Constructing Risk Neutral Densities

Goals:

- Review some of the most common stochastic processes and their properties.
- Discuss the relationship between stochastic processes and risk neutral densities.
- Explore methods for extracting risk neutral densities from a set of observed market prices.
- Examine the computational tools that are involved in this process.
- Discuss the differences between risk neutral and physical densities.

Review: Quadrature Methods in Options Pricing

- Earlier in the course, we learned how to use quadrature methods in order to approximate integrals that are difficult to compute analytically.
- In options pricing, we are particularly interested in applying quadrature methods to integrals of the form:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-\int_0^T r_u\,du} f(S_T)\right] \tag{1}$$

- Taking a standard European call, this can be written as:

$$c_0 = e^{-\int_0^T r_u\,du} \int_{-\infty}^{+\infty} (S_T - K)_+ \phi(S_T)\,dS_T \tag{2}$$

- In cases where the density, $\phi(x)$ is known, but the integral is unknown, this is a useful method.

Where does the density come from?

- Key question: Where do we obtain the density to put into the risk neutral valuation formula?
- Perhaps the most standard approach is to begin with a model that defines the SDE for the underlying asset and attempt to solve the SDE in order to obtain the density.
- This method relies on us choosing an SDE that represents the dynamics in the market well. (Why?)
- What models/SDE's work best?

Review of Common Stochastic Processes

- Some of the most commonly used stochastic processes are:
  - Black-Scholes (Log-Normal model)
  - Bachelier (Normal model)
  - CEV Model
  - Ornstein-Uhlenbeck (Mean Reverting model)
  - Cox-Ingersoll-Ross model
  - Heston model
  - SABR model
  - Variance Gamma model

Normal / Log-Normal Model

- **Black-Scholes Model (Log-Normal)**

$$dS_t = rS_t\,dt + \sigma S_t\,dW \tag{3}$$

  - Asset prices are log-normally distributed.
  - Model parameters are r and σ.

- **Bachelier Model (Normal)**

$$dS_t = rS_t\,dt + \sigma\,dW \tag{4}$$

  - Asset prices are normally distributed.
  - Model parameters are r and σ.
  - Model is most commonly used in interest rate markets.

- Question: What is the difference in the σ's between the two models?

CEV Model

$$dS_t = rS_t\,dt + \sigma S_t^{\beta}\,dW \tag{5}$$

- The CEV model is a generalization of the log-normal and normal models.
- Model parameters r, σ and β
- The additional parameter, β can be used to account for some degree of skew in the volatility surface.
- Cases $\beta = 1$ and $\beta = 0$ reduce back to the log-normal/normal models respectively.

Ornstein-Uhlenbeck Process / CIR Model

- **Ornstein-Uhlenbeck Process**

$$dS_t = \kappa(\theta - S_t)dt + \sigma dW_t \tag{6}$$

  – Asset prices are normally distributed. (Why?)

  – Model parameters are κ, θ and σ.

    κ represents the **speed of mean reversion**

    θ represents the **level of mean reversion**

- **CIR Model**

$$dS_t = \kappa(\theta - S_t)dt + \sigma\sqrt{S_t}dW_t \tag{7}$$

  – Square root in volatility term prevents S_t from being negative.

  – Otherwise, same as OU process.

- Models are most commonly used in interest rate and volatility modeling.

Local Volatility Model

$$dS_t = r(t)S_t\,dt + \sigma(S_t,t)S_t\,dW \tag{8}$$

- Asset prices are log-normally distributed at every increment.
- Volatility and drift are no longer constant and are now parameterized by state.
- Model parameters are the functions $r(t)$ and $\sigma(S_t,t)$.
- Imposes relatively minimal restrictions on the process of the underlying asset, because of the flexibility we have in choosing the function $\sigma(S_t,t)$.
- Enables exact fit to volatility surfaces, in the absense of arbitrage and data issues (which we will see later).

