TA Review

Autumn 2024

Tentative Schedule

1. Review of MV Optimization
2. Introduction to RF and new MV Optimization
3. Homework Review
    1. Negative Sharpe in the Tangency Portfolio?
    2. Result instability and extreme allocation
4. Pandas
5. Virtual Environment Set Up

Lecture 1: Diversification and Mean-Variance

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FINM 36700: Portfolio Management

Return notation: one-period

notation	description	formula	example
r^i	return rate of asset i		
r^f	risk-free return rate		
$\tilde{r}^i$	excess return rate of asset i	$r^i - r^f$	

Two investments: bonds and stocks

Consider the following portfolio example

Table: Portfolio example

	return	allocation weight
bonds	r^b	w
stocks	r^s	$1-w$

Table: Return statistics notation

mean	variance	correlation
μ	σ^2	ρ

Portfolio return stats

Investment portfolio return r^p has mean and variance of

$$\mu^p = w\mu^b + (1-w)\mu^s$$

$$\sigma_p^2 = w^2\sigma_b^2 + (1-w)^2\sigma_s^2 + 2w(1-w)\rho\sigma_s\sigma_b$$

Matrix Notation

$$\mu^p = \mathbf{w}' \, \boldsymbol{\mu}$$

$$\sigma_p^2 = \mathbf{w}' \Sigma \mathbf{w}$$

When do we have diversification?

Perfect correlation

Suppose that $\rho = 1$.

- Then the volatility (standard deviation) of the portfolio is proportional to the asset allocation weights:

$$\sigma_p = w\sigma_b + (1-w)\sigma_s$$

- Thus, both mean and volatility are linear in the allocations.

Imperfect correlation

Suppose that $\boxed{\rho < 1}$.

- The volatility function is convex,

$$\sigma_p < w\sigma_b + (1-w)\sigma_s$$

- Yet the mean return is still linear in the portfolio allocation:

$$\mu^p = w\mu^b + (1-w)\mu^s$$

When do we have a riskless portfolio?

A perfect hedge

For $\rho = -1$,

- The portfolio variance can be as small as desired, by choosing the appropriate allocation, w.
- In fact, $\sigma_p = 0$ if

$$w = \frac{\sigma_s}{\sigma_b + \sigma_s}$$

- Thus, a riskless portfolio can be formed from the two risky assets.

Riskless portfolios

- Above, we found that a riskless portfolio could be created if $\rho = -1$.
- Here, we found that a riskless portfolio can be created if $\rho = 0$.

Question:
How did the assumptions behind these conclusions differ?

