The Economics of FinTech

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Chapter 10

Additional Notes on the Wisdom of the Crowd and Prediction Markets

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10.1 Review of Bayesian Truth Serum

In this section, we shall review the BTS and the resulting estimator in Prelec et al. (2017).

10.1.1 The Bayesian Learning Setup

Consider humans live in a world where there are unknown events. Denote the value of such an unknown event to be Ω, which is a random variable in the human world. Without loss of generality, we assume there are only $K+1$ possible outcomes in Ω, i.e. $\Omega \in \{0, 1, \cdots, K\}$. The human world is different from a conceptually omniscient and eternal world, which is referred to as an objective world. In the objective world, there is no time elapse and the truth (e.g. the outcome of any future events, or correct answers to any questions) is revealed. Thus, the value of the event in the objective world, denoted by $\Omega_{obj} \in \{0, 1, \cdots, K\}$, is a deterministic constant but unknown to the human world.

People can learn about the objective world in certain ways by engaging in cognitive learning, e.g. observations, experiments, or reasoning. The consequence of such behavior leads to a signal. More precisely, suppose that there are N people learning about Ω, each receiving a signal S_i, $1 \leq i \leq N$, which is a sequence of independent identically distributed (i.i.d.) random variables with $S_i \sim f_{obj}(\cdot|\Omega_{obj})$ i.i.d., where $f_{obj}(\cdot|\Omega_{obj})$ is known as the objective likelihood function. For simplicity, we also assume $S_i \in \{0, 1, \cdots, K\}$. Thus, the joint objective likelihood function is $S_1, \cdots, S_N \sim \prod_{i=1}^{N} f_{obj}(S_i|\Omega_{obj})$.

Here we only consider the case with binary outcome, that is, Ω_{obj} and Ω take a value of either 0 or 1, and the multiple outcome case is relegated to Appendix.

After learning about the unknown event, an agent forms his subjective view about Ω based on the Bayesian updating rule. More precisely, the agent i updates the prior distribution $\mathbb{P}_i(\Omega = k) = \pi_i^k$, by using the private signal S_i, to get a posterior probability $\Omega|S_i \sim p_i(\cdot|S_i)$, according to the Bayesian formula and the subjective likelihood function $f_i(S_i|\Omega)$. Subjective probabilities are denoted by p_i and $1 - p_i$, where

$$p_i := p_i(1|S_i) = \frac{\pi_i f_i(S_i|\Omega = 1)}{\pi_i f_i(S_i|\Omega = 1) + (1 - \pi_i) f_i(S_i|\Omega = 0)}, \qquad 1 - p_i = p_i(0|S_i). \tag{10.1}$$

Here $\pi_i = \mathbb{P}_i(\Omega = 1)$, as we omit the superscript for simplicity in the binary case. Then the joint subjective likelihood function is $S_1, \cdots, S_N|\Omega \sim \prod_{i=1}^{N} f_i(S_i|\Omega)$. Note that different likelihood function means that different agents can interpret the signal in subjective and distinct ways.

For example, suppose in the human world, Ω takes values in 0 (lose) or 1 (win) with common prior probability 0.6 for winning. In the objective world, $\Omega_{obj} = 0$ (doomed to lose) and the objective likelihood function $f_{obj}(1|0) = 0.4$. Then approximately 40% of people would receive signal 1 and the rest receive signal 0. Suppose that all agents have the same subjective likelihood function $f_i(\cdot|0) = f_{obj}(\cdot|0)$, which happens to be the same as the objective likelihood, and that $f_i(1|1) = 0.8$, then those who receive signal 1 will have subjective posterior probability $p_i(1|1) = 0.75$, while those who receive signal 0 will have subjective posterior probability $p_i(1|0) = 1/3$.

