Fourier series and some of their applications

Goals:

- Review complex numbers, complex functions, and diagonalization of matrices.
- Learn about Fourier series, transforms, and some of their applications.

Complex numbers

A complex number c is usually represented by a pair of real numbers (x, y) like this:

$$c = x + iy \quad \text{where} \quad i = \sqrt{-1}. \tag{1}$$

Its *complex conjugate* is

$$\bar{c} = x - iy \tag{2}$$

and its length

$$|c| = \sqrt{c\bar{c}} = \sqrt{(x+iy)(x-iy)} = \sqrt{x^2 - i^2y^2} = \sqrt{x^2+y^2} \tag{3}$$

Complex numbers

Now consider the exponential $e^{i\theta}$ and expand it in Taylor series:

$$
\begin{aligned}
e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\
&= 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \cdots \\
&\quad +(i\theta) + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \cdots \\
&= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots \\
&\quad +i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\
&= \cos(\theta) + i\sin(\theta) \qquad (4)
\end{aligned}
$$

where in the last equality we recognized the Taylor series for $\sin(\cdot)$ and $\cos(\cdot)$. This is the Euler's formula.

Complex numbers

Euler's formula includes all of trigonometry. For example,

$$\cos(2\theta) + i\sin(2\theta) = e^{i2\theta} = \left(e^{i\theta}\right)^2 = (\cos(\theta) + i\sin(\theta))^2 \tag{5}$$

and all the double-angle formulas follow after collecting real and imaginary parts on the right-hand side.

It also follows from Euler's formula that

$$\overline{e^{i\theta}} = e^{-i\theta}.$$

and, therefore, the length of $e^{i\theta}$ is

$$|e^{i\theta}| = \sqrt{e^{i\theta}e^{-i\theta}} = 1.$$

In other words, $e^{i\theta}$ (for real θ) lies on the unit circle in the complex plane.

This, by the way, gives a polar representation of complex numbers: $re^{i\theta}$, where r is the length and θ is the angle with x-axis.

Eigenvalues and eigenvectors

Let M be an $n \times m$ matrix. We can (and do) view it as a linear transformation (also called linear operator) from $\mathbb{R}^m$ to $\mathbb{R}^n$, denoted $M : \mathbb{R}^m \to \mathbb{R}^n$.

If $n = m$, then M maps $\mathbb{R}^n$ back to itself. Sometimes M will leave some directions unchanged. More precisely, if there is $v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that

$$Mv = \lambda v$$

then v is called an eigenvector of M with eigenvalue λ. This does not always happen: for example, the rotation matrix

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{6}$$

doesn't have any (real) eigenvectors if θ is not a multiple of π.

Symmetric matrices (self-adjoint operators)

But there is an important special case. A matrix is symmetric if $M^T = M$ i.e, $M_{ij} = M_{ji}\ \forall i, j$. ($M^T$ denotes the transpose of M.)

If v and w are vectors in $\mathbb{R}^n$, then it's always true that

$$\langle v, Mw\rangle = \langle M^T v, w\rangle. \tag{7}$$

Therefore, if M is symmetric, then

$$\langle v, Mw\rangle = \langle Mv, w\rangle \qquad \forall v, w \in \mathbb{R}^n. \tag{8}$$

Theorem 1. *Suppose M is an $n \times n$ symmetric matrix. Then*

- *M has exactly n (not necessarily distinct) eigenvalues.*
- *There exists a set of n eigenvectors, one for each eigenvalue, that are mutually orthogonal.*

Eigenbasis

In particular, these eigenvectors (once normalized) will form an orthonormal basis of $\mathbb{R}^n$. We'll call it an eigenbasis, for short. Note that for every eigenbasis there has to be an operator M.

Let the vectors $e_1, \ldots, e_n$ form such an eigenbasis and let v be any vector in $\mathbb{R}^n$. Then projection of v onto each e_j equals $\langle v, e_j \rangle e_j$ and

$$v = \sum_{j=1}^{n} \langle v, e_j \rangle e_j. \tag{9}$$

Also,

$$Mv = \sum_{j=1}^{n} \langle v, e_j \rangle M e_j = \sum_{j=1}^{n} \langle v, e_j \rangle \lambda_j e_j, \tag{10}$$

where λ_j is the jth eigenvalue of M, i.e. $Me_j = \lambda_j e_j$.

