A Technique for Painless Derivation of Kalman Filtering Recursions

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Abstract

We develop a calculus for basic operations on Gaussian potentials and give technical details of several related intermediate results. Using basic operations, we demonstrate a very mechanical derivation of Kalman (forward and backward) filtering recursions in information form (Bar-Shalom and Li, 1993). The discussion herein is mainly inspired by (Murphy, 1998); we merely fill-in a few technical gaps. This note is not meant as a tutorial introduction to Kalman filtering but rather as a "lookup" reference.

1 The Gaussian Potential

Consider a Gaussian potential with mean μ and covariance Σ defined on some domain indexed by x.

$$\begin{aligned} \phi(x) &= \alpha \mathcal{N}(\mu, \Sigma) && (1) \\ &= \alpha |2\pi\Sigma|^{-\frac{1}{2}} \exp(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)) && (2) \end{aligned}$$

where $\int dx \phi(x) = \alpha$. If $\alpha = 1$ the potential is normalized. A general Gaussian potential ϕ need not to be normalized so α is in fact an arbitrary positive constant.

We note that the exponent is just a quadratic form. To see this explicitely we rewrite the potential as[1]

$$\begin{aligned}
\phi(x) &= \exp(\{\log\alpha - \frac{1}{2}\log|2\pi\Sigma| - \frac{1}{2}\mu^T\Sigma^{-1}\mu\} - \frac{1}{2}x^T\Sigma^{-1}x + \mu^T\Sigma^{-1}x) \\
&= \exp(g + h^T x - \frac{1}{2}x^T K x)
\end{aligned}$$

The second representation is an alternative to the conventional and intuitive moment form. Here we represent the potential by the polynomial coefficients h and K. We call this representation a *canonical* representation and refer to the coefficients h and K as natural parameters.

The moment parameters and canonical parameters are related by

$$K = \Sigma^{-1} \tag{3}$$

$$h = \Sigma^{-1}\mu \tag{4}$$

$$g = \log\alpha - \frac{1}{2}\log|2\pi\Sigma| - \frac{1}{2}\mu^T\Sigma^{-1}\Sigma\Sigma^{-1}\mu \tag{5}$$

$$= \log\alpha + \frac{1}{2}\log|\frac{K}{2\pi}| - \frac{1}{2}h^T K^{-1}h \tag{6}$$

For convenience, we will denote the part of g that is determined by h and K as $\bar{g}$

$$\bar{g} = \frac{1}{2}\log|\frac{K}{2\pi}| - \frac{1}{2}h^T K^{-1}h \tag{7}$$

$$g = \log\alpha + \bar{g} \tag{8}$$

For quick reference the inverse formulae are

$$\Sigma = K^{-1} \tag{9}$$

$$\mu = K^{-1}h \tag{10}$$

$$\alpha = \exp(g - \bar{g}) \tag{11}$$

$$= \exp(g - \frac{1}{2}\log|\frac{K}{2\pi}| + \frac{1}{2}h^T K^{-1}h) \tag{12}$$

To denote a potential in canonical form we will use the notation

$$\phi(x) = \alpha\mathcal{N}(\mu,\Sigma) \equiv [h, K, g]$$

When the potential is normalized, i.e. when $\alpha = 1$, we will drop α. Analogously, if g can be exactly computed from h and K, i.e. $g = \bar{g}$ we will write $[h, K]$.

[1] $|2\pi\Sigma|$ is a short notation for $(2\pi)^d \det\Sigma$, where Σ is $d \times d$.

1.1 Jointly Gaussian Vectors

Consider now two jointly Gaussian vectors x_1 and x_2. The moment form is given by

$$\begin{aligned}
\phi(x_1, x_2) &= \alpha \mathcal{N}(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}) \\
&= \alpha |2\pi\Sigma|^{-\frac{1}{2}} \exp(-\frac{1}{2} \begin{pmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix})
\end{aligned}$$

$$\phi(x_1, x_2) = \exp(g + \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix})$$

To find K, we need to find a parametric representation of $K = \Sigma^{-1}$ in terms of the partitions $\Sigma_{11}, \Sigma_{12}, \Sigma_{21}, \Sigma_{22}$.