Heston Model

$$\begin{aligned} dS_t &= rS_t\,dt + \sigma_t S_t\,dW_t^1 \\ d\sigma_t^2 &= \kappa(\theta - \sigma_t^2)\,dt + \xi\sigma_t\,dW_t^2 \\ Cov(dW_t^1, dW_t^2) &= \rho\,dt \end{aligned} \quad (9)$$

- The Heston model is a 2D SDE as it allows volatility to be stochastic.
  - Price process is lognormal.
  - Volatility process follows a mean-reverting CIR model.
- Model parameters are r, κ, θ, ρ and ξ
- As volatility is a latent variable, we must also choose its initial value, σ_0.

SABR Model

$$
\begin{aligned}
dS_t &= rS_t\,dt + \sigma_t S_t^{\beta}\,dW_t^1 \\
d\sigma_t &= \alpha\sigma\,dW_t^2 \\
Cov(dW_t^1, dW_t^2) &= \rho\,dt
\end{aligned} \tag{10}
$$

- The SABR model is also a 2D SDE as it allows volatility to be stochastic.
  - Price process follows a CEV model.
  - Unlike Heston, volatility is not mean-reverting.
- Model parameters are r, α, β and ρ
- As volatility is a latent variable, we must also choose its initial value, σ_0.
- Model is most commonly used in modeling rates and FX markets.

Comments on Heston vs. SABR Models & Mean Reversion of Volatility

- Heston and SABR are two very commonly used stochastic vol. models, and are worth familiarizing yourself with.
- Under certain sets of parameters, we can recover the Black-Scholes model from both processes. (How?) This is of crucial importance in practice because it gives us a set of natural tests of our code.
- As noted, the Heston model contains a mean reverting volatility process, whereas SABR's volatility process is lognormal.
  - This extra parameter in the Heston model enable us to fit the slope of the vol. surface in expiry, in addition to strike.
  - As a result, when calibrating a SABR model, we generally calibrate a set of calibration parameters **per expiry**.
  - Conversely, when calibrating a Heston model, we are able to calibrate all expiries in a **single calibration**.

Variance Gamma Model

$$X(t;\sigma,\nu,\theta) = \theta\gamma(t;1,\nu) + \sigma W(\gamma(t;1,v)) \tag{11}$$

$$\ln(S_t) = \ln(S_0) + (r - q + \omega)t + X(t;\sigma,\nu,\theta) \tag{12}$$

- Variance Gamma model uses a jump process.
- It can be seen as a Brownian motion process with a **random clock**.
- Time changes follow a Gamma distribution with mean 1 and variance ν.
- Model parameters are r, θ, σ, ν
- Allows us to control skewness and kurtosis of distribution.
- ω is selected to ensure the process is a martingale.

Comments on Variance Gamma and Other Jump Processes

- Jump processes are of particular use when trying to model extreme volatility skews of very short dated options.
- In these cases, implied volatility for out-of-the-money options will need to be raised to an unreasonable level in order to fit the market price.
- Variance Gamma model is comparable to SABR in that when are calibrating the model we generally pick a different set of parameters per option expiry. (Why?)

Comments on these Models

- By the end of this class, you should have the tools to use each of these models.
- You will also have a chance to see which models produce the best fit of market data.
- You will also have a chance to see how changing the parameters affects the models volatility surface, and risk neutral density, respectfully.

Practicalities of using these Models

- In practice, we are given a set of option prices for different strikes and expiries and need to find the optimal model. This involves:
  - Identifying the model which represents market dynamics the best. This decision is often subjective.
  - Finding the optimal model parameters that give us the best fit to the data. This requires an optimization procedure that we will discuss later in the course.
- The choice of a model, as well as its parameters define the risk neutral density in all states of the world.
- Generally, we have fewer model parameters than market prices. That is, we have an **overdetermined problem**

Alternative Approach: Extract the Risk Neutral Density Directly from Market Prices

- Consider the discretized form of (2), the risk neutral valuation formula for a European call:

$$\hat{c} \approx e^{-\int_0^T r_u \, du} \sum_{1}^{N} (S_{\tau_i} - K)_+ \phi(S_{\tau_i}) \, \Delta S_\tau$$

- Previously our approach to solving this problem has involved making assumptions about $\phi(S_{\tau_i})$ and then applying computational techniques to obtain a solution.
- Alternatively, we could parameterize the problem by the underlying probabilities of the terminal asset price, S_{τ_i}, and then find the probabilities that best fit the quoted market prices directly.
- In this lecture we will explore two approaches to calibrating the probabilities directly to market data.

