc_2 + c_2(\lambda_2 - iu) \ln\left(1 - \frac{iu}{\lambda_2}\right), \alpha_2 = 1. \end{cases}
\end{aligned}
\tag{5.90}
$$

6. Merton's model. Exponetial Lévy models

6.1. Stating the model

Merton adds a jump component to the Black-Scholes model

$$
dS_t = \mu S_t dt + \sigma S_t dB_t + S_t dM_t, \tag{6.1}
$$

where M_t is a compound Poisson process independent of the Brownian motion.

Definition 6.1 (Poisson process). *The Poisson process with intensity λ is a step process with jumps with size 1. The jump moments are exponentially distributed with rate λ. Due to that, the process is memoryless, i.e. the future jump is independent of the process history.*

Definition 6.2 (Compound Poisson process). *Let N_t be a Poisson process with intensity λ and Y_i be equal distributed random variables. Then the process*

$$M_t = \sum_{i=0}^{N_t} Y_i \tag{6.2}$$

is named a compound Poisson process.

In the Merton's model it is assumed that the jump size distributions are log-normal, i.e.

$$\log\left(1+Y\right) \sim N\left(\log\left(1+k^P\right) - \frac{\left(\delta^P\right)^2}{2}, \left(\delta^P\right)^2\right). \tag{6.3}$$

We have that

$$E\left[Y\right] = k^P + 1. \tag{6.4}$$

6.2. Changing the measure. Risk-neutral condition.

We change the Brownian motion in the same way as before – Girsanov theorem, theorem 3.1 –

$$dB_t^Q = \theta dt + dB_t^P. \tag{6.5}$$

The intensity of the Poisson process is changed freely. The jumps size condition is equivalence of the distribution, i.e. to have one and the same support. It is assumed that the compound Poisson process has the same structure under the new measure – log-normal jumps with new parameters λ, δ, and k.

Using (5.49) we see that the infinitesimal generator of SDE 6.1 is given by

$$\begin{aligned}\mathcal{A}f\left(x\right) = \mu\left(x\right) f_x\left(x\right) + \frac{1}{2}\sigma^2\left(x\right) f_{xx}\left(x\right) \\ +\lambda\left[E\left[f\left(x\right)Y\right] - f\left(x\right)\right].\end{aligned} \tag{6.6}$$

We have to check when $e^{-rt}S_t$ is a Q-martingale. We use proposition 3.5 applied to the function

$$f(t,x) = e^{-rt}x. \tag{6.7}$$

The expectation of a log-normal process with parameters a and b is

$$E[Y] = \exp\left(a + \frac{b^2}{2}\right). \tag{6.8}$$

Thus the expectation of the jump-sizes (6.3) turns to

$$E[Y] = \exp\left(\log(1+k) - \frac{\delta^2}{2} + \frac{\delta^2}{2}\right) = k+1. \tag{6.9}$$

Thus the infinitesimal generator is

$$\begin{aligned} 0 &= f_t + \mathcal{A}f \\ &= -re^{-rt}x + e^{-rt}\mu x + \lambda\left[E\left[e^{-rt}xY\right] - e^{-rt}x\right] \\ &= e^{-rt}x\left[\mu - r + \lambda(k+1-1)\right] \\ &= e^{-rt}x\left[\mu - r + \lambda k\right]. \end{aligned} \tag{6.10}$$

Thus the risk-neutral condition appears as

Condition 6.1 (Risk-neutral condition). *We have under the risk-neutral measure Q*

$$\mu = r - \lambda k. \tag{6.11}$$

6.3. Black-Scholes type equation

We suppose again that the option price can be presented as a function of time and the underlying assets – $f(t, S_t)$. Using once again proposition 3.5 we derive

$$\begin{aligned} &- re^{-rt}f + e^{-rt}f_t + e^{-rt}(r-\lambda k)xf_x + \frac{1}{2}e^{-rt}\sigma^2x^2f_{xx} \\ &+ \lambda\left[E\left[e^{-rt}f(t,xY)\right] - e^{-rt}f\right] = 0, \end{aligned} \tag{6.12}$$

which leads to the Black-Scholes PDE

$$\begin{aligned} &- rf + f_t + (r-\lambda k)xf_x + \frac{1}{2}\sigma^2x^2f_{xx} + \lambda\left[E\left[f(t,xY)\right] - f\right] = 0 \\ &f(T,x) = (x-K)^+. \end{aligned} \tag{6.13}$$

This PDE is solved by Merton in 1976 year.

6.4. Pricing

The SDE (6.1) can be solved as

$$S_t = S_0 \exp\left(X_t\right) \tag{6.14}$$

for

$$X_t = \left(r - \lambda k - \frac{\sigma^2}{2}\right) t + \sigma B_t + Z_t. \tag{6.15}$$

Z_t is another compound Poisson process with normal distributed jumps

$$N\left(\log\left(1+k\right) - \frac{\delta^2}{2}, \delta^2\right). \tag{6.16}$$

Denoting by $\tau = T - t$ and

$$H\left(x\right) = \left(x - K\right)^+, \tag{6.17}$$

we derive for the call option price

$$\begin{aligned} C_t &= e^{-r\tau} E\left[\left. H\left(S_T\right)\right| \mathcal{F}_t\right] \\ &= e^{-r\tau} E\left[\left. H\left(S_T\right)\right| S_t = S\right] \\ &= e^{-r\tau} E\left[H\left(Se^{\left(r-\lambda k-\frac{\sigma^2}{2}\right)t+\sigma B_t+\sum\limits_{i=1}^{N_t} Y_i}\right)\right]. \end{aligned} \tag{6.18}$$

Since the Brownian motion and the Poisson process are independent, we derive

$$C_t = e^{-r\tau} \sum_{n=0}^{\infty} Q\left(N_\tau = n\right) E\left[H\left(Se^{\left(r-\lambda k-\frac{\sigma^2}{2}\right)t+\sigma B_t+\sum\limits_{i=1}^{n} Y_i}\right)\right]. \tag{6.19}$$

Since the jump sizes are normally distributed (6.16), the sum

$$\xi = \sum_{i=1}^{n} Y_i \tag{6.20}$$

is normally distributed

$$N\left(n\alpha, n\delta^2\right) \tag{6.21}$$

for

$$\alpha = \log(1+k) - \frac{\delta^2}{2}. \tag{6.22}$$

Let us examine the term

$$E_n = E\left[H\left(Se^{\left(r-\lambda k-\frac{\sigma^2}{2}\right)\tau+\sigma B_\tau+\xi}\right)\right]. \tag{6.23}$$

Writing

$$\sigma_n^2 = \sigma^2 + n\frac{\delta^2}{t} \tag{6.24}$$

and using that $B_t \sim N(0,t)$ and

$$\frac{\delta^2}{2} + \alpha = \log(1+k) \tag{6.25}$$

we conclude

$$\begin{aligned} E_n &= E\left[H\left(Se^{\left(r-\lambda k-\frac{\sigma_n^2}{2}+n\frac{\delta^2}{2\tau}+\frac{n\alpha}{\tau}\right)\tau+\sigma_n\overline{B}_\tau}\right)\right] \\ &= E\left[H\left(Se^{\left(r-\lambda k-\frac{\sigma_n^2}{2}+\frac{n}{\tau}\log(k+1)\right)\tau+\sigma_n\overline{B}_\tau}\right)\right] \\ &= C_{BS}\left(S,\tau,K,\sigma_n^2,r_n\right), \end{aligned} \tag{6.26}$$

for

$$r_n = r - \lambda k + \frac{n}{\tau}\log(k+1). \tag{6.27}$$

We use above the notation C_{BS} for the option price in the Black-Scholes model. Since N_τ is exponentially distributed with rate λ we have

$$Q(N_\tau = n) = \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}. \tag{6.28}$$

Therefore we can write the Merton's option price as a sum of Black-Scholes prices

$$C_t = e^{-(r+\lambda)\tau} \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} C_{BS}\left(S, \tau, K, \sigma_n^2, r_n\right). \tag{6.29}$$

6.5. Exponential Lévy models

Definition 6.3. *A stochastic process Y_t is called a Lévy process if*

1. $Y_0 = 0$.
2. *The process has independent increments, i.e. if $t_1 < t_2 < ... < t_n$, then the random variables $Y_{t_1}, Y_{t_2} - Y_{t_1}, ..., Y_{t_n} - Y_{t_{n-1}}$ are independent.*
3. *The increments are stationary, i.e $Y_t - Y_s$is distributed as Y_{t-s}.*

We can see that the difference with the Brownian motion is that we do not impose a normality of the increments.

