MF 796: Computational Methods of Mathematical Finance

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Solutions to Problem set # 2

Due: Wednesday, February 5, 2020 by 8 am.

Problem 1: Evaluation of a known integral using various quadratures: In this problem we are going to compute the price of a European call option with 3 month expiry, strike 12, and implied vol 20, Assume the underlying is 10 now and the interest rate is 4%.

1. Use Black-Scholes formula to compute the price of the call analytically.

   Solution to 1 The values of the parameters as:

$$
\begin{aligned}
S_0 &= 10 \\
K &= 12 \\
\tau &= 0.25 \\
r &= 0.04 \\
\sigma &= 0.2
\end{aligned}
$$

   Plugging into the Black-Scholes formula, the call price is $C \approx 0.0188$

2. Calculate the price of the call numerically using the following 3 quadrature methods:

   (a) Left Riemann rule

   (b) Midpoint rule

   (c) Gauss nodes of your choice (say explicitly why you made that choice)

   with the number of nodes $N = 5, 10, 50, 100$ and compute the calculation error as a function of N for each of the methods.

   Solution to 2 Method 1 (Not recommended): Direct computation of the pricing formula

   First notice that we have the call price

$$C = e^{-r\tau} \int_{\ln K}^{+\infty} (x - K)^{+} \phi(x) d \ln x \tag{1}$$

   Where $\phi(x)$ is the density function of $\ln S_T$. As $\ln S_T$ is normally distributed, and normal density is concentraed within 3 times of the standarddeviation, using 5 times of the standard deviation in place of $+\infty$ is enough to approximate the price to our precision. We can rewrite (1) as

$$C \approx e^{-r\tau} \int_{\ln K}^{\ln S_0 + (r - \frac{1}{2}\sigma^2)\tau + 5\sigma\sqrt{\tau}} \frac{e^{\ln x} - K}{\sigma\sqrt{2\pi\tau}} \exp\left(\frac{[\ln x - (\ln S_0 + (r - \frac{1}{2}\sigma^2))]^2}{2\sigma^2 T} \right) d \ln x \tag{2}$$

We need to approximate the above equation by

$$C = e^{-r\tau} \sum_{i=1}^{N} w_i f(x_i) \tag{3}$$

for various quadrature rules.

Method 2 (Recommended): Compute the integrals in the Black-Scholes formula

The Black-Scholes formula has the form:

$$C_0 = S_0 N(d_1) - K e^{-rT} N(d_2) \tag{4}$$

where

$$N(d_1) = \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{5}$$

We can apply quadrature to the above integral directly. For Left-Riemann and Midpoint method it is straightforward (see the attached code for details).

For Gaussian Nodes, let's consider Gaussian-Legendre method first. The algorithm works as:

(1) A lower bound should be in place of the $-\infty$. As $\exp\left(-\frac{x^2}{2}\right)$ tends to zero super-exponentially as $x \to -\infty$, we will use -10 as the lower bound, which will suffice for machine precision.

(2) We need to change the integral domain from $(-10, d_1)$ to $(-1, 1)$. We will introduce a new variable y, such that when $x = -10$, $y = -1$, and when $x = d_1$, $y = 1$. This could be done by setting:

$$y = \frac{2}{d_1 + 10} x + \frac{10 - d_1}{d_1 + 10} \tag{6}$$

(3) Let $a = \frac{2}{d_1+10}$, $b = \frac{10-d_1}{d_1+10}$, so that we have $y = ax + b$, and $\frac{1}{a} dy = dx$. Then the original intergal becomes:

$$N(d_1) = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\frac{y-b}{a})^2}{2}\right) \frac{1}{a} dy \tag{7}$$

(4) Now we can apply Gaussian-Legendre quadrature to the above integral with weight $w(x) = 1$. [1]

(5) Use the same approach for $N(d_2)$.

Table 1 summarizes the errors of different methods. (See the attached code for detailed algorithm).

3. Estimate the experimental rate of convergence of each method and compare it with the known theoretical estimate.

[1]Python provides a built-in algorithm for Gaussian-Legendre which is very convenient to use: numpy.polynomial.legendre.leggauss

N	Left-Riemann	Midpoint	Gauss-Legendre
5	0.0033	0.0016	0.0000
10	0.0008	0.0004	0.0000
50	0.0000	0.0000	0.0000
100	0.0000	0.0000	0.0000

Table 1: Table for solution to 3: Error of each method for different N

Solution to 3 See the attached separate figures.

Figure 3 and Figure 4 show the absolute error of Left-Riemann and Midpoint methods, with the plot of $N^{-i},\ i = 1, 2, 3$. As could be seen the error of Midpoint method is of order between $O(N^{-2})$ to $O(N^{-3})$. This confirms what we see in the lecture slides.

