Inverse Optimization: A New Perspective on the Black-Litterman Model

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Paper we studied

- Paper: "Inverse Optimization: A New Perspective on the Black-Litterman Model"
  - Dimitris Bertsimas
  - Vishal Gupta
  - Ioannis Ch. Paschalidis

Content

- Introduction
- Inverse Optimization
- Example View in the Paper — Factor Model
- Simulation in the paper
- Our Simulation
- Our Example Views
- Comments & Improvements

Introduction

- Black Litterman
  - Assumes that covariance matirx is easy to estimate and focus on return estimation
  - Used for broad asset allocation
- This paper focuses on Covariance estimation

Introduction

Main Content of the Paper

- Inverse Optimization
  - MV-IO
  - RMV-IO
- Simulation with an Example View

Introduction

Our Contribution

- Explanation of the simulation result in the paper
- New Simulation to better evaluate the model
- Comments on the Application of the model

Theoretical Framework

What is Black Litterman?

- Market Equilibrium: $\boldsymbol{\mu} = r_f \mathbf{e} + 2\delta \boldsymbol{\Sigma} \mathbf{x}^{\mathrm{mkt}}$
- Private Information: $\mathbf{P}\boldsymbol{\mu} = \mathbf{q}$
- Updating Rules:

$$\boldsymbol{\mu}^{\mathrm{BL}} = \boldsymbol{\Sigma}_2^{-1} \begin{pmatrix} \mathbf{I} \\ \mathbf{p} \end{pmatrix}' \boldsymbol{\Omega}^{-1} \begin{pmatrix} \hat{\mathbf{r}} \\ \mathbf{q} \end{pmatrix}$$

$$\boldsymbol{\Sigma}_2 = \left[\begin{pmatrix} \mathbf{I} \\ \mathbf{P} \end{pmatrix}' \boldsymbol{\Omega}^{-1} \begin{pmatrix} \mathbf{I} \\ \mathbf{P} \end{pmatrix} \right], \quad \boldsymbol{\Sigma}^{\mathrm{BL}} = \boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2$$

where $\Sigma_1 = \hat{\boldsymbol{\Sigma}}$, $\quad \boldsymbol{\Omega} = \begin{pmatrix} \tau_0 \hat{\boldsymbol{\Sigma}} & \mathbf{0} \\ \mathbf{0} & \mathrm{diag}\left(\tau_1, \ldots, \tau_m\right) \end{pmatrix}$

Why Inverse Optimization

Inverse Optimization Problem

$$\min_{\mathbf{x}\in(\zeta)} f(\mathbf{x};\zeta)$$
$$\mathbf{x}^* = \arg\min_{\mathbf{x}_{\mathbf{k}\in\mathcal{G}(\mathbf{z})}} f(\mathbf{x};\zeta)$$

Given x^*, inverse optimization is to find a set of parameters ζ to make x^* is the solution to the above optimization problem

Advantages

The key insight is to use inverse optimization to characterize the BL estimate as the solution to a particular convex optimization problem, thereby eliminating the need for a statistical model

Under Inverse Optimization

Formation of Problem

$$\min_{\boldsymbol{\mu},\overline{\boldsymbol{\Sigma}},t} \left\{ t : \left\| \begin{pmatrix} \boldsymbol{\mu} - \boldsymbol{r}_f \mathbf{e} - \overline{\boldsymbol{\Sigma}} \mathbf{x}^* \\ \mathbf{P}\boldsymbol{\mu} - \mathbf{q} \end{pmatrix} \right\| \leqslant t, \overline{\boldsymbol{\Sigma}} \succeq \mathbf{0} \right\} \tag{1}$$

We here transform out need of solving problems like BL to the above minimization problem, which aims to find a set of parameter $\mu, \overline{\Sigma}, t$ to minimize the violation of BL conditions.

