For the second exercise (25th/26th of February)

Basic Graph Properties

1. Consider the graph G represented by the adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where the corresponding set of nodes is $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, with node i corresponding to the i-th row/column of A for all $i \in V$.

(a) Is G a simple graph? Explain your answer by using arguments exclusively relating to A.

(b) Is G connected? If yes, what is its diameter? If not, how many connected components does G consist of?

(c) Does a path from node 1 to node 5 exist? If yes, how many shortest paths between these nodes exist? What is its/their length?

(d) Compute the average degree $\langle k \rangle$ of the nodes in G.

(e) Compute the density of graph G. Is G sparse? Explain your answer.

Tree Graphs

2. Consider a tree graph consisting of N nodes. How many edges does it have? Explain your claim!

Bipartite Networks

3. Consider a bipartite network with N_1 and N_2 nodes in the two subsets.
   (a) What is the maximum number of links L^{bp}_{max} the network can have?
   (b) How many links cannot occur in the bipartite network compared to a non-bipartite network with $N = N_1 + N_2$ nodes?

Gatekeepers

4. For any given graph $G = (V, E)$, we say that a node v is a *gatekeeper* if for some other two nodes u and w, every path from u to w passes through v, with v not being equal to u or w.
   In addition, we say that a node v is a *local* gatekeeper if there are two neighbors of v, say u and w, that are not connected by an edge.
   (a) Give an example of a graph in which more than half of all nodes are gatekeepers.
   (b) Give an example of a graph in which there are no gatekeepers, but in which every node is a local gatekeeper.