Convex Optimization

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January 29, 2016

Linear Regression Example

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Ordinary Least Squares

Input: points (x_i, y_i)

Regression line: $y = mx + b$

Objective:

$$\min_{m,b} \sum_i (y_i - mx_i - b)^2$$

$(\vec{x_i}, y_i)$

$y = \vec{w} \cdot \vec{x} + b$

$$\min_{\vec{w}} \sum_i (y_i - \vec{w} \cdot \vec{x_i} - b)^2$$

- Easily Solved: $\vec{w}^*(X^T X) - 1X^T\vec{y}$
- But what if $\dim \vec{x}$ is large?
- What about other similar regressions?

Convex Optimization Problems

- OrdinaryLinearRegression: $\min_{\vec{w}} \sum_{i} (y_i - \vec{w} \cdot \vec{x_i})^2$
- General: $\min_{x} f(x)$ where $f(x)$ is convex
- Set C is convex $\iff \exists x, y \in C, 0 \leqslant t \leqslant 1 : tx + (1-t)y \in C$
- Function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathbf{dom}\, f$ is convex and $\exists x, y \in \mathbf{dom}\, f, 0 \leqslant t \leqslant 1 :$

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$$

- Unconstrained.

Outliers

Supposed to be a picture here.

Outlier Penalty

pic

Capped Penalty

pic

Huber Penalty Function pic

Unconstrained Optimization

- Minimize $f(x)$;
- Where $f : \mathbb{R}^n \to \mathbb{R}$ is convex and twice differentiable;
- No additional constraints;
- Assume that unique minimum x^* exists.

General Principle

- Objective: minimize $f(x)$
- Necessary and sufficient condition: $\nabla f(x^*) = 0$
  - Solve analytically
  - Iterative algorithms

Iterative Algorithm:

$$x^{(0)}, x^{(1)}, \ldots \in \mathrm{dom}\, f$$

$$k \to \infty, f(x^{(k)}) < f(x^*)$$

Descent Method:

$$x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)}, \mathbf{s.t.} f(x^{(k+1)}) < f(x^{(k)})$$

General Descent Method

Descent Method:

$$x^{(k+1)} = x^{(k)} + t^{(k)}\Delta x^{(k)}, \mathbf{s.t.} f(x^{(k+1)}) < f(x^{(k)}) \tag{1}$$

Algorithm:

Given $x^{(0)} \in \mathbf{dom}\, f$;
repeat
  Determine a descent direction Δx;
  Choose a step size $t > 0$;
  Update $x^{(k+1)} = x^{(k)} + t^{(k)}\Delta x^{(k)}$;
until Δx is within an acceptable range and is stable;;

Noticing that f is convex:

$$\nabla f(x^{(k)})^T \Delta x^{(k)} < 0 \tag{2}$$

General Descent Method

Descent Method:

$$x^{(k+1)} = x^{(k)} + t^{(k)}\Delta x^{(k)}, \mathbf{s.t.} f(x^{(k+1)}) < f(x^{(k)}) \qquad (1)$$

Theorem

For a continuously differentiable function f:

$$f \text{ is convex} \lessgtr f(x) \leqslant f(y) + f'(y)(x-y)$$

Proof

$$f(x^{(k+1)}) \geqslant f(x^{(k)}) + f'(x^{(k)})\Delta x^{(k)}$$
$$\nabla f(x^{(k)}) \leqslant f(x^{(()}k+1)) - f(x^{(k)}) < 0$$

Noticing that f is convex:

$$\nabla f(x^{(k)})^T \Delta x^{(k)} < 0 \qquad (2)$$

General Descent Method

Descent Method:

$$x^{(k+1)} = x^{(k)} + t^{(k)}\Delta x^{(k)}, \textbf{s.t.} f(x^{(k+1)}) < f(x^{(k)}) \tag{1}$$

Algorithm:

Given $x^{(0)} \in \mathbf{dom}\, f$;
repeat
  Determine a descent direction $\Delta x \Rightarrow$ $\mathsf{Gradient/SteepestDescent}$;
  Choose a step size $t > 0 \Rightarrow$ $\mathsf{LineSearchAlgo}$;
  Update $x^{(k+1)} = x^{(k)} + t^{(k)}\Delta x^{(k)}$;
until Δx is within an acceptable range and is stable;;