Definition 6.4. *Let A be a subset of $\mathbb{R}/\left\{0\right\}$. We define the measure $\nu\left(A\right)$ as the expected number of jumps with size in the set A till the moment 1:*

$$\nu\left(A\right) = E\left[\#t \in (0,1):\ \Delta Y_t \in A\right]. \tag{6.30}$$

Theorem 6.1 (Decomposition). *A Lévy process Y_t can be decomposed as a sum of a Brownian motion with drift and a pure jump Lévy process*

$$Y_t = \mu t + \sigma B_t + Z_t. \tag{6.31}$$

Definition 6.5 (Characteristic function). *The characteristic function of a Lévy process can be written as*

$$\Psi\left(x\right) = E\left[e^{ixY_t}\right] = e^{-t\psi(x)} \tag{6.32}$$

for

$$\psi\left(x\right) = -i\mu x + \frac{1}{2}\sigma^2 x^2 + \int_{-\infty}^{\infty} \left(e^{ixy} - 1 - I_{(-1,1)}\left(y\right) ixy\right) \nu\left(dy\right). \tag{6.33}$$

In fact the characteristic function is the Fourier transform of the probability density of the process

$$E\left[e^{ixY_t}\right] = \int_{-\infty}^{\infty} e^{ixy} f(t,y)\, dy. \tag{6.34}$$

Proposition 6.1. *The infinitesimal generator of a Lévy process is*

$$\begin{aligned}\mathcal{A}f(x) &= \mu f_x(x) + \frac{\sigma^2}{2} f_{xx}(x) \\ &+ \int_{-\infty}^{\infty} \left(f(x+y) - f(x) - y f_x(x) I_{(-1,1)}(y)\right) \nu(dy).\end{aligned} \tag{6.35}$$

Now, suppose that the asset price is presented as an exponent of a Lévy process

$$S_t = S_0 e^{Y_t}. \tag{6.36}$$

First, we have to derive the risk-neutral condition. Using proposition 3.5 and the form of the infinitesimal generator (6.35) applied to the function $f(t,x) = e^{-rt}e^x$ we derive

$$\begin{aligned}0 &= -re^{-rt}e^x + \mu e^{-rt}e^x + \frac{\sigma^2}{2} e^{-rt}e^x \\ &+ \int_{-\infty}^{\infty} \left(e^{-rt}e^{x+y} - e^{-rt}e^x - ye^{-rt}e^x I_{(-1,1)}(y)\right) \nu(dy)\end{aligned} \tag{6.37}$$

which is equivalent to

$$0 = -r + \mu + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty} \left(e^y - 1 - yI_{(-1,1)}(y)\right) \nu(dy). \tag{6.38}$$

Note that equation (6.38) is equivalent to

$$r + \psi(-i) = 0. \tag{6.39}$$

Note that the Merton's model is a particular case of exponential Lévy models. We turn to the option pricing. We shall use theorem 3.2 to see that the option price can be written as

$$C_t = S_t R\left(S_T > K\right) - K e^{-r(T-t)} Q\left(S_T > K\right). \tag{6.40}$$

We have for the second probability

$$Q\left(S_T > K\right) = \int_{\ln K}^{\infty} d_Q\left(x\right) dx, \tag{6.41}$$

where $d_Q\left(x\right)$ is the Q-density of the Lévy process. Using the inverse Fourier transform we derive

$$d_Q\left(x\right) = F^{-1}\left(e^{-t\psi(x)}\right). \tag{6.42}$$

Let us turn to the first probability in equation (6.40). We shall derive the characteristic function of the Lévy process w.r.t. the measure R. We have that the Radon-Nikodym derivative is

$$\left.\frac{dR}{dQ}\right|_t = \frac{e^{-rt} S_t}{S_0}. \tag{6.43}$$

Therefore

$$\begin{aligned}
\Psi^R\left(x\right) &= E^R\left[e^{ixY_t}\right] \\
&= E^R\left[e^{ix\log\left(\frac{S_t}{S_0}\right)}\right] \\
&= E^Q\left[e^{ix\log\left(\frac{S_t}{S_0}\right)}\frac{e^{-rt}S_t}{S_0}\right] \\
&= e^{-rt}E^Q\left[e^{i(x-i)\log\left(\frac{S_t}{S_0}\right)}\right] \\
&= e^{-rt}E^Q\left[e^{i(x-i)Y_t}\right] \\
&= e^{-rt}\Psi^Q\left(x-i\right).
\end{aligned} \tag{6.44}$$

Knowing the characteristic function w.r.t. the measure R we can continue in the same way as above (for the measure Q).

7. Stochastic volatility models of Heston and Bates

7.1. Heston's model

Heston add a stochastic element in the Black-Scholes model

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dB_t^1 \\ dV_t &= \xi\left(\eta - V_t\right) dt + \theta \sqrt{V_t} dB_t^2 \\ dB_t^1 dB_t^2 &= \rho dt. \end{aligned} \tag{7.1}$$

The Brownian motions are correlated with coefficient ρ. The process V_t is known as a Cox-Ingersoll-Ross process. The meaning of the parameters are (1) η is the level around which the process V_t oscillates, (2) ξ is the speed of reversion, and (3) θ is the volatility of the volatility. This process is non-negative, since when it is near to zero the volatility vanishes and the drift steers it in the up direction. The behavior around the zero is

1. If $\theta^2 \leq 2\xi\eta$ the process never touches the zero.
2. Otherwise, if $\theta^2 > 2\xi\eta$, the process can take the zero value after which to go up.

We shall assume that $\theta^2 \leq 2\xi\eta$ hereafter. We change the measure in a way which keeps the model structure

$$\begin{aligned} dB_t^{1,Q} &= -\alpha_t dt + dB_t^1 \\ dB_t^{2,Q} &= -\beta_t dt + dB_t^2 \end{aligned} \tag{7.2}$$

for

$$\begin{aligned} \alpha_t &= \frac{\gamma}{\sqrt{V_t}} \\ \beta_t &= \delta_1 \sqrt{V_t} + \frac{\delta_2}{\sqrt{V_t}}. \end{aligned} \tag{7.3}$$

Note that we have assumed that the process V_t is strictly positive. Hence, under the new measure the model transforms to

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{V_t} S_t dB_t^1 \\
&= \mu S_t dt + \sqrt{V_t} S_t \frac{\gamma}{\sqrt{V_t}} dt + \sqrt{V_t} S_t dB_t^{1,Q} \\
&= (\mu + \gamma) S_t dt + \sqrt{V_t} S_t dB_t^{1,Q} \\
dV_t &= \xi (\eta - V_t) dt + \theta \sqrt{V_t} dB_t^2 \\
&= \xi (\eta - V_t) dt + \theta \delta_1 V_t dt + \theta \delta_2 dt + \theta \sqrt{V_t} dB_t^{2,Q} \\
&= -(\xi - \theta \delta_1) V_t dt + (\xi \eta + \theta \delta_2) dt + \theta \sqrt{V_t} dB_t^{2,Q} \\
&= (\xi - \theta \delta_1) \left(\frac{\xi \eta + \theta \delta_2}{\xi - \theta \delta_1} - V_t \right) dt + \theta \sqrt{V_t} dB_t^{2,Q}.
\end{aligned} \tag{7.4}
$$

Note that we have to impose $\xi - \theta\delta_1 > 0$. Hence

$$
\begin{aligned}
\xi^Q &= \xi - \theta \delta_1 \\
\eta^Q &= \frac{\xi \eta + \theta \delta_2}{\xi - \theta \delta_1} \\
\theta^Q &\equiv \theta.
\end{aligned} \tag{7.5}
$$

Now we shall work under the risk-neutral measure and we shall omit the index Q. Note that the value of γ has to be

$$
\gamma = r - \mu. \tag{7.6}
$$

Hence,

$$
\begin{aligned}
dS_t &= r S_t dt + \sqrt{V_t} S_t dB_t^1 \\
dV_t &= \xi (\eta - V_t) dt + \theta \sqrt{V_t} dB_t^2 \\
dB_t^1 dB_t^2 &= \rho dt.
\end{aligned} \tag{7.7}
$$

We have to prove the following proposition before to continue with option pricing.

Proposition 7.1. *Let T be fixed and $g(\cdot)$ be some function. Let us define the function $f(\cdot,\cdot)$ as*

$$
f(t,x) = E^{t,x}\left[g(X_T) \right]. \tag{7.8}
$$

Then $f(t, X_t)$ is a martingale.

Proof: We have

$$f(t, X_t) = E^{t,X_t}[g(X_T)] = E[g(X_T)|\, \mathcal{F}_t]. \tag{7.9}$$

Let $s < t$. Then

$$\begin{aligned} E[f(t, X_t)|\, \mathcal{F}_s] &= E[E[g(X_T)|\, \mathcal{F}_t]|\, \mathcal{F}_s] \\ &= E[g(X_T)|\, \mathcal{F}_s] \\ &= f(t, X_s). \end{aligned} \tag{7.10}$$

Therefore $f(t, X_t)$ is a martingale. □

We continue with the option pricing. If $X_t = \log(S_t)$, then the Itô differential rule leads to

$$dX_t = \left(r - \frac{V_t}{2}\right) dt + \sqrt{V_t} dB_t^1. \tag{7.11}$$

We want to derive the Fourier transform

$$f(t, x, v; u) = E\left[e^{iuX_T}\middle|\, X_t = x, V_t = v\right]. \tag{7.12}$$

Proposition 7.1 gives us that $f(t, X_t, V_t)$ is a martingale. Thus we can use proposition 3.5 to see that the function $f(t, x, v; u)$ satisfies the PDE

$$\begin{aligned} &f_t + \mathcal{A}f = 0 \\ &f(T, x, v; u) = e^{iux}, \end{aligned} \tag{7.13}$$

which in the current case is

$$\begin{aligned} &f_t + \left(r - \frac{v}{2}\right) f_x + \frac{1}{2} v f_{xx} + \xi(\eta - v) f_v + \frac{\theta^2 v}{2} f_{vv} + \rho\theta v f_{xv} = 0 \\ &f(T, x, v; u) = e^{iux} \end{aligned} \tag{7.14}$$

Let $\tau = T - t$. We search the solution in the form

$$f(\tau, x, v; u) = e^{iux + C(\tau) + vD(\tau)}. \tag{7.15}$$

In such a way we derive the following system of the ODEs for the functions $C(\cdot)$ and $D(\cdot)$

$$\begin{aligned} D'(s) &= \frac{1}{2}\theta^2 D^2(s) + (i\rho\theta u - \xi) D(s) - \frac{u^2 + iu}{2} \\ C'(s) &= \xi\eta D(s) + iur \\ D(0) &= C(0) = 0, \end{aligned} \tag{7.16}$$

which solution is given by

$$\begin{aligned} D(s) &= \frac{u^2 + iu}{\gamma \coth\left(\frac{\gamma s}{2}\right) + \xi - i\rho\theta u} \\ C(s) &= iusr + \xi\eta s(\xi - i\rho\theta u) \\ &\quad - \frac{2\xi\eta}{\theta^2} \ln\left(\cosh\left(\frac{\gamma s}{2}\right) + \xi - \frac{i\rho\theta u}{\gamma} \sinh\left(\frac{\gamma s}{2}\right)\right), \end{aligned} \tag{7.17}$$

where γ is given by the equation

$$\gamma(u) = \sqrt{\theta^2(u^2 + iu) + (\xi - i\rho\theta u)^2}. \tag{7.18}$$

After we derived the Fourier transform, we can proceed in the way we present for the exponential Lévy models.

