MF 796: Computational Methods

Professors Eugene Sorets and Christopher Kelliher
Boston University Questrom School of Business, 595 Comm Ave

Office hours: Mon & Wed, 10:45 - 11:45am in HAR 514A
Email: sorets@bu.edu & cmk44@bu.edu

Teaching Assistants:

- Hannan "Frank" Zheng. Email: zhannan@bu.edu
- Boyan Cheng. Email: bycheng@bu.edu
  Office hours: Tue, 9:40am - 10:40am in HAR 518L
- Dapeng Shang. Email: dpshang@bu.edu
  Office hours: Tue, 12:40pm - 1:40pm in HAR 518L

Course Goals

- This course will introduce you to the most common computational methods used in mathematical finance.
- It will provide you with hands-on experience implementing these methods as well as commonly used financial models.
- It will help you to understand the strengths and weaknesses of each of the models & methods and enable you to choose optimally when faced with these problems in your career.
- It will provide you with tools that you can use to evaluate approximations in practical financial applications.

Requirements for this course

- This course assumes knowledge and fluency in linear algebra, stochastic calculus, statistics, and probability.
- This includes: random variables, probability distributions and densities, characteristic functions, Fourier Inversions, measure changes, Taylor series and Ito's formula.
- This course will also require a significant amount of programming and we expect you to be familiar with either R, Matlab, or Python.
- Additionally, we assume that you have been exposed to the concepts of derivative pricing and risk neutral valuation.

Computational Problems Quants Face

Quants face many computational problems in financial institutions. Below we list a few broad categories of what they do.

- Sell-side desk quant

  Pricing & Greeks for Exotics

- Buy-side desk quant

  Regresssion / Machine learning / Time Series

- Risk quant

  VaR Modeling / Model validation

- Asset Management quant

  Optimization / Portfolio Construction

- Research quant

  Alpha generation / Estimation / Filtering / Model design

What do these computational problems have in common?

- Underlying each such problem is a density, or set of densities.
- If we know this density, then by taking expectations against various functions we can:

  Price complex derivatives.

  Estimate Risk metrics, such as VaR.
- Where do these densities come from?

  Market Prices (Risk Neutral Density)

  Time Series / Econometric Analysis (Physical Density)
- IMPORTANT: Think through when each type of density is appropriate as a lot of money has been lost confusing the two.

Computational methods for pricing

- If you know the distribution Φ_X of X and you know the payoff function f, you can evaluate the security's price: it is:

$$\mathbb{E}[f(X)]$$

- In practice, you often know the payoff function f and the price of the security, but need to find Φ_X. For example, it's often the case that you know the prices of a number of vanilla (simple) options, which you use to find Φ_X. Once you have Φ_X, you use it to price an exotic security, i.e., one with a complicated payoff f. [Is Φ_X risk-neutral or physical here?]

- We look at a few examples below and see what techniques we will need to learn in this course in order to be able to handle the computations that these examples demand.

FTAP

- The first fundamental theorem of asset pricing states that there is no arbitrage if and only if there exists a risk-neutral measure.
- In this case, asset prices can be written as

$$X_0 = \tilde{\mathbb{E}}\left[e^{-\int_0^T r_u \, du} X_T\right]$$

  for a risk-neutral measure $\tilde{\mathbb{P}}$ and a short rate process r.
- Under $\tilde{\mathbb{P}}$, the current stock price reflects the discounted risk-neutral expectation of all future cashflows.
- What is the connection between $\tilde{\mathbb{P}}$ and Φ_X on the previous slide?

Pricing derivatives

- The price of a European derivative that pays $X_T = f(S_T)$ at maturity T is given by

$$c_0 = \tilde{\mathbb{E}}\left[e^{-\int_0^T r_u\,du} f(S_T)\right] \tag{1}$$

- The price of an American derivative that pays $X_\tau = g(S_\tau)$ if exercised at the stopping time τ is

$$C_0 = \sup_{s \in \mathcal{T}} \tilde{\mathbb{E}}\left[e^{-\int_0^s r_u\,du} g(S_s)\right], \tag{2}$$

  where $\mathcal{T}$ is a set of admissible stopping times s and τ is the maximizing stopping time among them.

- NOTE: Pricing a European derivative depends only on the distribution of the terminal value of S_T whereas pricing an American derivative depends on the distrubution of S_τ at all stopping times.