Figure 5 shows the absolute error of Gauss-Legendre method, with the plot of $N^{-iN},\ i = 1, 2$.

4. Which method is your favorite and why?

Solution to 4 This is an open-ended question. Generally, the following guidelines will be considered when I grade your assignment:

(1) Riemann rule should not be your favorite (as discussed in class it is dominated by other methods such as the Midpoint rule)

(2) Midpoint rule is easy to implement and advantageous when N is small.

(3) Guassian nodes is efficient when N is large in general. However, to use Gaussian nodes one needs to do proper transformation and in some cases it would be quite complicated.

(4) It's important that the function you are integrating is smooth over the integration domain, otherwise, you lose the advantage of Gauss nodes.

(5) For example, If you are going to use Gauss-Legendre (which a lot of students choose to use), you need to change variables so that $(-\infty, d_1]$ becomes $[-1, 1]$ or rescale so that the integrand is zero near the ends of [-1,1]. We recommend changing variables and avoid transforming Gauss nodes. Rescaling would work with Gauss-Legendre, but would be quite problematic with the others, because they have non-trivial weights.

Problem 2: Calculation of Contingent Options: Let S_1 be a random variable that takes on the value of SPY one year from now and let S_2 take on the values of SPY 6 months from now. Assume that they are jointly normally distributed with

$$\begin{aligned}\sigma_1 &= 20\\ \sigma_2 &= 15\\ \rho &= 0.95\end{aligned}$$

By ρ here we mean correlation between S_1 and S_2. Also, assume that interest rate is zero. Please specify where you got the current price of the underlying.

1. Evaluate the price of the one year call on SPY with the strike $K_1 = 370$. This is an example of a vanilla option.

   Solution to 1 Suppose the initial price is $S_0 = 324.12$ (the value of SPY at the end of Feb 3rd). Also assume the mean of (S_1, S_2) are $(\mu_1, \mu_2) = (324.12, 324.12)$.[2] By plugging in the Black-Scholes formula of vanilla option we have $C = 0.0745$ (See the attached code, I use midpoint rules to approximate the integral here).

2. Evaluate the price of the one year call on SPY with the strike $K_1 = 370$, contingent on SPY at 6 months being below 365. This is a contingent option.

   Solution to 2 We have the following pricing formula:

$$\begin{aligned}C &= \mathbb{E}^{\mathbb{Q}}[e^{-rT}(S_{1T} - K_1)^{+}\mathbf{1}_{\{S_{2T}<365\}}]\\ &= \int_{-\infty}^{365}\int_{370}^{\infty} e^{-rT}(s_1 - K_1)\psi(s_1, s_2)ds_1 ds_2 \end{aligned} \tag{8}$$

   where $\psi(s_1, s_2)$ is the joint density of S_{1T}, S_{2T}:

$$\begin{aligned}\psi(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times\\ &\quad \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right)\right\}\end{aligned}$$

   Plugging in the value of the parameters in the above formula, we see that the price of the contingent call is 0.0365.

3. Calculate the contingent option again, but with $\rho = 0.8$, $\rho = 0.5$, and $\rho = 0.2$.

   Solution to 3 See the attached code for calculation. The results are:

4. Does dependence on ρ make sense?

[2]Because interest rate is zero we could assume the means of both assets are same

ρ	Value of the contingent option
0.8	0.0537
0.5	0.0687
0.2	0.0734

Table 2: Table for solution to 3

Solution to 4 Yes. For a contingent call, when the correlation is (positively) higher, the contingent option value is lower. Intuitively, when correlation is high, there is a higher chance that both the SPY and SPY 6 month are lower than 365, or both are higher than 365, and in either case, the contingent option expires out of money.

5. Calculate the contingent option again, but with SPY at 6 months below $360, 350$, and 340.

   Solution to 5 See the attache code for calculation. The results are:

6 month SPY	Value of the contingent option
360	0.0163
350	$4.723 * 10^{-4}$
340	$4.211 * 10^{-7}$

Table 3: Table for solution to 5

6. Does the dependence on the 6 month value make sense?

   Solution to 6 Yes. When the threshold increases, the contingent condition is less strict, there is a higher chance that the option survives until maturity.

7. Under what conditions do you think the price of the contingent option will equal the price of the vanilla one?

   Solution to 7 To make contigent equal to vanilla, we would need to relax the contigency to zero. That would mean requiring that $K_2 = +\infty$ instead of $K_2 = 365$. We can also see that by examining (8).
   Also, from (8) we can see that making $\rho = 0$ will make the contigency independent from vanilla, and would make the contingent option equal to vanilla times the probability of contigency occuring.