7.2. Bates model

Bates refines the Heston's model in the same way as Merton improves the Black-Scholes model by adding a jump component:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dB_t^1 + S_t dZ_t \\ dV_t &= \xi(\eta - V_t) dt + \theta\sqrt{V_t} dB_t^2 \\ dB_t^1 dB_t^2 &= \rho dt. \end{aligned} \tag{7.19}$$

Z_t is an independent of the Brownian motions compound Poisson process with intensity λ and log-normal distributed jumps

$$\log(Y) \sim N\left(\log(1+k) - \frac{\delta^2}{2}, \delta^2\right). \tag{7.20}$$

We derive in the same way as in the Merton's model that the risk-neutral condition is

$$\mu^Q = r - \lambda k. \tag{7.21}$$

We shall derive again the Fourier transform (characteristic function) of the log-price. We use that the Brownian motion and the compound Poisson process are independent. We shall denote by X_t^c the continuous part of the log-price. We have

$$E\left[e^{iuX_t}\right] = E\left[e^{iu(X_t^c+Z_t)}\right] = E\left[e^{iuX_t^c}e^{iuZ_t}\right] = E\left[e^{iuX_t^c}\right]E\left[e^{iuZ_t}\right]. \tag{7.22}$$

The first expectation is derived as in the Heston's case changing r by $r-\lambda k$. Since the jump process Z_t is compound Poisson, the second expectation is given by

$$E_2 = \exp\left(t\lambda\left(e^{-\frac{\delta^2u^2}{2}+i\left(\log(1+k)-\frac{\delta^2}{2}\right)u} - 1\right)\right). \tag{7.23}$$

Finally, we derived

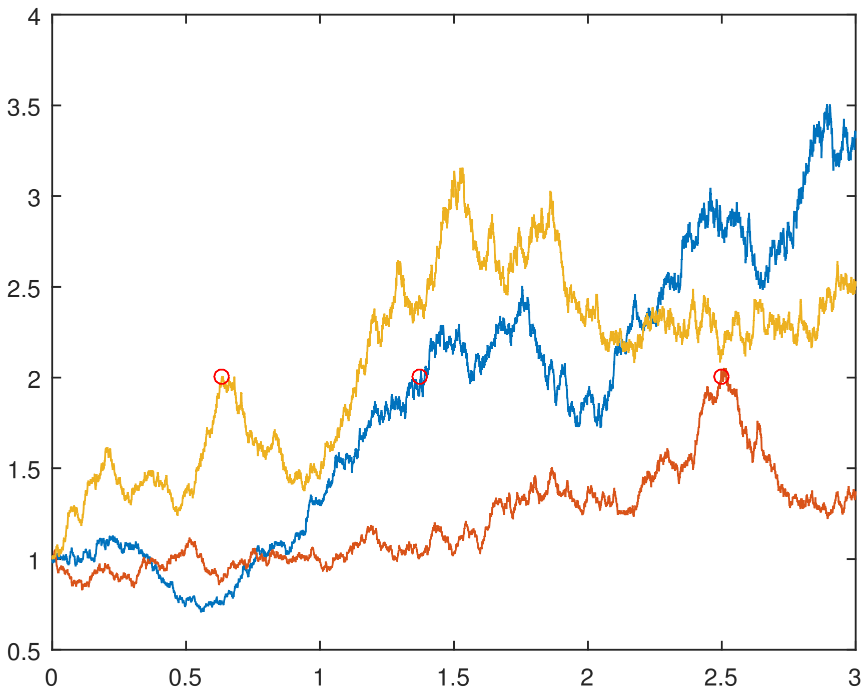
$$E\left[e^{iuX_t}\right] = \exp\left\{\begin{array}{l} iux + iu\tau\left(r-\lambda k\right) + \xi\eta s\left(\xi - i\rho\theta u\right) \\ -\frac{2\xi\eta}{\theta^2}\ln\left(\cosh\left(\frac{\gamma\tau}{2}\right) + \xi - \frac{i\rho\theta u}{\gamma}\sinh\left(\frac{\gamma\tau}{2}\right)\right) \\ +v\frac{u^2+iu}{\gamma\coth\left(\frac{\gamma\tau}{2}\right)+\xi-i\rho\theta u} \\ \tau\lambda\left(e^{-\frac{\delta^2u^2}{2}+i\left(\log(1+k)-\frac{\delta^2}{2}\right)u} - 1\right) \end{array}\right\}. \tag{7.24}$$

The option price is derived as in the Heston or exponential Lévy models. Zaevski generalize the Bates model changing the compound Poisson jump process by an arbitrary Lévy process.

8. American style derivatives

The American derivatives differ to their European equivalents by the existing early exercise right of the derivative holder. While the buyer of a European derivative can exercise the contract only at the maturity, the holder of an American derivative can exercise at every moment before the maturity. This leads to an optimal stopping problem – the derivative holder has to choose the optimal moment in which to exercise.

Figure 12: First hitting time of a log-normal process to the value 2



8.1. Stopping times

The stopping times are in some sense stochastic moments.

Definition 8.1. *A random variable τ is a stopping time w.r.t. the filtration $\mathcal{F}_t$ if the event $\tau < t$ is $\mathcal{F}_t$-measurable for every t.*

The meaning of the definition 8.1 is that at every moment t we know whether the stooping time is happened before t or not. A typical example of a stopping time is first hitting of some stochastic process to some boundary. This stopping time for a log-normal process can be seen at figure 12. There exist many other more complicated stopping times – for example the first moment at which the process is above some level for previously defined time.

8.2. Exercise region

Let $\mathcal{T}_{[t,T]}$ be the set of all stopping times τ with values between t and T, $t \leq \tau \leq T$. The set $\{(t, x) : 0 \leq t \leq T, x \geq 0\}$ can be divided into two parts

– continuation and exercise regions. In the first one the derivative holder prefers to keep the derivative, whereas in the second one the immediate exercise is optimal for him. Let the exercise region be denoted by Υ, the continuation region by $\overline{\Upsilon}$, and the exercise boundary by $c(t)$. Let $N(t,x)$ be the function which determines the payment structure, i.e the derivative seller has to pay amount $N(t,x)$ if the buyer exercises at the moment t provided that $S_t = x$. We define below the continuation and exercise regions.

Definition 8.2. *1. The point $(t,x) \in \Upsilon$ if for all stopping times $\zeta \in \mathcal{T}_{[t,T]}$,*

$$N(t,x) \geq E^{t,x}\left[e^{-r(\zeta-t)} N(\zeta, S_\zeta)\right]. \tag{8.1}$$

2. The point $(t,x) \in \overline{\Upsilon}$ if there exists a stopping time $\zeta \in \mathcal{T}_{[t,T]}$, such that

$$N(t,x) < E^{t,x}\left[e^{-r(\zeta-t)} N(\zeta, S_\zeta)\right]. \tag{8.2}$$

At figure 13 we show how the regions look for a put option. The parameters are $r = 0.08$, $\sigma = 0.2$, $T = 3$, and $K = 100$.

8.3. Pricing as a free boundary problem

Let $f(t,x)$ be the derivative pricing function, i.e. the derivative price is $f(t,S_t)$. Assume first that $(t,x) \in \Upsilon$ and therefore $f(t,x) = N(t,x)$. Using the first point of definition 8.2 we conclude

$$N(t,x) \geq E^{t,x}\left[e^{-r(u-t)} N(u, S_u)\right] \tag{8.3}$$

or equivalently

$$N(t,x) \geq E^{t,x}\left[e^{-r\epsilon} N(t+\epsilon, S_{t+\epsilon})\right] \tag{8.4}$$

for $u = t+\epsilon$. Moving $N(t,x)$ to the right, dividing by ϵ, and taking the limit $\epsilon \to 0$, we derive

$$0 \geq -rN(t,x) + N_t(t,x) + \mathcal{A}N(t,x). \tag{8.5}$$

Since $f(t,x) = N(t,x)$ in the whole region Υ, the derivatives of the functions $f(t,x)$ and $N(t,x)$ are equal too. Therefore

$$0 \geq -rf(t,x) + f_t(t,x) + \mathcal{A}f(t,x). \tag{8.6}$$

Inequality (8.6) is known as a variational inequality. Suppose now that $(t,x) \in \overline{\Upsilon}$ and let τ be the optimal strategy (stopping time) for the derivative holder. We have

$$f(t,x) = E^{t,x}\left[e^{-r(\tau-t)}N(\tau, S_\tau)\right] \tag{8.7}$$

Let ϵ be small enough. Then

$$f(t,x) = E^{t,x}\left[e^{-r\epsilon}f(t+\epsilon, S_{t+\epsilon})\right] \tag{8.8}$$

and therefore

$$0 = -rf(t,x) + f_t(t,x) + \mathcal{A}f(t,x). \tag{8.9}$$

Hence, we conclude that the price function $f(t,x)$ solves the following *free boundary probelm*, also known as an obstacle problem

$$\begin{aligned} &\max\{-rf + f_t + \mathcal{A}f, N - f\} = 0 \\ &f(T,x) = N(T,x). \end{aligned} \tag{8.10}$$

In such style differential equations we have to derive the solution of the equation as well as the region in which it holds – look again at figure 13.

