Inverse Optimization: A New Perspective on the Black-Litterman Model

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May 4, 2020

1 Introduction

1.1 Introduction of our reference paper

Our project is based on the paper: "Inverse Optimization: A New Perspective on the Black-Litterman Model" by Dimitris Bertsimas, Vishal Gupta, and Ioannis Ch. Paschalidis. As what we learned in class, the Black Litterman model is used for asset allocation problems with small number of assets, and it assumes that the covariance matrix is easy to estimate. However, in this paper, the main focus is the estimation of return.

The main idea of this paper is to breakdown the restriction that is set by Black Litterman model. They argue that Black Litterman only allows investors to specify views on asset returns, but not on volatilities or market dynamics. Thus, they modified the original Black Litterman model to allow more type of private information and investor views could be analyzed in the model.

The details of the paper is discussed into two parts. In the first part, they theoretically introduce the concept of inverse optimization, which is a novel way to construct equilibrium-based estimation of the mean returns and the covariance matrix. Under this concept, they derived two models which are "mean-variance inverse optimization" approach (MV-IO) , and robust mean variance inverse optimization approach (RMV-IO). MV-IO allows investors to put their own views on volatility in the model, and RMV-IO accommodates volatility uncertainty. In the second main part of their paper, they give an example view on volatility and conduct simulation and back-testing based on this view to compare the performance of market portfolio, the Black Litterman model and their own model.

1.2 Our Contributions

In our project, our main contributions are as the following:

First, since the original paper only simply talked about the result of their simulation and do not give us the financial explanation of their results, we give a further explanation and comment about their simulation.

Second, based on the shortcomings of their simulation, we made other simulations to test more scenarios to have a deeper understanding of their model.

Third, the paper do not discuss the scope of application of this model. Therefore, we discuss the difficulties in its application, the scope of applicaiton, and how this model could be applied in more generalized problem when the assumptions of Black Litterman fail.

2 Theoretical Framework

2.1 Inverse Optimization

In this section, we introduce a generalization of the BL model. The key here is to use inverse optimization to characterize the BL estimate as the solution to a particular convex optimization problem.To begin with, we define an usual optimization problem:

$$\min_{\mathbf{x}\in(\zeta)} f(\mathbf{x};\zeta)$$

where $f(\mathbf{x};\zeta)$ is the objective function. Given the data parameters ζ, we want to find an optimal solution $\mathbf{x}^*$ to the function f from certain domain. While in the inverse optimization proble,, we are given a candidate portfolio $\mathbf{x}^*$ and we need to seek a set of parameters which make this portfolio optimal.Now, what we are facing is only a convec optimization instead of a statistic model. The form can be shown like:

$$\mathbf{x}^* = \arg\min_{\mathbf{x}_{\mathbf{k}\in\mathcal{G}(\mathbf{z})}} f(\mathbf{x};\zeta)$$

Consider a market with n risky assets and one risk free asset, investors usually solve the Markowitz portfolio allocation problem: (Markowitz)

$$\max_{\mathbf{x}} \left\{ \boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})\, r_f : \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq L \right\}$$

where $\mathbf{r} \in R^n$ is the vector of the risky asset returns, $\boldsymbol{\mu} \equiv E[\mathbf{r}]$ is the vector of mean asset returns, $r_f \in R_+$ is the return on the riskless asset, $\boldsymbol{\Sigma} \in R^{nxn}$ is the covariance matrix of asset returns, $\mathbf{x} \in R^n$ is the fraction of wealth invested in each risky asset, and L is an investor-specific threshold level of risk. To solve this problem, we need to estimate the $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