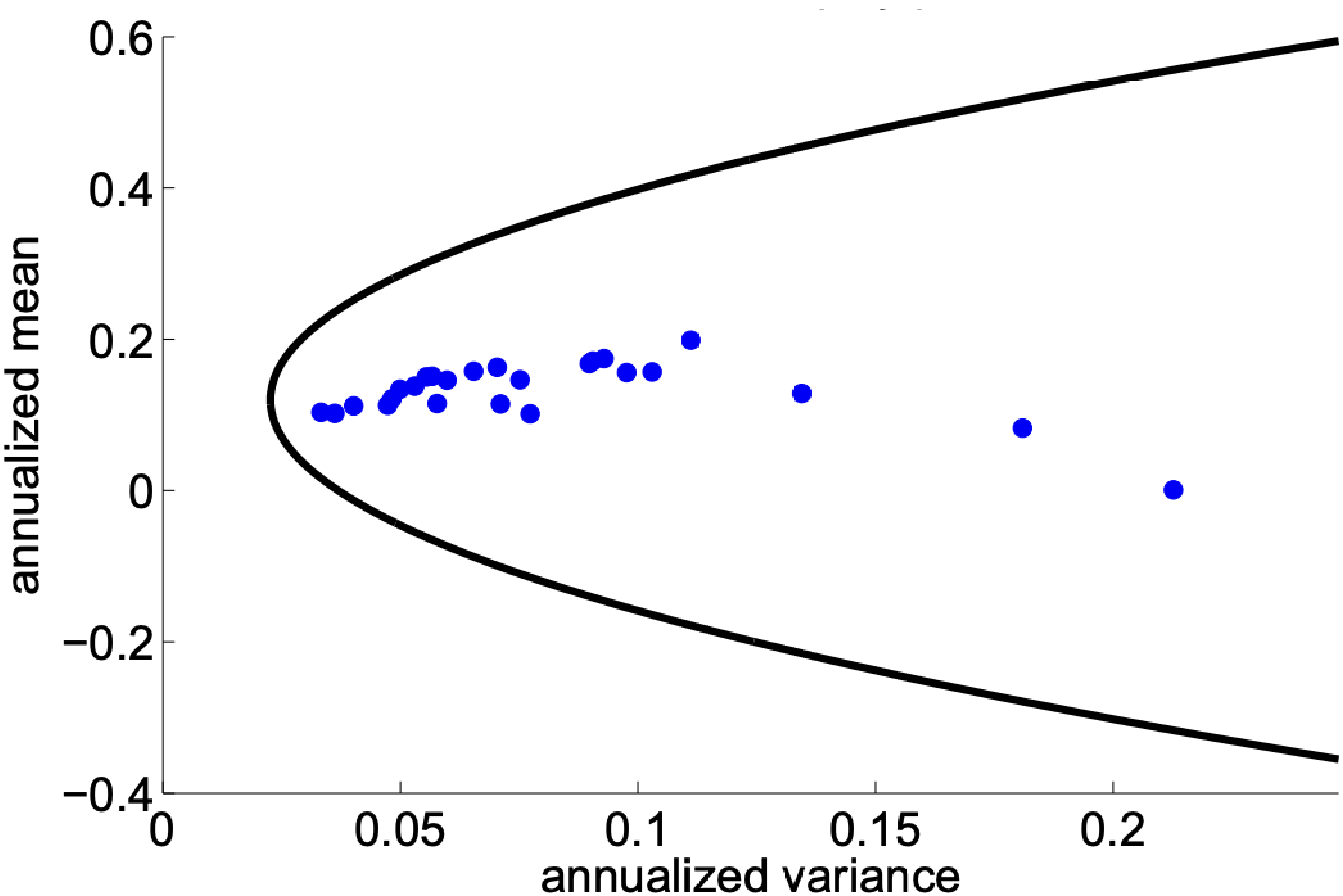

Figure: Mean-variance frontier formed by 25 U.S. equity portfolios, sorted by size and and book/market.

MV solution

Thus, a portfolio $\boldsymbol{\omega}^*$ is MV iff exists $\delta \in (-\infty, \infty)$ such that

$$\boldsymbol{\omega}^* = \delta \boldsymbol{\omega}^{\mathtt{t}} + (1-\delta)\boldsymbol{\omega}^{\mathtt{v}}$$

$$\boldsymbol{\omega}^{\mathtt{t}} \equiv \underbrace{\left(\frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}\right)}_{\text{scaling}} \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, \qquad \boldsymbol{\omega}^{\mathtt{v}} \equiv \underbrace{\left(\frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\right)}_{\text{scaling}} \boldsymbol{\Sigma}^{-1}\mathbf{1}$$

$\boldsymbol{\omega}^{\mathtt{t}}$ and $\boldsymbol{\omega}^{\mathtt{v}}$ are themselves MV portfolios ($\delta = 0, 1$)

