Exercise 1. Consider a geometric Brownian motion

$$Z_t = \exp\{\gamma t + \beta W_t\}, \quad \gamma > 0.$$

Define

$$\tau = \inf\{t \geq 0 : Z_t = 1 + \lambda\}, \quad \lambda > 0.$$

Show that

$$E[\tau] = \frac{1}{\gamma} \ln(1 + \lambda).$$

Note that we need the assumption $\gamma > 0$ in the above statement; otherwise, if $\gamma \leq 0$, then $E[\tau] = \infty$.

$Z_t = e^{rt+\beta W_t}$. and $T = \inf\{t \geq 0 : Z_t = 1+\lambda\}$

$$T = \inf\{t \geq 0 : e^{rt+\beta W_t} = 1+\lambda\}.$$

$$T = \inf\{t \geq 0 \quad rt + \beta W_t = \ln^{(1+\lambda)}\}$$

$$T = \inf\{t \geq 0 : t = \tfrac{1}{r}(\ln^{1+\lambda} - \beta W_t)\}.$$

$$E(\tau) = E(\inf\{\tfrac{1}{r}(\ln^{(1+\lambda)} - \beta W_t)\})$$

if $r > 0$ $\Rightarrow$ $= \inf E(\frac{1}{r}\ln^{(1+\lambda)} - \beta W_T)$

$$= \frac{1}{r}\ln(1+\lambda)$$

if $r \leq 0$ $E(\tau) = E(\sup(-\frac{1}{r}\ln(1+\lambda) - \beta W_T))$

$$= \infty$$