The Omega Test

- Goal: Decide satisfiability of conjunction of linear constraints over integers

$$\sum_{0 \leq i \leq n} a_i x_i \geq 0$$

- Original application:
  Program optimizations done by a compiler
- Extension of Fourier-Motzkin variable elimination:
  - Pick one variable and eliminate it
  - Continue until all variables but one are eliminated

The Omega Test

- Preprocessing (1): Normalize Coefficients
  - Divide by GCD

$$8x + 6y \le 0 \longrightarrow 4x + 3y \le 0$$

$$4y \ge 1 \longrightarrow y \ge \lceil 1/4 \rceil$$

Tightening

$$3x + 3y = 2 \longrightarrow x + y = 2/3 \longrightarrow \mathsf{UNSAT}$$

The Omega Test

- Preprocessing (2): Eliminate equalities
  - Let the variables be denoted by x_i with coefficients a_i
  - Use substitution if there is a variable with coefficient a_i=1
  - Otherwise, pick variable x_k, make a_k positive
  - Define $a \,\widehat{\mathrm{mod}}\, b := a - b\lfloor a/b + 1/2 \rfloor$

$$m = a_k + 1$$

  - Note that $a_k \,\widehat{\mathrm{mod}}\, m = -1$

The Omega Test

- Preprocessing (2): Eliminate equalities
  - Create new variable σ and add:

$$m\sigma = \sum_i (a_i \,\widehat{\text{mod}}\, m) x_i$$

  - Solve for x_k:

$$x_k = -m\sigma + \sum_{i \neq k} a_k (a_i \,\widehat{\text{mod}}\, m) x_i$$

  - Q: What's the point if we add a constraint to eliminate one?