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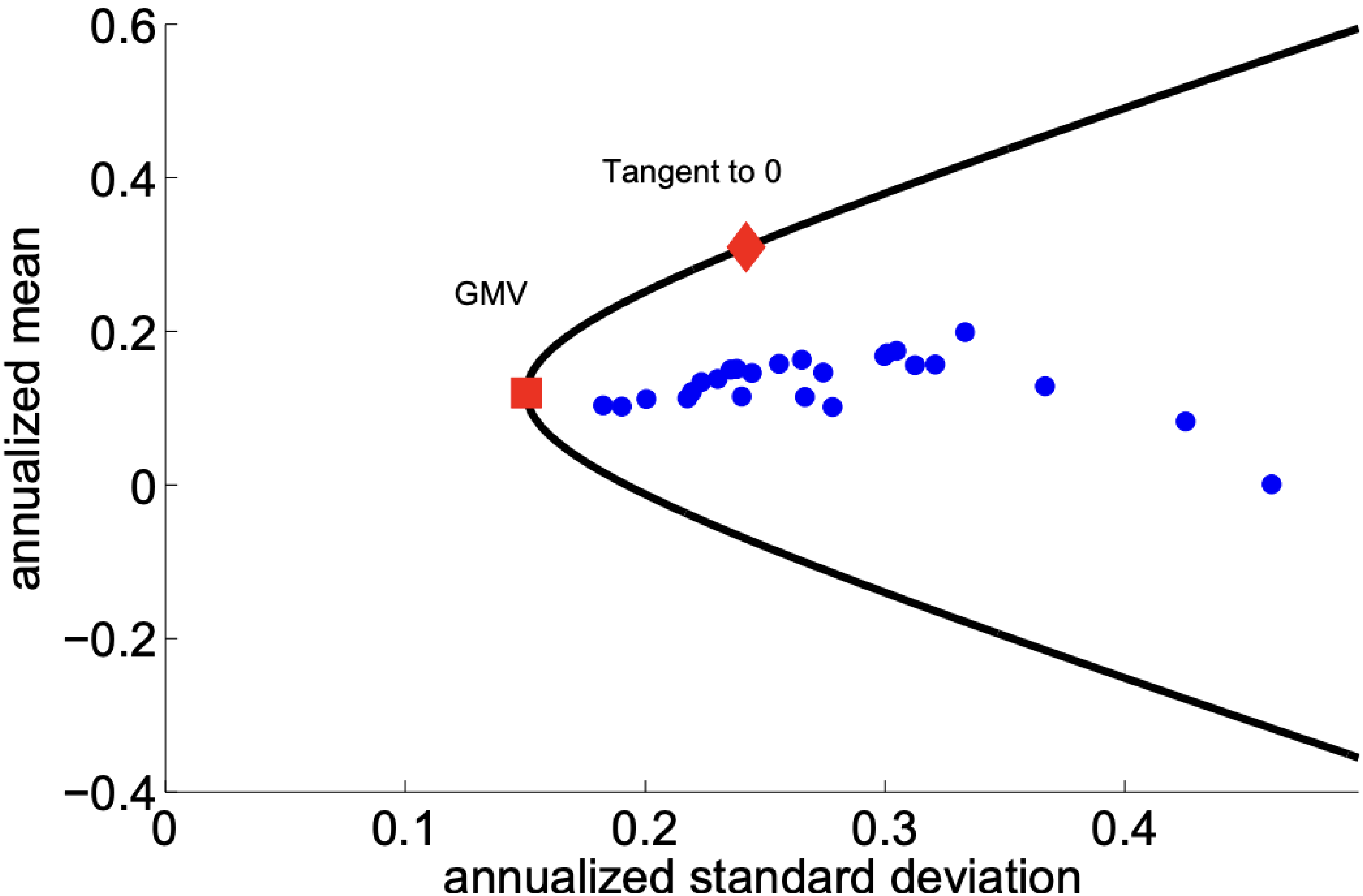

Figure: Illustration of two useful MV portfolios. The Global-Minimum-Variance portfolio as well as the zero-tangency portfolio.

MV investors

Consider MV investors, the investors for whom mean and variance of returns are sufficient statistics of the investment.

