The Economics of FinTech

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Chapter 9

Additional Notes on Dynamic Mean Variance Analysis for Robo-Advising

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9.1 Log Mean-Variance Criterion

In this section, we present two formulations for dynamic log-MV criterion, and show the connection between the two formulations.

9.1.1 Problem Setting

We start with a market in which there are two assets available for investment: a riskless asset (bond) with interest rate r_t, and a risky asset (stock). The stock price evolves according to

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t, \tag{9.1}$$

where the drift rate μ_t and volatility $\sigma_t > 0$, together with the interest rate r_t, are all adapted processes, and B_t is a standard Brownian motion. A self-financing wealth process W_t can be described by

$$dW_t = \left[r_t W_t + (\mu_t - r_t)u_t\right] dt + \sigma_t u_t dB_t, \tag{9.2}$$

where u_t is an adapted process representing the dollar amount invested in the stock at time t. In this paper, we do not allow investors to go bankrupt, i.e.,

$W_t > 0$ almost surely. As a result, we can rewrite the wealth process as follows:

$$\frac{dW_t}{W_t} = [r_t + (\mu_t - r_t)\pi_t]\, dt + \sigma_t \pi_t dB_t, \tag{9.3}$$

where $\pi_t := u_t/W_t$, the fraction of the total wealth in the stock, stands for a trading strategy which is admissible if π_t is adapted and $\mathbb{E}[\int_0^T |\sigma_t \pi_t|^2 dt] < +\infty$.

Let T be the investment horizon. Almost all of the existing literature on dynamic mean-variance portfolio choice focuses on the mean-variance criteria for terminal wealth W_T. In contrast, we propose a dynamic mean-variance criterion for *log-return of the portfolio*, that is, at any time $t < T$, we aim to maximize the objective

$$\mathbb{E}_t\big[\ln \frac{W_T}{W_t}\big] - \frac{\gamma_t}{2} Var_t\big[\ln \frac{W_T}{W_t}\big] \tag{9.4}$$

by choosing an admissible strategy π_t, subject to (9.3), where $\mathbb{E}_t$ and Var_t represent the conditional expectation and the conditional variance at time t, respectively, and $\gamma_t > 0$ can be regarded as the mean-variance preference parameter measuring the tradeoff between risk and return at time t.

It should be emphasized that dynamic portfolio choice (9.4) is time-inconsistent, and we are concerned with time-consistent strategies under a certain sense of optimality.[1] We will follow Björk, Khapko, and Murgoci [4] to introduce the concept of an equilibrium strategy to our dynamic mean-variance problem. In what follows, we assume an incomplete market setting with stochastic market parameters in which the riskfree rate, the stock return rate, and the stock volatility are all deterministic functions of time t and a stochastic state variable X_t, namely, $r_t = r(t, X_t)$, $\mu_t = \mu(t, X_t)$, $\sigma_t = \sigma(t, X_t)$, and X_t follows the dynamics:

$$dX_t = m\,(t, X_t)\, dt + \nu\,(t, X_t)\, dB_t^X, \tag{9.5}$$

where B_t^X is another standard Brownian motion correlated with B_t by $\mathbb{E}[dB_t^X dB_t] = \rho dt$ with constant $\rho \in [-1, 1]$, and $m(\cdot, \cdot)$ and $v(\cdot, \cdot)$ are two deterministic functions. All of these deterministic functions are assumed to be right continuous in t. Without extra effort, we can extend it to a more general case such as these deterministic functions depending on S_t as well.

To simplify notation, we denote $R_t = \ln W_t$ which satisfies

$$dR_t = \left[r_t + (\mu_t - r_t)\pi_t - \frac{1}{2}\sigma_t^2 \pi_t^2 \right] dt + \sigma_t \pi_t dB_t. \tag{9.6}$$

Due to the well-known Markovian property implied by (9.6) and (9.5), we restrict attention to feedback strategy $\pi_t = \pi(t, R_t, X_t)$. The maximization prob-

[1] We point out that the dynamic mean-variance criterion for log-return does not yield an analytical solution for the optimal pre-committed policy even under the geometric Brownian motion model. In contrast, Zhou and Li [10] present the explicit optimal pre-committed policy for the dynamic mean-variance criterion for terminal wealth, where the optimal dollar amount in stock is an affine function of the current wealth level.

lem with the objective function (9.4) can be rewritten as

$$\max_{\pi \in \mathcal{A}_t} \mathbb{E}_t\big[R_T\big] - \frac{\gamma_t}{2} Var_t\big[R_T\big] \quad t \in [0, T), \tag{9.7}$$

where the set of admissible strategies $\mathcal{A}_t$ is defined as

$$\mathcal{A}_t = \left\{ \pi_s = \pi(s, R_s, X_s) : \mathbb{E}_t \left[\int_t^T |\sigma_s \pi(s, R_s^\pi, X_s)|^2 ds \right] < +\infty \right\}.$$