We will derive the result by the following strategy: We will find two matrices X and Z such that W becomes block diagonal.

$$\begin{aligned}
X\Sigma Z &= W \\
\Sigma &= X^{-1} W Z^{-1} \\
\Sigma^{-1} &= Z W^{-1} X = K
\end{aligned}$$

This leads to following dual factorizations of Σ as

$$\begin{aligned}
\Sigma &= \begin{pmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_{11} - \Sigma_{12}(\Sigma_{22})^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} - \Sigma_{21}(\Sigma_{11})^{-1}\Sigma_{12} \end{pmatrix} \begin{pmatrix} I & -\Sigma_{11}^{-1}\Sigma_{12} \\ 0 & I \end{pmatrix}
\end{aligned}$$

We will introduce the notation

$$\begin{aligned}
\Sigma/\Sigma_{22} &= \Sigma_{11} - \Sigma_{12}(\Sigma_{22})^{-1}\Sigma_{21} \\
\Sigma/\Sigma_{11} &= \Sigma_{22} - \Sigma_{21}(\Sigma_{11})^{-1}\Sigma_{12}
\end{aligned}$$

$$\begin{aligned}
\Sigma^{-1} &= \begin{pmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{pmatrix} \begin{pmatrix} (\Sigma/\Sigma_{22})^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix} \begin{pmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} I & -\Sigma_{11}^{-1}\Sigma_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & (\Sigma/\Sigma_{11})^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} &= \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} (\Sigma/\Sigma_{22})^{-1} & -(\Sigma/\Sigma_{22})^{-1}\Sigma_{12}\Sigma_{22}^{-1} \\ -\Sigma_{22}^{-1}\Sigma_{21}(\Sigma/\Sigma_{22})^{-1} & \Sigma_{22}^{-1} + \Sigma_{22}^{-1}\Sigma_{21}(\Sigma/\Sigma_{22})^{-1}\Sigma_{12}\Sigma_{22}^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \Sigma_{11}^{-1} + \Sigma_{11}^{-1}\Sigma_{12}(\Sigma/\Sigma_{11})^{-1}\Sigma_{21}\Sigma_{11}^{-1} & -\Sigma_{11}^{-1}\Sigma_{12}(\Sigma/\Sigma_{11})^{-1} \\ -(\Sigma/\Sigma_{11})^{-1}\Sigma_{21}\Sigma_{11}^{-1} & (\Sigma/\Sigma_{11})^{-1} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} &= \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \\
h_1 &= K_{11}\mu_1 + K_{12}\mu_2 \\
h_2 &= K_{22}\mu_1 + K_{21}\mu_2
\end{aligned}
$$

Another important result is

$$
|\Sigma| = |\Sigma/\Sigma_{11}||\Sigma_{11}| = |\Sigma/\Sigma_{22}||\Sigma_{22}|
$$

We decompose the constant term g as

$$
\begin{aligned}
g &= \log\alpha - \frac{1}{2}\log|2\pi\Sigma| - \frac{1}{2}\mu^T\Sigma^{-1}\mu \\
&= \log\alpha - \frac{1}{2}\log|2\pi\Sigma_{11}| - \frac{1}{2}\log|2\pi\Sigma/\Sigma_{11}| \\
&\quad -\frac{1}{2}\begin{pmatrix} \mu_1 \\ \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1 \end{pmatrix}^T \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & (\Sigma/\Sigma_{11})^{-1} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1 \end{pmatrix}
\end{aligned}
$$

We observe that

$$
\begin{aligned}
(\Sigma/\Sigma_{11})^{-1} &= K_{22} = K_{22}K_{22}^{-1}K_{22} \\
-\Sigma_{21}\Sigma_{11}^{-1} &= K_{22}^{-1}K_{21}
\end{aligned}
$$

thus

$$
g = \log\alpha + \frac{1}{2}\log|\frac{K_{22}}{2\pi}| - \frac{1}{2}{h_2}^T{K_{22}}^{-1}h_2 - \frac{1}{2}\log|2\pi\Sigma_{11}| - \frac{1}{2}{\mu_1}^T{\Sigma_{11}}^{-1}\mu_1
$$

2 Operations on Gaussian Potentials

An important feature of Gaussian potentials is that standard operations on Gaussian potentials always result in Gaussian potentials (a closed algebraic system).

2.1 Multiplication and Division

The potentials are extended to the same domain by adding zeros to appropriate dimensions. Multiplying (dividing) exponential forms is equivalent to adding (substracting) the exponents.

Hence we have

$$\begin{aligned}[h_a, K_a, g_a] \times [h_b, K_b, g_b] &= [h_a + h_b, K_a + K_b, g_a + g_b] \\ [h_a, K_a, g_a]/[h_b, K_b, g_b] &= [h_a - h_b, K_a - K_b, g_a - g_b]\end{aligned}$$

One has to be more cautious with division since dividing arbitrary Gaussian potentials may result in non-normalizable potentials. This can be seen by considering $(K_a - K_b)^{-1}$ which has to be positive semidefinite. For example in one dimension, dividing a broad Gaussian by a narrower one will result in a non-normalizable potential since $K_a - K_b < 0$. For higher dimensions, all the eigenvalues of $K_a - K_b$ have to be non-negative.