If the $\overline{\Sigma} = 2\delta\hat{\Sigma}$, and the norm is defined to be $\|\mathbf{z}\|_2^{\boldsymbol{\Omega}} = \sqrt{\mathbf{z}'\boldsymbol{\Omega}^{-1}\mathbf{z}}$, we have the exactly same solution to BL model with respective to the estimate of μ

Under Inverse Optimization

Under Constraints

With the above framework, it is always easy to add on constraints, and private views on σ.
For example, if investor have some information on the σ of a basket of asset b, then what she needs is to only add on the following constraints.

$$\left\| \mathbf{b}'\overline{\mathbf{\Sigma}}\mathbf{b} - \sigma^2 \right\| \leqslant \epsilon \tag{2}$$

Example View in the Paper

Factor-like Views

$$\sum_{i=1}^{k} \lambda_i \geqslant \alpha \cdot \mathrm{trace}(\overline{\boldsymbol{\Sigma}}), \quad \left\| \overline{\boldsymbol{\Sigma}} \mathbf{v}^i - \lambda_i \mathbf{v}^i \right\| \leqslant \epsilon \quad i = 1, \ldots, k \tag{3}$$

Here we use factors from PCA formation, however, it is always welcome to impose economic factor model here like Fama-French Factors Model, etc, and the adaptation is intuitive and straight-forward.

MVIO & RMVIO

The above optimization with factor-like views on volatility is an example of the Mean-Variance Inverse Optimization formation in this paper.
Further, the paper also comes up with a Robust Mean Variance Inverse Optimization which in addition satisfies the requirement for the maximum of possible loss.

RMVIO

optmization problem

We start with following problem considering the maximum loss(VaR):

$$\max_{\mathbf{x}} \left\{ \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})\, r_f : (\mathbf{r} - r_f\mathbf{e})'\,\mathbf{x} \geqslant -L, \forall \mathbf{r} \in \mathcal{U} \right\}$$

$$\mathcal{U} = \Big\{ \mathbf{r} : \exists \boldsymbol{\Sigma} \text{ s.t. } \boldsymbol{\Sigma} \geq \mathbf{0}, (\mathbf{r} - r_f\mathbf{e})'\,\boldsymbol{\Sigma}^{-1}(\mathbf{r} - r_f\mathbf{e}) \leqslant 1$$

where

$$\left\| \boldsymbol{\Sigma}\mathbf{v}^i - \lambda_i \mathbf{v}^i \right\| \leqslant \epsilon \forall i = 1, \ldots, k, \operatorname{tr}(\boldsymbol{\Sigma}) \leqslant \frac{1}{\alpha}\sum_{i=1}^{k} \lambda_i \Big\}$$

The intuition of this form is that given any covariance matrix that is similar in factors, and any return vector that satisfies $(\mathbf{r} - r_f\mathbf{e})'\,\boldsymbol{\Sigma}^{-1}(\mathbf{r} - r_f\mathbf{e}) \leqslant 1$, we have the above optimization problem exactly the same as the Markovitz Mean-Variance problem, but meanwhile, it guarantee the maximum of loss will not exceed L by making the possible return vector r related to Σ

RMVIO

Inverse Optimization Form

First we set $L_{mkt} = max_{r \in \mathcal{U}} - rx_{mkt}$, this guarantee the market portfolio is in the feasible region. Then if market portfolio is the optimal weights, we have the following relationship hold:

$$H = \Big\{ \boldsymbol{\mu} : \exists \boldsymbol{\Sigma} \text{ s.t. } (\boldsymbol{\mu} - \boldsymbol{r}_f \mathbf{e})' \mathbf{x}^{\mathrm{mkt}} = zL_{\mathrm{mkt}}, \begin{pmatrix} \boldsymbol{\Sigma} & \boldsymbol{\mu} \\ \boldsymbol{\mu}' & z \end{pmatrix} \geq \mathbf{0}$$
$$\left\| \boldsymbol{\Sigma} \mathbf{v}^i - z\lambda_i \mathbf{v}^i \right\| \leqslant \epsilon \forall i = 1, \ldots, k, \mathrm{tr}(\boldsymbol{\Sigma}) \leqslant z \frac{1}{\alpha} \sum_{i=1}^k \lambda_i \Big\}$$

In this set, all the μ can make the market portfolio optimal.

adding BL type views

$$\min_{t, \boldsymbol{\mu}, \boldsymbol{\mu}^{eq}, \boldsymbol{t}} \left\{ t : \boldsymbol{\mu}^{eq} \in H, \left\| \begin{pmatrix} \boldsymbol{\mu} - \boldsymbol{r}_f \mathbf{e} - \boldsymbol{\mu}^{eq} \\ \mathbf{P}\boldsymbol{\mu} - \mathbf{q} \end{pmatrix} \right\|_{\boldsymbol{n}} \leqslant t \right\}$$

Here the return vector μ we are actually finding , is firstly close enough to the set H, which can make the market portfolio optimal, and is secondly satisfying the private views on return.

