SDEs and PDEs

Goals:

- Assess where we are in the course
- Find the connection between geometric Brownian Motion and Black-Scholes PDE
- Get first glimpse of numerical solutions of PDEs

Relevant literature:

- **Hirsa, Ch.3**

Methods we have seen so far: Quadrature

Quadrature Allows us to compute low-dimensional integrals, which we can use to evaluate

- European calls, puts, and more complex European payoffs on assets with known densities.
- European payoffs with European contingencies when we know the joint density of the underlying density.

While this method is useful, and reasonably efficient for low-dimensional problems, its applications are limited by the number of stochastic processes with known densities.

In practice, use of this method limits us to the Bachelier and Black-Scholes model, maybe with a few exceptions.

Methods we have seen so far: Fourier

Fourier Methods When the characteristic function is available, it enables us to evaluate

- European payoffs of more complex stochastic processes, e.g. Heston.
- Any instrument expressible as a sum of independent random variables with known densities (even when the number of underlying densities is large).

Use of Fourier greatly expands the breadth of stochastic processes that are accessible to us.

It does not, however, help when we are dealing with non-standard payoffs, especially those that are path dependent.

Methods we have seen so far: Optimization

Optimization Essential for calibrating models to data, forming portfolios, etc. It will become your best friend.

In particular, in cases where we have a stochastic process, a pricing method, for example quadrature or Fourier, and a set of European options prices, we can use optimization to calibrate our stochastic process to the set of observed market data.

Optimization also has applications that extend far beyond option pricing into almost all other areas of quantitative finance.

Methods we have seen so far: Summary

- We can now handle European payoffs on a single underlying that is governed by almost any stochastic process (e.g. Heston, SABR, Variance Gamma).
- Additionally, we can handle European payoffs that depend on a few underlying assets if we make certain assumptions of the underlying assets' dynamics.
  - Beyond calls & puts, what other payoffs might this include?
    - Basket options (Best-of, Worst-of, Average of Basket, Product of Basket)
    - Options with European Contingencies (e.g. Call Option with European Knock-Out)
  - Although the methods that we have discussed can be used to price these products, they won't necesarily provide prices in line with how markets trade (Why?)

Methods yet to learn

- We see basket type payoffs in futures contracts that have a "cheapest-to-deliver" feature.
- Equity derivatives markets have liquidly traded basket options as dealers are often trying to eliminate correlation risk.
- These products are also commonly traded in FX, where exotic options in general occur more frequently.
- Contingencies are also common in FX, and in cross-asset hybrid products (e.g. Buying an S&P Call that knocks out in the 10 year treasury rate is above 3% at expiry)

Methods yet to learn

- In a homework we learned how to price an option on one underlying, contingent on the value of another underlying at the same maturity.
- In particular, let's consider a call option with strike K_1 on the underlying asset S that only pays if another asset, X is above a given level, K_2 at expiry. That is:

$$\tilde{c} = \mathbb{E}\left[\max(0, S_T - K_1)\, 1_{\{X_T > K_2\}}\right] \tag{1}$$

$$\tilde{c} = \int_{K_2}^{\infty} \int_{K_1}^{\infty} (S_T - K_1)\, dS_T\, dX_T \tag{2}$$

- As you saw in your homework, we can compute the price of this option if we know the densities of S_T and X_T.

Methods yet to learn

- But suppose the option was contingent on the value of that other underlying at any point *prior* to maturity.
- For these options, we need to know the maximum value of X, M_T, rather than the terminal value of X_T. In this case, the expectation we would need to take would become:

$$\tilde{c} \quad = \quad \mathbb{E}\left[\max(0, S_T - K_1)\, 1_{\{M_T > K_2\}}\right] \tag{3}$$

- Pricing this option requires knowledge of the entire path of X, or knowledge of the density of the maximum of X, M_T.

Methods yet to learn

- These options exist and are called *barrier* options.
- As you can see, the techniques we have covered so far will fail us here. (Note that the target underlying and the contigent underlying can be—and often are—the same for the barrier option.)
- These barrier options, for example, "knock-in"s and "knock-out"s are common in FX.
- There are also American options and Bermudan options that exhibit "path-dependency". In other words, the value of these options depends not only on the value of the underlying at maturity, but also on the particular path that it took to arrive to maturity.

Methods yet to learn

In order to value such path-dependent instruments, we need new techniques.

Natural ones are the ones that arrive at the price of the option by starting at maturity and evolving the solution backward in time to the present.

This allows us to check at every step, for example, whether the barrier option has gotten "knocked-in" or the American option was worth exercising early.

Some of the methods that work by backwards induction are binomial and trinomial trees and numerical solutions of PDEs.

You have seen the trees in your previous courses: we will focus on PDEs here.