In moment representation

$$\alpha' \mathcal{N}(\mu', \Sigma') = \alpha_a \mathcal{N}(\mu_a, \Sigma_a) \times \alpha_b \mathcal{N}(\mu_b, \Sigma_b)$$

where

$$\begin{aligned}\Sigma' &= (\Sigma_a^{-1} + \Sigma_b^{-1})^{-1} \\ \mu' &= \Sigma'(\Sigma_a^{-1}\mu_a + \Sigma_b^{-1}\mu_b)\end{aligned}$$

The formulas for the moment form implicitely convert to the canonical form and then convert back. For division, $+$ operator is replaced simply by $-$.

The normalization constant of a product in moment form is tricky. From canonical form we have $g' = g_a + g_b$ where

$$\begin{aligned}g_a &= \log \alpha_a + \bar{g}_a = \log \alpha_a - \frac{1}{2}\log|2\pi\Sigma_a| - \frac{1}{2}\mu_a^T \Sigma_a^{-1} \mu_a \\ g_b &= \log \alpha_b + \bar{g}_b = \log \alpha_b - \frac{1}{2}\log|2\pi\Sigma_b| - \frac{1}{2}\mu_b^T \Sigma_b^{-1} \mu_b \\ g' &= \log \alpha_a + \log \alpha_b + \bar{g}_a + \bar{g}_b\end{aligned}$$

On the other hand we have by definition

$$\begin{aligned}g' &= \log \alpha' + \bar{g}' \\ &= \log \alpha' - \frac{1}{2}\log|2\pi\Sigma'| - \frac{1}{2}\mu'^T \Sigma'^{-1} \mu'\end{aligned}$$

So

$$\begin{aligned}
\alpha' &= \alpha_a \alpha_b \exp(\bar{g}_a + \bar{g}_b - \bar{g'}) \\
&= \alpha_a \alpha_b \left(\frac{|2\pi\Sigma'|}{|2\pi\Sigma_a||2\pi\Sigma_b|} \right)^{1/2} \exp(-\frac{1}{2}(\mu_a^T \Sigma_a^{-1} \mu_a + \mu_b^T \Sigma_b^{-1} \mu_b - \mu'^T \Sigma'^{-1} \mu'))
\end{aligned}$$

Similarly for division we have

$$\begin{aligned}
\alpha' &= \alpha_a \alpha_b \exp(\bar{g}_a - \bar{g}_b - \bar{g'}) \\
&= \alpha_a \alpha_b \left(\frac{|2\pi\Sigma'||2\pi\Sigma_b|}{|2\pi\Sigma_a|} \right)^{1/2} \exp(-\frac{1}{2}(\mu_a^T \Sigma_a^{-1} \mu_a - \mu_b^T \Sigma_b^{-1} \mu_b - \mu'^T \Sigma'^{-1} \mu'))
\end{aligned}$$

2.2 Marginalization

Consider the potential

$$\phi(x_1, x_2) = \alpha \mathcal{N}(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}) \tag{13}$$

$$\equiv \left[\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, g \right] \tag{14}$$

The marginals have a very simple form in moment representation, namely

$$\phi(x_1) = \int dx_2 \phi(x_1, x_2) = \alpha \mathcal{N}(\mu_1, \Sigma_{11})$$

Marginalization is more involved in the canonical form since we have to convert to moment representation, marginalize and then convert back

We first find

$$\begin{aligned}
\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}^{-1} &= \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \\
&= \begin{pmatrix} (K_{11} - K_{12}(K_{22})^{-1} K_{21})^{-1} & -(K_{11} - K_{12}(K_{22})^{-1} K_{21})^{-1} K_{12}(K_{22})^{-1} \\ -(K_{22} - K_{21}(K_{11})^{-1} K_{12})^{-1} K_{21}(K_{11})^{-1} & (K_{22} - K_{21}(K_{11})^{-1} K_{12})^{-1} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\phi(x_1) &\equiv [h', K', g'] \\
K' &= \Sigma_{11}^{-1} = K_{11} - K_{12} K_{22}^{-1} K_{21}
\end{aligned}$$

To find h' we use the basic relation

$$\begin{aligned}
\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} &= \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\
\mu_1 &= \Sigma_{11} h_1 + \Sigma_{12} h_2 \\
&= \Sigma_{11} h_1 + \Sigma_{11} K_{12} K_{22}^{-1} h_2 \\
h' &= \Sigma_{11}^{-1} \mu_1 \\
&= h_1 - K_{12} K_{22}^{-1} h_2
\end{aligned}$$

The new normalization constant is

$$g' = \log \alpha - \frac{1}{2} \log |2\pi \Sigma_{11}| - \frac{1}{2} \mu_1^T \Sigma_{11}^{-1} \mu_1$$