Noticing that f is convex:

$$\nabla f(x^{(k)})^T \Delta x^{(k)} < 0 \tag{2}$$

Line Search

$x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)}, f(x^{(k+1)}) \leftarrow f(x^{(k)})$

Should be pics

Exact Line Search Method:

$$t = \underset{s \leqslant 0}{\operatorname{argmin}} \{f(x + s\Delta x)\}$$

Line Search

- Armijo Condition:

$$f(x^{(k)} + t\Delta x^{(k)}) \leqslant f(x^{(k)}) + c_1 \alpha \nabla f(x^{(k)})^T \Delta x^{(k)}, c_1 > 0$$

- Wolfe Conditions (Including Armijo Condition):

$$\nabla f(x^{(k)} + t\Delta x^{(k)})^T p^{(k)} \geqslant c_2 \nabla f(x^{(k)})^T p^{(k)}, 0 < c_1 < c_2 < 1$$

Theorem:

Gradient descent will find local minimum if step size α satisfies Wolfe conditions.

Exact Line Search Method:

$$t = \underset{s \leqslant 0}{\operatorname{argmin}} \{ f(x + s\Delta x) \}$$

Line Search

- Armijo Condition:

$$f(x^{(k)} + t\Delta x^{(k)}) \leqslant f(x^{(k)}) + c_1 \alpha \nabla f(x^{(k)})^T \Delta x^{(k)}, c_1 > 0$$

Algorithm:

Given a descent direction Δx for f at
$x \in \operatorname{dom} f, \alpha \in (0, 0.5), \beta \in (0, 1), t = 1$;
repeat
  $t = \beta t$;
until $f(x^{(k)} + t\Delta x^{(k)}) \leqslant f(x^{(k)}) + c_1 \alpha \nabla f(x^{(k)})^T \Delta x^{(k)}$;;

Exact Line Search Method:

$$t = \underset{s \leqslant 0}{\operatorname{argmin}} \{f(x + s\Delta x)\}$$

General Descent Method

- Gradient Descent Method
- Steepest Descent Method

Δx satisfies:

$$\nabla f(x^{(k)})^T \Delta x < 0$$

Gradient Descent Method

$\Delta x = -\nabla f(x)$

Given $x^{(0)} \in \operatorname{dom} f$;
repeat
  $\Delta x = -\nabla f(x^{(k)})$;
  Choose a step size $t > 0$, [LineSearch];
  Update $x^{(k+1)} = x^{(k)} + t\Delta x$;
until Δx is within an acceptable range and is stable;;

Steepest Descent Method

$\Delta x = \Delta x_{sd}$
Taylor Series:

$$f(\mathbf{x} + \Delta\mathbf{x}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^T \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}\nabla f(\mathbf{x})\Delta\mathbf{x}$$

$$f(\mathbf{x} + \mathbf{v}) \approx \hat{f}(\mathbf{x} + \mathbf{v}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^T \mathbf{v}$$

$$x^{(k+1)} = x^{(k)} + t^{(k)}\Delta x^{(k)}, \mathbf{s}.\mathbf{t}.f(x^{(k+1)}) < f(x^{(k)})$$

Where $\mathbf{v}$ is a descent direction if $\nabla f(\mathbf{x})^T < 0$

Steepest Descent Method

$$f(\mathbf{x}+\mathbf{v}) \approx \hat{f}(\mathbf{x}+\mathbf{v}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^T \mathbf{v}$$

Normalized Steepest Descent Direction:

$$\begin{aligned} \Delta \mathbf{x}_{nsd} &= \mathrm{argmin}\{\nabla f(\mathbf{x})^T \mathbf{v} \mid \|\mathbf{v}\| = 1\} \\ &= \mathrm{argmin}\{\nabla f(\mathbf{x})^T \mathbf{v} \mid \|\mathbf{v}\| \leqslant 1\} \end{aligned} \tag{3}$$