10.1.2 Assumptions in Prelec et al. (2017)

Assumption 1. *All agents agree on a common prior distribution* $\pi^k = \mathbb{P}(\Omega = k) \in (0, 1)$*, such that the subjective probability* $0 < p_i < 1$ *almost surely.*

This assumption of common prior first appeared in Prelec (2004). As suggested in Prelec et al. (2017), the common prior corresponds to the common knowledge shared by all people; for example, the prior knowledge might be that Philadelphia is a well-known U.S. city or Brazil is famous for football.

Assumption 2. *All agents agree on the subjective likelihood function, which is also equal to the objective likelihood function.*

Assumption 2 is an assumption made implicitly in Prelec et al. (2017). It means that all agents can perfectly understand the law of the signals generated and interpret the signals in the correct ways. For example, people conduct

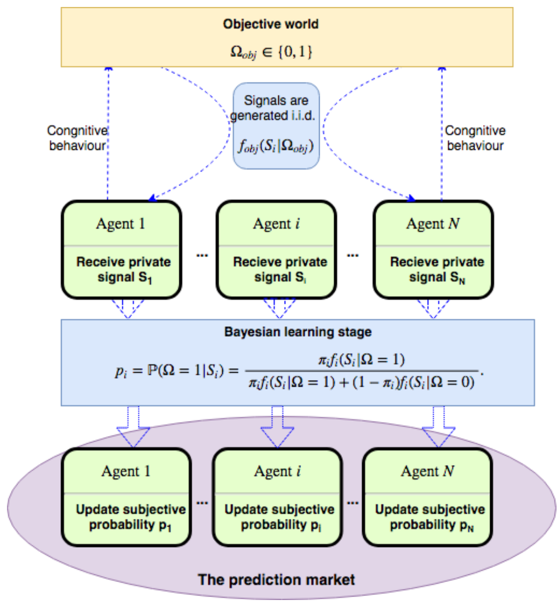

Figure 10.1: **Diagram for cognitive and decision making process under the BTS setting**. People can engage in cognitive activities to learn about the objective world by receiving a private but random signal, which can be used to update subjective probability by Bayesian updating. In addition, we assume that people make decisions based on their subjective probability. People's learning processes (in dotted arrows) are unobserved.

experiments to determine unknown objects and they are fully aware of the possible phenomenon conditioned on different objects. Assumptions 1 and 2 together also imply that the only reason for different beliefs among people is the signals observed.

Assumption 3. *$\mathbb{P}(\Omega = k|S_i = k) > \mathbb{P}(\Omega = k|S_i = j)$, for any $j \neq k$.*

Assumption 3 is called the truth sensitivity assumption in Prelec and Seung (2006), which means that the one who receives the message of true outcome will be the most confident in the true answer, no matter what the truth is.

Assumption 4. *$\mathbb{P}(\Omega = k|S_i = k) > \mathbb{P}(\Omega = j|S_i = k)$, for any $j \neq k$.*

Assumption 4 is called the signal preference assumption (Prelec and Seung, 2006) or the most likely voting rule (Prelec et al., 2017), which means that any agent who receives signal k will believe that $\Omega = k$ is the most likely outcome and vote for k. In the case of binary outcome, Assumption 3 is implied by Assumption 4, but is independent of Assumption 4 in the case of multiple outcomes.

10.1.3 Review of the Estimator in Prelec et al. (2017)

In contrast to a traditional survey question asking a particular question, an innovative idea in Prelec et al. (2017) is that an additional question is designed to ask respondents to predict the others' answer. For example, if the first survey question is "which one is more likely to happen, $\Omega = 1$ or $\Omega = 0$", then the additional question can be "what is the proportion of the whole population who think $\Omega = 1$ is more likely to happen".

Under Assumption 4, giving a signal $S_i = 1$, agent i will report $\Omega = 1$. Therefore, for agent i the answers to the above two questions are equivalent to $\mathbf{1}_{\{S_i=1\}}$ and $\mathbb{P}(S_j = 1|S_i)$, respectively. Then the survey designer can estimate $\mathbb{P}_{obj}(S_i = 1|\Omega_{obj})$ and $(M_{kl})_{k,l=0,1} = \mathbb{P}(S_j = l|S_i = k)$, known as metaknowledge matrix.