8.4. American style options. Closed form formula for a perpetual American put.

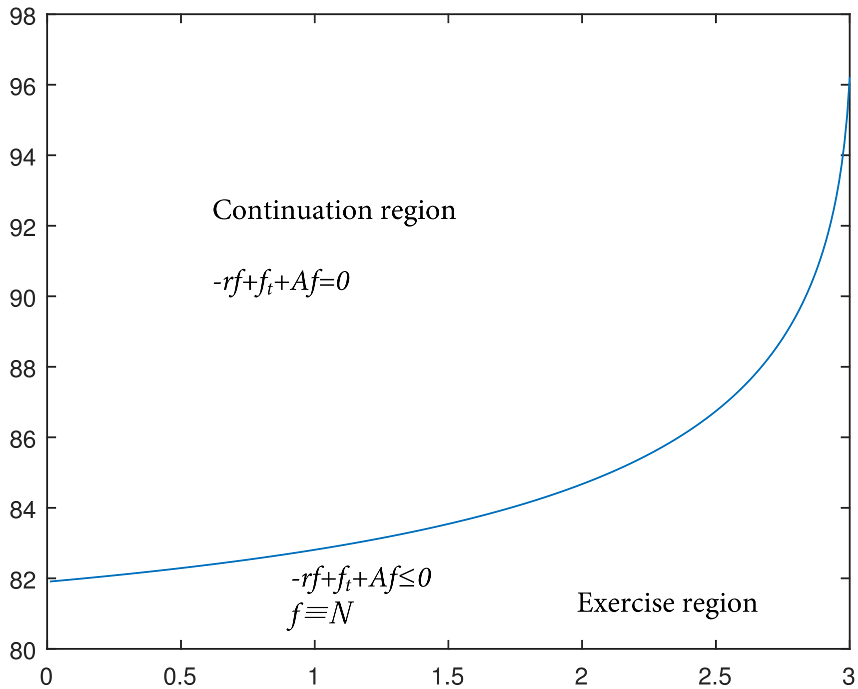
The following proposition states that early exercising is never optimal for American call options. This means that the price of an American call is equal to the price of the corresponding European call.

Proposition 8.1. *We have that $\Upsilon = \emptyset$, i.e. early exercising is never optimal.*

Proof: Suppose that $(t,x) \in \Upsilon$ and the stopping time $\zeta \in \mathcal{T}_{[t,T]}$ is arbitrary. Obviously $x = S_t > K$ and therefore, since $e^{-rt}S_t$ is a martingale, we have

$$\begin{aligned} E^{t,x}\left[e^{-r\zeta}N(\zeta, S_\zeta)\right] &= E^{t,x}\left[e^{-r\zeta}(S_\zeta - K)^+\right] \\ &\le e^{-rt}(x-K) \\ &= E^{t,x}\left[e^{-r\zeta}S_\zeta\right] - Ke^{-rt} \\ &< E^{t,x}\left[e^{-r\zeta}(S_\zeta - K)\right] \\ &\le E^{t,x}\left[e^{-r\zeta}(S_\zeta - K)^+\right]. \end{aligned} \tag{8.11}$$

Figure 13: Early exercise boundary of an American put



The contradiction finishes the proof. □

Note that if the underlying asset pays dividends, the proposition above is not true. Also, if the option is put, the early exercise can be optimal.

Consider now a perpetual put option, i.e. $T = \infty$. Since the underling asset is driven by a Markov process and the option holder is not time limited, the early exercise boundary is flat.

We shall use the following lemma related to the first hitting properties of a Brownian motion.

Lemma 8.1. *Let τ be the first hitting time of a Brownian motion with drift μ to the positive level a. Then the Laplace transform of its distribution is*

$$E\left[e^{-y\tau} I_{\tau<\infty}\right] = e^{-\left(\sqrt{\mu^2+2y}-\mu\right)a}. \tag{8.12}$$

If the value a is negative, then the Laplace transform is

$$E\left[e^{-y\tau}I_{\tau<\infty}\right]=e^{\left(\sqrt{\mu^2+2y}+\mu\right)a}. \tag{8.13}$$

We are ready to prove the theorem for the pricing problem of the perpetual American put options.

Theorem 8.1. *If the initial asset price is above the early exercise boundary, $S_0=x>c$, then the price of a discounted American perpetual put is*

$$B\left(x\right)=\left(\frac{K}{\gamma+1}\right)^{\gamma+1}\left(\frac{\gamma}{x}\right)^{\gamma}, \tag{8.14}$$

where

$$\gamma=2\frac{r}{\sigma^2}. \tag{8.15}$$

If the initial asset value is below the exercise boundary, then the option price is

$$B\left(x\right)=K-x. \tag{8.16}$$

The exercise boundary is

$$c=\frac{\gamma}{\gamma+1}K \tag{8.17}$$

and the optimal stopping time is the moment of first hitting to the interval $[0,c]$.

Proof: Suppose that the exercise boundary is the constant c and the asset starts above it $x>c$. Since the asset price can be written as

$$S_t=e^{\ln x+\left(r-\frac{\sigma^2}{2}\right)t+\sigma B_t}, \tag{8.18}$$

the stopping time can be viewed as the first hitting time of the Brownian motion with drift

$$\mu=\frac{r}{\sigma}-\frac{\sigma}{2} \tag{8.19}$$

to the value

$$a = \frac{\ln c - \ln x}{\sigma} < 0. \tag{8.20}$$

Note that it is negative. So we have to use equation (8.13) to derive

$$\begin{aligned}
&B(x;c) = E^x\left[e^{-r\tau}(K - S_\tau) I_{\tau<\infty}\right] \\
&= (K-c) E^x\left[e^{-r\tau} I_{\tau<\infty}\right] \\
&= (K-c)\exp\left\{\left(\sqrt{\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 + 2r} + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\right)\frac{\ln c - \ln x}{\sigma}\right\} \\
&= (K-c)\exp\left\{\left(\sqrt{\left(\frac{r}{\sigma} + \frac{\sigma}{2}\right)^2} + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\right)\frac{\ln c - \ln x}{\sigma}\right\} \\
&= (K-c)\exp\left\{\left(\left(\frac{r}{\sigma} + \frac{\sigma}{2}\right) + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\right)\frac{\ln c - \ln x}{\sigma}\right\} \\
&= (K-c)\exp\left\{\gamma(\ln c - \ln x)\right\} \\
&= (K-c)\left(\frac{c}{x}\right)^{\gamma}.
\end{aligned} \tag{8.21}$$

We have to derive which value of c maximizes function (8.21). Its c-derivative is

$$\begin{aligned}
B_c(x;c) &= -\left(\frac{c}{x}\right)^{\gamma} + (K-c)\gamma\frac{c^{\gamma-1}}{x^{\gamma}} \\
&= -\frac{c^{\gamma-1}}{x^{\gamma}}\left[c(\gamma+1) - K\gamma\right].
\end{aligned} \tag{8.22}$$

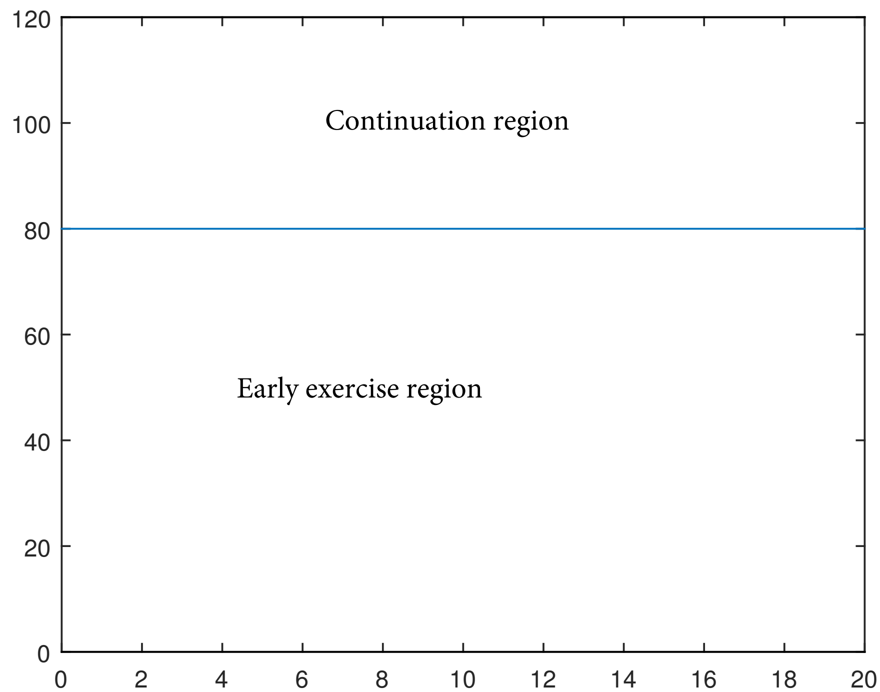
Derivative (8.22) has a root for c as in equation (8.17), and this root leads to the maximum of the price function (8.21). Therefore equation (8.21) turns to equation (8.14). $\square$

Using the parameters used for figure 13 ($r = 0.08$, $\sigma = 0.2$, and $K = 100$), we derive that $\gamma = 4$ and hence the optimal boundary is $c = 80$. The regions can be seen at figure 15.