Diagonalization

Finding an eigenbasis for M is called diagonalization of M, because in that basis the matrix is diagonal. To see why, let B be the matrix whose columns are the eigenvectors of M:

$$B = \left[e_1 \,\middle|\, \cdots \,\middle|\, e_n \right] \tag{11}$$

Then

$$MB = M \left[e_1 \,\middle|\, \cdots \,\middle|\, e_n \right] = \left[\lambda_1 e_1 \,\middle|\, \cdots \,\middle|\, \lambda_n e_n \right] = \left[e_1 \,\middle|\, \cdots \,\middle|\, e_n \right] D = BD \tag{12}$$

where D is the diagonal matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$ on the diagonal.

Therefore, M in the B basis is diagonal:

$$B^{-1}MB = D. \tag{13}$$

Some facts from Linear Algebra

Everything above about Linear Algebra is still true if we replace $\mathbb{R}^n$ with $\mathbb{C}^n$, a vector space of complex numbers. (Everything except the rotation matrix not having eigenvectors.) One difference is that the inner product in complex vector spaces is

$$\langle v, w \rangle = \sum_{j=1}^{n} v_j \overline{w_j}, \tag{14}$$

where $\overline{w_j}$ is the complex conjugate of w_j.

Spaces of Functions

Now consider the space of all (reasonably nice) complex-valued periodic functions on $[0, 2\pi]$. We put the standard inner product on this space:

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx \tag{15}$$

What could be a good basis for this space? We have met orthogonal polynomials, so they could be a candidate. But how would it help us to decompose a function into orthogonal polynomials? Which operation would be easier to do? Ideally, the orthonormal basis that we choose would be an eigenbasis of some useful operator, because the action of that operator on its eigenfunctions is particularly simple: it just multiplies them by a number.

Spaces of Functions

Consider the basis of functions

$$e_n(x) := e^{inx} \qquad n = -\infty, \dots, -1, 0, 1, \dots, \infty. \tag{16}$$

(Here n is an index: it has nothing to do with the dimension n in (9)). They are orthonormal, since

$$\langle e_n, e_m \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{inx} \overline{e^{imx}} \, dx = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)x} \, dx = \delta_{nm}. \tag{17}$$

But which operator do they diagonalize? Recall that derivative of e^{inx} is its multiple:

$$\frac{d}{dx} \left(e^{inx} \right) = in \, e^{inx}. \tag{18}$$

But that means that e^{inx} is an eigenfunction of the derivative operator with eigenvalue in. If we define $\mathfrak{D}$ to be the derivative operator, we can rewrite (18) as

$$\mathfrak{D} e_n = in \, e_n. \tag{19}$$

Fourier Series

These e_ns span the whole function space (Parseval's theorem). That means that for every periodic function f on $[0, 2\pi]$ the following holds:

$$f = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n \tag{20}$$

The inner product

$$\langle f, e_n \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{e^{inx}} \, dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \, dx \tag{21}$$

is called the *nth Fourier coefficient of f* and is usually denoted $\hat{f}(n)$, and the sum in (20), more commonly written

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} \tag{22}$$

is called the *Fourier series of f*.

Fourier Series and Differential Equations

Why are Fourier Series important? For one, as we saw in (19) they diagonalize the derivative operator. This turns differential equations into algebraic ones!

Here is a quick example: suppose we are given a periodic function $g(x)$ and we want to find a function $f(x)$ that satisfies the following differential equation:

$$f^{(4)}(x) + f(x) = g(x) \tag{23}$$

(We'll ignore boundary conditions here.)

Fourier Series and Differential Equations

Suppose the Fourier series of f is as in (22). Then

$$\frac{d^4}{dx^4} f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \frac{d^4}{dx^4} e^{inx} = \sum_{n=-\infty}^{\infty} n^4 \hat{f}(n) e^{inx} \tag{24}$$

because $i^4 = 1$. Substituting this into (23) we get:

$$\sum_{n=-\infty}^{\infty} \left(n^4 + 1\right) \hat{f}(n) e^{inx} = \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{inx} \tag{25}$$

Since the two series must agree on every term in order to be equal,

$$\hat{f}(n) = \frac{\hat{g}(n)}{n^4 + 1} \qquad \forall n \tag{26}$$

and we can reconstruct f from its Fourier coefficients via (22):

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} = \sum_{n=-\infty}^{\infty} \frac{\hat{g}(n)}{n^4 + 1} e^{inx}. \tag{27}$$

