HWK Problem.

1. Let X_1,..., X_n be random samples from an exponetial distribution with rate λ. Derive the posterior distribution of the Bayesian estimator $\hat{\lambda}$, if the prior distribution of λ is Gamma (α, β) with density

$$\frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\lambda} \lambda^{\alpha-1}.$$

$f(x) = \lambda e^{-\lambda x} \quad x \geq 0$

$f(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\lambda} \lambda^{\alpha-1}$

$f(x|\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$

$= \lambda^n e^{-\lambda(\sum_{i=1}^{n} X_n)}$

$f(\lambda|x) \propto f(x|\lambda) f(\lambda)$

let $Y_n = \sum_{i=1}^{n} X_n$

$\propto \lambda^n e^{-\lambda Y_n} e^{-\beta\lambda} \lambda^{\alpha-1}$

$\propto \lambda^{n+\alpha-1} e^{-\lambda(Y_n+\beta)}$

and let $\alpha^* = n+\alpha \quad \beta^* = Y_n + \beta$

$\Rightarrow$ and we know the mean of Gamma distribution

$E(X) = \frac{\alpha}{\beta}$

$\Rightarrow \hat{\lambda} = \frac{n+\alpha}{Y_n+\beta} = \frac{n+\alpha}{\sum_{i=1}^{n} X_i + \beta}$

HWK:

Using the data in this lecture notes, compute the optimal asset allocation in the Black-Litterman model for US, Euro, and Japanese stocks for $\tau = 0.1$, 1, and 10.

$\tau = 0.1, 1, 10$

calculate in MATLAB.

$$C = \begin{pmatrix} 7.344\% & 2.015\% & 3.309\% \\ 2.015\% & 4.410\% & 1.202\% \\ 3.309\% & 1.202\% & 3.497\% \end{pmatrix}$$

$$\xi = 0.42667$$

$$w = \left(\frac{35}{50}, \frac{10}{50}, \frac{5}{50}\right)^T$$

so $\pi = \frac{1}{\xi} C w = (0.1377, 0.0565, 0.0681)^T$

$$\Sigma = \tau\, C_{N\times N}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \quad q = \begin{pmatrix} 2.5\% \\ 2\% \end{pmatrix} \quad \Omega = \begin{pmatrix} (1\%)^2 & \\ & (1.5\%)^2 \end{pmatrix}$$

so. we put all the things into formula

$$\bar{R} = \pi + \Sigma P^T (P \Sigma P^T + \Omega)^{-1} (q - P\pi)$$

$$\bar{C} = C + \Sigma - \Sigma P^T (P \Sigma P^T + \Omega)^{-1} (P \Sigma)$$

$\Rightarrow w^* = \xi \bar{C}^{-1} \bar{R}$

$\Rightarrow$ when $\tau = 0.1$ $\quad w^* = (-0.0018, 0.4089, 0.0904)^T$

$\tau = 1$ $\quad w^* = (0.0091, 0.4145, 0.0500)^T$

$\tau = 10$ $\quad w^* = (0.0259, 0.4129, 0.0091)$