- Such investors will hold an MV portfolio, $\boldsymbol{\omega}^*$.
- Thus, these investors are holding linear combination of just two risky portfolios, $\boldsymbol{\omega}^{\mathrm{t}}$ and $\boldsymbol{\omega}^{\mathrm{v}}$.
- So if in real markets all investors were MV investors, everyone would simply invest in two funds.
- Those wanting higher mean returns would hold more in the high-return MV, $\boldsymbol{\omega}^{\mathrm{t}}$, while those wanting safer returns would hold more in the low-return MV, $\boldsymbol{\omega}^{\mathrm{v}}$.

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Pywidget

With a riskless asset

Now consider the existence a risk-free asset with return, r^f.

- Suppose there are still n risky assets available, still notating the risky returns as $\boldsymbol{r}$
- Let $\boldsymbol{w}$ denote a $n \times 1$ vector of portfolio allocations to the n risky assets.
- Since the total portfolio allocations must add to one, we have

$$\text{allocation to the risk-free rate} = 1 - \boldsymbol{w}'\mathbf{1}$$

Mean excess returns

$\boldsymbol{\mu}$ denotes the vector of mean returns of risky assets, $\mathbb{E}[\boldsymbol{r}]$.

Let μ^p denote the mean return on a portfolio.

$$\mu^p = \left(1 - \boldsymbol{w}'\boldsymbol{1}\right) r^f + \boldsymbol{w}'\boldsymbol{\mu}$$

Use the following notation for excess returns:

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - \boldsymbol{1} r^f$$

Thus the mean return and mean excess return of the portfolio are

$$\begin{aligned}\mu^p =& r^f + \boldsymbol{w}'\tilde{\boldsymbol{\mu}}\\ \tilde{\mu}^p =& \boldsymbol{w}'\tilde{\boldsymbol{\mu}}\end{aligned}$$

The $\tilde{\text{MV}}$ problem with a riskless asst

A Mean-Variance portfolio with risk-free asset ($\tilde{\text{MV}}$) is a vector, $\boldsymbol{w}^*$, which solves the following optimization for some mean excess return number $\tilde{\mu}^p$:

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} \\ \text{s.t.} \quad & \boldsymbol{w}'\tilde{\boldsymbol{\mu}} = \tilde{\mu}^p \end{aligned}$$

- In contrast to the MV problem, there is only one constraint.
- The allocation weight vector, $\boldsymbol{w}$ need not sum to one, as the remainder is invested in the risk-free rate.

Tangency portfolio and the Sharpe ratio

For an arbitrary portfolio, $\boldsymbol{w}^p$,

$$\mathrm{SR}\left(\boldsymbol{w}^p\right) = \frac{\mu^p - r^f}{\sigma^p} = \frac{\tilde{\mu}^p}{\sigma^p}$$

The tangency portfolio, $\boldsymbol{w}^{\mathrm{t}}$, is the portfolio on the risky MV frontier with maximum Sharpe ratio.

$$\mathrm{SR}\left(\boldsymbol{w}^*\right) = \pm\sqrt{(\tilde{\boldsymbol{\mu}})'\,\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}}$$

The SR magnitude is constant across all $\tilde{\text{MV}}$ portfolios.
(Sign depends on whether part of the efficient or inefficient frontier.)