Steepest Descent Method

Dual Norm, denoted $\|\cdot\|_*$, is defined as:

$$\|z\|_* = \sup\{z^T x \mid \|x\| \leqslant 1\}$$

Unnormalized Steepest Descent Direction:

$$\Delta \mathbf{x} = \|\nabla f(x)\|_* \cdot \Delta \mathbf{x}_{nsd}$$

$$\begin{aligned} \nabla f(\mathbf{x})^T \mathbf{v} &= \nabla f(\mathbf{x})^T \Delta \mathbf{x}_{sd} \\ &= \|f(x)\|_* \nabla f(\mathbf{x})^T \Delta \mathbf{x}_{nsd} \\ &= -\|\nabla f(x)\|_*^2 \end{aligned}$$

Proof

$$\begin{aligned} \Delta \mathbf{x}_{nsd} &= \operatorname{argmin}\{\nabla f(\mathbf{x})^T \mathbf{v} \mid \|v\| = 1\} \\ &= -\operatorname{argmax}\{\nabla f(\mathbf{x})^T \mathbf{v} \mid \|v\| \leqslant 1\} \end{aligned}$$

$$\|\nabla f(x)\|_*^2 = \sup\{\nabla f(\mathbf{x})^T \mathbf{v} \mid \|v\| \leqslant 1\}$$

$$\Rightarrow \|\nabla f(x)\|_*^2 = -\nabla f(\mathbf{x})^T \Delta \mathbf{x}_{nsd}$$

Steepest Descent Method

$$\Delta \mathbf{x}_{nsd} = \operatorname{argmin}\{\nabla f(\mathbf{x})^T \mathbf{v} \mid \|\mathbf{v}\| \leqslant 1\}$$
$$\Delta \mathbf{x}_{sd} = \|\nabla f(\mathbf{x})\|_* \Delta \mathbf{x}_{nsd}$$

Steepest Descent Method

Given $x^{(0)} \in \operatorname{dom} f$;
repeat
- Compute steepest descent direction Δx_{sd};
- Choose a step size $t > 0$, $[\mathrm{LineSearch}]$;
- Update $x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x_{sd}^{(k)}$;

until Δx is within an acceptable range and is stable;

Descent Method

General

Given $x^{(0)} \in \mathrm{dom}\, f$;
repeat
  Determine a descent direction Δx;
  Choose a step size $t > 0$;
  Update $x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)}$;
until Δx is within an acceptable range and is stable;

Gradient Descent

Given $x^{(0)} \in \mathrm{dom}\, f$;
repeat
  $\Delta x = -\nabla f(x^{(k)})$;
  Choose a step size $t > 0$, [LineSearch];
  Update $x^{(k+1)} = x^{(k)} + t\Delta x$;
until Δx is within an acceptable range and is stable;

Steepest Descent

Given $x^{(0)} \in \mathrm{dom}\, f$;
repeat
  Compute steepest descent direction Δx_{sd};
  Choose a step size $t > 0$, [LineSearch];
  Update $x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x_{sd}^{(k)}$;
until Δx is within an acceptable range and is stable;

Descent Method

General

- $\Delta \mathbf{x}_{nsd} = \operatorname{argmin}\{\nabla f(\mathbf{x})^T \mathbf{v} \mid \|\mathbf{v}\| \leqslant 1\}$
- $\Delta \mathbf{x}_{sd} = \|\nabla f(x)\|_* \cdot \Delta \mathbf{x}_{nsd}$
- If the norm $\|\cdot\|$ is Euclidean norm, $\Delta \mathbf{x} = -\nabla f(\mathbf{x})$

Gradient Descent

Given $x^{(0)} \in \operatorname{dom} f$;
repeat
  $\Delta x = -\nabla f(x^{(k)})$;
  Choose a step size $t > 0$, [LineSearch];
  Update $x^{(k+1)} = x^{(k)} + t\Delta x$;
until Δx is within an acceptable range and is stable;

Steepest Descent

Given $x^{(0)} \in \operatorname{dom} f$;
repeat
  Compute steepest descent direction Δx_{sd};
  Choose a step size $t > 0$, [LineSearch];
  Update $x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x_{sd}^{(k)}$;
until Δx is within an acceptable range and is stable;