Consider the Markowitz problem, and assume a candidate portfolio $\mathbf{x}^*$ and suppose that there is known a prior that $\boldsymbol{\Sigma} \geq \mathbf{0}$ and $L > 0$. Then, $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, r_f, L)$ solve the inverse Markowitz problem if and only if there exists a δ such that:

$$\mathbf{x}^{*'}\boldsymbol{\Sigma}\mathbf{x}^* \leq L^2, \quad \delta\left(L^2 - \mathbf{x}^{*'}\boldsymbol{\Sigma}\mathbf{x}^*\right) = 0$$

$$\boldsymbol{\mu} - r_f\mathbf{e} - 2\delta\boldsymbol{\Sigma}\mathbf{x}^* = \mathbf{0}, \quad \delta 0, L > 0, \boldsymbol{\Sigma} \succeq \mathbf{0}$$

Under the framework of CAPM and the condition above, there must exist an L such that $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, r_f, L)$ can make the x^{mkt} fulfill the above equation. And therefore, there are values of $\left(\boldsymbol{\mu}, \overline{\boldsymbol{\Sigma}}\right)$ such that:

$$\boldsymbol{\mu} - r_f\mathbf{e} - \overline{\boldsymbol{\Sigma}}\mathbf{x}^{\text{mkt}} = \mathbf{0}, \quad \overline{\boldsymbol{\Sigma}} \geq \mathbf{0}$$

Since it is a generalization of BL model, we are able to add the investors' private view into the optimization problem, which leads to:

$$\min_{\boldsymbol{\mu},\overline{\boldsymbol{\Sigma}},t} \left\{ t : \left\| \begin{pmatrix} \boldsymbol{\mu} - \boldsymbol{r}_f\mathbf{e} - \overline{\boldsymbol{\Sigma}}\mathbf{x}^* \\ \mathbf{P}\boldsymbol{\mu} - \mathbf{q} \end{pmatrix} \right\| \leq t, \overline{\boldsymbol{\Sigma}} \succeq \mathbf{0} \right\}$$

Here we transform out need of solving problems like BL to the above minimization problem, which aims to find a set of parameter $\mu, \overline{\Sigma}, t$ to minimize the violation of BL conditions. If the $\overline{\Sigma} = 2\delta\hat{\Sigma}$, and the norm is defined to be $\|\mathbf{z}\|_2^{\boldsymbol{\Omega}} = \sqrt{\mathbf{z}'\boldsymbol{\Omega}^{-1}\mathbf{z}}$, we have the exactly same solution to BL model with respective to the estimate of μ.

With the above framework, it is always easy to add on constraints, and private views on σ.For example, if investor have some information on the σ of a basket of asset b, then what she needs is to only add on the following constraints:

$$\left\|\mathbf{b}'\overline{\boldsymbol{\Sigma}}\mathbf{b} - \sigma^2\right\| \leq \epsilon$$

As a much more interesting application of Inverse Optimization problem, here we introduce the factor model incorporated with Inverse Optimization. Factor models are common in the financial industry and are closely related to the arbitrage pricing theory. One of advantage of the Mean Variance Inverse Optimization is that we can form a factor-like view from PCA, and meanwhile, it is always welcome to impose economic factor model like Fama-French Factor Model,etc, and the adaptation is intuitive and straight-forward as stated below.

$$\sum_{i=1}^{k} \lambda_i \geq \alpha \cdot \operatorname{trace}(\overline{\boldsymbol{\Sigma}}), \quad \left\|\overline{\boldsymbol{\Sigma}}\mathbf{v}^i - \lambda_i \mathbf{v}^i\right\| \leq \epsilon \quad i = 1, \ldots, k$$

The above optimization with factor-like views on volatility is an example of the Mean-Variance Inverse Optimization formation in this paper.

Here are some of our comments about the factor like model used as a view in the MV-IO model.

First, in the paper they argued that k is generally be chosen as 3, then if we have only 3 or 4 assets, this view will not be suitable. As we what is discussed in the MF 740 class, the Black Litterman model deals with problem with small number of assets, such as N equals to 3 or 4 or 5, the factor model is not suitable for these kind of cases. Thus, the situation to use this view may be different with the situation of the traditional Black Litterman model.