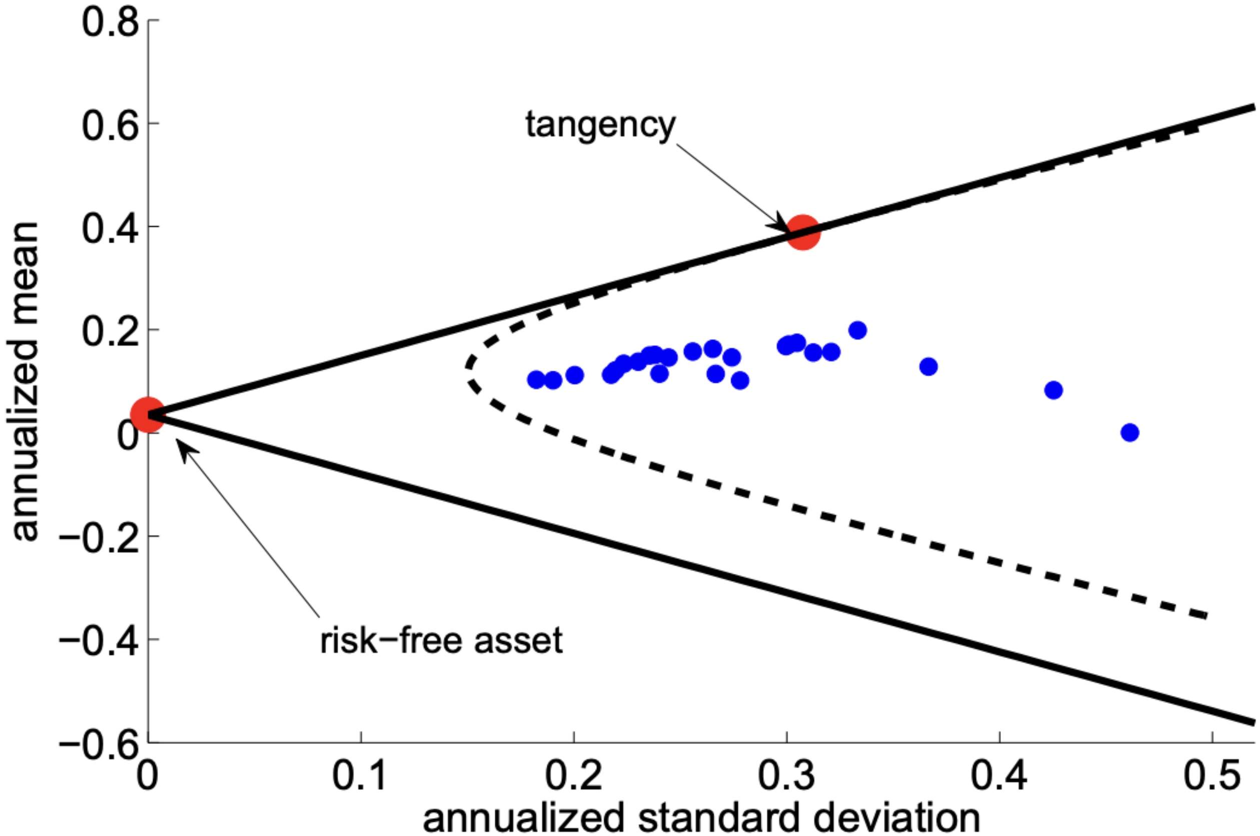

Figure: Illustration of the $\tilde{\text{MV}}$ frontier when a riskless asset is available. In this case, the $\tilde{\text{MV}}$ portfolio frontier consists of two straight lines. The curved frontier is the MV frontier when a riskless asset is unavailable.

Capital Market Line

The Capital Market Line (CML) is the efficient portion of the $\tilde{\text{MV}}$ frontier.

- The CML shows the risk-return tradeoff available to $\tilde{\text{MV}}$ investors.
- The slope of the CML is the maximum Sharpe ratio which can be achieved by any portfolio.
- The inefficient portion of the $\tilde{\text{MV}}$ frontier acheives the minimum (negative) Sharpe ratio by shorting the tangency portfolio.

In which assets/portfolios should the MV investor allocate?

Two-fund separation

Two-fund separation. Every $\tilde{\text{MV}}$ portfolio is the combination of the risky portfolio with maximal Sharpe Ratio and the risk-free rate.