9. Data analyzing

Let we have a time series $S_1, S_2, ..., S_n$. The log-returns are defined as

Figure 15: Early exercise boundary of a perpetual American put



$$r_i = \log\left(\frac{S_{i+1}}{S_i}\right) = \log\left(S_{i+1}\right) - \log\left(S_i\right). \tag{9.1}$$

If we use the Black-Scholes model, then

$$S_i = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t_i + \sigma B_{t_i}\right). \tag{9.2}$$

Therefore the log-returns turn to

Figure 17: Daily values of the S&P index between 1928 – 1947

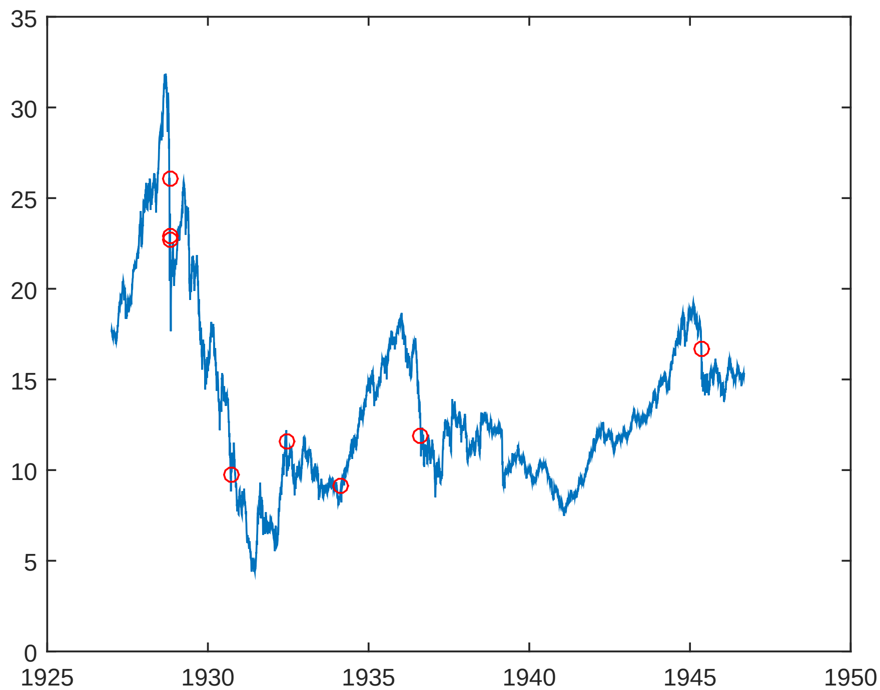


Figure 18: Daily values of the S&P index between 1983 – 1991

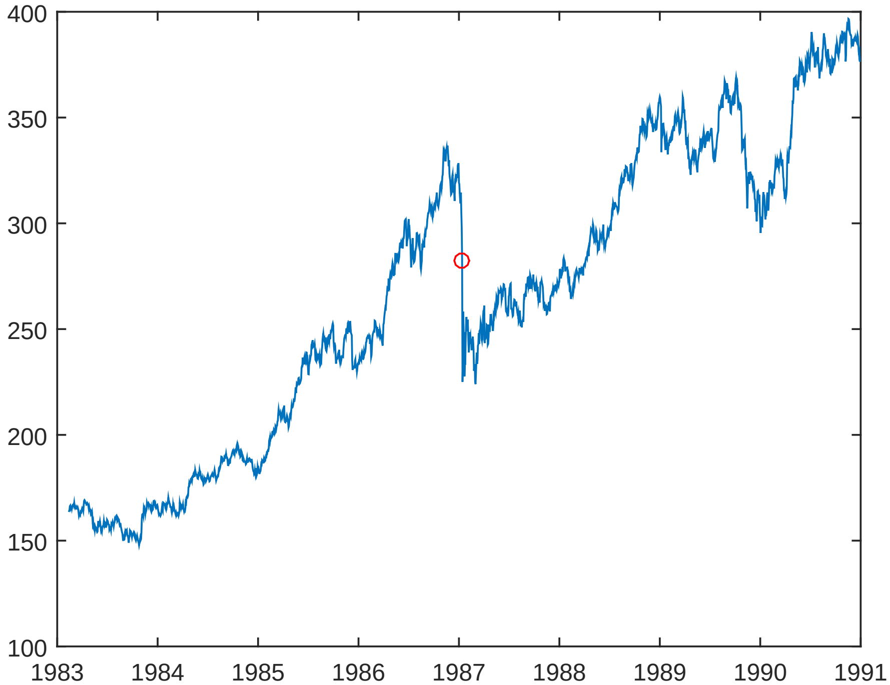


Figure 19: Daily values of the S&P index between 2006 – 2020

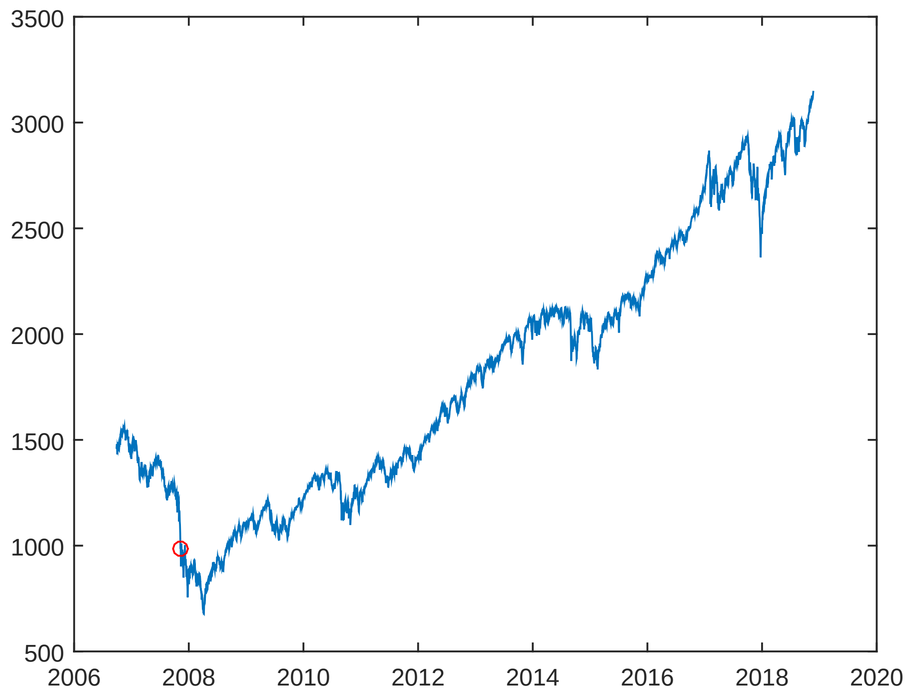


Figure 20: Daily log-returns of the S&P index between 1928 – 2020

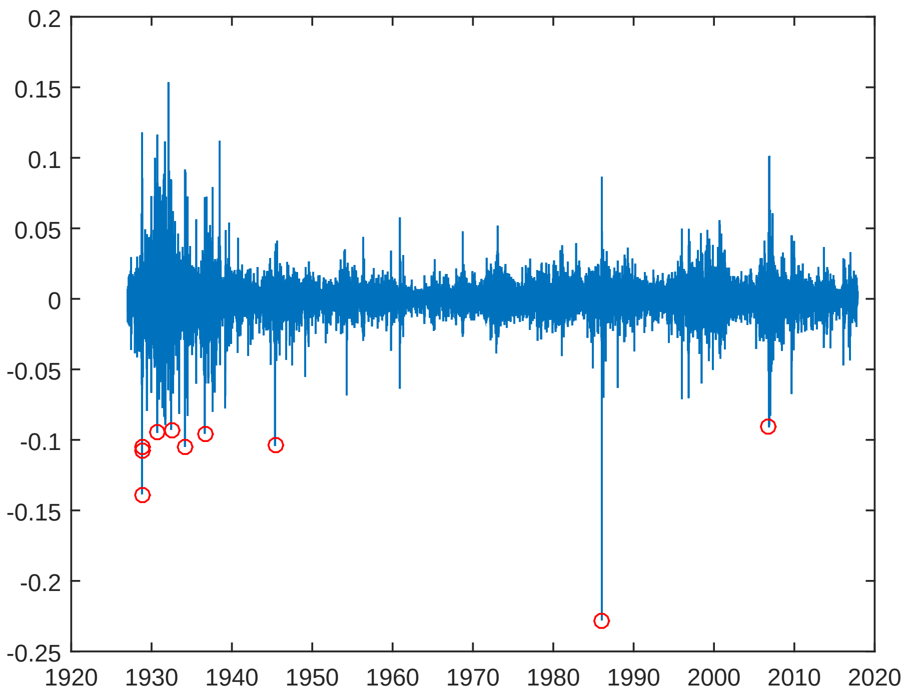
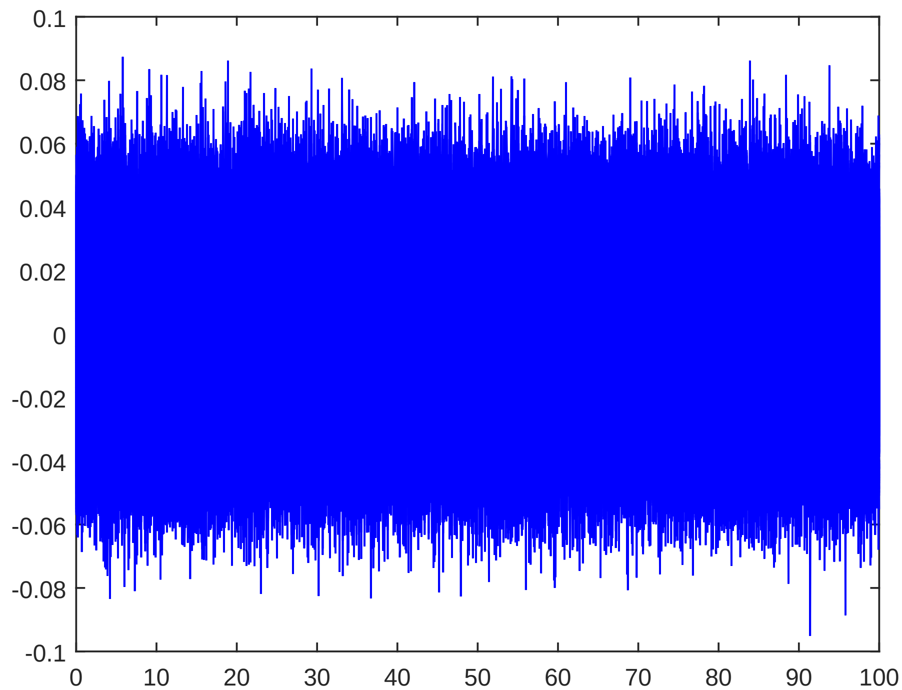


Figure 21: Log-returns generated by the Black-Scholes model



$$
\begin{aligned}
r_i &= \log\left(S_{i+1}\right) - \log\left(S_i\right) \\
&= \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\left(B_{t_n} - B_{t_n}\right) \\
&= \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \xi,
\end{aligned}
\tag{9.3}
$$

where ξ is normally distributed random variable. Hence

$$
r_i \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, \sigma\Delta t\right). \tag{9.4}
$$

Comparing figures 20 and 21 we can observe that

1. The large jumps (upward and downward) happen more often than the normal distribution implies. This means that the empirical distribution has significant heavier tails than Gaussian.

