# Convex Optimization

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# Linear Regression Example

There should be a picture here.

# Ordinary Least Squares

Input: points $(x_i, y_i)$

Regression line: $y = mx + b$

Objective:

$$\min_{m,b} \sum_i (y_i - mx_i - b)^2$$

$(\vec{x_i}, y_i)$

$y = \vec{w} \cdot \vec{x} + b$

$$\min_{\vec{w}} \sum_i (y_i - \vec{w} \cdot \vec{x_i} - b)^2$$

- Easily Solved: $\vec{w}^*(X^T X) - 1 X^T \vec{y}$
- But what if $\dim \vec{x}$ is large?
- What about other similar regressions?

# Convex Optimization Problems

- OrdinaryLinearRegression: $\min_{\vec{w}} \sum_i (y_i - \vec{w} \cdot \vec{x_i})^2$
- General: $\min_x f(x)$ where $f(x)$ is convex
- Set $C$ is convex $\iff \exists x, y \in C, 0 \leqslant t \leqslant 1 : tx + (1-t)y \in C$
- Function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathbf{dom}\, f$ is convex and $\exists x, y \in \mathbf{dom}\, f, 0 \leqslant t \leqslant 1 :$

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$$

- Unconstrained.

# Outliers

Supposed to be a picture here.

# Outlier Penalty

pic

# Capped Penalty

pic

Huber Penalty Function pic

# Unconstrained Optimization

- Minimize $f(x)$;
- Where $f : \mathbb{R}^n \to \mathbb{R}$ is convex and twice differentiable;
- No additional constraints;
- Assume that unique minimum $x^*$ exists.

# General Principle

- Objective: minimize $f(x)$
- Necessary and sufficient condition: $\nabla f(x^*) = 0$
  - Solve analytically
  - Iterative algorithms

**Iterative Algorithm:**

$$x^{(0)}, x^{(1)}, \ldots \in \mathbf{dom}\, f$$

$$k \to \infty, f(x^{(k)}) < f(x^*)$$

**Descent Method:**

$$x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)}, \mathbf{s.t.} f(x^{(k+1)}) < f(x^{(k)})$$

# General Descent Method

## Descent Method:

$$x^{(k+1)} = x^{(k)} + t^{(k)}\Delta x^{(k)}, \textbf{s.t.} f(x^{(k+1)} < f(x^{(k)}) \tag{1}$$

## Algorithm:

Given $x^{(0)} \in \mathbf{dom}\, f$;
repeat
    Determine a descent direction $\Delta x$;
    Choose a step size $t > 0$;
    Update $x^{(k+1)} = x^{(k)} + t^{(k)}\Delta x^{(k)}$;
until $\Delta x$ is within an acceptable range and is stable;;

## Noticing that f is convex:

$$\nabla f(x^{(k)})^T \Delta x^{(k)} < 0 \tag{2}$$

# General Descent Method

## Descent Method:

$$x^{(k+1)} = x^{(k)} + t^{(k)}\Delta x^{(k)}, \textbf{s.t.} f(x^{(k+1)} < f(x^{(k)}) \tag{1}$$

## Theorem

For a continuously differentiable function f:

$$f \text{ is convex} \curlyeqsucc f(x) \leqslant f(y) + f'(y)(x-y)$$

## Proof

$$f(x^{(k+1)}) \geqslant f(x^{(k)}) + f'(x^{(k)})\Delta x^{(k)}$$
$$\nabla f(x^{(k)}) \leqslant f(x^{(()}k+1)) - f(x^{(k)}) < 0$$

## Noticing that f is convex:

$$\nabla f(x^{(k)})^T \Delta x^{(k)} < 0 \tag{2}$$

# General Descent Method

**Descent Method:**

$$x^{(k+1)} = x^{(k)} + t^{(k)}\Delta x^{(k)}, \textbf{s.t.} f(x^{(k+1)} < f(x^{(k)}) \quad (1)$$