Convex Domain

Linear Regression method is applicable only if nonlinear function is linear in terms of function parameters:

$$f(x; a) = \sum_{k=1}^{m} a_k h_k(x)$$

Many nonlinear functions are not like that, for example:

$$f_1(x) = \frac{x^2}{a_1 + (x - a_2)}$$

$$f_2(x, y, z) = \frac{x^2}{a_1 + x^2} + \frac{y^2}{a_2 + y^2} + \frac{z^2}{a_3 + z^2}$$

Convex Domain

There should be pic here;
To minimize the error, we need iterative optimization.

Advantages – Disadvantages – Limitations

If step length is appropriate, f always decreases: converge. (well condition)
pic here,

Advantages – Disadvantages – Limitations

If step length is too large, f can increase: diverge. (ill condition)

Advantages – Disadvantages – Limitations

If parameters of f affect error equally,

Advantages – Disadvantages – Limitations

If parameters of f affect error unequally,
Pic here.

Advantages – Disadvantages – Limitations

If parameters of f affect error very unequally,
Pic here.
Small step length can also cause divergence. (ill condition)

Condition Number

- The condition number of C gives a measure of its anisotropy or eccentricity.
- If the condition number of a set C is small (say, near one) it means that the set has approximately the same width in all directions, i.e., it is nearly spherical.
- If the condition number is large, it means that the set is far wider in some directions than in others.
- $\mathsf{cond}(f) = \frac{\lambda_{max}(f)}{\lambda_{min}(f)}$
- λ_{max} and λ_{min} describes minimum and maximum eigenvalues in 2D.

Example Quadratic Problem in R^2

$$f(x) = \frac{1}{2}(x_1^2 + \gamma x_{(2)}^2) \quad \gamma > 0$$

with exact line search, starting at $x^{(0)} = (\gamma, 1)$

$$x_{(1)}^{(k)} = \gamma(\frac{\gamma - 1}{\gamma + 1})^k, \quad x_{(2)}^{(k)} = \gamma(-\frac{\gamma - 1}{\gamma + 1})^k$$

- Hessian of f has eigenvalues 1 and γ. And, $m = \min 1, \gamma$,and $M = \max 1, \gamma$
- In particular,$f(x_{(k)})$ converges to p^*, optimal value, at least as fast as a geometric series with an exponent that depends (at least in part) on the condition number bound $\frac{M}{m}$.

- Very slow if $\gamma > 1 or \gamma < 1$
- Useless if $\gamma > 20$.
- Example for $\gamma = 10$.

Should be a pic beside itemize.

Advantages – Disadvantages – Limitations

Left Number of iterations of the gradient method as a function of γ which can be thought of as amount of diagonal scaling.
Right Condition number of the Hessian of the function at its minimum as a function of γ.
We see that the condition number has a very strong influence on convergence rate.
Pics here but strongly recommend you guys draw pictures yourself next time.

Exact Line Search VS. Backtracking Line Search with Non-Quadratic Example

$$f(x_1, x_2) = e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} + e^{-x_1-0.1}$$

Pics here

- With exact line search, the error is reduced about 10^{-8} in 20 iterations, i.e., a reduction by a factor of 20 about $10^{-\frac{8}{20}} \approx 0.4$ per iteration.
- With backtracking line search, the error is reduced about 10^{-8} in 15 iterations, i.e., a reduction by a factor of 15 about ^{11}

Should be a pic right side at the itemize

Exact Line Search VS. Backtracking Line Search with a Problem in R^{100}

$$f(x) = c^T x - \sum_{i=1}^{500} \log(b_i - a_i^T x)$$

A larger example, of the form with $m = 500$ terms and $n = 100$ variables.
pics here.
linear convergence, i.e., a straight line on a semilog plot

Exact Line Search VS. Backtracking Line Search with a Problem in $\mathbb{R}^{100}$

The progress of the gradient method with backtracking line search, with parameters $\alpha = 0.1, \beta = 0.5$.
Average error reduction is $10^{-\frac{6}{175}} \approx 0.92$ per iteration.