An important observation made by Prelec and Seung (2006) and Prelec et al. (2017) is that $M_{kl}\mathbb{P}(S_i = k) = M_{lk}\mathbb{P}(S_i = l)$. Therefore, the ratio of two marginal probabilities $\mathbb{P}(S_i = k)/\mathbb{P}(S_i = l)$ can be estimated from M without knowing prior distribution or likelihood function via

$$\frac{\mathbb{P}(S_i = k)}{\mathbb{P}(S_i = l)} = \frac{M_{lk}}{M_{kl}}. \tag{10.2}$$

To summarize, denote agent i's answer to the first question by $v_i \in \{0, 1\}$ and answers to the second question by $\xi_i^k = \mathbb{P}(S_j = k|S_i)$, $k = 0, 1$. The quantity to be estimated is

$$\mathbf{1}_{\left\{\mathbb{P}(\Omega=\Omega_{obj}|S_i=1)>\mathbb{P}(\Omega=\Omega_{obj}|S_i=0)\right\}} = \mathbf{1}_{\left\{\frac{\mathbb{P}_{obj}(S_i=1|\Omega_{obj})}{\mathbb{P}(S_i=1)} > \frac{\mathbb{P}_{obj}(S_i=0|\Omega_{obj})}{\mathbb{P}(S_i=0)}\right\}}; \tag{10.3}$$

note that the left side equals 1 if $\Omega_{obj} = 1$, and 0 otherwise, given Assumptions 2 and 3. Then using (10.2) the estimator proposed by Prelec et al. (2017) is

$$\mathbf{1}_{\left\{\frac{\sharp\{i:v_i=1\}}{m_{01}} > \frac{\sharp\{i:v_i=0\}}{m_{10}}\right\}}, \tag{10.4}$$

where $m_{kl} = \dfrac{1}{\sharp\{i : v_i = k\}} \displaystyle\sum_{i:v_i=k} \xi_i^l$. It is worthwhile pointing out that Prelec and Seung (2006) also construct a more complicated estimator. We refer to (10.4) as the BTS survey estimator.

Here is an illustration of the BTS estimator. We ask people two questions. (1) Is Philadelphia the capital of Pennsylvania? (2) What is your guess of the percentage of the respondents who answer yes to the first question? Suppose 70% of respondents say yes to the first question; among the people who answer yes to the first question the average of their answers to the second question is 70%, and among the people who answer no to the first question the average of their answers to the second question is 75%. Assume that $\Omega = 1$ means that Philadelphia is the capital of Pennsylvania and $\Omega = 0$ means otherwise. Note that

$$\sharp\{i : v_i = 1\} = 0.7N, \quad \sharp\{i : v_i = 0\} = 0.3N,$$

$$m_{01} = 0.75,\ m_{10} = 0.3.$$

Thus, the estimator is $\mathbf{1}_{\{\frac{0.7}{0.75} > \frac{0.3}{0.3}\}} = 0$. In short, the estimator is No, Philadelphia is not the capital of Pennsylvania.

10.2 Market Prices in Prediction Markets

10.2.1 The Market Setup

We restrict attention to a one-period prediction market, in which two winner-take-all securities (H and T) are traded, to bet on the outcome of a coin tossing. The holder of one share of security H (T) gets 1 dollar if the head (tail) shows up, and gets nothing otherwise. We assume that market is an ideal market, e.g. with no transaction costs, no bid-ask spread, and short sale being allowed (unless otherwise specified). Suppose that there are N agents in the market with their subjective probabilities (p_i, q_i), $i = 1, 2, \cdots, N$, where $p_i + q_i = 1$ and $p_i, q_i > 0$. We also assume that the amount of money traded in the prediction market is small enough so that an agent will not link the payoff of the market to his overall investment strategy. In other words, the prediction market is not a hedging market, and the agents only maximize the expected utility within this prediction market. In addition, without loss of generality, the interest rate in this prediction market is zero.

