Option Pricing via Fast Fourier Transform: Part 1

Goals:

- Discuss Characteristic Functions & Fourier transforms
- Derive the FFT Option pricing formulas for European Options

Hirsa, Chapter 2 is a very good reference for this material, and this lecture will follow the book closely.

Review: Option Pricing via Quadrature

- Earlier in the course, we saw that one method for computing the price of a European option is to numerically integrate the density of the terminal asset price against the payoff function.
- This is a useful method when the density of the terminal asset price is known, but there is no closed-form solution for the integral.
- In this class we will see that we can also make use of the characteristic function to price options when the density is unknown or expensive to evaluate.
- It turns out that for many stochastic processes the characteristic function is known, but the pdf is unknown.
- Let's have one last look at the quadrature approach to options pricing before moving on to pricing via transform techniques.

Review: Option Pricing via Quadrature

- Recall that the price of a European call can be written as:

$$
\begin{aligned}
c &= \tilde{\mathbb{E}}\left[e^{-\int_0^T r_u\,du}(S_T-K)^+\right] && (1)\\
&= e^{-\int_0^T r_u\,du}\int_{-\infty}^{+\infty}(x-K)^+\phi(x)\,dx && (2)\\
&\approx e^{-\int_0^T r_u\,du}\int_{K}^{B}(x-K)\phi(x)\,dx && (3)
\end{aligned}
$$

- Note that we can get rid of the max in the integral, as $\int_{-\infty}^{K}(x-K)^+\phi(x)\,dx = 0$.
- We also replace the upper integral limit with some upper bound B.
- We could then apply our favorite quadrature rule to (3) in order to compute the integral numerically.

Review: Option Pricing via Quadrature

- Instead of calculating this integral directly, let's apply a change of variable in (3) to formulate the call price as a function of the log of the asset price.
- That is, let $s = \ln(x)$, $k = \ln(K)$ and $b = \ln(B)$.
- The pricing equation then becomes:

$$c = e^{-\int_0^T r_u\,du} \int_k^b (e^s - e^k) q(s) ds \tag{4}$$

- We can equivalently apply our favorite quadrature method to this integral.
- (3) and (4) are equal; however, the change of variables to define our formula in terms of the log of the asset price becomes important once we try to derive the expression in terms of the characteristic function.

Review: Option Pricing via Trapezoidal & Mid-Point Rules

- To begin, let's define our quadrature grid, that is:

$$\Delta s = (b - k)/n \tag{5}$$

$$s_i = k + i\Delta s \tag{6}$$

- Using the trapezoidal rule, our approximation becomes:

$$\hat{c} \approx e^{-\int_0^T r_u\, du} \sum_0^{n-1} \frac{1}{2} \left\{ (e^{s_{i+1}} - e^k) q(s_{i+1}) + (e^{s_i} - e^k) q(s_i) \right\} \Delta s \tag{7}$$

- Using the mid-point rule, our approximation would become:

$$\hat{c} \approx e^{-\int_0^T r_u\, du} \sum_0^{n-1} \left(e^{\frac{s_i + s_{i+1}}{2}} - e^k \right) q\left(\frac{s_i + s_{i+1}}{2} \right) \Delta s \tag{8}$$

Option Pricing via Quadrature: Conclusions

- If we know the density of the underlying asset, or the log of the underlying asset price, we can apply quadrature methods to numerically integrate the call (or put) pricing formula.
- While this method is useful, the number of practical applications are somewhat limited by the number of stochastic processes that have a known and analytically tractable density function.
- In cases where this density is unknown, or intractable to evaluate, we can try to use Fourier Transform techniques to solve options pricing problems.
- We discuss the details of Fourier Transform techniques and their application to option pricing next.

Options Pricing via Transform Techniques: Definitions

Let's review some definitions, which you learned in our last lecture. Note change of sign in the exponential between the last lecture and this one.

- Fourier Transform

$$\psi(u) = \int e^{iux} h(x)\, dx$$

- Inverse Fourier Transform

$$\psi(u) = \frac{1}{2\pi} \int e^{-iux} h(x)\, dx$$

- Characteristic Function

$$\psi(u) = \mathbb{E}\left[e^{iuX}\right] = \int e^{iux} \phi(x)\, dx$$

Characteristic Functions

- The **Characteristic Function** is the **Fourier transform** of a function if the function, $h(x)$, is a probability density function.
- Because of this, we can use **Fourier Inversion** to obtain the probability density function from a given **Characteristic Function**.

