CS5214 Design of Optimizing Compilers

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Assignment One

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Abstract

This is my solutions for assignment three problem set. This is the answer presented by Pan An. My student number is A0134556A.

1 Loop Analysis

The flow dependency of the given loop psudo-code concerns only variable list X. Here we denote the only value assignment in the loop as

$$X[\vec{f}(i,j,k)] = X[\vec{g}(i,j,k)]$$

1.1 Formulation

So in this case if we need to take care of the formulation there are three equations that we need to satisfy:

$$\begin{aligned} 3j_s + k_s =& \quad 2i_t + 4k_t \\ i_s + 2j_s + k_s =& \quad 5j_t + 7k_t \\ 2j_s + 3k_s =& \quad 3i_t + 5j_t \end{aligned} \tag{1}$$

And i, j, k subject to:

$$\begin{aligned} 0 < i_s, i_t < 11 \\ 0 < j_s, j_t < 51 \\ 0 < k_s, k_t < 21 \end{aligned} \tag{2}$$

So here I'm gonna go with formulating the problem with distance vector. The problem defined can be expressed in matrix form:

$$\boldsymbol{A} \cdot \boldsymbol{X} < \boldsymbol{b}$$

where:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 2 & 2 \\ 1 & 1 & 3 \\ -2 & 0 & -3 \\ 0 & -5 & -5 \\ -4 & -7 & 0 \end{bmatrix} \tag{3}$$

Which has to satisfy:

$$[i_s, j_s, k_s, i_t, j_t, k_t] \begin{bmatrix} 0 & 1 & 0 \\ 3 & 2 & 2 \\ 1 & 1 & 3 \\ -2 & 0 & -3 \\ 0 & -5 & -5 \\ -4 & -7 & 0 \end{bmatrix} = [0, 0, 0] \tag{4}$$

And after Echelon Reduction we have(proof reading has been done with a lot of A4 papers):

$$\boldsymbol{E} = \left\{ \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\} \quad and \quad \boldsymbol{U} = \begin{bmatrix} -\frac{4}{7} & \frac{3}{7} & -\frac{2}{7} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{7} & -\frac{1}{7} & \frac{3}{7} & 0 & 0 & 0 \\ -\frac{11}{7} & -\frac{3}{7} & 0 & \frac{5}{7} & 0 & 0 \\ \frac{30}{7} & -\frac{5}{7} & \frac{15}{7} & 0 & 1 & 0 \\ \frac{33}{7} & \frac{12}{7} & -\frac{8}{7} & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

And in order to do finish the back substitution have to let $\boldsymbol{t} \cdot \boldsymbol{E} = \mathbf{0}$. So we have:

$$t_1, t_2, t_3 = 0$$

which leads to the following:

$$\begin{aligned} i_s &= -\frac{11}{7}t_4 + \frac{30}{7}t_5 + \frac{33}{7}t_6 \\ j_s &= \frac{3}{7}t_4 - \frac{5}{7}t_5 + \frac{13}{7}t_6 \\ k_s &= \frac{5}{7}t_4 + \frac{15}{7}t_5 - \frac{8}{7} \\ i_t, j_t, k_t &= t_4, t_5, t_6 \end{aligned} \tag{6}$$

1.2 Fourier-Motzkin Elimination

For the previous section we have the following:

$$\begin{aligned} 0 < -\frac{11}{7}t_4 + \frac{30}{7}t_5 + \frac{33}{7}t_6 &\leq 10 \\ 0 < \frac{3}{7}t_4 - \frac{5}{7}t_5 + \frac{12}{7}t_6 &\leq 50 \\ 0 < \frac{5}{7}t_4 + \frac{15}{7}t_5 - \frac{8}{7}t_6 &\leq 20 \\ 0 < t_4 &\leq 10 \\ 0 < t_5 &\leq 50 \\ 0 < t_6 &\leq 20 \end{aligned} \tag{7}$$

However, real solutions does not exist when I ran it with code.

After the the elimination we have:

$$
\begin{aligned}
t_4 &\geq -\frac{122}{91} \\
\frac{3}{5}t_4 + t_5 &\geq \frac{1}{7} \\
-\frac{11}{30}t_4 + t_5 &\geq -\frac{659}{30} \\
-\frac{11}{25}t_4 + t_5 &\geq -\frac{3846}{175} \\
\frac{11}{105}t_4 + t_5 &\geq \frac{41}{735} \\
\frac{1}{3}t_4 + t_5 &\leq 20 \\
\frac{3}{5}t_4 + t_5 &\leq 32 \\
-\frac{3}{5}t_4 + t_5 &\leq \frac{239}{5} \\
-\frac{11}{25}t_4 + t_5 &\leq \frac{269}{175} \\
\frac{11}{105}t_4 + t_5 &\leq \frac{148}{21} \\
-\frac{5}{8}t_4 - \frac{15}{8}t_5 + t_6 &\geq -\frac{35}{2} \\
\frac{1}{4}t_4 - \frac{5}{12}t_5 + t_6 &\geq \frac{1}{12} \\
-\frac{1}{3}t_4 + \frac{10}{11}t_5 + t_6 &\geq \frac{1}{33} \\
-\frac{1}{4}t_4 + \frac{5}{12}t_5 - t_6 &\geq -\frac{175}{6} \\
\frac{1}{3}t_4 - \frac{10}{11}t_5 - t_6 &\geq -\frac{70}{33} \\
\frac{5}{8}t_4 + \frac{15}{8}t_5 - t_6 &\geq \frac{1}{8}
\end{aligned}
\tag{8}
$$

which is a lot of math. I generated it with C++, I did it with hands and got it wrong many times then I finally gave up since it took too much time.