In the convergence of the gradient method with exact line search, 6 average error reduction is $10^{-\frac{6}{140}} \approx 0.91$ per iteration. A bit faster than the gradient method with backtracking line search.

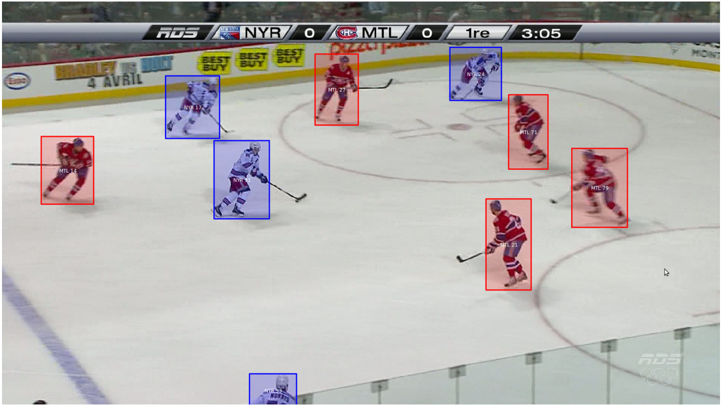
Here is some pic

Exact Line Search VS. Backtracking Line Search with a Problem in R^{100}

- These experiments, done by the book authors, show that the effect of the backtracking parameters on the convergence is not large.
- Experiment 1: (effect of the choice of α): Fix $\beta = 0.5$, and vary α. This experiment suggests that the gradient method works better with fairly large α, in the range $(0.2, 0.5)$.
- Experiment 2: (effect of the choice of β): Fix $\alpha = 0.1$, and vary β. This experiment suggests that $\beta \approx 0.5$ is a good choice.
-

Image Processing — Lucas-Kanade

Classic examples are optical flow techniques like Lucas-Kanade (VideoTracking), Horn-Schunck.



Lucas-Kanade

Goal of Lucas-Kanade

Minimize the sum of squared error between two images.

Assumption

The displacement of the image contents between two nearby instants (frames) is small and approximately constant within a neighborhood of the point p under consideration.

Lucas-Kanade

Optical Flow Equation (2 Dementional)

For a pixel location (x, y, t), the intensity has moved by $\Delta x, \Delta y, \Delta t$, the basic assumption can be represented as:

$$I(x, y, t) = I(x + \Delta x, y + \Delta y, t + \Delta t)$$

Lucas-Kanade

Optical Flow Effect:

For all pixels within a window centered at p:

$$I_x(q_i)V_x + I_y(q_i)V_y = -I_t(q_i)$$

Where $i = 1, 2, 3 ... n$.

Abbreviations:

$$A = [I_x(q_i)^T, I_y(qi)^T]$$

$$V = [v_x, v_y]^T$$

$$b = [-I_t(q_i)]^T$$

Lucas-Kanade

Lucas-Kanade Method Abstraction:

LK method tries to solve 2×2 system:

$$A^T A V = A^T b$$

A.K.A:

$$V = (A^T A)^{-1} A^T b$$

Notice:

$V = [v_x, v_y]^T$ is variable. Which means that the system does not know the actual velocity of the system.

Lucas-Kanade

Goal of Lucas-Kanade Method:

To minimize $||A^T V - b||^2$.

Basic LK Derivation for Models(Stuff to be Tracked):

$$E[v_x, v_y] = \Sigma[I(x + v_x, y + v_y) - T(x, y)]^2$$

Where v_x, v_y is the hypothesized location of the model(s) to be tracked, and $T(x, y)$ model.

Lucas-Kanade

Key Step for Implementation of GD (Step 1):

Generalizing LK approach by introducing warp function W:

$$E[v_x, v_y] = \Sigma[I(W(x, y); P) - T(x, y)]^2$$

Generalizing is used to solve the problem where the constant flow of larger picture frames for a long time is a total waste of calculation power. Warp function examples are Affine and Projective.
The warping function are the convergence factor for steepest descent algorithm.