Characteristic Function: Connection to Moment Generating Function

- You may have also seen *moment generating functions*, which are closely related.
- The **Moment Generating Function** is defined as:

$$\psi(u) = \mathbb{E}\left[e^{tX}\right] = \int e^{tx}\phi(x)\,dx$$

- Characterstics functions always exist, for all distributions, whereas the moment generating function might not.
- Aside: Like the moment generating function, characteristic function can be used to obtain moments of a distribution by taking its nth derivative.

Euler's Formula: Review

- Recall from last class the Euler's formula: the function $f(u) = e^{iu}$ satisfies:

$$f(u) = \cos(u) + i\sin(u)$$

- f is antiperiodic with period π:

$$f(u+\pi) = -f(u)$$

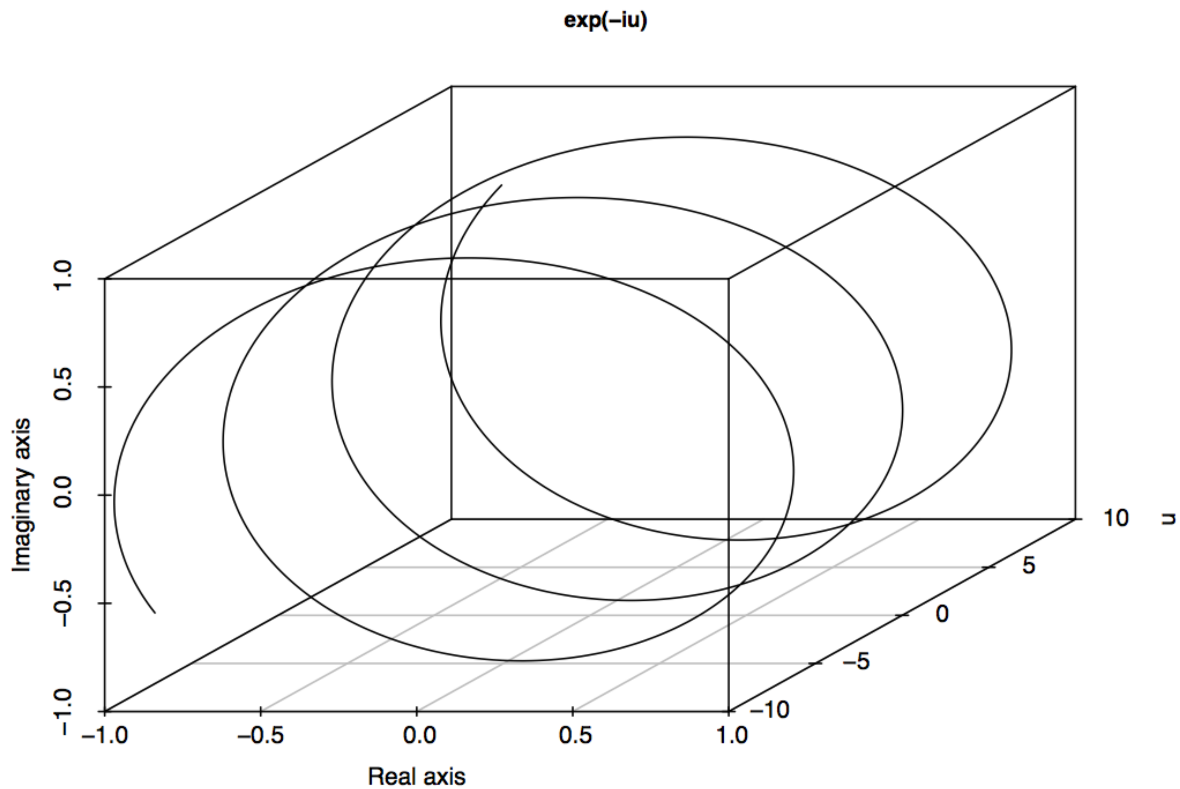
  for all $u \in \mathbb{R}$.

- Therefore, f is periodic with period 2π:

$$f(u+2\pi) = f(u)$$

  for all $u \in \mathbb{R}$

Euler's Formula: Review



Plot of the complex function $u \mapsto e^{-iu}$ for $u \in [-10, 10]$

- Note that because the function is periodic its limit as u approaches infinity & negative infinity is not well defined.