Methods yet to learn

Numerical solutions of PDEs are very effective for path-dependent derivatives, but they are limited to only a few underlyings.

This makes sense since numerical solution of differential equations is generalization of quadrature.

Therefore, in high-dimensional situations, like fixed income, in many cases (not always!) we are forced to abandon PDEs (and the trees) and move on to simulations.

We will cover simulations in the later part of the course.

Black-Scholes PDE

Before we start solving the Black-Scholes PDE, we should find out where it comes from.

Suppose a stock price process S_t evolves according to

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW \tag{4}$$

Let $c(s,t)$ be a function such that $c(S_t,t)$ is a derivative instrument with payoff at maturity T given by $c(s,T)$. Then by Ito's Lemma, we have:

$$\begin{aligned} dc(S_t,t) &= \frac{\partial c}{\partial s} \, dS_t + \frac{1}{2} \frac{\partial^2 c}{\partial s^2} \, (dS_t)^2 + \frac{\partial c}{\partial t} \, dt \\ &= \frac{\partial c}{\partial s} \mu S_t \, dt + \frac{\partial c}{\partial s} \sigma S_t \, dW + \frac{1}{2} \frac{\partial^2 c}{\partial s^2} \sigma^2 S_t^2 \, dt + \frac{\partial c}{\partial t} \, dt \\ &= \left(\frac{\partial c}{\partial t} + \mu S_t \frac{\partial c}{\partial s} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 c}{\partial s^2} \right) dt + \frac{\partial c}{\partial s} \sigma S_t \, dW \end{aligned} \tag{5}$$

Black-Scholes PDE

Now let's form a portfolio consisting of the derivative instrument $c(S_t, t)$ and the underlying S_t in the right proportions so as to make it riskless.

That means we want to eliminate the dW term in (5).

Let's take one unit of $c(S_t, t)$ and $-\frac{\partial c}{\partial s}$ units of S_t to form the portfolio

$$\Pi(S_t, t) = c(S_t, t) - \frac{\partial c}{\partial s} S_t. \tag{6}$$

The quantity $\frac{\partial c}{\partial s}$ is called Delta of $c(S_t, t)$ and this portfolio is called Delta-hedged portfolio.

Black-Scholes PDE

Let's compute the dW term of Π. A glance at (4) and (5) tells us that it is

$$\frac{\partial c}{\partial s}\sigma S_t - \left(\frac{\partial c}{\partial s}\right)\sigma S_t = 0$$

But this means that Π is a *riskless instrument*! Therefore, by the standard arbitrage argument its evolution equation must be

$$d\Pi = r\Pi\,dt \tag{7}$$

Of course, that it is only instantaneously delta-hedged: the hedge will be off in the next instant. But that's ok: we can rehedge it then.

Note that we are making an important assumption here: that hedging is available continuously and there are no transaction costs.

Black-Scholes PDE

But let's get back to work: our goal is to compute the dt term of $d\Pi$ using (4), (5) and (6):

$$\begin{aligned} & \left(\frac{\partial c}{\partial t} + \mu S_t \frac{\partial c}{\partial s} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c}{\partial s^2}\right) - \frac{\partial c}{\partial s}(\mu S_t) \\ = \quad & \frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c}{\partial s^2} \end{aligned} \tag{8}$$

According to (7), (8) must equal $r\Pi$, which is

$$rc(S_t, t) - r\frac{\partial c}{\partial s} S_t. \tag{9}$$

Putting (8) and (9) together, and renaming S_t to s (they are both just arguments here) we get the equation that $c(s,t)$ must satisfy:

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 c}{\partial s^2} + rs\frac{\partial c}{\partial s} - rc = 0, \tag{10}$$

which is the celebrated Black-Scholes equation.

Black-Scholes PDE: discussion

What does Black-Scholes PDE tell us?

- It connects three important Greeks:

$$\Theta + \frac{1}{2}\sigma^2 s^2 \Gamma + rs\Delta = rc \tag{11}$$

  So, if Gamma is large (which is good), then Theta decay is also large (which is not good). So, if you want to maintain a large Gamma position, you will be paying a lot of rent for it.

- Note that in the derivation we never used anything particular to the derivative instrument c_t. That means that every derivative instrument whose underlying follows Geometric Brownian Motion, must satisfy (10).

Black-Scholes PDE: discussion

- Since solutions to (10) are unique, it follows that different instruments are determined by their payoffs at maturity.
- There are a few special payoffs, like those for calls and puts that allow an analytic solution to (10). Those solutions are, of course, the famous Black-Scholes formula. But they are rare and we will need to resort to numerical methods for other instruments.
- An added benefit of the numerical methods is that a small modification of the numerical solution algorithm allows us to price American-style and other path-dependent instruments. We will learn more in the next few lectures when we investigate these algorithms.

