Sample Mathematica Problems

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1 Tips for Debugging

Note: most of these examples are from past sets I found, so you can't tell me you'll never use this stuff (although I did make some up when I got lazy of looking for examples). Feel free to email me if you get stuck on some code syntax! I also found and wrote this all up real quick, so if you get something that doesn't agree I could definitely be wrong.

1. Make sure you clear your variables!
2. Check for any missing spaces between variables
3. When defining functions, don't forget the _ after the variable. It has to be $x(t\_) =$ not $x(t) =$
4. Check parenthesis and brackets and make sure you're using them in the right places (although the latest Mathematica update looks like it generally does this for you)

2 Algebra and Calculus

2.1 Basic Derivative

Find the derivative of the following and evaluate it at $x = 5$:

$$\frac{x^3 e^x}{\log x \tan x}$$

Answer:

$$\boxed{-\frac{e^x x^2 \csc(x)(-((x+3)\log(x)) + x\log(x)\cot(x) + 1)}{\log^2(x)}}$$

$$\boxed{-209.793}$$

2.2 Finding a Maximum

Defining the functions

$$A(f) = \frac{1}{1 - \left(\frac{f}{f_{pend}}\right)^2}$$

$$S(f) = \frac{10^{-6}}{f^2}$$

For both $f_{pend} = \frac{1}{2\pi}$ and $f_{pend} = \frac{\sqrt{9.8}}{2\pi}$, find the local maximum value of the following for $f > 10^{-2}$:

$$ASD(f) = A(f)^2 S(f)$$

Answer:

$$\boxed{f_{max} = \frac{1}{2\pi\sqrt{3}}}$$

$$\boxed{f_{max} = 0.287}$$

2.3 Definite Integral

Evaluate the integral:

$$\int_{-\infty}^{\infty} x^6 e^{-x^2} \cos(x) \mathrm{d}x$$

Answer:

$$\boxed{\int_{-\infty}^{\infty} x^6 e^{-x^2} \cos(x) \mathrm{d}x = -\frac{31\sqrt{\pi}}{64\sqrt[4]{e}}}$$

2.4 Solve for x

Solve for x:

$$\left(ax + xe^{5t} + 2\right)(x+2) = \tan(2t)$$

Answer:

$$\boxed{x = -\frac{\pm\sqrt{\left(a + e^{5t} - 1\right)^2 + \left(a + e^{5t}\right)\tan(2t)} + a + e^{5t} + 1}{a + e^{5t}}}$$

2.5 Indefinite Integral

Solve the indefinite integral:

$$\int x^2 \cos(x) e^x \mathrm{d}x$$

Answer:

$$\boxed{\int x^2 \cos(x) e^x \mathrm{d}x = \frac{1}{2} e^x (x-1)((x-1)\sin(x) + (x+1)\cos(x))}$$

2.6 Solve for normalizing constant

Assume $a > 0$, find A such that

$$1 = A^2 \int_0^a \left(\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) \right) \mathrm{d}x$$

Answer:

$$\boxed{A = \pm \frac{1}{\sqrt{a}}}$$

2.7 Solve using Assumptions

Knowing that $a > 0$ and $n \in \mathbb{Z}$, simplify this as much as possible:

$$\frac{2}{a} \int_0^a \left(\sin\left(\frac{n\pi x}{a}\right) \left(\frac{1}{\sqrt{a}}\right) \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) \right) \mathrm{d}x$$

My best simplification:

$$\boxed{-\frac{8\left((-1)^n + 1\right) n}{\pi \sqrt{a} \left(n^4 - 10n^2 + 9\right)}}$$

Hint: Use our assumptions on a and n.

2.8 Solve complex integral problem

Take this wave function:

$$\psi(x) = \sqrt[1/4]{\frac{2a}{\pi}} e^{-\frac{amx^2}{1+2i\hbar at}} \sqrt{\frac{m}{1+2i\hbar at}}$$

You can assume $a, m, \hbar, t$ are real and > 0. Find:

$$\int_{-\infty}^{\infty} \psi^*(x)\psi''(x)\mathrm{d}x$$

where $\psi^*(x)$ is the complex conjugate of $\psi(x)$ and $\psi''(x) = \frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2}$.

Answer:

$$\boxed{\int_{-\infty}^{\infty} \psi^*(x)\psi''(x)\mathrm{d}x = -am^{3/2}}$$

Hints: This one is a bit difficult and I found it easiest to go one step at a time, copy paste, and adjust things as needed. Here is how I suggest to approach this:

1. Find $\psi^*(x)\psi''(x)$. I had to manually set $\sqrt{|a|} = \sqrt{a}$ and also

$$\left(\frac{1}{\sqrt{1+\frac{2ia\hbar t}{m}}}\right)^* = \frac{1}{\sqrt{1-\frac{2ia\hbar t}{m}}}$$

2. Integrate the expression. I got a conditional expression that has the conditions satisfied.

3. Simplify again using our assumptions

3 Taylor Series

3.1 Basic Taylor Series

Take the functions:

$$\begin{aligned} x(t) &= e^{-\frac{t}{2}} \\ y(t) &= \log(2t) \end{aligned}$$

Find the 2nd order Taylor Series around $t = \frac{1}{2}$ of

$$\frac{x(t)}{y(t)}$$

Answer:

$$\boxed{\frac{1}{2\sqrt[4]{e}\left(t-\frac{1}{2}\right)} + \frac{1}{4\sqrt[4]{e}} - \frac{17\left(t-\frac{1}{2}\right)}{48\sqrt[4]{e}} + \frac{29\left(t-\frac{1}{2}\right)^2}{96\sqrt[4]{e}} + O\left(\left(t-\frac{1}{2}\right)^3\right)}$$

3.2 Fun Plotting

Let's see why $\sin x = x$ and how Taylor Series' converge. Plot $\sin x$ as well as the Taylor Series of $\sin x$ around $x = 0$ with an increasing number of terms. The plot is in Fig. 1.