Secondly, relating to Principal Component Analysis of the variance of a number of assets, it is broadly accepted that the first largest component is the market volatility, and first largest component generally accounts for approximately 80% of the total variance. Thus, if the small eigen values are treated as noise and are removed, what is left is mainly the market volatility. Then when this view of volatility is added into the Black Litterman model to become the MV-IO model, the optimized weights will shift toward the market weights. Then the MV-IO optimized weights will simply become a combination of the market weights and the optimized weights of Black Litterman model. In this paper, the US stock sector data to calculated the optimized weight of different methods and the result is shown in table below, the data in which just verifies this point.

Further, this paper also comes up with a Robust Mean Variance Inverse Optimization which in addition satisfies the requirement for the maximum of possible loss. We start with following problem considering the maximum loss(VaR):

$$\max_{\mathbf{x}} \left\{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})\, r_f : (\mathbf{r} - r_f\mathbf{e})'\, \mathbf{x} \ \geq -L, \forall \mathbf{r} \in \mathcal{U}\right\}$$

$$\mathcal{U} = \Big\{\mathbf{r} : \exists \boldsymbol{\Sigma} \text{ s.t. } \boldsymbol{\Sigma} \geq \mathbf{0}, (\mathbf{r} - r_f\mathbf{e})'\, \boldsymbol{\Sigma}^{-1} (\mathbf{r} - r_f\mathbf{e}) \leq 1$$

$$\left\|\boldsymbol{\Sigma}\mathbf{v}^i - \lambda_i \mathbf{v}^i\right\| \leq \epsilon \forall i = 1, \ldots, k, \operatorname{tr}(\boldsymbol{\Sigma}) \leq \frac{1}{\alpha}\sum_{i=1}^{k} \lambda_i \Bigg\}$$

The intuition of this form is that given any covariance matrix that is similar in factors, and any return vector that satisfies $(\mathbf{r} - r_f\mathbf{e})'\, \boldsymbol{\Sigma}^{-1} (\mathbf{r} - r_f\mathbf{e}) \leq 1$, we have the above optimization problem exactly the same as the Markovitz Mean-Variance problem, but meanwhile, it guarantee the maximum of loss will not exceed L by making the possible return vector r related to Σ. In terms of the formation of the RMVIO problem.Firstly, we set $L_{mkt} = max_{r \in \mathcal{U}} - rx_{mkt}$, this guarantee the market portfolio is in the feasible region. Then if market portfolio is the optimal weights, we have the following relationship hold:

$$H = \Bigg\{ \boldsymbol{\mu} : \exists \boldsymbol{\Sigma} \text{ s.t. } (\boldsymbol{\mu} - \boldsymbol{r}_f \mathbf{e})' \mathbf{x}^{\text{mkt}} = z L_{\text{mkt}}, \begin{pmatrix} \boldsymbol{\Sigma} & \boldsymbol{\mu} \\ \boldsymbol{\mu}' & z \end{pmatrix} \geq \mathbf{0}$$

$$\left\| \boldsymbol{\Sigma} \mathbf{v}^i - z \lambda_i \mathbf{v}^i \right\| \leq \epsilon, \forall i = 1, \ldots, k, \operatorname{tr}(\boldsymbol{\Sigma}) \leq z \frac{1}{\alpha} \sum_{i=1}^{k} \lambda_i \Bigg\}$$

In this set, all the μ can make the market portfolio optimal. And then we are trying to add our BL type views:

$$\min_{t, \boldsymbol{\mu}, \boldsymbol{\mu}^{eq}, \boldsymbol{t}} \left\{ t : \boldsymbol{\mu}^{eq} \in H, \left\| \begin{pmatrix} \boldsymbol{\mu} - \boldsymbol{r}_f \mathbf{e} - \boldsymbol{\mu}^{eq} \\ \mathbf{P}\boldsymbol{\mu} - \mathbf{q} \end{pmatrix} \right\|_{\boldsymbol{n}} \leq t \right\}$$