To emphasize the dependence of the log-return process on π, we denote by R_t^π the log-return process associated with π. The reward function related to π is denoted by

$$J(t, R_t, X_t; \pi) := \mathbb{E}_t\big[R_T^\pi\big] - \frac{\gamma_t}{2} Var_t\big[R_T^\pi\big].$$

Following Björk, Khapko, and Murgoci [4], we now define an *equilibrium solution* to problem (9.7), which leads to an equilibrium strategy that is optimal locally at any time given that the strategy will be followed in the future.

Definition 1. *An admissible trading strategy $\hat{\pi}(\cdot, \cdot, \cdot)$ is called an (optimal) equilibrium strategy for problem (9.7) if, at any time t, for any admissible perturbation strategy $\pi^{h,v} \in \mathcal{A}$ defined by*

$$\pi^{h,v}(\tau, y, x) = \begin{cases} v, & \text{for } t \le \tau < t + h, \\ \hat{\pi}(\tau, y, x), & \text{for } t + h \le \tau \le T, \end{cases}$$

with any $h \in \mathbb{R}^+$ and $v \in \mathbb{R}$, the reward function $J(t, y, x; \hat{\pi})$ is locally better off, namely,

$$\liminf_{h \to 0^+} \frac{J(t, y, x; \hat{\pi}) - J(t, y, x; \pi^{h,v})}{h} \ge 0.$$

The equilibrium value function V generated by the equilibrium strategy $\hat{\pi}$ is thus defined as $V(t, y, x) = J(t, y, x; \hat{\pi})$.

This definition implicitly imposes time-consistency in the sense that, at any time $t < T$, given the trading strategy in the future the investor will not deviate from her current trading strategy. It should be pointed out in a discrete-time setting, this kind of time-consistency can also be achieved by imposing backward induction and a new optimization problem at any time t (see, e.g., [3] and [9]). However, in a continuous-time setting, because of the lack of "the previous time spot" and "the next time spot", the backward induction constraint cannot be easily adopted.

It is worthwhile pointing out that unlike in Björk, Murgoci, and Zhou [5], our formulation (9.7) does not lead to a dependence on the wealth value at time t. Therefore, we do not need the technique developed by Björk, Khapko, and Murgoci [4] for handling such dependence. Instead, we develop a BSDE approach to solve problem (9.7) in a general market setting, due to the fact that the log wealth process involves a quadratic control variable.

In the formulation (9.7), we need to prescribe the mean-variance preference parameter γ_t that is less intuitive. Let us consider an alternative dynamic mean variance formulation loyal to the original (one-period) Markowitz's model:

$$\min_{\pi \in \mathcal{A}_t} Var_t(R_T), \quad \text{subject to } \frac{1}{T-t}\mathbb{E}_t[R_T - R_t] \geq \hat{a}_t, \tag{9.8}$$

where $\hat{a}_t$ is a predetermined adaptive process, representing the investor's expected annual target return at time t. The formulation (9.8) indicates that the investor dynamically minimizes her risk subject to a predetermined target annual return, $\hat{a}_t$, which may depend on the investment horizon and the realized sample path.

We can similarly define equilibrium solution to problem (9.8). Indeed, we only need to replace J and $\mathcal{A}_t$ in Definition 1 by $\bar{J}(t, R_t, X_t; \pi) := -Var_t\left[R_T^\pi\right]$ and $\bar{\mathcal{A}}_t = \{\pi \in \mathcal{A}_t : \frac{1}{T-s}\mathbb{E}_s[R_T^\pi - R_s^\pi] \geq \hat{a}_s,\ s \in [t, T)\}$, respectively.

Recently He and Jiang [7] independently study the formulation (9.8), but focus on myopic strategies, by considering only deterministic drift and volatility. In contrast, we have a more general setting, resulting in an extra non-myopic term (known as intertemporal hedging demand) in markets with stochastic coefficients; in addition, similar to Basak and Chabakauri [1] and Björk, Khapko, and Murgoci [4], our focus is the formulation (9.7), and the formulation (9.8) will be employed to mainly identify the mean-variance preference parameter in a complete market.

9.1.2 Connection between Two Formulations

We have an interesting link from the formulation (9.7) to the formulation (9.8).