We can relate the new g' to g by considering the decomposition

$$\begin{aligned} g &= \log \alpha - \frac{1}{2} \log |2\pi \Sigma_{11}| - \frac{1}{2} {\mu_1}^T {\Sigma_{11}}^{-1} \mu_1 + \frac{1}{2} \log |\frac{K_{22}}{2\pi}| - \frac{1}{2} {h_2}^T {K_{22}}^{-1} h_2 \\ g' &= g - \frac{1}{2} \log |\frac{K_{22}}{2\pi}| + \frac{1}{2} {h_2}^T (K_{22})^{-1} h_2 \end{aligned}$$

2.3 Conditioning

Suppose x_2 is observed to be $\hat{x}_2$. Then, we have

$$\begin{aligned} \phi^*(x_1) &= \phi(x_1, x_2 = \hat{x}_2) \\ &= \exp(g + \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} x_1 \\ \hat{x}_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} x_1 & \hat{x}_2 \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ \hat{x}_2 \end{pmatrix}) \\ &= \exp(\{g + h_2^T \hat{x}_2 - \frac{1}{2} \hat{x}_2^T K_{22} \hat{x}_2\} + x_1^T (h_1 - K_{12} \hat{x}_2) - \frac{1}{2} x_1^T K_{22} x_1) \end{aligned}$$

Hence the resulting potential has the simple form

$$\phi^*(x_1) = [(h_1 - K_{12} \hat{x}_2), K_{11}, g + h_2^T \hat{x}_2 - \frac{1}{2} \hat{x}_2^T K_{22} \hat{x}_2]$$

We can derive above results using an alternative interpretation of conditioning.

$$\phi(x_1, x_2 = \hat{x}_2) = \int_{x_2} \phi(x_1, x_2) \delta(x_2 - \hat{x}_2)$$

In words, conditioning is equivalent to multiplication by a potential $[K\hat{x}_2, K, \frac{1}{2} \log |\frac{K}{2\pi}| - \frac{1}{2} \hat{x}_2 K \hat{x}_2]$, where $K \to \infty$, i.e. a delta dirac and then marginalizing over x_2.

We first extend the domain to x_1, x_2 and multiply

$$\begin{aligned} p(x_1, x_2) \delta(x_2 - \hat{x}_2) &= \left[\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \right] \times \left[\begin{pmatrix} 0 \\ K\hat{x}_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \right] \\ &= \left[\begin{pmatrix} h_1 \\ h_2 + K\hat{x}_2 \end{pmatrix}, \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} + K \end{pmatrix} \right] \end{aligned}$$

$$
\begin{aligned}
h' &= h_1 - K_{12}(K_{22}+K)^{-1}(h_2 + K\hat{x}_2) \\
&= h_1 - K_{12}\hat{x}_2 \\
K' &= K_{11} - K_{12}(K_{22}+K)^{-1}K_{21} = K_{11}
\end{aligned}
$$

We can find the moment characteristics $\alpha'\mathcal{N}(\mu', \Sigma')$ using the standard conversion formulas.

$$
\begin{aligned}
\Sigma' &= K'^{-1} = K_{11}^{-1} \\
&= \Sigma_{11} - \Sigma_{12}(\Sigma_{22})^{-1}\Sigma_{21} \\
\mu' &= \Sigma' h' = K_{11}^{-1}h_1 - K_{11}^{-1}K_{12}\hat{x}_2 \\
&= K_{11}^{-1}(K_{11}\mu_1 + K_{12}\mu_2) - K_{11}^{-1}K_{12}\hat{x}_2 \\
K_{11}^{-1}K_{12} &= -\Sigma_{12}\Sigma_{22}^{-1} \\
\mu' &= \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\hat{x}_2 - \mu_2)
\end{aligned}
$$

We will derive the new normalization constant α' from its definition as

$$
\begin{aligned}
\alpha' &= \int_{x_1} \phi(x_1, x_2 = \hat{x}_2) \\
&= \int_{x_1}\int_{x_2} \phi(x_1, x_2)\delta(x_2 - \hat{x}_2) \\
&= \int_{x_2} \phi(x_2)\delta(x_2 - \hat{x}_2)
\end{aligned}
$$

In words we have to just evaluate the marginal $\phi(x_2)$ at $\hat{x}_2$

$$
\begin{aligned}
\alpha' &= \alpha \int_{x_2} \mathcal{N}(\mu_2, \Sigma_{22})\delta(x_2 - \hat{x}_2) \\
&= \alpha|2\pi\Sigma_{22}|^{-\frac{1}{2}} \exp(-\frac{1}{2}(\hat{x}_2 - \mu_2)^T\Sigma_{22}^{-1}(\hat{x}_2 - \mu_2))
\end{aligned}
$$