Option Pricing via FFT: Formulation

- Consider again our standard pricing formula for a European Call:

$$
\begin{aligned}
C_T(K) &= \tilde{\mathbb{E}}\left[e^{-\int_0^T r_u\,du}(S_T - K)^+\right] && (9)\\
&= e^{-\int_0^T r_u\,du}\int_{-\infty}^{+\infty}(x-K)^+\phi(x)\,dx && (10)
\end{aligned}
$$

- Let's change variables again to formulate the call price as a function of the log of the asset price:

$$
\begin{aligned}
s &= \ln(x) && (11)\\
k &= \ln(K) && (12)
\end{aligned}
$$

$$
C_T(k) = e^{-\int_0^T r_u\,du}\int_{k}^{+\infty}(e^s - e^k)q(s)\,ds \qquad (13)
$$

Option Pricing via FFT: Intuition

- Let's assume that $q(s)$ is unknown, or intractable, otherwise we would just take the integral in (13) directly.
- But suppose that we do have access to the characteristic function of $q(s)$. This will turn out to be the true in many cases.

$$\Phi(u) = \mathbb{E}\left[e^{ius}\right] = \int e^{ius} q(s)\, ds \tag{14}$$

- In order to extract the probability density function from the characteristic function, we could use Fourier Inversion:

$$q(s) = \frac{1}{2\pi} \int e^{-ius} \Phi(u)\, du \tag{15}$$

Option Pricing via FFT: Intuition

- An inefficient approach would be to evaluate (13) by using (15) and using quadrature to calculate the combined integral numerically.
- This approach would entail calculating a 2D integral: we will see that we are able to compute one of these integrals analytically.
- In fact, with this method there would be no benefit to using transform techniques to solve option pricing problems. Luckily, as we will see next, this is not the case.

Option Pricing via FFT: Approach

Instead of the brute force approach described on the previous slide, we do the following:

- We begin by taking the **Fourier Transform** of the option prices $C_T(k)$ viewed as a function of k, and expressing the transform in terms of the **Characteristic Function of the density** of the log of the asset price, $q(s)$.

- Once we have obtained the **Fourier Transform** of the option prices, we can then apply **Fourier Inversion** in order to recover the option price.

Fourier Transform of Call Price: First Attempt

- Next, let's try to calculate the Fourier transform of the call price, which is defined as:

$$\Psi(\nu) = \int_{-\infty}^{+\infty} e^{i\nu k} C_T(k) dk \tag{16}$$

- Substituting in the definition of the call price, $C_T(k)$, we have:

$$\Psi(\nu) = e^{-\int_0^T r_u \, du} \int_{-\infty}^{+\infty} e^{i\nu k} \left\{ \int_{k}^{+\infty} (e^s - e^k) q(s) \, ds \right\} dk \tag{17}$$

- We can use **Fubini's theorem** to switch the order of the integrals, which leaves us with:

$$\Psi(\nu) = e^{-\int_0^T r_u \, du} \int_{-\infty}^{+\infty} q(s) \left\{ \int_{-\infty}^{s} e^{i\nu k} (e^s - e^k) \, dk \right\} ds \tag{18}$$

Fourier Transform of Call Price: First Attempt

- Looking at the inner integral, we have:

$$\int_{-\infty}^{s} e^{i\nu k}(e^s - e^k)\,dk = e^s \int_{-\infty}^{s} e^{i\nu k}\,dk - \int_{-\infty}^{s} e^{(i\nu+1)k}\,dk \quad (19)$$

- The first integral is:

$$\int_{-\infty}^{s} e^{i\nu k} dk = \frac{1}{i\nu} e^{i\nu s} - \frac{1}{i\nu} e^{i\nu(-\infty)} \quad (20)$$

- But the second term (at $-\infty$) is not well-defined, and, certainly does not go away. [Why?]

- Therefore, the integral does not converge and we cannot continue. Fortunately, as we will see next, incorporation of a damping factor α will force the integral to converge.