Black-Scholes PDE: discussion

- Numerical solutions of PDE require proper discretization and we will need to navigate past some hidden rocks.
- As you have learned in the homework on extracting densities, taking derivatives of noisy (i.e., realistic) data can lead to instabilities. These problems only magnify for higher derivatives.
- Even when those problems are solved, iteration along time axis can make the algorithm blow up. We will learn how to recognize and prevent these situations.

Specifying the PDE

In order to fully specify the solution of a parabolic PDE (Black-Scholes is an example of one; heat equation $c_t = c_{xx}$ is another) we need to specify some boundary conditions. In our case, they will be

1. The payoff at maturity T, i.e., $c(s, T)\ \forall s$.
2. Values along "the bottom", i.e., $c(0, t)\ \forall t \in [0, T]$.
3. Values along "the top", i.e., $\lim_{s\to\infty} c(s, t)\ \forall t \in [0, T]$

Once these boundary conditions are specified the equation (10) will have a unique solution.

Specifying the PDE

To fix ideas, let's suppose that the instrument $c(s,t)$ is a European call with strike K. The payoff condition will then be

$$c(s,T) = (s-K)^+ \tag{12}$$

Since the value of the underlying is non-negative, it's natural to specify the "bottom" condition at $s=0$. What happens when $s=0$? Look at the evolution equation (4): if S_t ever touches zero, dS_t must also be zero and S_t will never recover. In other words, once S_t touches zero, it stays there. It will not surprize you that this zero is called an "absorbing state". But if the underlying is zero, then a call on it is worth zero. For us, this means that the "bottom" condition is:

$$c(0,t) = (0-K)^+ = 0 \qquad \forall t \in [0,T]. \tag{13}$$

Specifying the PDE

What about the "top" condition?

If the underlying is very large compared to the strike, then the call is very likely to be exercised and the call is valued as if it is sure to be exercised. Another way of saying this is:

$$\lim_{s \to \infty} c(s,t) = s - Ke^{-r(T-t)} \tag{14}$$

Note the discounting of K. (Why?)

So now we have all three conditions on slide 19 specified.

Therefore, there will be a unique solution. (under some technical conditions, of course, but we won't worry about that here.)

Specifying the PDE

In the case of European call (or put or anything that can be made out of them), it is actually possible to solve the Black-Scholes PDE analytically. If you do, you will rediscover the Black-Scholes formula.

In the more interesting [a] cases, we will need to resort to numerical solutions.

[a] FYI: Mathematicians say "interesting" when they mean "difficult". And when something is really difficult, they say "deep".

Discretizing the PDE

A numerical solution involves discretizing the problem, i.e., replacing derivatives by finite differences and solving the resulting finite-difference equations.

Of course, someone had to prove that the solution to the finite-difference equation is an approximation to the solution of the original PDE. Turns out it is not always the case and that someone had to establish under which conditions we can be assured that as we refine the grid our solution will converge to the real one.

Strict analysis of the convergence is outside the scope of this course, but it is reassuring that the work has been done.

Discretizing the PDE

Let's Taylor expand c near the point (s,t) in t while holding s fixed:

$$c(s,t+h) = c(s,t) + \frac{\partial c}{\partial t}(s,t)\,h + O(h^2) \tag{15}$$

Solving for $\frac{\partial c}{\partial t}$, we get

$$\frac{\partial c}{\partial t}(s,t) = \frac{c(s,t+h) - c(s,t)}{h} + O(h) \tag{16}$$

Similarly, we get an approximation to the s derivative

$$\frac{\partial c}{\partial s}(s,t) = \frac{c(s+h,t) - c(s,t)}{h} + O(h) \tag{17}$$

These are called *forward* differences. *Backward* differences are

$$\frac{\partial c}{\partial s}(s,t) = \frac{c(s,t) - c(s-h,t)}{h} + O(h) \tag{18}$$

Discretizing the PDE

Next, let's write 2nd order Taylor expansion with h and $-h$:

$$c(s+h,t) = c(s,t) + \frac{\partial c}{\partial s}(s,t)\,h + \frac{1}{2}\frac{\partial^2 c}{\partial s^2}(s,t)\,h^2 + O(h^3) \tag{19}$$

$$c(s-h,t) = c(s,t) - \frac{\partial c}{\partial s}(s,t)\,h + \frac{1}{2}\frac{\partial^2 c}{\partial s^2}(s,t)\,h^2 + O(h^3) \tag{20}$$

If we subtract (20) from (19) and solve for $\frac{\partial c}{\partial s}$, we will get *central* differences approximation to the first derivative wrt s:

$$\frac{\partial c}{\partial s}(s,t) = \frac{c(s+h,t) - c(s-h,t)}{2h} + O(h^2). \tag{21}$$

Note the much improved $O(h^2)$, compared to $O(h)$ in (17) and (18).