First approach: Breeden-Litzenberger

- It turns out that, for a European call option, the density can also be accessed in another non-parametric manner. To see this, lets start with the payoff formula:

$$c = e^{-\int_0^T r_u\,du} \int_K^{+\infty} (S_T - K)\phi(S_T)\,dS_T \tag{13}$$

- Differentiating the payoff function we respect to strike we obtain:

$$\frac{\partial c}{\partial K} = e^{-\int_0^T r_u\,du} \int_K^{+\infty} -\phi(S_T)\;dS_T = e^{-\int_0^T r_u\,du}(\Phi(K) - 1) \tag{14}$$

- Differentiating again with respect to strike we get:

$$\frac{\partial^2 c}{\partial K^2} = e^{-\int_0^T r_u\,du}\phi(K) \tag{15}$$

Implementation of Breeden-Litzenberger in Practice

- To implement this technique in practice, we would need to employ **finite differences** to obtain a numerical approximation to $\frac{\partial^2 c_t}{\partial K^2}$.
- The standard approach is:

$$\frac{\partial^2 c}{\partial K^2} \approx \frac{c(K-h) - 2c(K) + c(K+h)}{h^2} \tag{16}$$

$$\phi(K) \approx e^{\int_0^T r_u \, du} \frac{c(K-h) - 2c(K) + c(K+h)}{h^2} \tag{17}$$

- If the traded price of the quantity on the right hand side is negative, that implies a negative probability in our density function, which implies an arbitrage opportunity.
- Question: What tradeable structure approximates the right hand side of this equation?

Natural Relationship between Market Implied Densities and Butterfly Prices

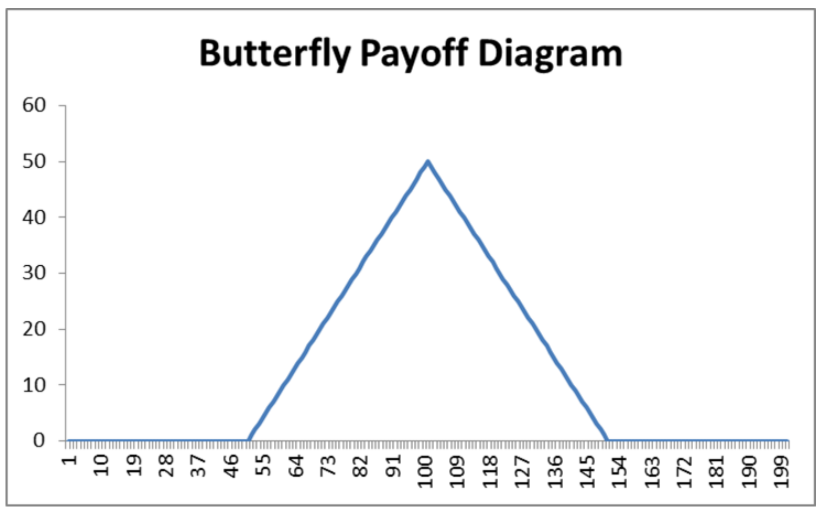
- We can replicate the density at a given point through the following tradeable instruments:

  Long one call at strike K - h

  Short two calls at strike K

  Long one call at strike K + h

- This structure is commonly known as a butterfly



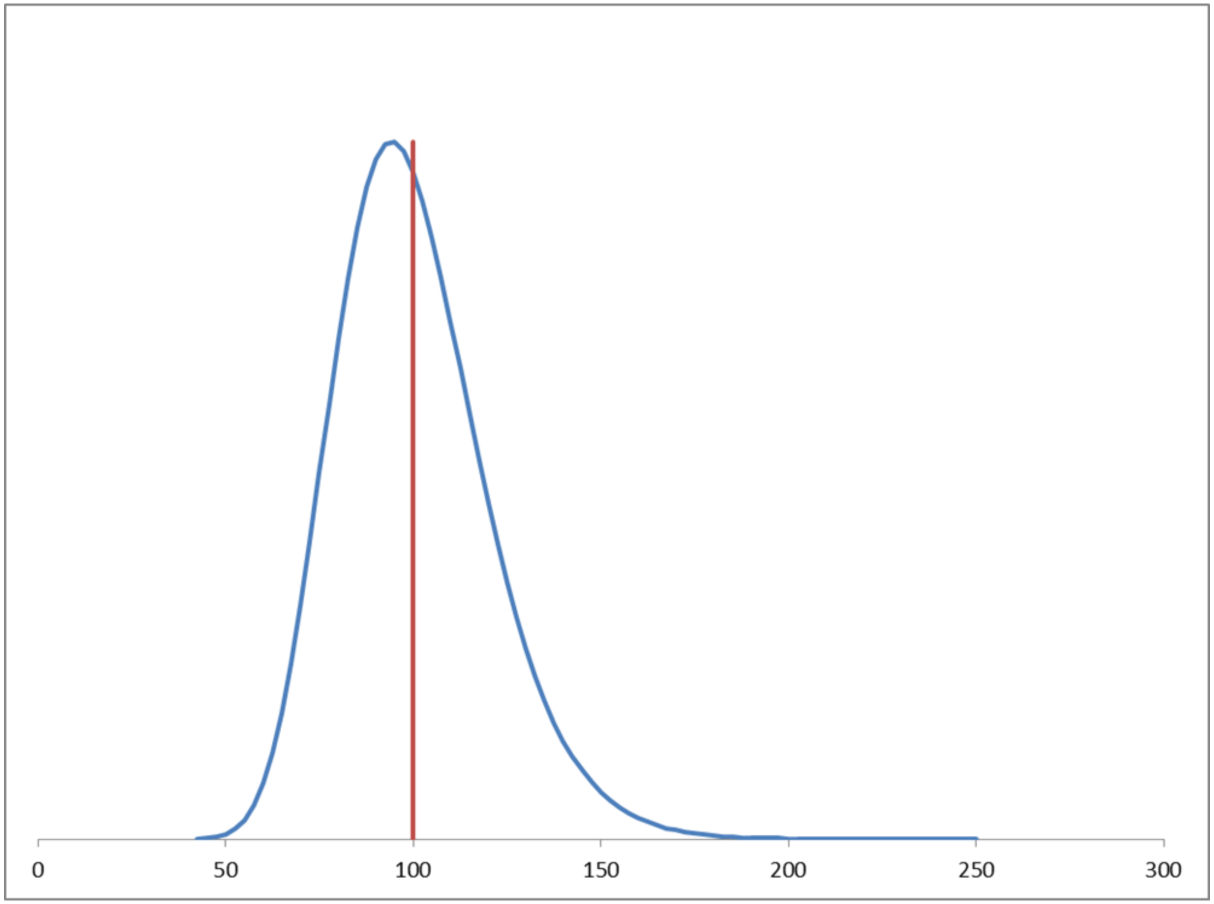
Natural Relationship between Market Implied Densities and Digital Prices

- A European digital call option pays \$1 if the asset is above K at expiry.
- It is easy to see the relationship between this instrument and the CDF of the asset:

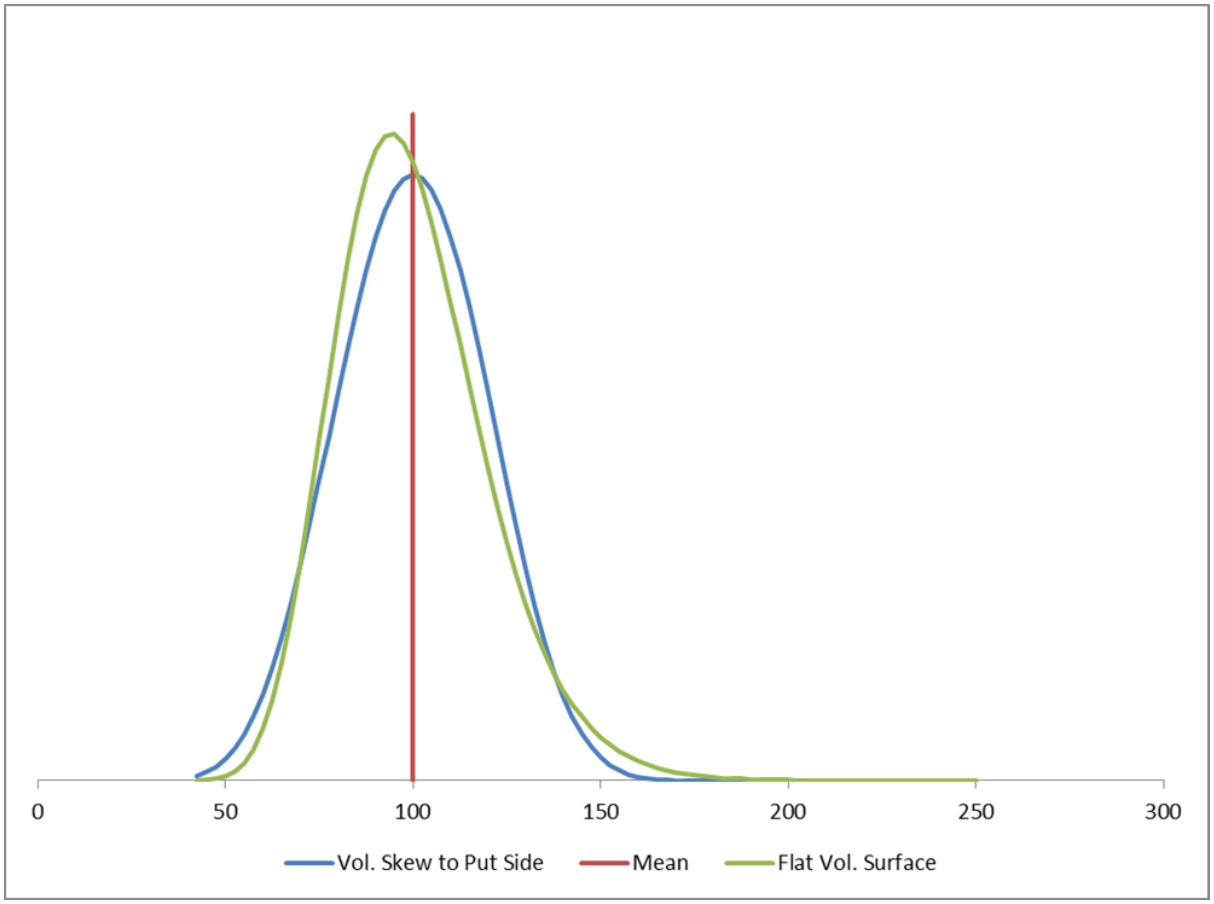
$$d = e^{-\int_0^T r_u \, du} \int_K^{+\infty} \phi(S_T) \, dS_T = e^{-\int_0^T r_u \, du}(1 - \Phi(K)) \quad (18)$$

- Therefore, when pricing a digital option, we can apply quadrature methods to obtain the CDF of the market implied density numerically.
- This relationship also holds for a **tight call spread**. (Why?)