2. The downward jumps are more and larger – the left tail is heavier than the right.

3. We can think that there are sudden jumps in the asset prices, although the observations are discrete.

4. The volatility is grouped in clusters – volatility clustering. There are periods with a large volatility and others with small.

5. Leverage effect – the trend is downward in the periods of a large volatility and vice versa.

6. Long range dependance – there is dependence in the log-returns.

The largest one day fall happens on October 19, 1987 – the black Monday. The fall is larger than 20%. The most of important series of falls is between October 25, 1929 and October 29, 1929 – begins the great depression.

9.1. Greeks

Definition 9.1. *The Greeks of the options are defined as the following derivatives*

1. *The delta, Δ, of the option is defined as the option derivative w.r.t. the initial asset value*

$$\Delta = \frac{dC}{dS_0}. \tag{9.5}$$

2. *The theta, Θ, of the option is the option derivative w.r.t. the time (or time to maturity)*

$$\Theta = \frac{dC}{dt}. \tag{9.6}$$

3. *The gamma, Γ, of the option is the derivative of the delta w.r.t. the initial asset value. Equivalently, gamma is second derivative of the option price w.r.t. the initial asset value*

$$\Gamma = \frac{d^2C}{d(S_0)^2} = \frac{d\Delta}{dS_0}. \tag{9.7}$$

4. *The vega, ν, of the option is the option derivative w.r.t. the volatility*

$$\nu = \frac{dC}{d\sigma}. \tag{9.8}$$

5. *The rho, ρ, of the option is the option derivative w.r.t. the risk free rate*

$$\rho = \frac{dC}{dr}. \tag{9.9}$$

We shall derive now the value of vega.

Proposition 9.1. *The vega of the European call via the Black-Scholes model is*

$$\nu = C'(\sigma) = S\sqrt{t}N'(d_1). \tag{9.10}$$

Remind that

$$d_1 = \frac{\ln \frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)(t)}{\sigma\sqrt{t}}. \tag{9.11}$$

Proof: Note that $N(x)$ means

$$N(x) = \frac{1}{2\pi} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du. \tag{9.12}$$

Using Black-Scholes formula for a European call (2.64),

$$C(\sigma) = SN(d_1) - Ke^{-r(T-t)} N(d_2), \tag{9.13}$$

we derive the vega as

$$\begin{aligned}
\nu \equiv C'(\sigma) &= \left(SN(d_1) - Ke^{-r(T-t)} N(d_2)\right)' \\
&= \frac{S}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{\sigma^2 t^{1.5} - \left(\log \frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right) t\right) \sqrt{t}}{\sigma^2 t} \\
&- \frac{Ke^{-rt}}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \frac{-\sigma^2 t^{1.5} - \left(\log \frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right) t\right) \sqrt{t}}{\sigma^2 t} \\
&= \frac{S}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{\sigma^2 t^{1.5} - \left(\log \frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right) t\right) \sqrt{t}}{\sigma^2 t} \\
&+ \frac{Ke^{-rt}}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \frac{\left(\log \frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right) t\right) \sqrt{t}}{\sigma^2 t}.
\end{aligned} \tag{9.14}$$

We use that

$$d_2 = d_1 - \sigma\sqrt{t} \tag{9.15}$$

to derive

$$
\begin{aligned}
e^{-\frac{d_2^2}{2}} &= e^{-\frac{d_1^2}{2}} e^{d_1 \sigma \sqrt{t}} e^{-\frac{\sigma^2 t}{2}} \\
&= e^{-\frac{d_1^2}{2}} e^{\log \frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right) t} e^{-\frac{\sigma^2 t}{2}} \\
&= \frac{S}{K} e^{rt} e^{-\frac{d_1^2}{2}}.
\end{aligned} \tag{9.16}
$$

Thus we conclude

$$
\begin{aligned}
\nu &= \frac{S}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{\sigma^2 t^{1.5} - \left(\log \frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right) t\right) \sqrt{t}}{\sigma^2 t} \\
&+ \frac{S}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{\left(\log \frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right) t\right) \sqrt{t}}{\sigma^2 t} \\
&= \frac{S}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \sqrt{t} \\
&= S \sqrt{t} N'\left(d_1\right).
\end{aligned} \tag{9.17}
$$

□

9.2. Implied volatility

Proposition 9.2. *The call option price is an increasing function of the volatility and*

$$
\begin{aligned}
C(0) &= \left(S - K e^{-rT}\right)^{+} \\
C(\infty) &= S.
\end{aligned} \tag{9.18}
$$

Proof: The fact that the call price increases w.r.t. the volatility is a corollary from proposition 9.1, since the derived value of vega is positive.

Let us see what happens when $\sigma \to 0$. We have to examine two cases

1. Suppose first that

$$
\log \frac{S}{K} + rT > 0 \tag{9.19}
$$

or equivalently

$$S > Ke^{-rT}. \tag{9.20}$$

We have that $d_1,\ d_2 \to +\infty$ and therefore $N(d_{1,2}) \to 1$. Hence

$$C(\sigma) \to S - Ke^{-rT} > 0. \tag{9.21}$$

2. Suppose now that

$$\log \frac{S}{K} + rT < 0 \tag{9.22}$$

or equivalently

$$S < Ke^{-rT}. \tag{9.23}$$

Thus $d_1,\ d_2 \to -\infty$ and therefore $N(d_{1,2}) \to 0$. Hence

$$C(\sigma) \to 0. \tag{9.24}$$

Combining the both statements above we conclude

$$\lim_{\sigma \to 0} C(\sigma) \to \left(S - Ke^{-rT}\right)^{+}. \tag{9.25}$$

Let us examine $\sigma \to \infty$. We have $d_1 \to +\infty$ and $d_2 \to -\infty$ or equivalently $N(d_1) \to 1$ and $N(d_2) \to 0$. Therefore

$$\lim_{\sigma \to +\infty} C(\sigma) \to S. \tag{9.26}$$

$\square$

Let us give some reasons why the interval $\left[\left(S - Ke^{-rT}\right)^{+}, S\right]$ is the largest possible for the option price.

1. The lower boundary

   (a) $C \geq 0$, since the option gives only rights.

Table 1: Models properties

	BS	Merton	Exp. Lévy	Heston	Bates/Zaevski
heavy tails	no	no	yes	no	yes
asymmetry	no	yes	yes	no	yes
jumps	no	yes	yes	no	yes
volatility clustering	no	no	no	yes	yes
leverage effect	no	no	no	yes	yes
long-range depend.	no	no	no	yes	yes
impl. vol. smiles/skews	no	yes	yes	yes	yes

(b) Suppose that $S > Ke^{-rT}$. Let us compose a portfolio L_t which consists of one share of S_t and Ke^{-rT} short positions of the risk free asset. Its value in the moment T is

$$L_T = S_T - e^{rT}e^{-rT}K = S_T - K. \tag{9.27}$$

Therefore $C_t > L_t$, since $(S_T - K)^+ \geq S_T - K$. Particularly, $C_0 > L_0$.

2. Upper boundary: since $(S_T - K)^+ < S_T$, we have $C_0 < S$.

Definition 9.2 (Implied volatility). *Let the observed option price be $\overline{C}$. The value $\overline{\sigma}$ which gives the same option price via the Black-Scholes model,*

$$C_{BS}(\overline{\sigma}) = \overline{C} \tag{9.28}$$

is called implied volatility.

Definition 9.3 (Implied volatility surface). *Let us examine the strike K as a parameter and $\sigma(K)$ be the function generated by the implied volatiles. It is known as implied volatility surface. Typically it is a smile or a skew.*

If we have some model, its implied volatility surface is obtained by calculating the corresponding option prices after which we derive the implied volatiles.

Note that the surface of the Black-Scholes model is a line.

Finally, at the following table we present the properties of different models.

10. Term structure of interest rates

Now we assume that the risk free rate is a stochastic process. We shall denote it by r_t. Let the price of a T-bond at a moment t be $\Lambda(t,T)$. It pays to the holder amount of \$1 at moment T. Note that it is a stochastic process w.r.t. the variable t, a differential function w.r.t. the variable T, and $\Lambda(T,T)=1$. Since $\Lambda(t,T)$ is an asset

$$\exp\left(-\int_0^t r_s ds\right)\Lambda(t,T), \tag{10.1}$$

is a martingale for every T. Therefore

$$\begin{aligned}\exp\left(-\int_0^t r_s ds\right)\Lambda(t,T) &= E^Q\left[\exp\left(-\int_0^T r_s ds\right)\Lambda(T,T)\middle|\mathcal{F}_t\right]\\ &= E^Q\left[\exp\left(-\int_0^T r_s ds\right)\middle|\mathcal{F}_t\right],\end{aligned} \tag{10.2}$$

which leads to

$$\Lambda(t,T) = E^Q\left[\exp\left(-\int_t^T r_s ds\right)\middle|\mathcal{F}_t\right]. \tag{10.3}$$

Definition 10.1. *The return, which we receive between moments t and T is called yield curve and it is obtained as*

$$Y(t,T) = -\frac{\ln\Lambda(t,T)}{T-t}. \tag{10.4}$$

If the risk-free rate is a constant, then

$$\Lambda(t,T) = e^{-r(T-t)}, \tag{10.5}$$

which leads to a flat yield curve

$$Y(t,T) = r. \tag{10.6}$$

Definition 10.2. *The forward rate of return is defined as the average rate of return between some future moments T_1 and T_2, $T_1 < T_2$. It is denoted by $f(t, T_1, T_2)$.*

Let us consider the following strategy

1. We sell shortly a unit of T_1-bond at moment t. We receive amount of $\$\Lambda(t, T_1)$.

2. We invest this amount in T_2-bonds. Therefore we bye

$$\frac{\Lambda(t, T_1)}{\Lambda(t, T_2)} \tag{10.7}$$

   shares.