We are still not done: we need a finite-difference version of the second derivative. Adding (19) and (20), we get

$$\frac{\partial^2 c}{\partial s^2}(s,t) = \frac{c(s+h,t) - 2c(s,t) + c(s-h,t)}{h^2} + O(h) \tag{22}$$

Discretizing the PDE

We could use the approximation (16) for Theta, (17) for Delta, (22) for Gamma, and plug them in to the Black-Scholes equation.

But how would that help us? The idea is to take the values that we know, namely the payoff $c(s, T)$ and use the propertly discretized Black-Scholes equation to deduce values of $c(s, t)$ for all s for $t = T - h$, where h is a small positive number.

We would then repeat the procedure to get the values of $c(s, T - 2h)$ and so on until we get to $c(s, 0)$, i.e., the values of c at present.

But, wait: c should only have one value at present! We'll get back to this point in a later lecture.

Discretizing the PDE

In order to do that, we must get organized.

Let t_j for $j = 0, \dots, N$ be evenly spaced points on the time axis so that $t_0 = 0$ and $t_N = T$. In other words,

$$t_j = \frac{T}{N} j. \tag{23}$$

For s mesh, we need to choose $s_{\max}$ so that the approximation for the "top" condition (14) is sufficiently close. Then we will assign the right-hand-side of (14) to be the boundary condition along $s_{\max}$ and define

$$s_i = \frac{s_{\max}}{M} i, \qquad i = 0, \dots, M \tag{24}$$

Discretizing the PDE

The discrete version of the boundary conditions on slide 19 for our European call becomes:

1. The payoff at maturity T: $c(s_i, t_N) = (s_i - K)^+ \quad \forall i$
2. "The bottom": $c(0, t_j) = 0 \quad \forall j$.
3. "The top": $c(s_M, t_j) = s_M - Ke^{-r(t_N - t_j)} \quad \forall j$

In our notation, we should distinguish between solutions to the continuous PDE and the discrete one: we'll denote by C the solution to the discrete PDE. The boundary conditions above are obeyed by both c and C, of course.

Discretizing the PDE

Let's rewrite (10), so we can see it as we discretize:

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 c}{\partial s^2} + rs\frac{\partial c}{\partial s} - rc = 0. \tag{25}$$

Let $h_t = t_{j+1} - t_j$ denote the size of the time mesh and let $h_s = s_{i+1} - s_i$ denote the price mesh. Then

$$\begin{aligned}
& \frac{C(s_i, t_j) - C(s_i, t_{j-1})}{h_t} \\
+ \quad & \frac{1}{2}\sigma^2 s_i^2 \frac{C(s_{i+1}, t_j) - 2C(s_i, t_j) + C(s_{i-1}, t_j)}{h_s^2} \\
+ \quad & rs_i \left(\frac{C(s_{i+1}, t_j) - C(s_i, t_j)}{h_s} \right) - rC(s_i, t_j) \\
= \quad & O(h_s) + O(h_t)
\end{aligned} \tag{26}$$

Discretizing the PDE

Why is (26) easier to solve than (25)?

How do we know that the solution C exists at all?

And if we do solve (26), how we know that C is in any way related to c?

Why can't people just trade something else?

We'll answer most of these questions.

To start, let's look at (26) again and let's pretend that $O(h_s) + O(h_t)$ is zero.

At maturity, i.e., when $j = N$, we know all the values of $C(s_i, t_j)$, and, therefore, we can use (26) to find $C(s_i, t_{j-1})$.

An algorithm for solving the discretized PDE

Let's try this again more carefully:

1. We know $C(s_i, t_N)$ for i $=$ 0, . . . , M. (Why?)
2. We can use (26) to compute $C(s_i, t_{N-1})$ for i $=$ 1, . . . , M-1. (Why?)
3. We can get $C(s_0, t_{N-1})$ from the bottom boundary condition and $C(s_M, t_{N-1})$ from the top boundary condition.
4. Now we know $C(s_i, t_{N-1})$ for i $=$ 0, . . . , M. Notice that we had to use all three boundary conditions.
5. But, look! We are now in the same situation at time t_{N-1} that we were in at t_N just a minute ago. So we can repeat the above steps to find $C(s_i, t_{N-2})$ for i $=$ 0, . . . , M.
6. We can continue until we get to the present, i.e., until we know $C(s_i, t_0)$ for i $=$ 0, . . . , M.

An algorithm for solving the discretized PDE

What could possibly go wrong?

We'll find out next week.

Enjoy your weekend!