Simplest Example: European Option on Asset Following Geometric Brownian Motion

What kind of dynamics of S_t would generate $\tilde{\mathbb{P}}$? The simplest one in use is the Geometric Brownian Motion:

$$dS_t = rS_t\,dt + \sigma S_t\,dW \tag{3}$$

Under these dynamics, a derivative instrument $c_t = c(S_t, t)$ must obey the following partial differential equation (PDE), known as Black-Scholes PDE:

$$\frac{\partial c_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c_t}{\partial S_t^2} + rS_t \frac{\partial c_t}{\partial S_t} - rc_t = 0, \tag{4}$$

If the instrument is a European call, i.e., its payoff at the expiry T is

$$c(S, T) = (S - K)_+ \tag{5}$$

then the Black-Scholes PDE has a closed-form solution.

Black-Scholes formula

- There exists an analytic solution to (4), known as the **Black-Scholes formula**, for the price of a European call:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT}\left(S_T - K\right)_+\right] = \Phi(d_1)S_0 - \Phi(d_2)Ke^{-rT} \tag{6}$$

- Here, Φ is the CDF of the standard normal distribution and

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T\right), \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned} \tag{7}$$

- In theory, we can price European options with constant short rate and constant volatility analytically using the Black-Scholes formula
- In reality, there is no closed form for Φ and we will need to learn *quadrature*, i.e., techniques for approximating definite integrals numerically.

Black-Scholes formula: Implied Volatility

- Next, suppose that we are given a price $c_{0,K}$ of a call with strike K: we need to find σ. Substituting (7) into (6), we can write

$$c_{0,K} = BS(K, \sigma) \tag{8}$$

  which tells us that σ is a function of K. It is called volatility (or vol, for short) of K.

- To compute vol of K from the price, we will need some root-finding method. One important example is Newton-Raphson.

- This process of fitting model parameters to market prices is called **calibration**.

Calibration

- Usually we are given prices of vanilla options for many strikes and expiries and, naturally, we would like to fit them all simultaneously as closely as possible.

- We can calibrate to them by the method of least squares. Specifically, we look for the minimizer $\sigma_{\min}$ defined by

$$\sigma_{\min} = \underset{\sigma}{\operatorname{argmin}} \left\{ \sum_K \left(\hat{c}(\tau, K, \sigma) - c_{\tau,K} \right)^2 \right\} \tag{9}$$

  where c is the market price, $\hat{c}$ is the model price and K runs over all the strikes for which we are given the prices $c_{0,K}$ of the options.

- And so we will need to learn how to solve optimization problems.

- Question: Under what circumstances would the dynamics on the previous slide lead to a good fit to market/observed data?

Multi-parameter models

- The log-normal distribution corresponding to $\sigma_{\min}$ in (9) will fit the option prices well if all σ_Ks are close to each other. (Why?)
- But usually they are not: the plot of σ vs K often exhibits a smile or a smirk and σ is generally not constant in expiry.
- To get a better fit, people came up with models with more flexibility (& more parameters). These models can be broken down into the following categories:

  Stochastic Volatility

  Local Volatility

  Jump Diffusion
- NOTE: Once we expand the model to account for volatility skews, we generally lose analytical pricing ability.

Heston model: definition

- Recall that Black-Sholes formula is a closed-form solution of the Black-Sholes PDE which comes from the geometric Brownian motion:

$$dS_t = rS_t\,dt + \sigma S_t\,dW \tag{10}$$

- Heston model generalizes this by allowing the volatility itself to be stochastic:

$$\begin{aligned} dS_t &= rS_t\,dt + \sigma_t S_t\,dW_t^1 \\ d\sigma_t^2 &= \kappa(\theta - \sigma_t^2)\,dt + \xi\sigma_t\,dW_t^2 \\ Cov(dW_t^1, dW_t^2) &= \rho\,dt \end{aligned} \tag{11}$$

- Instead of one this model has a few parameters: κ, θ, ξ, ρ and σ_0.
- What process does the variance follow?
- How do we compute prices under a model such as Heston?

Heston model: Pricing European Options

- Unsurprisingly, there is no closed form solution to the Heston SDE/PDE.
- One can still make a PDE out of the Heston process, which would be two-dimensional, and attempt to solve it numerically.
- We could also apply simulation methods.
- These methods would work well for pricing but would present problems trying to find an optimal set of parameters. (Why?)
- If we were able to compute the density function $\phi(S_T)$ for a Heston process then we could use quadrature methods to compute the pricing integral.
- Although we don't know the density function for the Heston model, we do know the characteristic function.