2.4 Example

Consider the following parametrization

$$
\begin{aligned}
x_2 &= Ax_1 + \varepsilon \\
\varepsilon &\sim \mathcal{N}(0, Q) \\
x_1 &\sim \mathcal{N}(\mu, P)
\end{aligned}
$$

We have

$$p(x_1, x_2) = \mathcal{N}(\begin{pmatrix} \mu \\ A\mu \end{pmatrix}, \begin{pmatrix} P & PA^T \\ AP & APA^T + Q \end{pmatrix})$$

The inverse of the covariance matrix is

$$\begin{pmatrix} P^{-1} + P^{-1}PA^T(APA^T + Q - AP(P)^{-1}PA^T)^{-1}APP^{-1} & -(P)^{-1}PA^T(APA^T + Q - AP(P)^{-1}PA^T)^{-1} \\ -(APA^T + Q - AP(P)^{-1}PA^T)^{-1}AP(P)^{-1} & (APA^T + Q - AP(P)^{-1}PA^T)^{-1} \end{pmatrix}$$

$$p(x_1, x_2) = \left[\begin{pmatrix} P^{-1}\mu \\ 0 \end{pmatrix}, \begin{pmatrix} P^{-1} + A^TQ^{-1}A & -A^TQ^{-1} \\ -Q^{-1}A & Q^{-1} \end{pmatrix}, \bar{g} \right]$$

where

$$\bar{g} = -\frac{d}{2}\log(2\pi) - \frac{1}{2}\log|P||APA^T + Q - AP(P)^{-1}PA^T| \tag{15}$$

$$-\frac{1}{2}\begin{pmatrix} \mu & A\mu \end{pmatrix}\begin{pmatrix} P^{-1} + A^TQ^{-1}A & -A^TQ^{-1} \\ -Q^{-1}A & Q^{-1} \end{pmatrix}\begin{pmatrix} \mu \\ A\mu \end{pmatrix} \tag{16}$$

$$= -\frac{1}{2}\log|2\pi P| - \frac{1}{2}\mu^T P^{-1}\mu - \frac{1}{2}\log|2\pi Q| \tag{17}$$

We can see that this is in fact the representation

$$p(x_1)p(x_2|x_1) = \left[\begin{pmatrix} P^{-1}\mu \\ 0 \end{pmatrix}, \begin{pmatrix} P^{-1} & 0 \\ 0 & 0 \end{pmatrix}, -\frac{1}{2}\log|2\pi P| - \frac{1}{2}\mu^T P^{-1}\mu \right]$$
$$* \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} A^TQ^{-1}A & -A^TQ^{-1} \\ -Q^{-1}A & Q^{-1} \end{pmatrix}, -\frac{1}{2}\log|2\pi Q| \right]$$

2.5 Summary

Two representations of a Gaussian potential

Moment Form

$$\phi(x_1, x_2) = \alpha\mathcal{N}(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix})$$

Canonical Form

$$\phi(x_1, x_2) = \left[\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, g \right]$$

The moment parameters and canonical parameters are related by

$$\begin{aligned}
K &= \Sigma^{-1} && (18)\\
h &= \Sigma^{-1}\mu && (19)\\
g &= \log\alpha - \frac{1}{2}\log|2\pi\Sigma| - \frac{1}{2}\mu^T\Sigma^{-1}\mu && (20)\\
&= \log\alpha + \frac{1}{2}\log|\frac{K}{2\pi}| - \frac{1}{2}h^T K^{-1}h && (21)\\
\bar{g} &= \frac{1}{2}\log|\frac{K}{2\pi}| - \frac{1}{2}h^T K^{-1}h && (22)\\
g &= \log\alpha + \bar{g} && (23)
\end{aligned}$$

$$\begin{aligned}
\Sigma &= K^{-1} && (24)\\
\mu &= K^{-1}h && (25)\\
\alpha &= \exp(g - \bar{g}) && (26)\\
&= \exp(g - \frac{1}{2}\log|\frac{K}{2\pi}| + \frac{1}{2}h^T K^{-1}h) && (27)
\end{aligned}$$

2.5.1 Multiplication

$$\phi_a(x) \times \phi_b(x)$$

$$\begin{aligned}
h' &= h_a + h_b\\
K' &= K_a + K_b\\
g' &= g_a + g_b
\end{aligned}$$

$$\begin{aligned}
\Sigma' &= (\Sigma_a^{-1} + \Sigma_b^{-1})^{-1}\\
\mu' &= \Sigma'(\Sigma_a^{-1}\mu_a + \Sigma_b^{-1}\mu_b)\\
\alpha' &= \alpha_a\alpha_b\left(\frac{|2\pi\Sigma'|}{|2\pi\Sigma_a||2\pi\Sigma_b|}\right)^{1/2}\exp(-\frac{1}{2}(\mu_a^T\Sigma_a^{-1}\mu_a + \mu_b^T\Sigma_b^{-1}\mu_b - \mu'^T\Sigma'^{-1}\mu'))
\end{aligned}$$