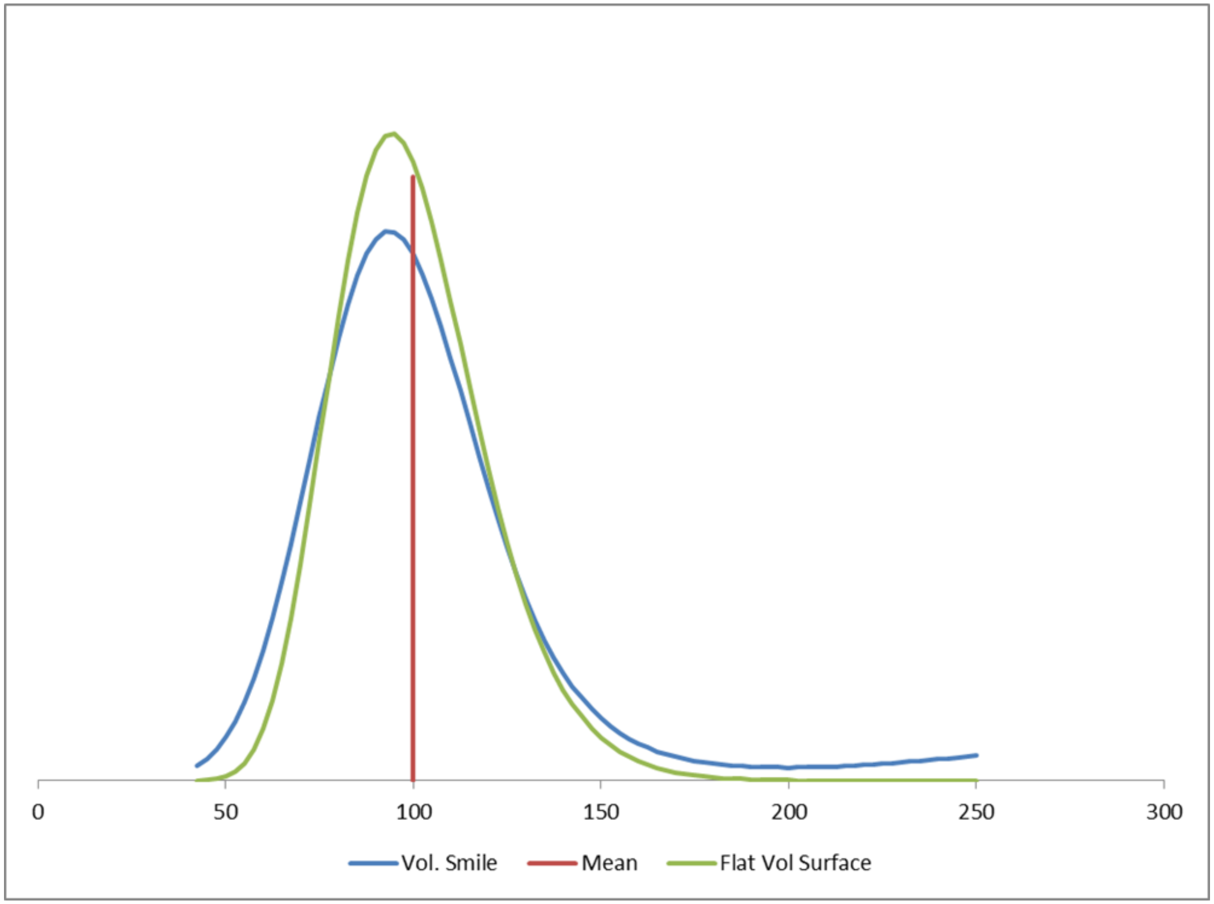
Risk Netural Density Example: Flat Vol Surface