**Algorithm:**

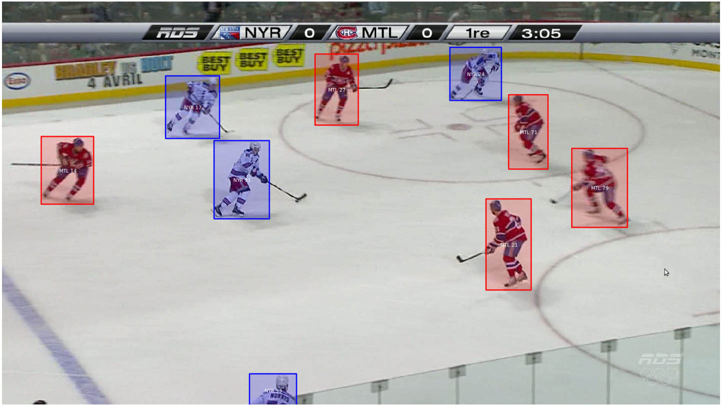
Given $x^{(0)} \in \mathbf{dom}\, f$;
repeat
  Determine a descent direction $\Delta x \Rightarrow$ Gradient/SteepestDescent;
  Choose a step size $t > 0 \Rightarrow$ LineSearchAlgo ;
  Update $x^{(k+1)} = x^{(k)} + t^{(k)}\Delta x^{(k)}$;
until $\Delta x$ is within an acceptable range and is stable;;

**Noticing that f is convex:**

$$\nabla f(x^{(k)})^T \Delta x^{(k)} < 0 \quad (2)$$

# Image Processing — Lucas-Kanade

Classic examples are optical flow techniques like Lucas-Kanade (VideoTracking), Horn-Schunck.



# Lucas-Kanade

**Goal of Lucas-Kanade**

Minimize the sum of squared error between two images.

**Assumption**

The displacement of the image contents between two nearby instants (frames) is small and approximately constant within a neighborhood of the point $p$ under consideration.

# Lucas-Kanade

## Optical Flow Equation (2 Dementional)

For a pixel location $(x, y, t)$, the intensity has moved by $\Delta x, \Delta y, \Delta t$, the basic assumption can be represented as:

$$I(x, y, t) = I(x + \Delta x, y + \Delta y, t + \Delta t)$$

# Lucas-Kanade

## Optical Flow Effect:

For all pixels within a window centered at p:

$$I_x(q_i)V_x + I_y(q_i)V_y = -I_t(q_i)$$

Where $i = 1, 2, 3...n$.

## Abbreviations:

$$A = [I_x(q_i)^T, I_y(qi)^T]$$

$$V = [v_x, v_y]^T$$

$$b = [-I_t(q_i)]^T$$

# Lucas-Kanade

**Lucas-Kanade Method Abstraction:**

LK method tries to solve $2 \times 2$ system:

$$A^T A V = A^T b$$

A.K.A:

$$V = (A^T A)^{-1} A^T b$$

**Notice:**

$V = [v_x, v_y]^T$ is variable. Which means that the system does not know the actual velocity of the system.

# Lucas-Kanade

**Goal of Lucas-Kanade Method:**

To minimize $||A^T V - b||^2$.

Basic LK Derivation for Models(Stuff to be Tracked):

$$E[v_x, v_y] = \Sigma[I(x + v_x, y + v_y) - T(x, y)]^2$$

Where $v_x, v_y$ is the hypothesized location of the model(s) to be tracked, and $T(x, y)$ model.

# Lucas-Kanade

**Key Step for Implementation of GD (Step 1):**

Generalizing LK approach by introducing warp function W:

$$E[v_x, v_y] = \Sigma[I(W(x,y);P) - T(x,y)]^2$$

Generalizing is used to solve the problem where the constant flow of larger picture frames for a long time is a total waste of calculation power. Warp function examples are Affine and Projective.
The warping function are the convergence factor for steepest descent algorithm.

# Lucas-Kanade

**Key Step for Implementation of GD (Step 2):**

The key to the derivation is Taylor series approximation:

$$I(W(x, y); P + \Delta P) \approx I(W([x, y]; P)) + \nabla I \frac{\partial W}{\partial P} \Delta P$$

- The approximation equation is actually the abstract of the basic assumption of optical flow described in the slides before.
- Derivation of this equation can be discussed in forum (Too long for slides).