2.5.2 Marginalization

$$\phi(x_1) = \int_{x_2}\phi(x_1, x_2)$$

$$
\begin{aligned}
h' &= h_1 - K_{12}K_{22}^{-1}h_2 \\
K' &= K_{11} - K_{12}K_{22}^{-1}K_{21} \\
g' &= g - \frac{1}{2}\log|\frac{K_{22}}{2\pi}| + \frac{1}{2}{h_2}^T(K_{22})^{-1}h_2
\end{aligned}
$$

$$
\begin{aligned}
\Sigma' &= \Sigma_{11} \\
\mu' &= \mu_1 \\
\alpha' &= \alpha
\end{aligned}
$$

2.5.3 Conditioning

$$
\phi(x_1, x_2 = \hat{x}_2)
$$

$$
\begin{aligned}
h' &= h_1 - K_{12}\hat{x}_2 \\
K' &= K_{11} \\
g' &= g + h_2^T\hat{x}_2 - \frac{1}{2}\hat{x}_2^T K_{22}\hat{x}_2
\end{aligned}
$$

$$
\begin{aligned}
\Sigma' &= \Sigma_{11} - \Sigma_{12}(\Sigma_{22})^{-1}\Sigma_{21} \\
\mu' &= \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\hat{x}_2 - \mu_2) \\
\alpha' &= \alpha|2\pi\Sigma_{22}|^{-\frac{1}{2}}\exp(-\frac{1}{2}(\hat{x}_2 - \mu_2)^T\Sigma_{22}^{-1}(\hat{x}_2 - \mu_2))
\end{aligned}
$$

3 The Kalman Filter

$$
\begin{aligned}
z_k &= Az_{k-1} + \zeta_k \\
y_k &= Cz_k + \epsilon_k
\end{aligned}
$$

where A and C are constant matrices, $\zeta_k \sim \mathcal{N}(0, Q)$ and $\epsilon_k \sim \mathcal{N}(0, R)$

The model encodes the joint distribution

$$
\begin{aligned}
p(z_{1:K}, y_{1:K}) &= \prod_{k=1}^{K} p(y_k|z_k)p(z_k|z_{k-1}) && (28) \\
p(z_1|z_0) &= p(z_1) && (29)
\end{aligned}
$$

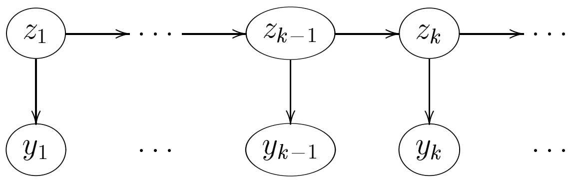

Figure 1: Standard Kalman Filter

$$
\begin{aligned}
p(z_1) &= [P^{-1}\mu, P^{-1}, -\frac{1}{2}\log|2\pi P| - \frac{1}{2}\mu^T P^{-1}\mu] \\
p(y_1|z_1) &= \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} C^T R^{-1} C & -C^T R^{-1} \\ -R^{-1}C & R^{-1} \end{pmatrix}, -\frac{1}{2}\log|2\pi R|\right] \\
p(y_1 = \hat{y}_1|z_1) &= [0 + C^T R^{-1}\hat{y}_1, C^T R^{-1} C, -\frac{1}{2}\log|2\pi R| - \frac{1}{2}\hat{y}_1^T R^{-1}\hat{y}_1] \\
p(z_2|z_1) &= \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} A^T Q^{-1} A & -A^T Q^{-1} \\ -Q^{-1}A & Q^{-1} \end{pmatrix}, -\frac{1}{2}\log|2\pi Q|\right] \\
&\cdots
\end{aligned}
$$

3.1 Forward Message Passing

Suppose we wish to compute the likelihood[2]

$$
\begin{aligned}
p(y_{1:3}) &= \int dz_{1:3}\, p(z_{1:3}, y_{1:3}) \\
&= \int_{z_3} p(y_3|z_3) \int_{z_2} p(z_3|z_2) p(y_2|z_2) \int_{z_1} p(z_2|z_1) p(y_1|z_1) p(z_1)
\end{aligned}
$$

We define forward "messages"

$$
\begin{aligned}
\alpha_{1|0} &= p(z_1) \\
\alpha_{1|1} &= p(y_1 = \hat{y}_1|z_1)\alpha_{1|0} \\
\alpha_{2|1} &= \int_{z_1} p(z_2|z_1)\alpha_{1|1} \\
\alpha_{2|2} &= p(y_2 = \hat{y}_2|z_y)\alpha_{2|1} \\
\alpha_{3|2} &= \int_{z_2} p(z_3|z_2)\alpha_{2|2} \\
&\cdots
\end{aligned}
$$