Theorem 1. *Let $\hat{\pi}$ be an equilibrium policy to (9.7) and $R_t^{\hat{\pi}}$ be the associated optimal return. Then $\hat{\pi}$ must be an equilibrium policy to (9.8) with*

$$\hat{a}_t = \frac{1}{T-t}\mathbb{E}_t[R_T^{\hat{\pi}} - R_t^{\hat{\pi}}].$$

Proof. If there exists a perturbation v at time t, such that

$$\liminf_{h \to 0^+} -\frac{\mathrm{V}ar_t(R_T^{\hat{\pi}}) - \mathrm{V}ar_t(R_T^{\pi^{h,v}})}{h} < 0.$$

then

$$\begin{aligned}
& \liminf_{h \to 0^+} \frac{J(t, y, x; \hat{\pi}) - J(t, y, x; \pi^{h,v})}{h} \\
= \quad & \liminf_{h \to 0^+} \frac{\left(\mathbb{E}_t[R_T^{\hat{\pi}}] - \mathbb{E}_t[R_T^{\pi^{h,v}}]\right) - \frac{\gamma}{2}\left(\mathrm{V}ar_t(R_T^{\hat{\pi}}) - \mathrm{V}ar_t(R_T^{\pi^{h,v}})\right)}{h} \\
\leq \quad & \liminf_{h \to 0^+} -\frac{\gamma}{2}\frac{\mathrm{V}ar_t(R_T^{\hat{\pi}}) - \mathrm{V}ar_t(R_T^{\pi^{h,v}})}{h} < 0,
\end{aligned}$$

which contradicts the fact that $\hat{\pi}$ is an equilibrium strategy. $\square$.

In general, there is no easy connection for the reverse direction, i.e. from an equilibrium policy for (9.8) to that for (9.7). Throughout the rest of this paper (unless otherwise stated), we always make the following assumption, and mainly focus on the formulation (9.7) in which the investor's mean-variance preference is characterized by a constant γ.

Assumption A: The mean-variance preference parameter γ for an individual investor is a positive constant and remains unchanged.

A natural and challenging question is how to estimate the mean-variance preference parameter γ that measures the trade-off between mean return and variance. It is well-known that in the single-period mean-variance model, there exists a one-one mapping between γ and the expected target return. As a consequence, one may instead request the investor to input a target return, from which one can infer γ. The advantage of this idea is that the target return is very intuitive to investors who are given the market information (e.g. the expected return level and variance level of risky assets, and the risk-free rate level in the market). Moreover, investors do not need to know the formula between γ and the target return, or to understand any optimization behind the mean-variance portfolio selection. Interestingly, as will be shown in (9.12), under the complete market with constant investment opportunity, our dynamic model also reveals a one-one mapping between γ and the annual target return. This allows us to borrow the idea used in the single-period model to estimate γ via a fictitious complete market. Thanks to Assumption A, the parameter γ will be used to solve our mean-variance problem in an incomplete market.

9.2 Portfolio Choices for a Complete Market

In this section, we study the dynamic log-MV portfolio choice under a complete market setting with constant market parameters $\mu_t \equiv \mu$, $r_t \equiv r$, and $\sigma_t \equiv \sigma$.

9.2.1 An Equilibrium Solution

The portfolio choice problem with reward function (9.7) yields a closed form equilibrium solution in the complete market as follows.

Theorem 2. *Consider the mean-variance criterion (9.7) subject to (9.6) under the complete market setting with constant market parameters.*

(i) An equilibrium strategy is given by

$$\hat{\pi} \equiv \frac{\mu - r}{(1+\gamma)\sigma^2}. \tag{9.9}$$

(ii) The target annual return associated with the equilibrium strategy is constant, namely,

$$\frac{1}{T-t}\mathbb{E}_t[R_T^{\hat{\pi}} - R_t^{\hat{\pi}}] \equiv \hat{a} =: r + \left(\frac{1}{1+\gamma} - \frac{1}{2}\frac{1}{(1+\gamma)^2}\right)\theta^2, \tag{9.10}$$

where $\theta = (\mu - r)/\sigma$. Moreover, $\hat{a} \in (r, r + \frac{1}{2}\theta^2)$.