Given the market price $\bar{p}$ for H and $\bar{q}$ for T and initial wealth w_i, agent i needs to determine the number of shares (x_i, y_i) in H and T respectively, to maximize the expected utility, with respect to his subjective probability, namely,

$$\max_{x,y:\ \bar{p}x+\bar{q}y=w_i} p_i U_i(x) + q_i U_i(y). \tag{10.5}$$

To preclude arbitrage, we need $\bar{p} + \bar{q} = 1$. Also, we can exchange one dollar for one share of H and one share of T, and holding the same shares of two securities is equivalent to investing in a risk-free money market account with zero interest rate.

Definition 1. *A competitive equilibrium is defined as a pair $(\bar{p}, \bar{q})$ and (x_i, y_i), $i = 1, \cdots, N$ such that $\bar{p} + \bar{q} = 1$, (x_i, y_i) is an optimal solution to (10.5), and market clears. Here, the market clearing condition means that the total positions in both assets are the same. That is, $\sum_{i=1}^N x_i = \sum_{i=1}^N y_i$.*[1]

Assumption 5. *The agents in the economy either all have constant relative risk aversion (CRRA) preferences or constant absolute risk aversion (CARA) preferences, where the risk averse coefficients $\gamma_i > 0$ are independent identically distributed (i.i.d.), are independent of signals and prior probabilities, and satisfy $0 < \mathbb{E}_{obj}[\frac{1}{\gamma_i^2}] < \infty$.*[2] *Initial wealth $w_i > 0$ are i.i.d. and independent of signals, prior probabilities, and risk aversion coefficients.*

[1] This indicates that the payment to the winners can be fully covered by the total asset value, i.e. $\sum_{i=1}^N x_i = \sum_{i=1}^N y_i = \sum_{i=1}^N (\bar{p}x_i + \bar{q}y_i)$, no matter what the outcome is.

[2] A sufficient condition for this condition is that there exist $l, u > 0$ such that $l \le \gamma_i \le u$ almost surely.

The case where agents are risk neutral can also be covered similarly.

Regularity Condition 1. *$0 < \mathbb{E}_{obj}[w_i] < \infty$.*

In practice, there is usually a wealth limit in a prediction market, so that this regularity condition is automatically satisfied.

10.2.2 The Equilibrium Market Price and Agents' Optimal Positions

Proposition 1. (Equilibrium market price and trading positions) *Suppose Assumption 5 and Regularity Condition 1 hold.*

(a) If agent i has utility function $U_i(c) = 1 - \frac{1}{\gamma_i} e^{-\gamma_i c}$, for $1 \leq i \leq N$, then the equilibrium price $\bar{p}$ exists and is the unique solution to

$$\sum_{i=1}^{N} \frac{1}{\gamma_i} \log \frac{p_i}{1-p_i} = \log \frac{\bar{p}}{1-\bar{p}} \sum_{i=1}^{N} \frac{1}{\gamma_i}, \tag{10.6}$$

and optimal strategy (x_i, y_i) for agent i satisfies

$$x_i - y_i = \frac{1}{\gamma_i}(\log \frac{p_i}{1-p_i} - \log \frac{\bar{p}}{1-\bar{p}}). \tag{10.7}$$

(b) If agent i has utility function $U_i(c) = \frac{c^{1-\gamma_i}}{1-\gamma_i}$ and initial wealth w_i, for $1 \leq i \leq N$, then the equilibrium price $\bar{p}$ exists and is the unique solution to

$$\sum_{i=1}^{N} \frac{1 - (\frac{\bar{p}(1-p_i)}{(1-\bar{p})p_i})^{\frac{1}{\gamma_i}}}{[(\frac{\bar{p}(1-p_i)}{(1-\bar{p})p_i})^{\frac{1}{\gamma_i}} - 1]\bar{q} + 1} w_i = 0, \tag{10.8}$$

and optimal strategy (x_i, y_i) for agent i satisfies

$$\log x_i - \log y_i = \frac{1}{\gamma_i}(\log \frac{p_i}{1-p_i} - \log \frac{\bar{p}}{1-\bar{p}}). \tag{10.9}$$