At each step messages $\alpha_{i|j} = [h_{i|j}, K_{i|j}, g_{i|j}]$ are computed from

[2] $\int dx \equiv \int_x$

previous ones using the basic operations

$$
\begin{aligned}
\alpha_{1|0} &= [h_{1|0}, K_{1|0}, g_{1|0}] \\
&= [P^{-1}\mu, P^{-1}, -\frac{1}{2}\log|2\pi P| - \frac{1}{2}\mu^T P^{-1}\mu] \\
\alpha_{1|1} &= [h_{1|1}, K_{1|1}, g_{1|1}] \\
&= [C^T R^{-1}\hat{y}_1 + h_{1|0}, C^T R^{-1} C + K_{1|0}, -\frac{1}{2}\log|2\pi R| - \frac{1}{2}\hat{y}_1^T R^{-1}\hat{y}_1 + g_{1|0}] \\
p(z_2|z_1)\alpha_{1|1} &= \left[\begin{pmatrix} 0 + h_{1|1} \\ 0 \end{pmatrix}, \begin{pmatrix} A^T Q^{-1} A + K_{1|1} & -A^T Q^{-1} \\ -Q^{-1}A & Q^{-1} \end{pmatrix}, -\frac{1}{2}\log|2\pi Q| + g_{1|1} \right] \\
\alpha_{2|1} &= \int_{z_1} p(z_2|z_1)\alpha_{1|1} = [h_{2|1}, K_{2|1}, g_{2|1}] \\
M_1 &= (A^T Q^{-1} A + K_{1|1})^{-1} \\
h_{2|1} &= Q^{-1} A M_1 h_{1|1} \\
K_{2|1} &= Q^{-1} - Q^{-1} A M_1 A^T Q^{-1} \\
g_{2|1} &= g_{1|1} - \frac{1}{2}\log|2\pi Q| + \frac{1}{2}\log|2\pi M_1| + \frac{1}{2} h_{1|1}^T M_1 h_{1|1} \\
\dots
\end{aligned}
$$

3.2 Backward Message Passing

We can compute the likelihood by starting from the end of the sequence

$$
\begin{aligned}
p(y_{1:3}) &= \int dz_{1:3} p(z_{1:3}, y_{1:3}) \\
&= \int_{z_1} p(z_1)p(y_1|z_1) \int_{z_2} p(z_2|z_1)p(y_2|z_2) \int_{z_3} p(z_3|z_2)p(y_3|z_3)
\end{aligned}
$$

We define backward "messages"

$$
\begin{aligned}
\beta_{3|3} &= p(y_3 = \hat{y}_3|z_3) \\
\beta_{2|3} &= \int_{z_3} p(z_3|z_2)\beta_{3|3} \\
\beta_{2|2} &= p(y_2|z_2)\beta_{2|3} \\
\beta_{1|2} &= \int_{z_2} p(z_2|z_1)\beta_{2|2}
\end{aligned}
$$

$$
\begin{aligned}
p(y_i|z_i) &= \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} C^T R^{-1} C & -C^T R^{-1} \\ -R^{-1} C & R^{-1} \end{pmatrix}, -\frac{1}{2}\log|2\pi R| \right] \\
p(y_i = \hat{y}_i|z_i) &= [0 + C^T R^{-1}\hat{y}_i, C^T R^{-1} C, -\frac{1}{2}\log|2\pi R| - \frac{1}{2}\hat{y}_i^T R^{-1}\hat{y}_i]
\end{aligned}
$$

$$
\begin{aligned}
\beta_{3|3} &= [h^*_{3|3}, K^*_{3|3}, g^*_{3|3}] \\
&= [0 + C^T R^{-1} \hat{y}_3, C^T R^{-1} C, -\frac{1}{2} \log |2\pi R| - \frac{1}{2} \hat{y}_3^T R^{-1} \hat{y}_3] \\
\beta_{2|3} &= \int_{z_3} p(z_3|z_2) \beta_{3|3} = [h^*_{2|3}, K^*_{2|3}, g^*_{2|3}] \\
&= \int_{z_3} \left[\begin{pmatrix} 0 \\ 0 + h^*_{3|3} \end{pmatrix}, \begin{pmatrix} A^T Q^{-1} A & -A^T Q^{-1} \\ -Q^{-1} A & Q^{-1} + K^*_{3|3} \end{pmatrix}, g^*_{3|3} - \frac{1}{2} \log |2\pi Q| \right] \\
M^*_3 &= (Q^{-1} + K^*_{3|3})^{-1} \\
h^*_{2|3} &= 0 + A^T Q^{-1} M^*_3 h^*_{3|3} \\
K^*_{2|3} &= A^T Q^{-1} A - A^T Q^{-1} M^*_3 Q^{-1} A = A^T Q^{-1} (Q - M^*_3) Q^{-1} A \\
g^*_{2|3} &= g^*_{3|3} - \frac{1}{2} \log |2\pi Q| + \frac{1}{2} \log |2\pi M^*_3| + \frac{1}{2} {h^*}^T_{3|3} M^*_3 h^*_{3|3} \\
\cdots
\end{aligned}
$$