Here the return vector μ we are actually finding , is firstly close enough to the set H, which can make the market portfolio optimal, and is secondly satisfying the private views on return.

	x^{mkt}	x^{BL}	x^{MV}
Energy	7.15	5.03	6.03
Materials	4.54	8.58	5.75
Industrials	10.92	10.94	10.93
Consumer discretionary	10.77	5.56	8.45
Consumers staple	13.47	17.36	16.16
Health care	13.57	9.20	11.91
Financial	15.81	16.93	17.26
information technology	15.16	16.09	15.41
communication service	5.88	2.61	3.47
Utilities	2.74	7.71	4.63

As is shown in the table, except for the weight of the financial sector, for all other sectors the weight of MV-IO is just a number between the market weight and Black Litterman weight. Thus, intuitively, we can treat this view as one believe the market portfolio is better than the Black Litterman model with a certain view, since he put more weights on the market portfolio than Black Litterman.

Therefore, since this view result in more weights in market portfolio, and because the market portfolio is always more diversified, it is possible that MV-IO portfolio under this view is less risky than BL portfolio with same view on return.

3 Simulation

3.1 Simulation conducted in the paper

The second main part of their paper is the numerical testing to compare the performance of the market portfolio, Black Litterman portfolio and the MV-IO portfolio. A simulation is made to compare the three models and in the MV-IO model the Factor-liked view mentioned above is used as the view of volatility. The historical covariance is used as the estimated covariance matrix in the Black Litterman model. In addition, MV-IO and Black Litterman share the same view of return. A scenario is simulated in the paper that CAPM equilibrium holds entirely and an entirely wrong view is imposed on return in Black Litterman model and MV-IO model. To be specific, p is randomly chosen as [-10%, 0%, 0%, -20%, 40%, -10%, 30%, 0%, -40%, 10%].

The real q, i.e., what is expected in equilibrium, is $p\hat{r} = 0.3862\%$, which is close to zero. The table below shows the result of the simulation.

	q	-10	-5	-2	-1	0	1	2	5	10
Return	Mkt	4.14	4.14	4.14	4.14	4.14	4.14	4.14	4.14	4.14
	BL	3.22	3.75	3.96	4.00	4.02	4.02	4.00	3.84	3.38
	MV-IO	4.08	4.11	4.12	4.13	4.14	4.11	4.10	4.05	3.92
	RMV-IO	3.99	4.09	4.13	4.13	4.15	4.07	4.05	3.97	3.80
Std dev	Mkt	12.87	12.87	12.87	12.87	12.87	12.87	12.87	12.87	12.87
	BL	14.26	13.03	12.54	12.48	12.48	12.55	12.67	13.31	14.67
	MV-IO	12.73	12.80	12.84	12.85	12.87	12.78	12.74	12.61	12.30
	RMV-IO	12.60	12.80	12.87	12.88	12.91	12.68	12.62	12.44	12.04
Sharpe ratio	Mkt	32.17	32.17	32.17	32.17	32.17	32.17	32.17	32.17	32.17
	BL	22.62	28.77	31.55	33.01	32.17	32.02	31.56	28.88	23.07
	MV-IO	32.07	32.10	32.11	32.11	32.17	32.16	32.15	32.09	31.90
	RMV-IO	31.67	31.96	32.08	33.11	32.15	32.09	32.05	31.91	31.56

Relating to this result, the authors of the paper have given their explanations. They argued that with the view of q = 0, the performance of the Black Litterman and MV-IO portfolios are similar to that of market portfolios, because the views are correct. Moreover, as $|q|$ increases, Black Litterman and MV-IO have increasingly incorrect view so they performed worse than the market. Additionly,BL has worse returns and Sharp ratio than MV-IO, which indicates that MV-IO is more robust when the view on return is inaccurate.