Fourier Transform of Call Price: Damping Factor

- Let's define the modified, or damped call price as:

$$\tilde{C}_T(k) = e^{\alpha k} C_T(k) \tag{21}$$

- Revisiting the Fourier Transform, this time of the modified call price, we have:

$$\begin{aligned}
\Psi(\nu) &= \int_{-\infty}^{+\infty} e^{i\nu k} \tilde{C}_T(k)\, dk \\
&= e^{-\int_0^T r_u\, du} \int_{-\infty}^{+\infty} e^{(i\nu+\alpha)k} \left\{ \int_k^{+\infty} (e^s - e^k) q(s)\, ds \right\} dk \\
&= e^{-\int_0^T r_u\, du} \int_{-\infty}^{+\infty} q(s) \left\{ \int_{-\infty}^{s} e^{(i\nu+\alpha)k} (e^s - e^k)\, dk \right\} ds
\end{aligned}$$

Fourier Transform of Call Price Damping Factor: Inner Integral Convergence

- Looking at the inner integral, we have:

$$\int_{-\infty}^{s} e^{(i\nu+\alpha)k}(e^s - e^k)\,dk = \int_{-\infty}^{s} e^{(i\nu+\alpha)k} e^s\,dk - \int_{-\infty}^{s} e^{(i\nu+\alpha+1)k}\,dk$$

- If we set $\alpha > 0$, both anti-derivative terms now disappear at $-\infty$ and the integral converges.

$$\begin{aligned} \int_{-\infty}^{s} e^{(i\nu+\alpha)k}(e^s - e^k)\,dk &= e^s \frac{e^{(\alpha+i\nu)s}}{(\alpha+i\nu)} - \frac{e^{(\alpha+i\nu+1)s}}{(\alpha+i\nu+1)} \\ &= \frac{e^{(\alpha+i\nu+1)s}}{(\alpha+i\nu)(\alpha+i\nu+1)} \end{aligned} \tag{22}$$

Fourier Transform of Modified Call Price

- Plugging the solution (22) of our inner integral back into our equation for $\Psi(\nu)$ leaves us with:

$$\begin{aligned}\Psi(\nu) &= e^{-\int_0^T r_u\,du}\int_{-\infty}^{+\infty} q(s)\left\{\frac{e^{(\alpha+i\nu+1)s}}{(\alpha+i\nu)(\alpha+i\nu+1)}\right\}ds \\ &= \frac{e^{-\int_0^T r_u\,du}}{(\alpha+i\nu)(\alpha+i\nu+1)}\int_{-\infty}^{+\infty} q(s)e^{(\alpha+i\nu+1)s}ds \qquad (23)\end{aligned}$$

- Notice that the remaining integral is closely related to the characteristic function of $q(s)$. Let's factor out an i to see this clearer. (Remember that $i^2=-1$)

$$\Psi(\nu) = \frac{e^{-\int_0^T r_u\,du}}{(\alpha+i\nu)(\alpha+i\nu+1)}\int_{-\infty}^{+\infty} q(s)e^{i(\nu-(\alpha+1)i)s}ds \qquad (24)$$

- We can now see that the remaining integral is just the characteristic function of $q(s)$ evaluated at $\nu-(\alpha+1)i$.

Fourier Transform of Call Price In Terms of Characteristic Function of Log Density

- We have now derived the Fourier transform of the call price in terms of the characteristic function $\Phi(\nu)$ of the density $q(s)$:

$$\Psi(\nu) = \frac{e^{-\int_0^T r_u\,du}}{(\alpha + i\nu)(\alpha + i\nu + 1)} \Phi(\nu - (\alpha + 1)i) \tag{25}$$

- In particular, if we know the characteristic function of $q(s)$, then we can compute the Fourier transform of the call price, $\Psi(\nu)$.

Call Price via Fourier Transform

- Of course, we want to calculate the call price $C_T(k)$, not the Fourier transform of the call price.
- In order to recover the call price, we recall that:

$$\begin{aligned}\Psi(\nu) &= \int_{-\infty}^{+\infty} e^{i\nu k}\tilde{C}_T(k)\,dk \\ &= \int_{-\infty}^{+\infty} e^{\alpha k}e^{i\nu k}C_T(k)\,dk \qquad (26)\end{aligned}$$

And then use Fourier Inversion:

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{+\infty} e^{-i\nu k}\Psi(\nu)\,d\nu, \qquad (27)$$

where $\Psi(\nu)$ has been computed to be:

$$\Psi(\nu) = \frac{e^{-\int_0^T r_u\,du}}{(\alpha+i\nu)(\alpha+i\nu+1)}\Phi(\nu-(\alpha+1)i) \qquad (28)$$