3.3 Summary

Forward Propagation

$$
\begin{aligned}
i &= 1 : T \\
\alpha_{1|0} &= p(z_1) \\
\alpha_{i|i} &= p(y_i = \hat{y}_i | z_i) \alpha_{i|i-1} \\
\alpha_{i+1|i} &= \int_{z_i} p(z_{i+1}|z_i) \alpha_{i|i}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{1|0} &= [h_{1|0}, K_{1|0}, g_{1|0}] \\
h_{1|0} &= P^{-1}\mu \\
K_{1|0} &= P^{-1} \\
g_{1|0} &= -\frac{1}{2}\log|2\pi P| - \frac{1}{2}\mu^T P^{-1}\mu \\
\alpha_{i|i} &= [h_{i|i}, K_{i|i}, g_{i|i}] \\
h_{i|i} &= C^T R^{-1}\hat{y}_i + h_{i|i-1} \\
K_{i|i} &= C^T R^{-1} C + K_{i|i-1} \\
g_{i|i} &= g_{i|i-1} - \frac{1}{2}\log|2\pi R| - \frac{1}{2}\hat{y}_1^T R^{-1}\hat{y}_i \\
\alpha_{i+1|i} &= [h_{i+1|i}, K_{i+1|i}, g_{i+1|i}] \\
M_i &= (A^T Q^{-1} A + K_{i|i})^{-1} \\
h_{i+1|i} &= Q^{-1} A M_i h_{i|i} \\
K_{i+1|i} &= Q^{-1} - Q^{-1} A M_i A^T Q^{-1} \\
g_{i+1|i} &= g_{i|i} - \frac{1}{2}\log|2\pi Q| + \frac{1}{2}\log|2\pi M_i| + \frac{1}{2}h_{i|i}^T M_i h_{i|i}
\end{aligned}
$$

Backward Propagation

$$
\begin{aligned}
i &= T : -1 : 1 \\
\beta_{i|i} &= p(y_i = \hat{y}_i | z_i) \\
\beta_{i-1|i} &= \int_{z_i} p(z_i | z_{i-1}) \beta_{i|i} \\
\beta_{0|1} &= \int_{z_1} p(z_1) \beta_{1|1}
\end{aligned}
$$

$$
\begin{aligned}
g^*_{T|T+1} &= 0 \\
\beta_{i|i} &= [h^*_{i|i}, K^*_{i|i}, g^*_{i|i}] \\
h^*_{i|i} &= C^T R^{-1} \hat{y}_i \\
K^*_{i|i} &= C^T R^{-1} C \\
g^*_{i|i} &= g^*_{i|i+1} - \frac{1}{2} \log |2\pi R| - \frac{1}{2} \hat{y}_i^T R^{-1} \hat{y}_i \\
\beta_{i-1|i} &= [h^*_{2|3}, K^*_{2|3}, g^*_{2|3}] \\
M^*_i &= (Q^{-1} + K^*_{i|i})^{-1} \\
h^*_{i-1|i} &= A^T Q^{-1} M^*_i h^*_{i|i} \\
K^*_{i-1|i} &= A^T Q^{-1} (Q - M^*_i) Q^{-1} A \\
g^*_{i-1|i} &= g^*_{i|i} - \frac{1}{2} \log |2\pi Q| + \frac{1}{2} \log |2\pi M^*_i| + \frac{1}{2} h^{*T}_{i|i} M^*_i h^*_{i|i}
\end{aligned}
$$

3.4 Kalman Smoothing

We combine forward and backward messages as

$$
\begin{aligned}
\gamma_i &= \alpha_{i|i} \times \beta_{i|i+1} \\
&= [h_{i|i} + h^*_{i|i+1}, K_{i|i} + K^*_{i|i+1}, g_{i|i} + g^*_{i|i+1}]
\end{aligned}
$$

References

Yaakov Bar-Shalom and Xiao-Rong Li. *Estimation and Tracking: Principles, Techniques and Software.* Artech House, Boston, 1993.

Kevin P. Murphy. Inference and learning in hybrid bayesian networks. Technical Report UCB/CSD-98-990, Dept. of Computer Science, University of California, Berkeley, 1998.