I used a package in Nodejs and it shows that there IS solutions for this one. However I performed the calculation by hand at first and unfortunately failed again and again since the math might get very dirty after a couple of eliminations. Code is also attached to the tarball.

1.3 Omega Test

With the constraints listed in the previous section, we can do omega test to check if there are integer solutions for the problem. Omega test involves a couple steps. First the previous inequalities can be used to derived the following facts:

$$\begin{aligned} 140t_5 &\geq 761 \\ 55t_5 &\leq 399 \\ -77t_4 + 175t_5 &\leq 269 \\ -77t_4 + 175t_5 &\geq -3846 \\ 77t_4 + 735t_5 &\leq 5180 \\ 77t_4 + 735t_5 &\geq 41 \\ t_4 + 15t_6 &\geq 1 \\ -21t_4 + 49t_6 &\geq -279 \\ t_4 + 15t_6 &\leq 310 \\ -21t_4 + 49t_6 &\leq 68 \end{aligned} \tag{9}$$

I hope I did not make any mistaking of the input here. I will be switching to Org-Mode later. Even Emacs has pretty good tools for literal stuff there will still be a lot of mistakes in those formulas when you input by hand.

With one step further:

$$\begin{aligned} 5t_5 &\leq 399 \\ 140t_5 &\geq 761 \\ 14t_6 &\leq 253 \\ 182t_6 &\geq -129 \end{aligned} \tag{10}$$

To avoid solving problems with meatball eyes, I followed the procedure and found the $\aleph$ and β for each of t_5, t_6.

So for t_5, we have $\beta = 761$, $b = 140$, $\alpha = 399$, $a = 5$. And we have:

$$b\alpha - a\beta = 140 \times 399 - 5 \times 761 = 52055$$

$$ba - a - b + 1 = 556$$

Obviously there exists integer solution.

And for t_6: $\beta = 761$, $b = 140$, $\alpha = 399$, $a = 5$

1.4 Brute Force

The test with Python shows the following case was the solution for dependency:

```
(10, 15, 9) (20, 22, 39)
    i=8,  j=3, k=1
(9, 17, 13) (32, 31, 40)
    i=10, j=2, k=3
```

2 Loop Tiling

Considering that in this problem X and Y has exactly the same size. There is no need to take into consideration the utilization of loop interchange.

Anyway, the idea of loop tiling is to increase cache. Consider the given case:

```
for (int i = 0; i < n; ++i) {
    for (int j = 0; j < n; ++j) {
        X[i, j] = Y[j, i];
    }
}
```

This however, will result in loading the data in X and Y unnecessary number of times. The program will miss every access to one of the array and every step of C on the other, where C is the size of the cache. And the total cache miss would be:

$$N^2 + \frac{N^2}{C}$$

And in this case it's $N^2 + \frac{N^2}{8}$.

Loop tiling code is shown as below(notice that both X and Y are $N \times N$ matrix):

```
for(ii=0; ii<N; ii+=block_size)
    for(jj=0; jj<N; jj+=block_size)
        for(i=ii; i<min(ii+block_size); i++)
            for(j=jj; j<min(jj+block_size); j++)
                X[i, j] = Y[j, i]
```

The loop will be excuted as the following graph shows:

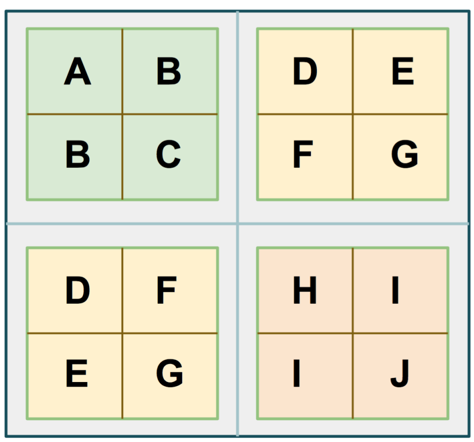

Figure 1: Loop tiling for 4×4matrix.

In 1 alphabetic letters represents values in each position of a $N \times N$ matrix. Blocks with the same color will be loaded into into cache at the same time(symmetrical in X and Y). And the position with the same letter will switch value.

The previous paragraph however, will be strongly criticized if this is an assignment in a CS6+ class since the whole thing is not clear and it leads to infinite way of understanding even for a native English speaker. So the process of loop tiling is shown below:

- Cache was empty at first and the first round will experience a missing hit;

- The tile is gonna be loaded and iterated without new cache misses;
- When operation on the tile is finished, there will be new miss hit, and new tile is gonna be loaded again;
- Same process goes on until program reaches the end.

So how much is the new cache misses. When going through this part before the deadline I actually was not very clear about every step. And after that I consulted a couple guys about cache mechanism.

Let M denote the total cache misses and b the block grid size, in this case $b = 2 \times 2 = 4$

Then we have:

$$M \approx 2\frac{N^2}{b} = \frac{N^2}{2}$$

In a word:

- No tiling: $N^2 + \frac{N^2}{8}$
- With tiling: $\frac{N^2}{2}$

Different methods and algorithms will have different results in for tiling cache miss.