Call Price via Fourier Transform

- Since we know $C_T(k)$ is a real-valued function, we know that its Fourier Transform is even in its real part and odd in its imaginary part.
- We only care about the real part of the option price, so we treat it as even.
- As a result, we treat $C_T(k)$ as even, which turns (27) into:

$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^{+\infty} e^{-i\nu k} \Psi(\nu)\, d\nu, \tag{29}$$

where $\Psi(\nu)$ is:

$$\Psi(\nu) = \frac{e^{-\int_0^T r_u\, du}}{(\alpha + i\nu)(\alpha + i\nu + 1)} \Phi(\nu - (\alpha + 1)i) \tag{30}$$

- Remember that $\alpha > 0$ is a damping factor that is a parameter of the FFT Option Pricing Method.

Fourier Transform of Call Price: Conclusions

- In this lecture we derived the relationship between the call price and the characteristic function of the density of the log of the asset.
- A similar derivation exists for put options. In particular, we simply need to choose a damping factor $\alpha < 0$.
- If we wish to incorporate additional payoffs into this framework, we would need to rederive the relationship between the payoff and the underlying characteristic function. In some cases this may be doable, e.g., for digital options; however in other cases, such as path-dependent options, it will be impossible.

Option Pricing via FFT: Calculating the Pricing Integral

- The Call pricing formula in (29) still requires we compute an integral over ν for every k.
- In theory, we can compute these integrals numerically in many ways, including using the quadrature methods we discussed in earlier lectures.
- However, it turns out that applying quadrature methods is not the most efficient means of computing this integral. Instead, we will employ a technique called **Fast Fourier Transform**, which we will discuss in detail in our next class.
- Even though it is less efficient, in order to build our intution let's take a quick look at how we would apply our favorite quadrature technique here.

Option Pricing via FFT: Calculating the Pricing Integral via Quadrature

- In order to compute the integral in (29), we first need to choose an upper bound B for our integral. The pricing formula then becomes:

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \int_0^B e^{-i\nu k} \Psi(\nu)\, d\nu \tag{31}$$

- We also need to choose a quadrature rule and define our nodes and node weights. In particular, let's divide our interval 0 to B into N equal intervals.

- The nodes would then be defined as $\nu_i = (j-1)\Delta\nu$ and the node spacing would be: $\Delta\nu = \frac{B}{N}$.

Option Pricing via FFT: Calculating the Pricing Integral via Quadrature

- If we were to use the trapezoidal rule, our quadrature approximation would become:

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} \frac{1}{2} \left[e^{-i\nu_j k} \Psi(\nu_j) + e^{-i\nu_{j+1} k} \Psi(\nu_{j+1}) \right] \Delta\nu \quad (32)$$

- NOTE: Expanding this sum, we can see that if $j \neq 1$ and $j \neq (N+1)$ the term will appear twice, and will appear once for $j = 1$ and $j = N+1$.

Option Pricing via FFT: Calculating the Pricing Integral via Quadrature

- In other words, the function we are integrating numerically is:

$$f(x) = e^{-i\nu k}\Psi(\nu) \tag{33}$$

- The nodes that we are using in our approximation are:

$$\nu_i = (j-1)\Delta\nu \tag{34}$$

$$\Delta\nu = \frac{B}{N} \tag{35}$$

- And finally, for $1 < j < N+1$ the weight is:

$$w_j = \Delta\nu \tag{36}$$

- When $j = 1$ or $j = (N+1)$ the weight is:

$$w_j = \frac{\Delta\nu}{2} \tag{37}$$

Option Pricing via FFT: Calculating the Pricing Integral via Quadrature

- We could of course apply a different quadrature rule, such as mid-point or Gauss nodes. However, it turns out that a more efficient approach is to use Fast Fourier Transform to take the integral.
- In the next class, we will talk about the Fast Fourier Transform algorithm in more detail, and will show how to formulate the above integral in a manner suitable for pre-built FFT algorithms.

Summary

- We can express the price of an option in terms of the Characteristic Function of the density of the log of the asset price.
- Evaluating option prices in this setting reduces to the problem of inverting the Fourier transform
- We needed to introduce a damping factor in order to ensure that our integrals converged. The damping factor α becomes a parameter for our technique.
- There are many stochastic processes with known Characteristic Functions, which make this technique quite useful in practice.
- This technique is very general and easy to implement via abstract code. In particular, we can implement an FFT Pricing technique that takes different Characteristic Functions for different models.