Simulation in the Paper (Factor Model)

Comment on the Factor-like Views

- May not be suitable in small number of asset(i.e 3, 4)
- First Principal Component is Market Volatility
- MV-IO Optimized Weights shift towards Market Weights
- Simply becomes a Combination of Market weights and BL weights
- Can be explained as Market weights is better than BL weights
- More diversified, lower volatility

Simulation in the Paper (Factor Model)

	x^{mkt}	x^{BL}	x^{MV}
Energy	7.15	5.03	6.03
Materials	4.54	8.58	5.75
Industrials	10.92	10.94	10.93
Consumer discretionary	10.77	5.56	8.45
Consumers staple	13.47	17.36	16.16
Health care	13.57	9.20	11.91
Financial	15.81	16.93	17.26
information technology	15.16	16.09	15.41
communication service	5.88	2.61	3.47
Utilities	2.74	7.71	4.63

Simulation in the Paper (Factor Model)

- The CAPM equilibrium holds entirely
- The imposed view on return is entirely incorrect
  - p=[-10%, 0%, 0%, -20%, 40%, -10%, 30%, 0%, -40%, 10%]
  - Expected in equilibrium: $p\hat{r}$ =0.3862 % (real q)
- View on MV-IO Volatility: Factor Model
- Results: Table 2 in the paper

Table 2. Sensitivity to accuracy of the view under CAPM assumptions.

	q	−10	−5	−2	−1	0	1	2	5	10
Return	Mkt	4.14	4.14	4.14	4.14	4.14	4.14	4.14	4.14	4.14
	BL	3.22	3.75	3.96	4.00	4.02	4.02	4.00	3.84	3.38
	MV-IO	4.08	4.11	4.12	4.13	4.14	4.11	4.10	4.05	3.92
Std dev	Mkt	12.87	12.87	12.87	12.87	12.87	12.87	12.87	12.87	12.87
	BL	14.26	13.03	12.54	12.48	12.48	12.55	12.67	13.31	14.67
	MV-IO	12.73	12.80	12.84	12.85	12.87	12.78	12.74	12.61	12.30
Sharpe ratio	Mkt	32.17	32.17	32.17	32.17	32.17	32.17	32.17	32.17	32.17
	BL	22.62	28.77	31.55	32.01	32.17	32.02	31.56	28.88	23.07
	MV-IO	32.07	32.10	32.11	32.11	32.17	32.16	32.15	32.09	31.90

Simulation in the Paper (Factor Model)

Explanation in the Paper

- q=0
  - BL, MV-IO, mkt are close
- As $|q|$ increases
  - BL & MV-IO underperform relative to market (because the view is increasingly incorrect)
  - BL has worse returns and Sharpe ratio than MV-IO
  - MV-IO more robust to inaccuracy in the views

Simulation in the Paper (Factor Model)

Our Explanation on the Result

- $q = 0$: Correct but Useless View
- Factor model coincide with the simulated world

Our Comment on the Simulation

- Lack of scenario of Correct and Useful BL View
- Lack of scenario of Wrong View of Volatility in MV-IO

Our Simulation

$q = 1$ means our view of return is right

	q	-1	0	1	2	3
Return	Mkt	0.559	0.559	0.559	0.559	0.559
	BL	-220.95	11.880	10.614	11.250	11.502
	MV-IO	-7.247	55.519	10.620	23.307	22.307
Std dev	Mkt	17.925	17.925	17.925	17.925	17.925
	BL	52.158	18.232	17.960	18.001	18.017
	MV-IO	17.303	19.702	17.954	18.260	17.800
Sharpe ratio	Mkt	0.031	0.031	0.031	0.031	0.031
	BL	-4.236	0.651	0.59	0.624	0.618
	MV-IO	-0.418	2.817	0.591	1.276	1.253

Our Simulation

When our view of return is right but view of volatility is wrong

	mean	std	SR
mkt	0.559615	17.925170	0.031219
BL	10.614730	17.960760	0.590996
MV-IO	-237.808815	40.829848	-5.824387

Comments & Improvements

Issue in Practical Application

- Sensitivity of SDP solution
- Convex optimization package not work well

Comments & Improvements

Scope of Application

- Too Complex when BL is Suitable(Covariance matrix, asset number)
  - Covariance of BL model in class vs this paper
- Flexibility: could be applied in problems that traditional BL cannot be solved (in this paper: PCA)
- Inverse Optimization could be used in improving accuracy (i.e. High moment parameters)

Thank You!