Mathematical Thinking, Proof and Logic

12th June 2013

Abstract

Arithmetic and Number Theory - Patterns of counting.
Calculus - patterns of motion
Logic - patterns of reasoning
Probability - patterns of chance
Topology - patterns of closeness and position
Fractal Geometry - similarity in the natural world

During the 19th century, the key maths techniques went from methods and calculation to abstraction, relationship and generalisation to handle complicated results that boggle intuition, such as Banach-Taski paradox "You can cut up one sphere and resemble it in such a way to create 2 sphere of equal size to the first sphere."

1 Logic

1.1 The need for Mathematical statement

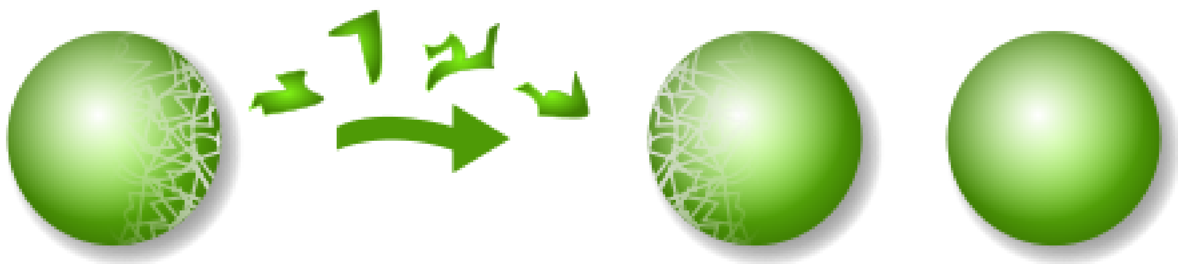
Everyday language is not exact and often relies on context:

"July is a summer month" - True for an Englishman, false for an Australian
"Sisters reunited after ten years in checkout line at Tesco."
"In case of fire, do not use lift."
"This page is intentionally left blank."

1.2 All Mathematical Statements can be expressed in one or a mixture of:

1. Object a has Property P
   (a) 3 is a prime number.
2. Every object of type T has property P.
   (a) Every polynomial equation has a complex root.
3. There is an object of type T having property P.
   (a) There is a prime number between 20 and 25
4. If statement A, then statement B.
   (a) If p is a prime of the form $4n + 1$, then p is a sum of two squares.

Remark. We use the joining words:



1. "and" $\wedge$
   (a) True if **both** conjuncts (parts) are true.
2. "(inclusive) or" $\vee$
3. "not" $\neg$
   (a) "All foreign cars are badly made."
   Clearly "At least one domestic car is not badly made" is **not** the negation, as this new sentence is not even about foreign cars any more.
   The real negation is "At least one foreign car is not badly made".
   (b) NOTE: "All foreign cars are badly made." is a FALSE statement; that means the negation must be TRUE:

ϕ	$\neg\phi$
T	F
F	T

4. "implies"
   (a) Implication involves causality; consider ϕ : "$\sqrt{2}$ is irrational", and ψ : "$0 < 1$".
   Clearly ϕ and ψ are true, however the statement "$\phi \implies \psi$" is not true since there is no relation between them.
5. "for all"
6. "here exists"

1.3 Logical Equivalence

Definition. Statement

$$P \implies Q$$

Remark. $P \implies Q \equiv (\neg P) \vee Q$

Equivalent ϕ only if ψ [NOT the same as "If ψ then ϕ"; Considerϕ : "*Ride in Tour de France*" and ψ : *Own a bicycle*" then they are clearly different "You can ride in Tour de France only if you own a bicycle", "If you own a bicycle then you ride in Tour de France".

Definition. Converse

$$Q \implies P$$

Definition. Inverse

$$\neg P \implies \neg Q$$

Definition. Contrapositive

$$\neg Q \implies \neg P$$

Remark. A False Statement implies anything is true.

Definition.

$$\text{Tautology: } (A \vee \neg A) = A$$

Theorem 1.1. *Statement* $\iff$ *Contrapositive.*

Example (Wiki - Logical Equivalence)**.** If Lisa is in France, then she is in Europe
If Lisa is not in Europe, then she is not is France.

Proof.

P	Q	$P \implies Q$	$\neg Q \implies \neg P$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

□

Theorem 1.2. *Inverse* $\iff$ *Converse*

Example. As above, (replace Q with P).

Proof.

P	Q	$Q \implies P$	$\neg P \implies \neg Q$
T	T	T	T
T	F	T	T
F	T	F	F
F	F	T	T

□

2 Proofs

Abstract

Template:

1. State the assumptions
2. What to these imply
   (a) (Body: the logical arguments)
3. Conclude the result.

2.1 Direct Proofs

1. Work backwards and forwards and meet in the middle?

2.2 Proof by Contradiction:

Definition.

$$\begin{aligned} \neg P &\Rightarrow (Q \wedge \neg Q) \qquad \text{a contradiction!} \\ \text{Thus} &\Rightarrow P\ (true) \end{aligned}$$

- is a good way to prove that **something does not existence**.
- provides a **good way to start**.
- - all we need is **any contradiction** to the assumptions.