# Lucas-Kanade

## Some Explanations:

- Gradient image $\nabla I$
- Image error $I_E = T(x, y) - I(W[x, y]; P)$
- Jacobian matrix $\frac{\partial W}{\partial P}$
- Steepest image $I_S = \nabla I \frac{\partial W}{\partial P}$
- Hessian Matrix $\Sigma(\nabla I \frac{\partial W}{\partial P})^T (\nabla I \frac{\partial W}{\partial P})$
- Iteration step $\Delta P = \Sigma I_S^T I_E$

# Lucas-Kanade

**Algorithms:**

- Warp image and get $I(W[x,y];P)$;
- Get image error $I_E$;
- Warp gradient image $\nabla I$;
- Evaluate Jacobian;
- Compute steepest descent image $I_S = \nabla I \frac{\partial W}{\partial P}$;
- Compute Hessian matrix $\Sigma I_S^T I_S$;
- Get warping step $\Delta P = I_S I_E$;
- Update warping parameter $P = P + \Delta P$;
- Repeat until $\Delta P$ is negligible.

# APPLICATIONS – MACHINE LEARNING

## Generalized Utilization of Convex: Delta Rule

- The delta rule is derived by attempting to minimize the error in the output of the neural network through gradient descent.
- Gradient Descent optimization is the most basic principle for training neurons even with different activation functions.
- Delta rule, can also be modified, if possible, with steepest descent method.

# APPLICATIONS – MACHINE LEARNING

**Delta Rule:**

$$\Delta w_{ji} = \alpha(t_j - y_j)g'(h_j)x_j$$

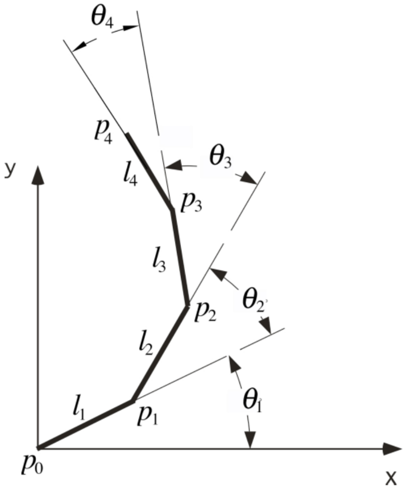
Where $\alpha$ is the learning rate, $g(x)$ is the neuron's activation function. $t_j$ and $y_j$ is the target and actual output of the neuron. $h_j$ is the weighted sum of the neuron's inputs. And $x_i$ is the $i_{th}$ input.

**The above equation holds the following:**

$$h_j = \Sigma x_i w_{ji}$$

$$y_j = g(h_j)$$

# APPLICATIONS – INVERSE KINEMATICS



# APPLICATIONS – INVERSE KINEMATICS

**Goal of Inverse Kinematics**

Given a position in the space, calculate a way for a robot hand to reach a place.

**Problem Abstract:**

$$\vec{e} = R_1 T_1 R_2 T_2 R_3 T_3 R_4 T_4 \vec{e_0}$$

Where $T_i$ is a series of translation transformation and $R_i$ is a series of rotation translation.

# APPLICATIONS – INVERSE KINEMATICS

**Abstraction for Convex Optimization:**

$$\Delta\vec{\theta} = \alpha J^{T}\vec{e}$$

.
The target for the optimization is to achieve $|\vec{e_p} - \vec{e_t}| = 0$, where $\vec{e_p}$ is th original position of the tip of the robotic arm and $\vec{e_t}$ is the target position. $J$ is the jacobian matrix in terms of $\vec{\theta}$, which is the vector of all the spatial angles of all joints. $\alpha$ is the convergence rate and $\vec{e}$ is the position derivation (step size).

# APPLICATIONS – INVERSE KINEMATICS

**About Inverse Kinematics**

- Jacobian transpose is the implementation of gradient descent in the real physical world.
- It can actually achieve near linear solution for robotic arms with a fast convergence rate.