Convolution

There is another important use of Fourier series: convolutions. Before we define them, suppose we want to compute the pdf of a sum of two independent random variables X and Y when we know their individual pdfs, ϕ_X and ϕ_Y. Let $Z = X + Y$, let Φ_Z be its cdf and ϕ_Z its pdf. Then we have:

$$\begin{aligned}
\Phi_Z(t) &= P(X+Y \leq t) = \iint\limits_{\{x+y\leq t\}} \phi_{X,Y}(x,y)\,dydx \\
&= \iint\limits_{\{x+y\leq t\}} \phi_X(x)\phi_Y(y)\,dydx \\
&= \int_{-\infty}^{\infty} \phi_X(x) \int_{-\infty}^{t-x} \phi_Y(y)\,dydx \\
&= \int_{-\infty}^{\infty} \phi_X(x)\Phi_Y(t-x)\,dx. \qquad (28)
\end{aligned}$$

Convolution

From (28) it follows that

$$
\begin{aligned}
\phi_Z(t) &= \frac{d}{dt}\Phi_Z(t) = \int_{-\infty}^{\infty} \phi_X(x)\frac{d}{dt}\Phi_Y(t-x)\,dx \\
&= \int_{-\infty}^{\infty} \phi_X(x)\phi_Y(t-x)\,dx =: (\phi_X * \phi_Y)(t) \qquad (29)
\end{aligned}
$$

where the last equality is the definition of convolution (denoted "$*$") of ϕ_X and ϕ_Y.

So we proved that the pdf of a sum of two independent random variables is a convolution of their pdfs. If we wanted to compute the pdf of a sum of N independent random variables, we would have to compute an $(N-1)$-dimensional integral, which, of course, would be intractable for $N > 3$, say. There are situations in finance when $N > 100$.

What to do? We'll get to that on slide 18.

Fourier Transform

First, we need to discuss the Fourier Transform. Fourier Transform is analogous to Fourier Series, except its domain in the whole real line, rather than the interval $[0, 2\pi]$. Also, its frequencies are not integers, but real numbers:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx \tag{30}$$

Here $\hat{f}(\xi)$ is the Fourier coefficient of f, corresponding to the frequency $\xi \in \mathbb{R}$.

Expression (30) is referred to as the Forward Fourier Transform.

If all $\hat{f}(\xi)$ are known, then $f(x)$ can be recovered via the Inverse Fourier Transform:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x}\, d\xi. \tag{31}$$

Convolution

To appreciate the power of Fourier methods, let's try to compute Fourier transform of the convolution in (29).

The ξ-th Fourier coefficient of the convolution is

$$
\begin{aligned}
\widehat{\phi_X * \phi_Y}(\xi) &= \int_{-\infty}^{\infty} (\phi_X * \phi_Y)(t) e^{-i\xi t}\, dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_X(x)\phi_Y(t-x) e^{-i\xi t}\, dx\, dt \\
&= \int_{-\infty}^{\infty} \phi_X(x) \int_{-\infty}^{\infty} \phi_Y(t-x) e^{-i\xi t}\, dt\, dx \\
&= \int_{-\infty}^{\infty} \phi_X(x) \int_{-\infty}^{\infty} \phi_Y(s) e^{-i\xi(s+x)}\, ds\, dx \\
&= \int_{-\infty}^{\infty} \phi_X(x) e^{-i\xi x} \int_{-\infty}^{\infty} \phi_Y(s) e^{-i\xi s}\, ds\, dx \\
&= \int_{-\infty}^{\infty} \phi_X(x) e^{-i\xi x} \widehat{\phi_Y}(\xi)\, dx \\
&= \widehat{\phi_X}(\xi)\widehat{\phi_Y}(\xi). \qquad (32)
\end{aligned}
$$

Convolution

- Calculation (32) tells us that Fourier transform of a convolution is a *product* of Fourier transforms of the individual densities.
- And convolution of densities, remember, is the density of the sum of random variables (if they are independent).
- So, if we need to compute a density of a sum of many random variables, quadrature will not be tractable, but Fourier will save the day. Indeed in the next lecture, you will learn about FFT, the algorithm that computes all N Fourier coefficients in $O(N \log N)$ instead of $O(N^2)$.
- Computing convolution (29) by direct method is $O(N^2)$, since you have N integrals to compute. (One for every t.) Using FFT, however, is $O(N \log N)$ to calculate the Fourier coefficients of each density, another $O(N)$ to multiply them, and another $O(N \log N)$ for inverse transform for a total of $O(N \log N)$.