In our perspective, however, the explanation given in the paper is not thorough. Thus, further explanations of this result are discussed in our group and shown as following .

First, the reason why the performance of the three models are similar is that even though the view is correct, it is an useless view. In other words, this view generally does not affect the optimized weights in an asset allocation strategy. The reason behind is that a view(pr=q) with positive q will result in longing more of the portfolio p while a view with negative q will result in shorting more of the portfolio p, whereas a q close to zero would generally result in neither longing nor shorting more of the portfolio p as long as the original optimized weight is far from the portfolio p. Therefore, this simulation have not given a comprehensive evaluation of the MV-IO approach. First they do not have a scenario that the view of return in Black Litterman and MV-IO are both Correct and Useful. Secondly, they have not shown what will happen when the view on volatility in the MV-IO approach is wrong.

Secondly, the an explanation about why MV-IO is more robust than Black Litterman when the view is incorrect is given as following. As I talked before, this factor-liked view can be explained as one believe CAPM holds in the real world, since this simulated world is also a CAPM world, this means the view they put in the volatility in MV-IO is a correct view. Thus, putting a correct view will definitely result in a good performance. So this result is not an evidence that MV-IO is more robust than Black Litterman, but is an evidence that MV-IO can perform better than Black Litterman when the view on volatility is correct.

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3.2 New Simulations We Conducted

As is mentioned above, the paper only simulated two scenarios. The first scenario is that the view of return in both Black Litterman model and MV-IO model is wrong and the view on volatility in the MV-IO model is correct, and the result is that when the view of return is wrong in both models, MV-IO model will be much more robust as long as its view on volatility is nearly correct. The second is that the view of return in both Black Litterman model and MV-IO model is correct but is useless, where the performance of the portfolios in both models are similar to the market portfolio.

In order to have a more comprehensive evaluation of the MV-IO model, we have conducted two more simulations as a complement of the original paper.

3.2.1 Scenario with Correct and Useful view on Return and Volatility

In our first simulation, we aim to figure out what would happen if both the Black Litterman model and the MV-IO model have correct but also useful view of return. For simplicity, we used multivariate normal distribution to generate random numbers as our return of the 10 Global Industrial Classification Standard (GICS) sectors for 1000 time periods. The mean return and covariance matrix we used as the parameters in the multivariate normal distribution are the mean and covariance of the recent years' historical data. For the view of return, we tested q=-1, 0, 1, 2, 3 with the same p. When q=1, the view of return is exactly correct. Thus, for q=2 and 3, the view successfully predicted the direction of the portfolio p but the return in the prediction is not that precise as q = 1. For the case of q=-1, the view predicted the return of portfolio p wrongly, which is just opposite to what will really happen. In addition, since we now the real distribution of the simulated data, we imposed a view of volatility in the MV-IO model which is just the exactly correct one.

	q	−1	0	1	2	3
Return	Mkt	0.559	0.559	0.559	0.559	0.559
	BL	−220.95	11.880	10.614	11.250	11.502
	MV-IO	−7.247	55.519	10.620	23.307	22.307
Std dev	Mkt	17.925	17.925	17.925	17.925	17.925
	BL	52.158	18.232	17.960	18.001	18.017
	MV-IO	17.303	19.702	17.954	18.260	17.800
Sharpe ratio	Mkt	0.031	0.031	0.031	0.031	0.031
	BL	−4.236	0.651	0.59	0.624	0.618
	MV-IO	−0.418	2.817	0.591	1.276	1.253

The result of our first simulation is displayed in the table above. As is shown in the table, when the view on return is correct and useful (q=1) in both Black Litterman model and MV-IO model, both the two models perform much better than the market portfolio, which is different from the case in the original paper that the two models performed similarly with the market portfolio. Besides, when the view on return imposed in the Black Litterman model is correct, the performance of MV-IO is just similar to that of Black Litterman, even though MV-IO has a better view on volatility. This indicates that if one is able to predict the return precisely, he may not need to use the MV-IO model to express his view on volatility. He can just use the Black Litterman model because the performance of Black Litterman model will not be worse than MV-IO as long as the view on return is precise.