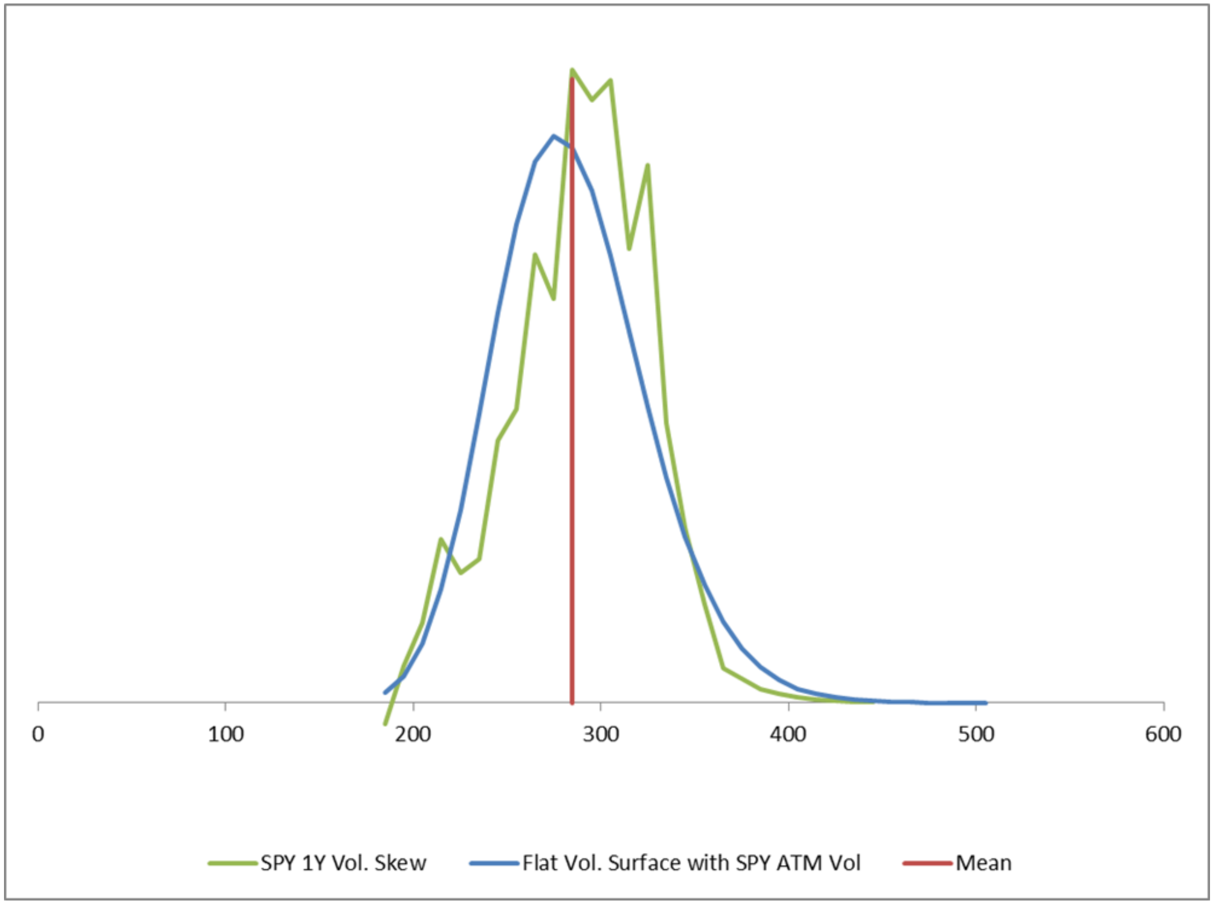
Risk Neutral Density Example: Vol Surface with Put Skew



Risk Neutral Density Example: Vol Surface with Smile



Risk Netural Density Example: S&P Options Vol. Surface



Breeden-Litzenberger: Limitations

- The Breeden-Litzenberger method only works for a 1D density (at a single expiry).
- This approach also assumes that prices are *available for every strike* and are *exact for every strike*.
  - In reality, prices are only available at finite strikes.
  - Additionally, prices are not exact but instead as a range (bid-offer)
- This means we need to infer prices at non quoted strikes. This can be done by employing **interpolation methods**, among other ways.

Breeden-Litzenberger: Interpolation

- If this interpolation is not done carefully, it can result in introducing arbitrage into the model.
- Even in the absence of arbitrage, interpolation may introduce unrealistic assumptions about the market implied density.
- To obviate this problem, we might first calibrate a stochastic process, and then extract the risk neutral density.

Second Approach: Non-Parametric Risk Neutral Density Formulation

- The second approach for calibrating probabilities directly to market data is to setup an optimization. In particular, we look for a set of probabilities, p_i that satisfy the following:

$$\vec{p}_{\min} = \underset{\vec{p}}{\operatorname{argmin}} \left\{ \sum_j \left(\hat{c}(\tau_j, K_j) - c_{\tau_j, K_j} \right)^2 \right\} \tag{19}$$

where c_j is the market price of the j^{th} instrument and $\hat{c}$ satisfies:

$$\hat{c} \approx e^{-\int_0^T r_u \, du} \sum_1^N (S_{\tau_i} - K)_+ p_i(S_{\tau_i})$$