Proof. For any deterministic trading policy $\pi.$, we can get $\mathbb{E}_t[R_T] = R_t + \int_t^T \left(r + \pi_s(\mu - r) - \frac{1}{2}\pi_s^2\sigma^2\right) ds$ and $\mathrm{V}ar_t(R_T) = \int_t^T \pi_s^2\sigma^2 ds$. Since $\hat{\pi}$ is deterministic,

$$J(t, R_t, X_t; \hat{\pi}) = R_t + \int_t^T \left(r + \hat{\pi}_s(\mu - r) - \frac{1+\gamma}{2}(\hat{\pi}_s)^2\sigma^2 \right) ds.$$

Similarly, for any perturbation $\pi_s^{h,v} := v\mathbf{1}_{s\in[t,t+h)} + \hat{\pi}_s\mathbf{1}_{s\in[t+h,T]}$, which is also deterministic, we also have

$$\begin{aligned} & J(t, R_t, X_t; \pi^{h,v}) = R_t + \int_t^T \left(r + \pi_s^{h,v}(\mu - r) - \frac{1+\gamma}{2}(\pi_s^{h,v})^2\sigma^2 \right) ds \\ = \quad & J(t, R_t, X_t; \hat{\pi}) + \int_t^{t+h} \left(\left[v(\mu - r) - \frac{1+\gamma}{2}v^2\sigma^2 \right] - \left[\hat{\pi}(\mu - r) - \frac{1+\gamma}{2}(\hat{\pi}_s)^2\sigma^2 \right] \right) ds. \end{aligned}$$

Since $\hat{\pi}_s = \mathrm{argmax}_{v\in\mathbb{R}} \left\{ v(\mu - r) - \frac{1+\gamma}{2}v^2\sigma^2 \right\}$, we know $J(t, R_t, X_t; \pi^{h,v}) \leq J(, R_t, X_t; \hat{\pi})$. Hence $\hat{\pi}$ is an equilibrium solution. Part (ii) then follows by a direct calculation. $\square$

Basak and Chabakauri [1] link a certain dynamic mean variance asset allocation to CARA preferences, generalizing the well-known connection between mean variance and utility maximization in a one-period setting. In contrast, part (i) of the above theorem establishes an equivalence between dynamic mean-variance and CRRA preferences in the complete market. More precisely, the dynamic mean-variance equilibrium strategy as given by (9.9) is the same as the optimal solution for CRRA utility maximization (Merton [8])

$$\max_{\pi} \mathbb{E}_t \left[\frac{W_T^{1-\tilde{\gamma}} - 1}{1 - \tilde{\gamma}} \right]$$

subject to the self-finance process (9.3), where the relative risk aversion parameter of the CRRA optimizer $\tilde{\gamma} = 1 + \gamma$. Note that CRRA utility is effectively a moment generating function of the log return:

$$\begin{aligned} & \mathbb{E}_t \left[\frac{W_T^{1-\tilde{\gamma}} - 1}{1 - \tilde{\gamma}} \right] \\ = \quad & \mathbb{E}_t \left[\frac{e^{(1-\tilde{\gamma})\ln W_T} - 1}{1 - \tilde{\gamma}} \right] = \mathbb{E}_t \left[\frac{1 - e^{-\gamma \ln W_T}}{\gamma} \right] \\ = \quad & \mathbb{E}_t \left[\ln W_T - \frac{\gamma}{2}(\ln W_T)^2 + \frac{\gamma^2}{3!}(\ln W_T)^3 - \frac{\gamma^3}{4!}(\ln W_T)^4 + \cdots \right], \quad (9.11) \end{aligned}$$

which indicates that CRRA preferences make use of all moments of log-return, whereas mean-variance uses only the first two.[2] It is easy to see that when $\gamma \to 0$

[2] Based on this observation, the risk sensitive asset management model, e.g. in Bielecki and Pliska [2] and Davis and Lleo [6], is also relevant to our dynamic mean-variance model. However, there is no literature on estimating the risk aversion parameter in the risk sensitive asset management model.

($\tilde{\gamma} \to 1$), the mean-variance optimization and the CRRA utility maximization are identical and thus yield the same optimal solution. For $\gamma > 0$ ($\tilde{\gamma} > 1$), the two optimization problems have different implications, and the CRRA utility maximization is naturally time consistent whereas the mean-variance optimization is not. However, they share the same optimal strategy, despite that the optimality is achieved in different senses. Notice that the equivalence holds only for $\gamma > 0$ and the mean-variance optimization does not make sense for $\gamma < 0$, which coincides with the fact that the estimated value of the CRRA parameter $\tilde{\gamma}$ is usually larger than 1.

Later we will see that in incomplete markets, our dynamic mean-variance optimization and the CRRA utility maximization are not equivalent in general; instead, they are linked via a measure transformation. Since (9.11) remains valid in any markets, it is not surprising that our mean-variance policy still mimics the Merton's policy with CRRA preferences in incomplete markets.

Part (ii) of Theorem 2 reveals that under a complete market setting with constant market parameters, the expected annual return of the equilibrium strategy is constant. It is easy to see that the return is monotonically decreasing with γ, which coincides with the intuition that the higher the mean-variance preference parameter, the lower the expected target return. Part (ii) also suggests that the mean-variance maximizer cannot expect an unreasonable target return: At time t, any target return that is higher than $r + \frac{1}{2}\theta^2$ is never attainable.