If all agents are in continuum type, then the results in Proposition 1 coincide with Wolfers and Zitzewitz (2006) and Ottaviani and Sørensen (2015). Here we focus on finite number of agents, because our goal is to develop a statistical estimator based on market prices. Proposition 1 indicates that the trading positions of risk averse agents depends on subjective probability, and becomes more aggressive when the subjective probability deviates further from market price. Therefore, the net position of an agent contains additional information about the subjective probability. If all agents have CRRA preference, it is remarkable that short selling and leveraging are always not optimal.

10.3 Proof of Proposition 1

Before proving Proposition 1, we first recall a result obtained by Wolfers and Zitzewitz (2006) and Ottaviani and Sørensen (2015), which is summarized by the following lemma.

Lemma 10.3.1. *For agent i, consider the following constrained expected utility maximization problem*

$$\max_{\bar{p}x+\bar{q}y=w_i} p_i U_i(x) + q_i U_i(y), \tag{10.10}$$

where $\bar{p}+\bar{q}=1$, $p_i+q_i=1$, $0<p_i<1, 0<\bar{p}<1$. We have the following conclusions.

(a) If the agent has CARA preference with risk aversion coefficient $\gamma_i>0$, i.e. $U_i(c)=1-\frac{1}{\gamma_i}e^{-\gamma_i c}$, then the optimal solution to (10.10) is given by a pair of (x_i,y_i) such that

$$x_i - y_i = \frac{1}{\gamma_i}(\log\frac{p_i}{1-p_i} - \log\frac{\bar{p}}{1-\bar{p}}). \tag{10.11}$$

Moreover, the unique solution exists and is given by

$$x_i = w_i + \frac{1-\bar{p}}{\gamma_i}(\log\frac{p_i}{1-p_i} - \log\frac{\bar{p}}{1-\bar{p}}),\ y_i = w_i - \frac{\bar{p}}{\gamma_i}(\log\frac{p_i}{1-p_i} - \log\frac{\bar{p}}{1-\bar{p}}). \tag{10.12}$$

(b) If the agent has CRRA preference with risk aversion coefficient $\gamma_i>0$, i.e. $U_i(c)=\frac{c^{1-\gamma_i}}{1-\gamma_i}$, then the optimal solution to (10.10) is given by a pair of (x_i,y_i) such that

$$x_i - y_i = w_i \frac{e^{\frac{1}{\gamma_i}(\log\frac{p_i}{1-p_i} - \log\frac{\bar{p}}{1-\bar{p}})} - 1}{\bar{p}e^{\frac{1}{\gamma_i}(\log\frac{p_i}{1-p_i} - \log\frac{\bar{p}}{1-\bar{p}})} + 1 - \bar{p}}. \tag{10.13}$$

Moreover, the unique solution exists and is given by

$$x_i = w_i \frac{e^{\frac{1}{\gamma_i}(\log\frac{p_i}{1-p_i} - \log\frac{\bar{p}}{1-\bar{p}})}}{\bar{p}e^{\frac{1}{\gamma_i}(\log\frac{p_i}{1-p_i} - \log\frac{\bar{p}}{1-\bar{p}})} + 1 - \bar{p}},\ y_i = w_i \frac{1}{\bar{p}e^{\frac{1}{\gamma_i}(\log\frac{p_i}{1-p_i} - \log\frac{\bar{p}}{1-\bar{p}})} + 1 - \bar{p}}. \tag{10.14}$$

Proof. A descriptive proof is available in Wolfers and Zitzewitz (2006) and Ottaviani and Sørensen (2015). For completeness, we give the details.

