Some Takeaways - Tangency Portfolio Factor Model: Derivation of In-Sample $R^2 = 1$ in the Cross-Sectional Regression

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When we use the tangency portfolio for a given set of assets as the factor model for the same assets, the tangency portfolio explains 100% of the variability in the cross-section. Meaning that, in-sample, it is the perfect pricing model ($R^2 = 1$).

Why is that the case?

1 Reminders

Weights in the tangency portfolio:

$$\mathbf{w}_p = \frac{\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1}\boldsymbol{\mu}}$$

The denominator, $\mathbf{1}^\top \Sigma^{-1}\boldsymbol{\mu}$, serves as a scaling factor to normalize the portfolio weights.

The β value in a regression with a single feature:

$$\beta = \frac{\mathrm{Cov}(y, x)}{\mathrm{Var}(y)}$$

In our case:

$$\beta = \frac{\mathrm{Cov}(r_i, r_p)}{\mathrm{Var}(r_p)}$$

Where:

- r_p is the time-series of returns of the tangency portfolio.
- r_p is the time-series of returns of the asset i.

2 Covariance between r_i and Tangency Portfolio

The covariance between the return of asset i and the tangency portfolio r_p is:

$$\mathrm{Cov}(r_i, r_p) = \mathrm{Cov}\left(r_i, \mathbf{w}_p^\top \mathbf{r}\right) = \mathbf{w}_p^\top \mathrm{Cov}(r_i, \mathbf{r}) = \mathbf{w}_p^\top \Sigma_{\cdot i}$$

where $\mathbf{w}_p^\top \Sigma_{\cdot i}$ represents the i column of Σ or the i value of the vector $\mathbf{w}_p^\top \Sigma$.
Since:

$$\Sigma \mathbf{w}_p = \frac{\boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}},$$

we have:

$$\mathrm{Cov}(r_i, r_p) = \frac{\mu_i}{\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}}.$$

3 Variance of the Tangency Portfolio

The variance of the tangency portfolio r_p is:

$$\mathrm{Var}(r_p) = \mathbf{w}_p^\top \Sigma \mathbf{w}_p = \left(\frac{\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}}\right)^\top \Sigma \left(\frac{\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}}\right).$$

Which simplifies to:

$$\mathrm{Var}(r_p) = \frac{\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}}{\left(\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}\right)^2}.$$

4 Beta of r_i to the Tangency Portfolio

The beta of asset i relative to the tangency portfolio is:

$$\beta_i = \frac{\mathrm{Cov}(r_i, r_p)}{\mathrm{Var}(r_p)} = \frac{\frac{\mu_i}{\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}}}{\frac{\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}}{\left(\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}\right)^2}}$$

Simplifying:

$$\beta_i = \frac{\mu_i \cdot \mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}}{\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}}.$$

5 Expected Returns as a Function of Beta

Rearranging the expression for β_i, we find:

$$\mu_i = \beta_i \cdot \frac{\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}}.$$

This shows a perfect linear relationship between μ_i and β_i, with the slope given by:

$$\frac{\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}}$$

Thus, the relationship between the asset and its β, for any asset i, is given by this **same** slope.

6 Cross-Sectional Regression

When performing a cross-sectional regression:

$$\mu_i = \eta + \lambda \beta_i + \nu_i$$

From the relationship $\mu_i = \beta_i \cdot \frac{\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}}$, we find:

$$\hat{\lambda} = \frac{\boldsymbol{\mu}^\top \Sigma^{-1} \boldsymbol{\mu}}{\mathbf{1}^\top \Sigma^{-1} \boldsymbol{\mu}}$$

$$\hat{\eta} = 0$$

Which connects perfectly all the μ_i, leadind to $\nu_i = 0 \quad \forall \quad i$. Consequently:

$$R^2 = 1.$$

Thus, in-sample the tangency portfolio is the perfect factor pricing model.