Heston model: Pricing European Options

- Luckily, thanks to the work of Carr and Madan, we can make use of the characteristic function to obtain prices under the Heston model reasonably efficiently.
- Thanks to them, we will have to learn about the Fourier Transform and the attendant algorithm Fast Fourier Transform.
- FFT pricing methods work well for standard European options but don't necesarilly extend to other structures.

Solving PDEs numerically

- So far we had only discussed pricing European options. But what if the option is American?
- Then the Fourier method will not work. In order to price American options, we will need to learn to solve PDEs numerically.
- We will learn how to keep the algorithms stable, prove that they converge to the actual solution, and measure their efficiency.
- One commonly used algorithm is Crank-Nicolson.

Simulation

- Numerical solutions of PDEs work well when there is only one underlying asset.
- But what if we need to price an option on a basket of assets? Or what if we encounter options on fixed-income instruments, which depends on entire curves of assets? Then the computational load of numerical solutions of PDEs becomes daunting. (Why?)
- In these multi-asset cases we will have to resort to simulations.
- In particular, we will examine various methods for making simulations more efficient.
- And what if the multi-asset option is American?

Finding Optimal Model Parameters

- Don't forget that in all the situations that we had described above, we also need to be able to find the parameters from the market prices, i.e., solve the inverse problem.
- In other words, our job is generally to make densities out of market prices. Then, we can use these densities for pricing, hedging, etc.
- Generally, we do this by calibrating the model parameters to all available European option prices and then use the optimized parameters to price more complex derivatives.

Course Topics

In this course we will cover the following topics:

1. Common Stochastic models used in finance
2. Option Pricing via Numerical integration (Quadrature methods)
3. Option Pricing via Fourier Methods
4. Option Pricing via Numerical Solutions to Partial Differential Equations
5. Option Pricing via Simulation
6. Calibration and Optimization
7. Construction of Risk Neutral Densities

Grading

- Your final grade for this course will be assigned based on your performance on:
  1. The Exam (35% of your final grade)
  2. The final project (35% of your final grade)
  3. Homework (30% of your final grade)

Exam

- The exam will be held on April 8th.
- No materials (calculators/cheat sheets, etc.) are allowed during the exam.
- We will provide a practice exam and will dedicate the class on April 6th to review for the exam.

Final Project

- You will be required to build teams of 4-6 people to work with on a project of practical relevance.
- Your team will be asked to give a 10-minute presentation of your results. The presentations will be held during the last two sessions.
- In addition, your team will be required to submit a written summary of at most 10 pages containing your method and results. You will have to submit your summary paper two days after you give your presentation.
- We expect all group members to contribute to their team's final project. We also expect all group members to be able to discuss all aspect's of their teams work. We reserve the right to ask any group or individual member about any aspect of their project and incorporate these answers into the individual's project grade.

Final Project

- For your final project we expect you to:
  1. Choose a topic/technique directly related to computational methods in finance.
  2. Research this topic/technique thoroughly, highlighting its applications in finance and potential uses.
  3. Apply this topic to a real-world finance problem. This includes implementing the topic/technique in R/Matlab/Python and obtaining substantive results.
  4. Analyze the strengths and weaknesses of the topic/technique based on your results as well as external research.

Final Project Proposal

- You will be required to choose a topic and submit a project proposal to us by February 19th.
- We will give you a significant amount of flexibility in choosing your topic for the final project; however, it must be directly related to the topics that we discuss in class.
- Your proposal should describe the topic that you are planning to research, and include at least 3 references that you plan to leverage in your research.

Homework

- Homework will be assigned every week.
- The assignments will be a combination of theoretical and computational questions, leaning toward computation.

  Some assigments will require a significant amount of coding.

  Students may choose to program in R, Matlab or Python.
- Students are allowed to discuss their homework assignments; however, all solutions and any code must be unique and written by the student submitting the assignment.

Homework

- Homework will be due at 8am on the specified due date, usually Wednesdays.
- Please upload your homework solutions to your class website.
- Late homework submissions will result in a loss of 10 points for each day past the due date.

First Homework

- Your first assignment is a questionnaire designed to give us more information on your background and interests.
- These questions will be graded pass / fail. If you answer the questions with some level of thought you will receive full credit.
- Please answer questions related to your interests and goals thoughtfully and honestly as they will help inform the examples we use in subsequent lectures.
- The first assignment has been posted to the course website and is due next Wednesday January 29th by 8am.

Good luck!

It is going to be a fun course.