3. In moment T_1, the T_1-bond expires and therefore we have to pay amount of \$1.

4. In moment T_2, the T_2-bond expires and therefore we receive amount of

$$\$\frac{\Lambda(t, T_1)}{\Lambda(t, T_2)}1. \tag{10.8}$$

The forward rate is this value which leads to this result, i.e

$$1e^{f(t,T_1,T_2)(T_2-T_1)} = \frac{\Lambda(t, T_1)}{\Lambda(t, T_2)}1. \tag{10.9}$$

Thus we derive the following formula for the forward rate.

Proposition 10.1. *We have*

$$f(t, T_1, T_2) = -\frac{\ln \Lambda(t, T_2) - \ln \Lambda(t, T_1)}{T_2 - T_1}. \tag{10.10}$$

Definition 10.3. *The instantaneous forward rate of return is defined as the forward rate for an infinitesimal future time interval and is denoted by $F(t, T)$*

$$F(t, T) = \lim_{\varepsilon \to 0} f(t, T, T + \varepsilon). \tag{10.11}$$

Proposition 10.2. *We have for the instantaneous forward rate*

$$F(t,T) = -\frac{1}{\Lambda(t,T)}\Lambda_T(t,T). \tag{10.12}$$

Proof: We have

$$\begin{aligned} F(t,T) &= \lim_{\varepsilon\to 0} f(t,T,T+\varepsilon) \\ &= -\lim_{\varepsilon\to 0}\frac{\ln\Lambda(t,T+\varepsilon)-\ln\Lambda(t,T)}{\varepsilon} \\ &= -\frac{\partial(\ln\Lambda(t,T))}{\partial T} \\ &= -\frac{1}{\Lambda(t,T)}\Lambda_T(t,T) \end{aligned} \tag{10.13}$$

□

We have the following pricing formula for the T-bond

Proposition 10.3.

$$\Lambda(t,T) = \exp\left(-\int_t^T F(t,u)\,du\right). \tag{10.14}$$

Proof: Integrating equation (10.12) w.r.t the second variable we derive

$$\begin{aligned} \int_t^T F(t,u)\,du &= -\int_t^T \frac{1}{\Lambda(t,u)}\Lambda_T(t,u)\,du \\ &= -\int_t^T \frac{1}{\Lambda(t,u)}d\Lambda(t,u) \\ &= -(\ln\Lambda(t,T)-\ln\Lambda(t,t)) \\ &= -\ln\Lambda(t,T). \end{aligned} \tag{10.15}$$

Therefore

$$\Lambda(t,T) = \exp\left(-\int_t^T F(t,u)\,du\right). \tag{10.16}$$

$\square$

The following proposition gives the relation between the yield curve and the instantaneous forward rate.

Proposition 10.4.

$$F(t,T) = Y(t,T) + (T-t)\,Y_T(t,T). \tag{10.17}$$

Proof: We have

$$Y(t,T) = -\frac{\ln\Lambda(t,T)}{T-t} = \frac{1}{T-t}\int_t^T F(t,u)\,du \tag{10.18}$$

or equivalently

$$\int_t^T F(t,u)\,du = (T-t)\,Y(t,T). \tag{10.19}$$

After differentiation we derive formula (10.17). $\square$

The next proposition gives the relation between the short rate and the yield curve.

Proposition 10.5.

$$r_t = \lim_{T\to t} Y(t,T) = Y(t,t) = F(t,t). \tag{10.20}$$

Proof: Obviously $Y(t,t) = F(t,t)$. We have

$$
\begin{aligned}
\Lambda_T(t,t) &= \lim_{\varepsilon\to 0}\frac{E\left[\exp\left(-\int\limits_t^{t+\varepsilon} r_u du\right)-1\right]}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}\left[\exp\left(-\int\limits_t^{t+\varepsilon} r_u du\right)(-r_t)\right] \\
&= -r_t.
\end{aligned} \tag{10.21}
$$

Using proposition 10.2 and $\Lambda(t,t)=1$ we finish the proof. □

10.1. Affine models

Definition 10.4. *In the affine models we define the risk free rate r_t and assume that the bond prices can be presented as*

$$
\Lambda(t,T) = e^{a(t,T)-b(t,T)r_t} \tag{10.22}
$$

for some functions $a(t,T)$ and $b(t,T)$.

Suppose that the short rate is defined by the SDE

$$
dr_t = \mu(t,r_t)\,dt + \sigma(t,r_t)\,dB_t. \tag{10.23}
$$

Assume that the bond prices can be presented as the function $\Lambda(t,r_t,T)$. To avoid the arbitrage opportunities the process

$$
\exp\left(-\int\limits_0^t r_u du\right)\Lambda(t,r_t,T) \tag{10.24}
$$

has to be a martingale for every T. Using the Itô formula we derive

$$
\begin{aligned}
&d\left[\exp\left(-\int_0^t r_u du\right)\Lambda\left(t,r_t,T\right)\right]= \\
&=\exp\left(-\int_0^t r_u du\right)\left(-r_t\Lambda+\Lambda_t+\mu\left(t,r_t\right)\Lambda_x+\frac{1}{2}\sigma^2\left(t,r_t\right)\Lambda_{xx}\right)dt \\
&+\exp\left(-\int_0^t r_u du\right)\sigma\left(t,r_t\right)\Lambda_x dB_t.
\end{aligned} \tag{10.25}
$$

Hence, the process is a martingale when the drift vanishes and therefore we reach to the terminal value problem

$$
\begin{aligned}
&-x\Lambda\left(t,x,T\right)+\Lambda_t\left(t,x,T\right)+\mu\left(t,x\right)\Lambda_x\left(t,x,T\right)+\frac{1}{2}\sigma^2\left(t,x\right)\Lambda_{xx}\left(t,x,T\right)=0 \\
&\Lambda\left(T,x,T\right)=1.
\end{aligned} \tag{10.26}
$$

Since the model is affine we can use equation (10.22) to derive

$$
\begin{aligned}
&a_t\left(t,T\right)-\left[b_t\left(t,T\right)+1\right]x-\mu\left(t,x\right)b\left(t,T\right)+\frac{1}{2}\sigma^2\left(t,x\right)b^2\left(t,T\right)=0 \\
&a\left(T,T\right)=b\left(T,T\right)=0
\end{aligned} \tag{10.27}
$$

10.2. Vašíček model

Vašíček assumes that the short rate follows an Ornstein-Uhlenbeck process under the risk neutral measure

$$
dr_t=\xi\left(\eta-r_t\right)dt+\theta dB_t, \tag{10.28}
$$

which solution is

$$
r_t=\eta+e^{-\xi t}\left(r_0-\eta\right)+\theta\int_0^t e^{-\xi(t-s)}dB_s. \tag{10.29}
$$

We can see that

1.
$$E\left[r_t\right] = \eta + e^{-\xi t}\left(r_0 - \eta\right). \tag{10.30}$$

2.
$$\lim_{t\to\infty} E\left[r_t\right] = \eta. \tag{10.31}$$

3. The OU-process admits negative values.

We shall solve equation (10.27) which now has the form

$$x\left(\xi b\left(t,T\right) - b_t\left(t,T\right) - 1\right) + a_t\left(t,T\right) - \xi\eta b\left(t,T\right) + \frac{\theta^2 b(t,T)^2}{2} = 0. \tag{10.32}$$

Hence, we reach to the following system of Riccati differential equations

$$\begin{aligned} &b_t\left(t,T\right) - \xi b\left(t,T\right) + 1 = 0 \\ &a_t\left(t,T\right) - \xi\eta b\left(t,T\right) + \frac{\theta^2 b(t,T)^2}{2} = 0 \\ &a\left(T,T\right) = b\left(T,T\right) = 0, \end{aligned} \tag{10.33}$$

which solution is given by

$$\begin{aligned} b\left(t,T\right) &= \frac{1 - e^{-\xi(T-t)}}{\xi} \\ a\left(t,T\right) &= \left(\eta - \frac{\theta^2}{2\xi^2}\right)\left(b\left(t,T\right) - T + t\right) + \frac{\theta^2}{4\xi} b^2\left(t,T\right). \end{aligned} \tag{10.34}$$

10.3. Cox-Ingersoll-Ross model

Now the risk free rate follows a CIR process

$$dr_t = \xi\left(\eta - r_t\right)dt + \theta\sqrt{r_t}dB_t. \tag{10.35}$$

Thus equation (10.27) turns to

$$a_t\left(t,T\right) - \left[b_t\left(t,T\right) + 1\right]x - \xi\left(\eta - x\right)b\left(t,T\right) + \frac{\theta^2 x b^2\left(t,T\right)}{2} = 0 \tag{10.36}$$

or

$$x\left[\xi b\left(t,T\right)+\frac{\theta^{2}b^{2}\left(t,T\right)}{2}-b_{t}\left(t,T\right)-1\right]+a_{t}\left(t,T\right)-\xi\eta b\left(t,T\right)=0. \quad (10.37)$$

Hence, we have the following system

$$\begin{aligned} a_{t}\left(t,T\right)&=\xi\eta b\left(t,T\right) \\ b_{t}\left(t,T\right)&=\xi b\left(t,T\right)+\frac{\theta^{2}b^{2}\left(t,T\right)}{2}-1. \end{aligned} \quad (10.38)$$

Its solution is

$$\begin{aligned} a\left(t,T\right)&=\frac{2\xi\eta}{\theta^{2}}\left[\begin{array}{l} \frac{(h+\xi)(T-t)}{2}+\ln\left(2h\right) \\ -\ln\left(2h+\left(h+\xi\right)\left(e^{h(T-t)}-1\right)\right) \end{array}\right] \\ b\left(t,T\right)&=\frac{2\left(e^{h(T-t)}-1\right)}{2h+\left(h+\xi\right)\left(e^{h(T-t)}-1\right)} \end{aligned} \quad (10.39)$$

for

$$h=\sqrt{\xi^{2}+2\theta^{2}}. \quad (10.40)$$

10.4. Heath-Jarrow-Morton model

In this model we use the instantaneous forward rate instead the short rate. We assume that under the risk-neutral measure the instantaneous forward satisfies

$$dF\left(t;T\right)=\mu_{F}\left(t;T\right)dt+\sigma_{F}\left(t;T\right)dB_{t}, \quad (10.41)$$

where for every T, $\mu\left(t;T\right)$ and $\sigma\left(t;T\right)$ are stochastic processes. Since $\Lambda\left(t;T\right)$ are market assets, they have to be martingales after discounting. Hence,

$$d\Lambda\left(t;T\right)=r_{t}\Lambda\left(t;T\right)dt+\sigma_{B}\left(t;T\right)\Lambda\left(t;T\right)dB_{t}. \quad (10.42)$$