2.2.1 Uses

1. Existence:
2. Showing a number is **equal** to **zero**:

$$\text{Suppose } \underline{a}, \underline{b} \neq 0, \underline{a} \not\parallel \underline{b}, \lambda, \mu \in \mathbb{R} \text{ then:}$$
$$\lambda \underline{a} + \mu \underline{b} = 0 \Rightarrow \lambda = 0 = \mu$$
$$\textit{Proof.}\ \text{Suppose } \lambda \neq 0, \text{ then}$$
$$\underline{a} = -\frac{\mu}{\lambda} \underline{b} \qquad \text{(the contradiction we need)}$$

3. **Uniqueness**: e.g. uniqueness of components of vectors

$$\text{Assume } \exists \text{ another representation}$$
$$\underline{a} = a_1' \underline{i} + a_2' \underline{j} + a_3' \underline{k} \qquad \text{for } a_1', a_2', a_3' \in \mathbb{R} \text{ wlog } a_1' \neq a_1 \text{ etc.}$$
$$\text{Subtracting } (a_1 - a_1')\, \underline{i} + (a_2 - a_2')\, \underline{j} + (a_3 - a_3')\, \underline{k} = \underline{a} - \underline{a} = \underline{0}$$
$$\text{Thus } \underline{i} = \frac{(a_2 - a_2')\, \underline{j} + (a_3 - a_3')\, \underline{k}}{(a_1 - a_1')}$$
$$\text{But then } \underline{i} \parallel \underline{j} \parallel \underline{k} \text{ contradicting } \underline{i} \perp \underline{j} \perp \underline{k}$$

2.3 Proof by Contrapositive

- Is it clearer; e.g.

  $$\sin\theta = 0 \Longrightarrow \forall n \in \mathbb{N} : \theta \neq n\pi$$

  then $\exists n \in \mathbb{N} : \theta = n\pi \Longrightarrow \sin\theta = 0$ is much much easier to prove.
- Perhaps, if there are more assumptions in the results.

1. You want to prove some statement ϕ.
2. You start by assuming $\neg\phi$.
3. You reason until you reach a conclusion that is false.
   (a) often by deducing $\neg\psi \wedge \psi$ for some ψ

   (b) e.g. "p, q have no common factors" and "p, q are both even"

4. We know a true assumption cannot lead to a false conclusion.

5. Hence the assumption $\neg\phi$ must be false; i.e. ϕ is true.

ϕ	ψ	$\phi \Rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

2.4 Proving a conditional statement $\phi \Rightarrow \psi$

- We know this is true if ϕ is false (**truth tables**), so assume it's true.
- Then use logical reasoning to deduce ψ.

2.5 Indirect proofs

Example (Proof by Cases)**.** There are irrationals r, s such that r^s is rational.

Proof. We consider 2 cases:

Case 1. If $\sqrt{2}^{\sqrt{2}}$ is rational, take $r = s = \sqrt{2}$.

Case 2. If $\sqrt{2}^{\sqrt{2}}$ is irrational, take $r = \sqrt{2}^{\sqrt{2}}$ and $s = \sqrt{2}$; then

$$r^2 = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$$

□

2.6 Proving $\forall x A(x)$

- One way is to take arbitrary x and show it satisfies $A(x)$.
  - e.g. to prove $\forall n \exists m : m > n^2$, take $m = n^2 + 1$, "Note m clearly has to depend on n; i.e. $m(n)$"
- Another approach is by contradiction
  - Suppose $\exists x \neg A(x)$

2.7 Prove by induction (Dominoes - Show that something is true $\forall n \in \mathbb{N}$)

Proving a statement of the form $\forall n A(n)$, where $n \in \mathbb{N}$.

1. Let $\mathbb{P}(n)$ denote the statement.
2. Check $\mathbb{P}(1)$ holds (maybe need to check more base cases; $\mathbb{P}(2), \mathbb{P}(3)$)
3. Assume $\mathbb{P}(k)$ holds $\forall k \geq 1$ to deduce $\mathbb{P}(k+1)$ is true.
   (a) $\forall n : \mathbb{P}(n) \Rightarrow \mathbb{P}(n+1)$, which makes sense since $\mathbb{P}(1) \Rightarrow \mathbb{P}(2) \Rightarrow \mathbb{P}(3) \Rightarrow ...$
4. Conclude that $\mathbb{P}(n)$ holds $\forall n \in \mathbb{N}$ **by induction**.

Remark (An equivalent, but often more convenient method of proof using *well-ordering principle*). 1. Suppose not, for contradiction.

2. **Then $\exists n \in \mathbb{N}$ smallest, for which it fails.**
3. But this leads to a contradiction because . . . .
4. We conclude by contradiction the statement in question is true $\forall n \in \mathbb{N}$.