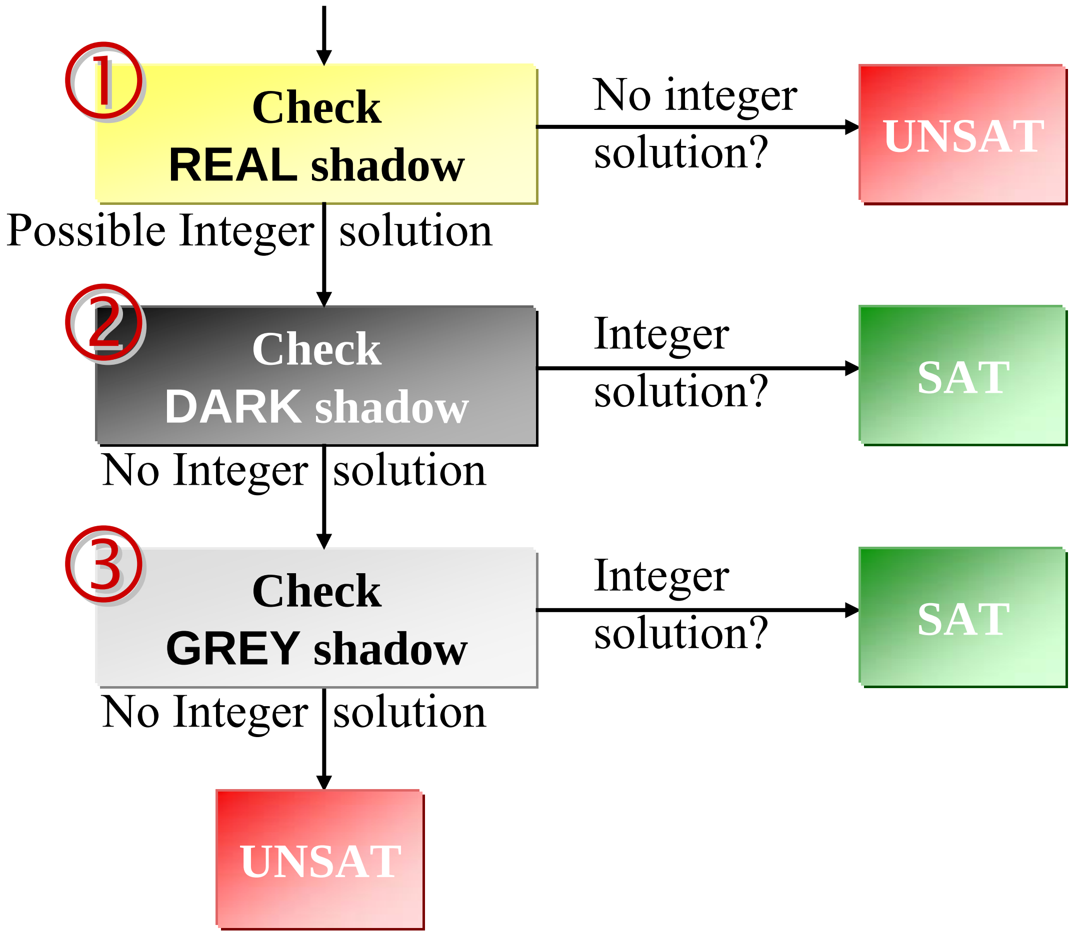
The Omega Test

- Preprocessing (3): Remove unbounded variables
  - Iteratively remove variables that are not bounded in both ways (and all the constraints that use them)
  - The new problem has a solution iff the old problem has one

$$
\begin{array}{rcl}
\cancel{8x} & \cancel{\geq} & \cancel{7y} \\
\cancel{x} & \cancel{\geq} & \cancel{3} \\
y & \geq & z \\
z & \geq & 10 \\
20 & \geq & z
\end{array}
\longrightarrow
\begin{array}{rcl}
\cancel{y} & \cancel{\geq} & \cancel{z} \\
z & \geq & 10 \\
20 & \geq & z
\end{array}
\longrightarrow
\begin{array}{rcl}
z & \geq & 10 \\
20 & \geq & z
\end{array}
$$

The Omega Test

- Overview



The Omega Test

① **Check REAL shadow**

- Assume we eliminate variable *z*
- For each pair of upper/lower bound:

$$\beta \leq bz \qquad az \leq \alpha$$

$$a\beta \leq abz \qquad abz \leq b\alpha$$

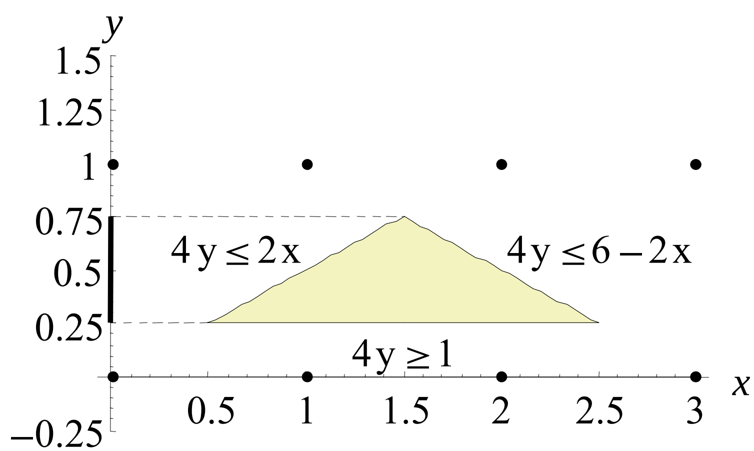
- Constraint for real shadow:

$$a\beta \leq b\alpha$$

The Omega Test

1 Check
REAL shadow

- Example:



$$4y \leq 2x$$
$$4y \leq -2x + 6$$
$$4y \geq 1$$

- Eliminate *x*:

$$4y \leq 2x \qquad 2x \leq 6 - 4y$$

$$4y \leq 6 - 4y \quad | + 4y$$

$$8y \leq 6$$

Real shadow:

$$8y \leq 6 \qquad y \leq 0.75$$
$$4y \geq 1 \qquad y \geq 0.25$$

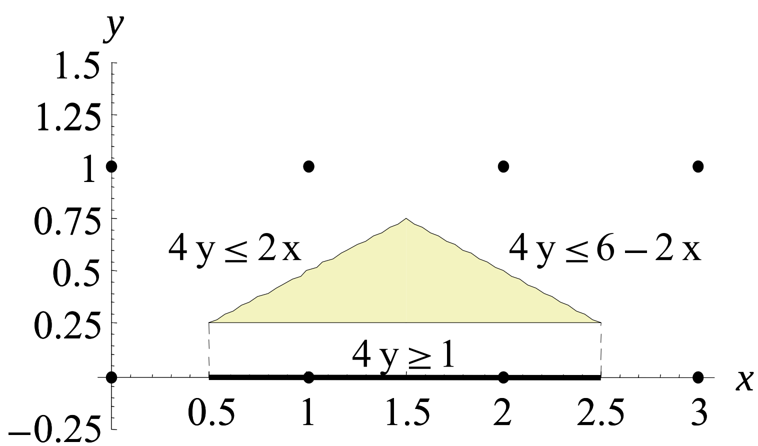
No integer solution

$\Rightarrow$ Original problem has no solution

The Omega Test

① **Check REAL shadow**

- Example:



$$4y \leq 2x$$
$$4y \leq -2x+6$$
$$4y \geq 1$$

- Assume we eliminated *y* instead:

$1 \leq 4y$, $4y \leq 2x$ $\rightarrow$ $1 \leq 2x$

$1 \leq 4y$, $4y \leq -2x+6$ $\rightarrow$ $1 \leq -2x+6$

Real shadow:

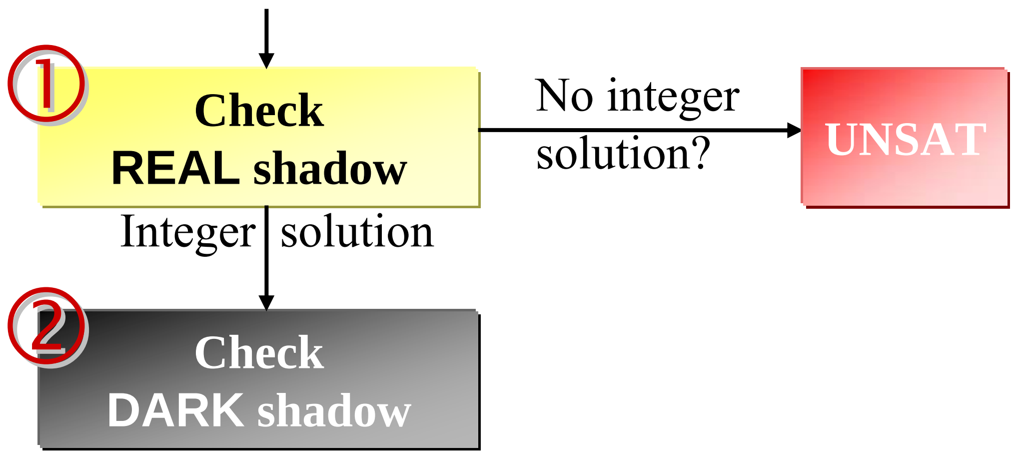
$$\begin{aligned} 1 &\leq 2x \\ 1 &\leq -2x+6 \end{aligned} \Rightarrow \begin{aligned} x &\geq 0.5 \\ x &\leq 2.5 \end{aligned}$$

This has integer solutions!

$\Rightarrow$ But original problem has no solution!

The Omega Test

- Overview



- An integer solution for the REAL shadow does not guarantee that there is an integer solution for the original problem
- Thus, we check the DARK shadow next

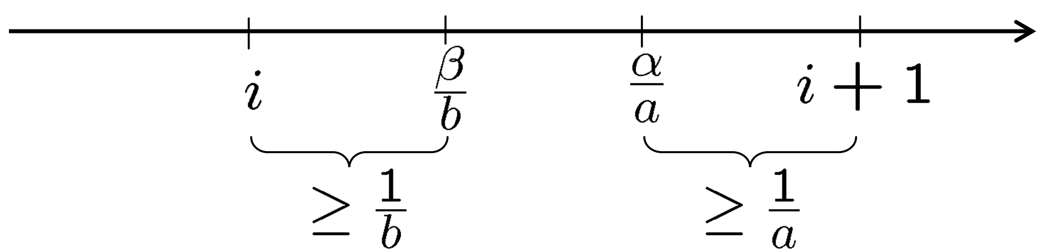
The Omega Test

- Idea of the DARK shadow:

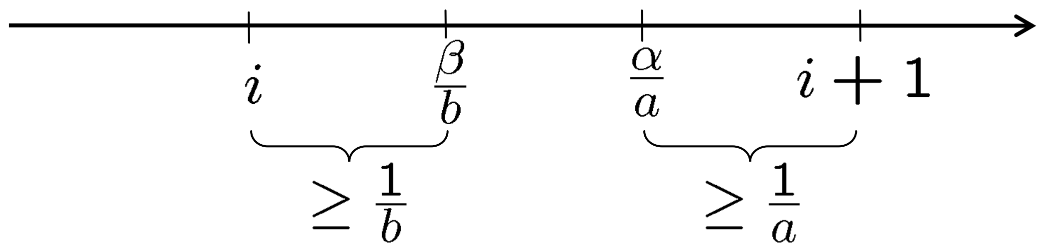
$$\begin{array}{lllll} \beta \le bz & \mid : b & \qquad az \le \alpha & \mid : a & \\ \frac{\beta}{b} \le z & & \qquad z \le \frac{\alpha}{a} & & z \in \mathbb{N} \end{array}$$

- How to compute the dark shadow?
- Show that there an integer z between β/b and α/a!
- Assume there is none. Then

$$i := \lfloor \tfrac{\beta}{b} \rfloor \qquad i \in \mathbb{N}$$



The Omega Test



$$\begin{array}{rcl}
\frac{\beta}{b} - i & \geq & \frac{1}{b} \\
i + 1 - \frac{\alpha}{a} & \geq & \frac{1}{a} \\
\hline\hline
\frac{\beta}{b} + 1 - \frac{\alpha}{a} & \geq & \frac{1}{b} + \frac{1}{a} \qquad | \cdot a \cdot b \\
a\beta + ab - b\alpha & \geq & a + b \\
a\beta - b\alpha & \geq & -ab + a + b \qquad | \cdot (-1) \\
b\alpha - a\beta & \leq & ab - a - b
\end{array}$$

The Omega Test

- From previous slide:

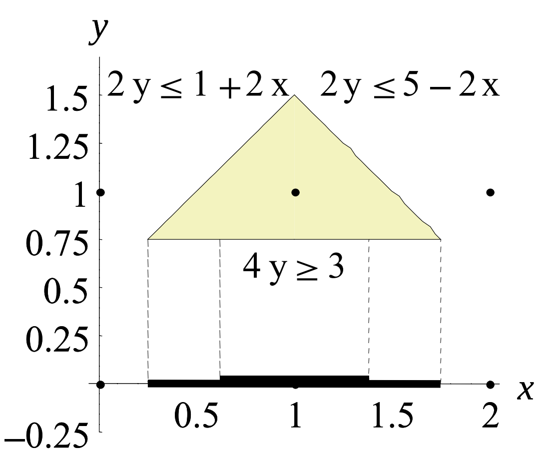
$$
\begin{array}{rcl}
 & b\alpha - a\beta \;\leq & ab - a - b \\
\iff & \neg(b\alpha - a\beta \;> & ab - a - b) \\
\iff & \neg(b\alpha - a\beta \;\geq & ab - a - b + 1) \\
\iff & \neg(\underbrace{b\alpha - a\beta \;\geq & (a-1)(b-1)}_{*})
\end{array}
$$

- Thus, if * holds, we know that there must be an integer solution.
- If a=1 or b=1, then this is the same as the real shadow
- This case is called an exact projection.

The Omega Test

② Check DARK shadow

- Example:



$$\begin{aligned} 2y &\leq 2x+1 \\ 2y &\leq -2x+5 \\ 4y &\geq 3 \end{aligned}$$

- Dark shadow:

$2y \leq 2x+1$ $\quad 4y \geq 3$

$$4(2x+1) - 2\cdot 3 \geq (2-1)(4-1)$$

$4y \geq 3$ $\quad 2y \leq -2x+5$

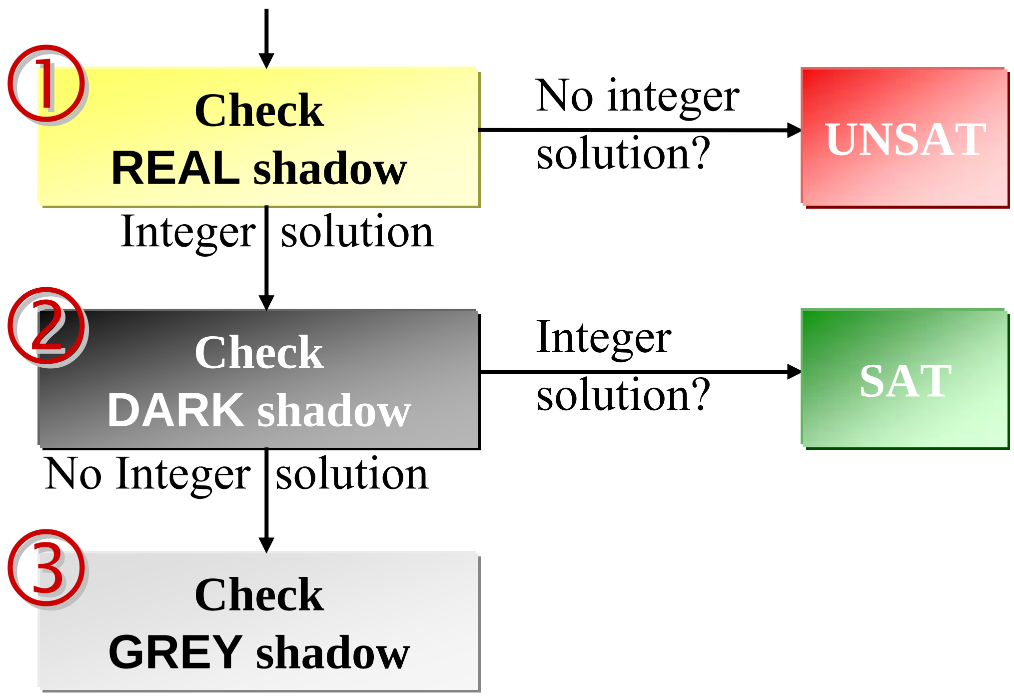
$$4(-2x+5) - 2\cdot 3 \geq (2-1)(4-1)$$

$$\Rightarrow \quad \begin{aligned} x &\geq 5/8 \\ x &\leq 11/8 \end{aligned}$$

Integer solution $x=1$

The Omega Test

- Overview



The Omega Test

- Idea of the Grey shadow:

  If the real shadow R has integer solutions, but the dark shadow D does not, search R-D

- In R: $b\alpha \geq abz \geq a\beta$

  Not in D: $ab - a - b \geq b\alpha - a\beta$

  $\iff \quad ab - a - b + a\beta \geq b\alpha$

$$\Rightarrow \quad ab - a - b + a\beta \geq abz \geq a\beta \quad | : a$$

$$(ab - a - b)/a + \beta \geq bz \geq \beta$$

The Omega Test

- Try all values of z such that

$$(ab - a - b)/a + \beta \geq bz \geq \beta$$

- This is done by finding the largest coefficient a in any upper bound and trying for each lower bound $bz \geq \beta$:

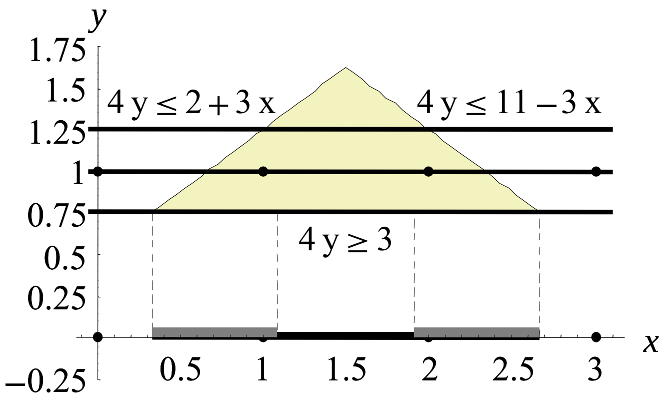
$$bz = \beta + i \quad \text{for } (ab - a - b)/a \geq i \geq 0$$

- This constraint is combined with the original problem, and eliminates z

The Omega Test

③ **Check**
GREY shadow

- Example:



$$\begin{aligned} 4y &\leq 3x+2 \\ 4y &\leq -3x+11 \\ 4y &\geq 3 \end{aligned}$$

- Eliminating y

$$a=4, b=4, \beta=3$$

- New constraint:
$4y = 3 + i$
for $2 \geq i \geq 0$

$$4y = 3$$
$$4y = 4$$
$$4y = 5$$

- Integer solution with
$4y$=4