(a) Let us first introduce an unconstrained counterpart of (10.10):

$$\max_{x,y} p_i U_i(w_i + x - \bar{p}x - \bar{q}y) + q_i U_i(w_i + y - \bar{p}x - \bar{q}y). \tag{10.15}$$

Observe that (10.15) can be written as

$$\max_z u(z) = \max_z p_i U_i(w_i + (1-\bar{p})z) + (1-p_i)U_i(w_i - \bar{p}z), \tag{10.16}$$

by letting $z = x - y$. Apparently $U_i(\cdot)$ is a concave function. By the first order condition,

$$p_i(1-\bar{p})U_i'(w_i + (1-\bar{p})z) = (1-p_i)\bar{p}U_i'(w_i - \bar{p}z). \tag{10.17}$$

A direct calculation gives (10.11). It is easy to verify that (10.12) satisfies (10.11) and is thus the solution to the unconstrained optimization problem (10.15). Note that (10.12) satisfies the constraint $\bar{p}x + \bar{q}y = w_i$, which leads to the desired result, and the uniqueness follows from the strict concavity.

(b) The proof for Part (b) is similar, and is left as an exercise.

□

Exercise 1. Prove Part (b).

We are ready to prove Proposition 1.

Proof. Recall that the market clearing condition is

$$\sum_{i=1}^{N}(x_i - y_i) = 0. \tag{10.18}$$

(a) By the market clearing condition (10.18) and the optimal strategy (10.11), we immediate infer that the equilibrium price satisfies

$$\sum_{i=1}^{N}\frac{1}{\gamma_i}\log\frac{p_i}{1-p_i} = \log\frac{\bar{p}}{1-\bar{p}}\sum_{i=1}^{N}\frac{1}{\gamma_i}. \tag{10.19}$$

The existence of the equilibrium price follows by the fact that the $f_1(x) = \log\frac{x}{1-x}$ is continuous and monotone in $(0,1)$, and $f_1(0^+) = -\infty$ and $f(1^-) = +\infty$. The uniqueness follows from the strict monotonicity of f_1.

(b) A combination of (10.18) and (10.13) gives

$$\sum_{i=1}^{N}\frac{1-(\frac{\bar{p}(1-p_i)}{(1-\bar{p})p_i})^{\frac{1}{\gamma_i}}}{[(\frac{\bar{p}(1-p_i)}{(1-\bar{p})p_i})^{\frac{1}{\gamma_i}}-1]\bar{q}+1}w_i = 0. \tag{10.20}$$

Denote

$$f_2(x) = \sum_{i=1}^{N}\frac{1-(\frac{x(1-p_i)}{(1-x)p_i})^{\frac{1}{\gamma_i}}}{[(\frac{x(1-p_i)}{(1-x)p_i})^{\frac{1}{\gamma_i}}-1](1-x)+1}w_i.$$

Note that the denominator is always positive for $0 < x < 1$ since

$$[(\frac{x(1-p_i)}{(1-x)p_i})^{\frac{1}{\gamma_i}}-1](1-x)+1 = (\frac{x(1-p_i)}{(1-x)p_i})^{\frac{1}{\gamma_i}}(1-x)+x \geq 0. \tag{10.21}$$

Moreover, we claim

$$f_2(0^+) \geq 0, f_2(1^-) \leq 0. \tag{10.22}$$

Indeed, when $0 < x \le \min_{1\le i\le N}\{p_i\}$, we have $\frac{x(1-p_i)}{(1-x)p_i} \le 1$ for all $1 \le i \le N$. When $\max_{1\le i\le N}\{p_i\} \le x < 1$, we have $\frac{x(1-p_i)}{(1-x)p_i} \ge 1$ for all $1 \le i \le N$. We then deduce (10.22). The existence of the equilibrium price follows by the continuity of $f_2(x)$ and (10.22). Next, to prove the uniqueness, it suffices to verify $f_2(x)$ is strictly decreasing. This is left as an exercise problem.

□

Exercise 2. Show that the function $f_2(x)$ is strictly decreasing.

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