9.2.2 Recovery of mean-variance preference

Note that (9.10) can be rewritten as

$$\gamma = \frac{1}{1 - \sqrt{1 - 2(\hat{a} - r)/\theta^2}} - 1, \tag{9.12}$$

which indicates a one-one mapping between the mean-variance preference parameter γ and the expected target annual return $\hat{a}$. By Theorem 2, we infer the following result.

Corollary 1. *Assume a complete market setting with constant market parameters. Consider the mean-variance criterion* (9.8) *with constant target annual return* $\hat{a}_t \equiv \hat{a} \in (r, r + \frac{1}{2}\theta^2]$*. Then an equilibrium strategy* $\hat{\pi}$ *is given by* (9.9) *and* γ *is as given by* (9.12).

With constant target annual return, problem (9.8) is intuitively more appealing. The above corollary indicates that under a complete market setting with constant market parameters, problem (9.8) with a given constant target annual return $\hat{a}$ shares the same equilibrium policy as problem (9.7) with constant mean-variance preference parameter γ computed by (9.12). This observation suggests a simple and intuitive way, which can be employed to identify the mean-variance preference parameter γ used in our dynamic mean-variance criterion (9.7): Given a "fictitious" complete market with (exogenously given) constant parameters μ, σ and r, an individual investor is asked in a questionnaire to input his/her constant target annual return $\hat{a}$; then the investor's mean-variance preference parameter γ used in (9.7) can be identified through (9.12).

The one-to-one mapping in (9.12) between $\hat{a}$ and γ is special for the complete market with constant market parameters (μ, r, σ). In an incomplete market or a complete market with time-varying market parameters, in which investment opportunity is varying, investors are unlikely to maintain a constant annual target return $\hat{a}$. As a result it may be difficult to infer investors' mean-variance preference parameter via their target returns. That is why we propose a fictitious complete market in the questionnaire to recover the preference parameter; and then, thanks to Assumption A, we can use the parameter γ to solve the dynamic mean-variance problem in incomplete markets.

It should be pointed out that as in the single-period Markowitz framework, the estimation suffers a drawback: it may not be robust in the sense that a change in the expected target return may lead to a different γ. To partially overcome this disadvantage, we may repeat the estimation with several sets of market parameters and provide an average estimate of λ.

More precisely, in practice we can show investors in the questionnaire several constant investment environments, e.g. (μ_i, r_i, σ_i), $i = 1, 2, 3, 4$, and ask investors to input their target returns $\hat{a}_i$ within the range between r_i and $r_i + \frac{1}{2}\theta_i^2$, $i = 1, 2, 3, 4$, respectively. Then we can get γ_i, $i = 1, 2, 3, 4$ using (9.12) and recover bounds on the mean-variance preference parameter. Note that estimating the mean-variance preference parameter is a special case of estimating the risk profile of an investor. In general, how to estimate the risk profile in a robust way is a difficult problem in decision science.

Bibliography

[1] Basak, S., and Chabakauri, G. (2010). Dynamic mean–variance asset allocation. *Review of Financial Studies*, 23, 2970-3016.

[2] Bielecki, T., and Pliska, H. (1999). Risk-sensitive dynamic asset management. *Applied Mathematics and Optimization* 39, 337-360.

[3] Björk, and Murgoc, A. (2014). A theory of Markovian time-inconsistent stochastic control in discrete time. *Finance and Stochastics*, 18, 545-–592.

[4] Björk, T., Khapko, M., and Murgoc, A. (2017). On time-inconsistent stochastic control in continuous time. *Finance and Stochastics*, 21, 331-360.

[5] Björk, T., Murgoci, A., and Zhou, X. Y. (2014). Mean–variance portfolio optimization with state-dependent risk aversion. *Mathematical Finance*, 24, 1-24.

[6] Davis, M., and Lleo, S. (2014). *Risk-sensitive investment Management.* World Scientific Publishing Company.

[7] He, X. and Jiang, Z. (2018). Dynamic mean-risk asset allocation and myopic strategies: A universal portfolio rule. Working paper, Chinese University of Hong Kong.

[8] Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3, 373-413.

[9] Shapiro, A. (2012). Time consistency of dynamic risk measures. *Operations Research Letters,* 40, 436-439.

[10] Zhou, X. Y., and Li, D. (2000). Continuous-time mean-variance portfolio selection: A stochastic LQ framework. *Applied Mathematics and Optimization*, 42, 19-53.