Second Approach: Non-Parametric Risk Neutral Density Formulation

- In order to ensure we had a valid density we would also need to include the following constraints:

$$p_i > 0 \qquad \forall p_i \tag{20}$$

$$\sum_i p_i = 1 \tag{21}$$

Non-Parametric Risk Neutral Density Approach: Comments

- In order to implement this technique we need to be able to apply quadrature methods, which we discussed last class, and solve optimization problems, which we will discuss later in the semester.
- NOTE: This approach helps us to find the density from today until option expiry. It does not naturally define conditional probabilities from one intermediate state to another (as an SDE would)
- However, the approach requires no assumptions about the underlying stochastic process.

Non-Parametric Risk Neutral Density Approach: Option Prices as Constraints

- Generally, there will be far more p_i's than option prices, that is, the problem is **underdetermined**. This means:
  - We can hit option prices exactly (in the absence of arbitrage)
  - There will be many densities that match market prices.
- As a result, the term in the objective function in (22) is unnecessary and can instead be incorporated through equality constraints (or inequality constraints if we want to incorporate bid-offer).
- Question: If we did this, what would be objective function in our optimization become, and how do we choose between the many densities that fit the market prices?

Second Approach: Non-Parametric Risk Neutral Density Formulation with Constraints

- We could alternatively formulate the objective function as follows:

$$\vec{p}_{\min} = \underset{\vec{p}}{\operatorname{argmin}} \left\{ \sum_i g(p_i(S_{\tau_i})) \right\} \tag{22}$$

- With the following constraints:

$$p_i > 0 \qquad \forall p_i \tag{23}$$

$$\sum_i p_i = 1 \tag{24}$$

$$\hat{c}(\tau_j, K_j) = c_{\tau_j, K_j} \qquad \forall j \tag{25}$$

$$\tag{26}$$

- But what should we ask of $g(p_i(S_{\tau_i}))$?

Non-Parametric Risk Neutral Density Approach: Smoothing Methods

- Among the solutions that fit market prices, we want to choose the **best**. This is subjective and could be defined as the smoothest solution or the one that most closely matches some prior distribution.
- In particular, we use the $g(p_i)$ term in order to impose some preferred structure to the density, although we of course allow this structure to be overriden in order to hit market prices.

Non-Parametric Risk Neutral Density Approach: Smoothing Methods

- It would be desireable to choose a convex function for $g(p_i)$, as this would ensure our optimization problem has a unique minimum.
- Notice, however, that $g(p_i)$ only depends on the probabilities and not on the option prices. This is because we know that, if our optimization converges, then we will hit market prices exactly.
- Also notice, that when we do this, the pricing constraints are *linear* with respect to the probabilities. This is an important property and makes the optimization problem more tractable.

Risk Neutral Density Approach: Uses

- What can we do with this density?
  - Match market prices exactly in the absence of arbitrage in a completely non-parametric manner.
  - Price any European derivative via **quadrature methods** evaluated against extracted risk neutral density
  - Better understand the way that options markets trade and the densities that underlie them.
  - Measure the properties of the density (i.e. skewness / kurtosis) without imposing model assumptions
  - Analyze these densities as a time series or across expiries in an attempt to make relative value judgments.
  - Compare these densities to *physical densities* in order to inform trading decisions.

Risk Neutral Density Approach: Limitations

- What are the limitations of this approach?
  - Difficult to extend to multiple expiries & path dependent options
    - In order to account for multiple expiries we need to not only extract the density from today until a given set of expiries, but also the conditional transition densities from any time & state to any other.
  - Difficult to apply to markets with sparse data
    - In these cases, where the market doesn't provide enough information to accurately represent the density, we need to use interpolation methods or impose additional constraints.
  - Parameters & sensitivities may be less intuitive than Greeks implied by standard SDE models.

Risk Neutral Density Approach: Conclusions

- Each of these limitations can be overcome, however, and in practice this ends up being a very useful, albeit challenging method.
- The flexibility and lack of assumptions about the underlying process is particularly appealing.
- Using densities is a helpful tool for understanding the underlying markets that we are modeling.