Proposition 10.1 gives us for the forward rate

$$f\left(t,T_{1},T_{2}\right)=-\frac{\ln\Lambda\left(t,T_{2}\right)-\ln\Lambda\left(t,T_{1}\right)}{T_{2}-T_{1}}. \quad (10.43)$$

Using the Itô's rule for the dynamics of the log-bond prices we derive

$$\begin{aligned} d\left(\ln\Lambda\left(t;T\right)\right) &= \left[\frac{1}{\Lambda\left(t;T\right)}r_t\Lambda\left(t;T\right) - \frac{1}{2\Lambda^2\left(t;T\right)}\sigma_B^2\left(t;T\right)\Lambda\left(t;T\right)\right]dt \\ &+ \frac{1}{\Lambda\left(t;T\right)}\sigma_B(t;T)_t\Lambda\left(t;T\right)dB_t \\ &= \left[r_t - \frac{1}{2}\sigma_B^2\left(t;T\right)\right]dt + \sigma_B(t;T)_t dB_t \end{aligned} \tag{10.44}$$

or equivalently

$$\ln\Lambda\left(t;T\right) = \ln\Lambda\left(0;T\right) + \int\limits_0^t r_u - \frac{1}{2}\sigma_B^2\left(u;T\right)du + \int\limits_0^t \sigma_B\left(u;T\right)dB_u. \tag{10.45}$$

Thus we can derive for the instantaneous forward rate

$$\begin{aligned} F\left(t,T\right) &= \lim_{\varepsilon\to 0} F\left(t,T,T+\varepsilon\right) \\ &= -\lim_{\varepsilon\to 0}\frac{\ln\Lambda\left(t;T+\varepsilon\right) - \ln\Lambda\left(t;T\right)}{\varepsilon} \\ &= -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\begin{array}{l} \ln\Lambda\left(0;T+\varepsilon\right) - \ln\Lambda\left(0;T\right) \\ -\frac{1}{2}\int\limits_0^t \sigma_B^2\left(u;T+\varepsilon\right) - \sigma_B^2\left(u;T\right)du \\ +\int\limits_0^t \sigma_B\left(u;T+\varepsilon\right) - \sigma_B\left(u;T\right)dB_u \end{array}\right] \\ &= -\frac{\Lambda_T\left(0;T\right)}{\Lambda\left(0;T\right)} + \int\limits_0^t \sigma_B\left(u;T\right)\frac{\partial\sigma_B\left(u;T\right)}{\partial T}du - \int\limits_0^t \frac{\partial\sigma_B\left(u;T\right)}{\partial T}dB_u \\ &= F\left(0;T\right) + \int\limits_0^t \sigma_B\left(u;T\right)\frac{\partial\sigma_B\left(u;T\right)}{\partial T}du - \int\limits_0^t \frac{\partial\sigma_B\left(u;T\right)}{\partial T}dB_u. \end{aligned} \tag{10.46}$$

In that way we prove the following theorem

Theorem 10.1. *The model is consistent when*

$$\begin{aligned}
\sigma_F(t;T) &= -\frac{\partial \sigma_B(t;T)}{\partial T} \\
\mu_F(t;T) &= \sigma_B(t;T)\frac{\partial \sigma_B(t;T)}{\partial T} \equiv \sigma_F(t;T)\int_0^t \sigma_F(t;u)\,du.
\end{aligned} \tag{10.47}$$

We can conclude that the Heath-Jarrow-Morton model works in the following way

1. We have to know the initial instantaneous forwards $F(0;T)$ as well as the volatility structure $\sigma(t;T)$.

2. In such a way the drift turns to

$$\mu(t;T) = \sigma(t;T)\int_0^t \sigma(t;u)\,du. \tag{10.48}$$

3. The forward rate for a fixed maturity T can be obtained as the solution of the SDEs

$$dF(t;T) = \mu(t;T)\,dt + \sigma(t;T)\,dB_t. \tag{10.49}$$

4. Using proposition (10.3) we derive the bond prices as

$$\Lambda(t,T) = \exp\left(-\int_t^T F(t,u)\,du\right). \tag{10.50}$$

   We have for the integral

$$\int_t^T F(t,u)\,du = \int_t^T F(0,u)\,du + \int_0^t\int_t^T \mu(s,u)\,duds + \int_0^t\int_t^T \sigma(s,u)\,dudB_s. \tag{10.51}$$

11. The inverse problem – calibration of the models

Let we have a data $S_1, S_2, ..., S_n$ which leads to the log-returns $l_1, l_2, ..., l_{n-1}$

$$l_i = \ln S_i - \ln S_{i-1}. \tag{11.1}$$

Let γ be the set of model parameters for some model and Υ be the set of all admissible gamma's.

1. Black-Scholes model – $\gamma = \{\mu, \sigma\}$.
2. Merton model – $\gamma = \{\mu, \sigma, \lambda, \kappa, \delta\}$.
3. Heston model – $\gamma = \{\mu, \xi, \eta, \theta, \rho, V_0\}$.
4. Bates model – $\gamma = \{\mu, \lambda, \kappa, \delta, \xi, \eta, \theta, \rho, V_0\}$.
5. Exponential Lévy models – γ is closely related to the particular form of the Lévy process.

11.1. Maximum likelihood estimation

Let $f(x;\gamma)$ be the probability density function of the model log-returns with parameter's set γ. Our cost function is

$$F(\gamma) = \prod_{i=1}^{n-1} f(l_i;\gamma). \tag{11.2}$$

and we want to maximize it:

$$\begin{aligned} &\max_{\gamma\in\Upsilon} F(\gamma) \\ &\overline{\gamma} = \arg\max_{\gamma\in\Upsilon} F(\gamma). \end{aligned} \tag{11.3}$$

To avoid some very large values, we take a logarithm in formula (11.2). Thus the cost function turns to

$$F(\gamma) = \ln\left(\prod_{i=1}^{n-1} f(l_i;\gamma)\right) = \sum_{i=1}^{n-1} \ln f(l_i;\gamma). \tag{11.4}$$

11.2. Least square error method

1. This step is not required. We may cut some lowest and highest observations from the log-return data, reckoning them as a statistical error.

2. The interval which contains the log-returns is divided into equal subintervals with length d. Let their number be N. The choice $N = 50$ is appropriate.

3. We count the number of observations, which belong to each interval. We shall denote it by N_i.

4. The statistical distribution is obtained as

$$p_{st,i} = \frac{N_i}{Nd}. \tag{11.5}$$

5. Let x_i be the center of the i-th interval. We shall denote by $p_{th}(x;\gamma)$ the probability density function of the distribution with parameters γ. Let

$$p_{th,i} = p_{th}(x_i;\gamma). \tag{11.6}$$

6. The cost function is defined as

$$F(\gamma) = \sqrt{\sum_{i=1}^{N} \left(p_{th,i} - p_{st,i}\right)^2}, \tag{11.7}$$

   which we have to minimize.

11.3. Calibration of the risk neutral measure

To calibrate the risk-neutral measure we have to know some prices of derivatives. Usually, this derivatives are options with different strikes and maturities. Let us denote by C_{st} the observed prices and by $C_{th}(\gamma)$ the price calculated by the use of some model with parameters γ. Let us denote by N_1 and N_2 the numbers of the strikes and maturities, respectively. An appropriate choice for the cost function is

$$F(\gamma) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_j} \left| \frac{C_{th,i,j}(\gamma) - C_{st,i,j}}{C_{st,i,j}} \right|. \tag{11.8}$$

1. Black-Scholes model – $\gamma = \{\sigma\}$. Note that the drift vanishes, since it is set to be equal to the risk-free rate.

2. Merton's model – $\gamma = \left\{\sigma, \lambda^Q, \kappa^Q, \delta^Q\right\}$. Note that now the risk-neutral condition turns to $\mu^Q = r - \lambda^Q \kappa^Q$.

3. Heston's model – $\gamma = \left\{\xi^Q, \eta^Q, \theta, \rho, V_0\right\}$.

4. Bates model – $\gamma = \left\{\lambda^Q, \kappa^Q, \delta^Q, \xi^Q, \eta^Q, \theta, \rho, V_0\right\}$.

5. Exponential Lévy models – it is closely related to the way of changing measures.

11.4. Joint calibration

It is preferable to use LSqE method when we calibrate jointly the real world and risk neutral measures. Now, the cost function turns to

$$F(\gamma) = \sqrt{\sum_{i=1}^{N} \left(p_{th,i} - p_{st,i}\right)^2 + \sum_{i=1}^{N_1} \sum_{j=1}^{N_j} \left| \frac{C_{th,i,j}(\gamma) - C_{st,i,j}}{C_{st,i,j}} \right|}. \tag{11.9}$$

In this case the sets γ look in the following way

1. Black-Scholes model – $\gamma = \{\mu, \sigma\}$.

2. Merton's model – $\gamma = \left\{\mu, \sigma, \lambda, \kappa, \delta, \lambda^Q, \kappa^Q, \delta^Q\right\}$.

3. Heston's model – $\gamma = \left\{\mu, \xi, \eta, \xi^Q, \eta^Q, \theta, \rho, V_0\right\}$.

4. Bates model – $\gamma = \left\{\mu, \lambda, \kappa, \delta, \xi, \eta, \lambda^Q, \kappa^Q, \delta^Q, \xi^Q, \eta^Q, \theta, \rho, V_0\right\}$.

5. Exponential Lévy models – it is closely related to the way of changing measures.