However, when it comes to the scenario that one is not able to estimate the return precisely, which as always the general case, the ability to have a correct view on volatility would help. In the case above, when q=2,3, where the view on return is not precise, MV-IO with a correct view on volatility have a better return. For q=-1, where the view on return is totally wrong, MV-IO with correct view on volatility helps reduce the loss and successfully controls the volatility. This case can be seen as another example of the simulation conducted by the original paper. This result indicates that when an investor is not very confident in his estimation of the return, which is always general in the real world, MV-IO would make great contribution if the investor can successfully estimate the covariance. We believe this is also the most important reason why the MV-IO model is good.

3.2.2 Scenario with Correct and Useful View on Return but Wrong View on Volatility

Our second simulation is about the case that the view on volatility is wrong in the MV-IO model. We simulated a scenario that the view on return in both Black Litterman model and the MV-IO model are correct and useful but the view on volatility is extremely wrong. The result is displayed in the table below.

	mean	std	SR
mkt	0.559615	17.925170	0.031219
BL	10.614730	17.960760	0.590996
MV-lO	−237.808815	40.829848	−5.824387

As is shown in the table, when the view on volatility is wrong, the MV-IO portfolio may perform much more worse even if the view on return is correct. It indicates that the MV-IO is not always robust as the paper argued, it would be a good model only when the view of volatility imposed in the model is correct. This also remind us that it is better not to use MV-IO to impose a view on volatility if an investor does not have any good idea about the volatility.

4 Conclusion

4.1 Practical use related to coding

Due to the complexity of the Inverse-Optimization process, the MV-IO approach requires more time and effort to use and the one may face some issues during the procedure of coding. For example in our coding process, we encountered mainly two hardships.

First, we found that the sensitivity of SDP solution is really high. If we change the parameter a little the final solution would change rapidly.

Secondly, the convex optimization package we used did not work very well. We use Coxpy and sco package in Python. Coxpy package cannot get a good solution every time. However, the sco package's solution is influenced largely by the initial value we choused.

Thus, the convenience and simplicity is also an important part to take into consideration when investors are trying to choose between the Black Litterman model and the MV-IO model.

4.2 Comment on scope of application of the models in the paper

The authors of our reference paper believe that MV-IO, and RMV-IO are more robust than Black Litterman since they could express personal views on volatility. However, according to what we have learnt in class, Black Litterman model mainly deals with portfolio problems with small number of assets easily estimated covariance matrix. Thus, BL model becomes better than MV-IO and RMV-IO when the assumptions of BL holds (i.e., very small number of assets and the covariance matrix is easy to estimate) because MV-IO and RMV-IO are complicate to conduct without outperformed results .

Because of the flexibility of MV-IO and RMV-IO, they could be applied in problems that traditional BL cannot solve. For example, these two models could be used in estimating portfolios with large number of assets when investors have accurate estimations on portfolio volatility.

And in our reference paper, they applied factor model to remove noise of the variance. However, we have to emphasize that MV-IO and RMV-IO are worth to use only when we are confident with our views on portfolio volatility. As in our simulation, if the view on volatility is opposite to the reality, people will get a worse result in MV-IO than BL, even though they make a lot of effort to apply a more complex model.

We think the model in our reference paper could be explored further to meet more individualized needs. One application that we could come up with is that if investors need to control the VaR or CVaR of some fat-tail assets, they could put VaR or CVaR in their objective function and add some higher-moments parameters to the Inverse-Optimize part to have a more accurate estimation.

References

[1] Dimitris Bertsimas, Vishal Gupta, Ioannis Ch. Paschalidis. (2012). Operations Research Vol.60, No. 60, November - December 2012, pp.1389-1403