Thus, for an $\tilde{\text{MV}}$ investor the asset allocation decision can be broken into two parts:

1. Find the tangency portfolio of risky assets, $\boldsymbol{w}^{t}$.
2. Choose an allocation between the risk-free rate and the tangency portfolio.

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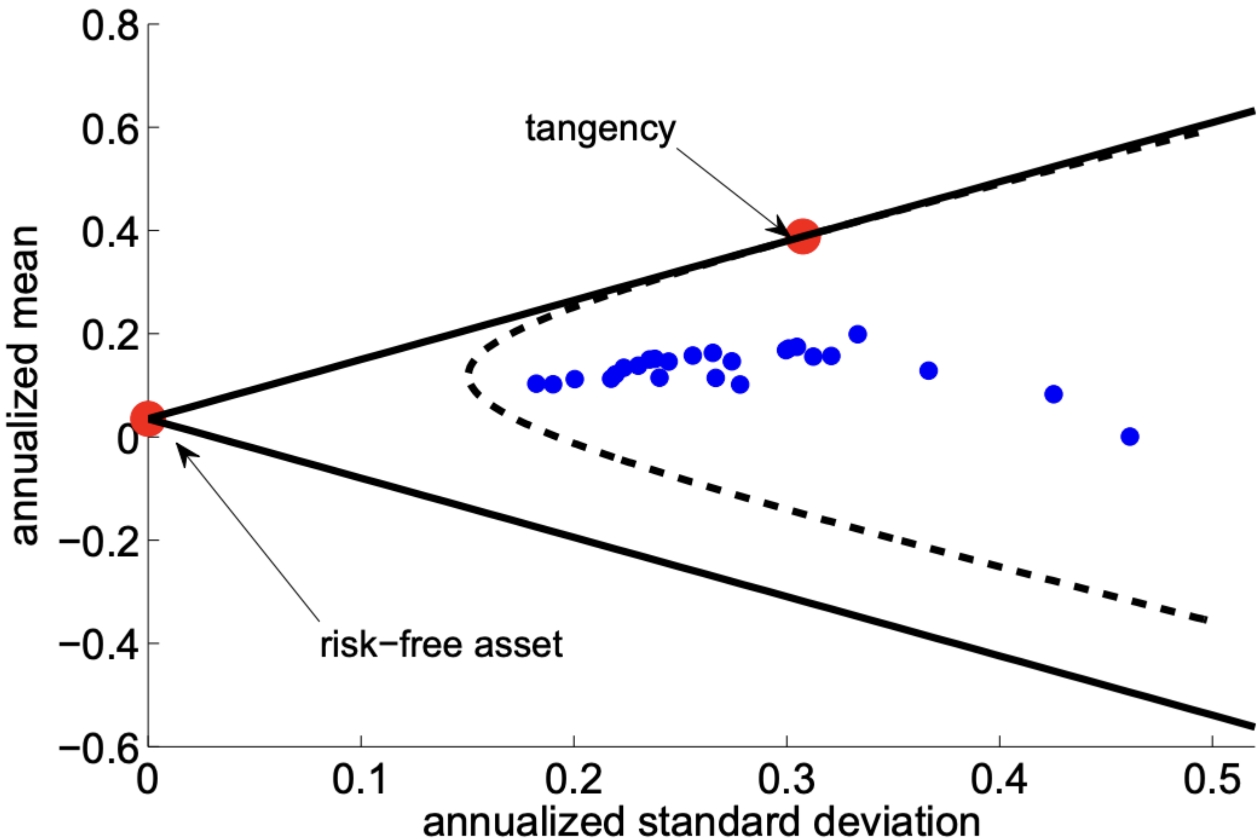
The investor should choose a combination of the tangency portfolio and the risk-free rate and get the “X” expected return for a lower risk when compared to a portfolio with of risky assets that give the same “X” expected return.



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Homework

Tangency Allocation

- 1. Have higher mean returns.
- 2. Have lower volatility (variance).
- 3. Have lower covariance with other assets.