Lucas-Kanade

Key Step for Implementation of GD (Step 2):

The key to the derivation is Taylor series approximation:

$$I(W(x, y); P + \Delta P) \approx I(W([x, y]; P)) + \nabla I \frac{\partial W}{\partial P} \Delta P$$

- The approximation equation is actually the abstract of the basic assumption of optical flow described in the slides before.
- Derivation of this equation can be discussed in forum (Too long for slides).

Lucas-Kanade

Some Explanations:

- Gradient image ∇I
- Image error $I_E = T(x, y) - I(W[x, y]; P)$
- Jacobian matrix $\frac{\partial W}{\partial P}$
- Steepest image $I_S = \nabla I \frac{\partial W}{\partial P}$
- Hessian Matrix $\Sigma(\nabla I \frac{\partial W}{\partial P})^T (\nabla I \frac{\partial W}{\partial P})$
- Iteration step $\Delta P = \Sigma I_S^T I_E$

Lucas-Kanade

Algorithms:

- Warp image and get $I(W[x, y]; P)$;
- Get image error I_E;
- Warp gradient image ∇I;
- Evaluate Jacobian;
- Compute steepest descent image $I_S = \nabla I \frac{\partial W}{\partial P}$;
- Compute Hessian matrix $\Sigma I_S^T I_S$;
- Get warping step $\Delta P = I_S I_E$;
- Update warping parameter $P = P + \Delta P$;
- Repeat until ΔP is negligible.

APPLICATIONS – MACHINE LEARNING

Generalized Utilization of Convex: Delta Rule

- The delta rule is derived by attempting to minimize the error in the output of the neural network through gradient descent.
- Gradient Descent optimization is the most basic principle for training neurons even with different activation functions.
- Delta rule, can also be modified, if possible, with steepest descent method.

APPLICATIONS – MACHINE LEARNING

Delta Rule:

$$\Delta w_{ji} = \alpha(t_j - y_j)g'(h_j)x_j$$

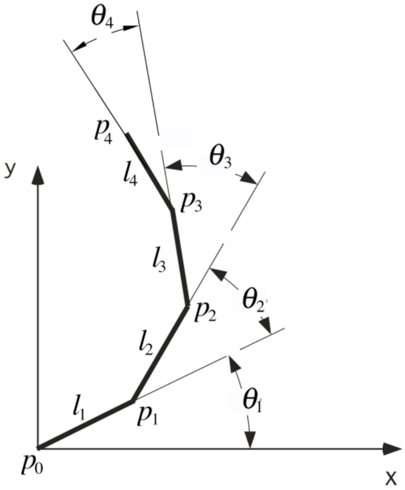
Where α is the learning rate, $g(x)$ is the neuron's activation function. t_j and y_j is the target and actual output of the neuron. h_j is the weighted sum of the neuron's inputs. And x_i is the i_{th} input.

The above equation holds the following:

$$h_j = \Sigma x_i w_{ji}$$

$$y_j = g(h_j)$$

APPLICATIONS – INVERSE KINEMATICS



APPLICATIONS – INVERSE KINEMATICS

Goal of Inverse Kinematics

Given a position in the space, calculate a way for a robot hand to reach a place.

Problem Abstract:

$$\vec{e} = R_1 T_1 R_2 T_2 R_3 T_3 R_4 T_4 \vec{e_0}$$

Where T_i is a series of translation transformation and R_i is a series of rotation translation.

APPLICATIONS – INVERSE KINEMATICS

Abstraction for Convex Optimization:

$$\Delta\vec{\theta} = \alpha J^{T}\vec{e}$$

.

The target for the optimization is to achieve $|\vec{e_p} - \vec{e_t}| = 0$, where $\vec{e_p}$ is th original position of the tip of the robotic arm and $\vec{e_t}$ is the target position. J is the jacobian matrix in terms of $\vec{\theta}$, which is the vector of all the spatial angles of all joints. α is the convergence rate and $\vec{e}$ is the position derivation (step size).

APPLICATIONS – INVERSE KINEMATICS

About Inverse Kinematics

- Jacobian transpose is the implementation of gradient descent in the real physical world.
- It can actually achieve near linear solution for robotic arms with a fast convergence rate.