4 Differential Equations

4.1 Diff Eq with Boundary Conditions

Solve:

$$\begin{aligned} t^2 y''(t) - 2y(t) &= 3t^2 - 1 \\ y'(1) &= 0 \\ y(1) &= 3 \end{aligned}$$

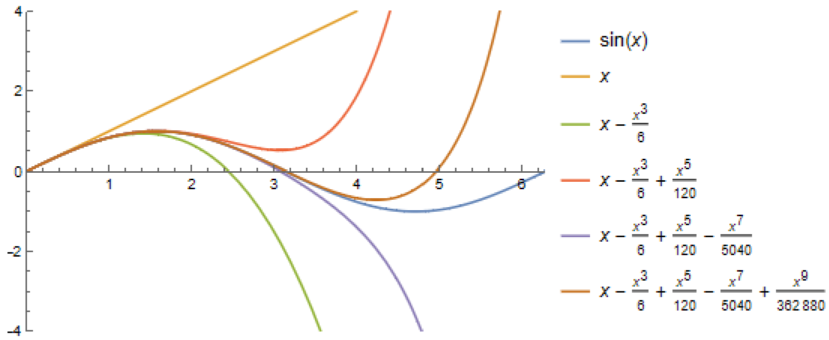

Figure 1: Taylor Series of $\sin x$ plot.

Answer:

$$\boxed{y(t) = \frac{4 + t + t^3 + 2t^3 \log(t)}{2t}}$$

4.2 Solve for General Solution

Find the general solution:

$$y'''(x) - 3y''(x) + 3y'(x) - y(x) = 0$$

Answer:

$$\boxed{y(x) = c_3 e^x x^2 + c_2 e^x x + c_1 e^x}$$

4.3 Solve and Plot Differential Equation

Solve the following differential equation and then plot for a couple of the constant.

$$y'(x) = 1 - y(x)^2$$

Answer and plot:

$$\boxed{y(x) = \frac{e^{2x} - e^{2c}}{e^{2x} + e^{2c}}}$$

Plot in Fig. 2.

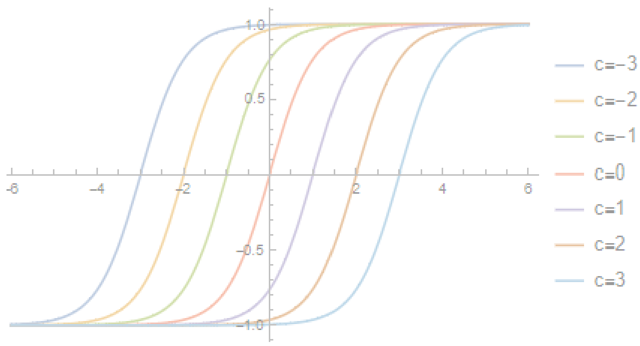

Figure 2: Solution to 3.2 for a few c values.

5 Linear Algebra

5.1 Basic Eigenvalue and Eigenvector Problem

Find the eigenvalues and eigenvectors of A:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Answer:

$$\begin{aligned} \lambda_{1,2,3} &= 1, 2, 3 \\ \vec{u}_1 &= (-1, 1, 0) \\ \vec{u}_2 &= (0, 0, 1) \\ \vec{u}_3 &= (1, 1, 0) \end{aligned}$$

5.2 Finding Eigenvalues and Eigenvectors

Find the eigenvectors and eigenvalues of

$$\begin{pmatrix} \cos(\theta) & e^{-i\phi}\sin(\theta) \\ e^{i\phi}\sin(\theta) & -\cos(\theta) \end{pmatrix}$$

Answer:

$$\lambda_1 = 1 \qquad\qquad \lambda_2 = -1$$

$$\vec{u}_1 = \begin{pmatrix} -e^{-i\phi}\tan\left(\frac{\theta}{2}\right) \\ 1 \end{pmatrix} \qquad\qquad \vec{u}_2 = \begin{pmatrix} e^{-i\phi}\left(\cot(\theta) + \csc(\theta)\right) \\ 1 \end{pmatrix}$$

5.3 Matrix Multiplication and Powers

This problem I made up, but it has all the useful matrix manipulations you'll need. Take this matrix A:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & 2 \\ 5 & 1 & 1 \end{pmatrix}$$

$A^\top$ is the transpose of A. Find:

$$A^2 A^\top A^{-1}$$

Answer:

$$\boxed{A^2 A^\top A^{-1} = \begin{pmatrix} \frac{143477}{961} & -\frac{163787}{961} & \frac{183539}{961} \\ -\frac{73682}{31} & -\frac{15835}{31} & \frac{61301}{31} \\ -\frac{1200445}{961} & -\frac{503218}{961} & \frac{1378561}{961} \end{pmatrix}}$$

This is a lot grosser than I expected, but hopefully you get the idea.

5.4 Solving System of Equations

You can also use matrices in Solve[] to find values in the matrices. Let $x+y+z = 1$, find x, y, z:

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{9} \\ \frac{1}{4} & \frac{1}{3} & \frac{5}{9} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Answer:

$$x = \frac{4}{13}$$
$$y = \frac{27}{91}$$
$$z = \frac{36}{91}$$

Hint: use the other equation that $x + y + z = 1$

5.5 Matrix Exponential

Find e^{At} where A is given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Answer:

$$\boxed{e^{At} = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}}$$

6 Cool Plotting

I will let you figure these out for yourself, but they make really cool looking plots!

6.1 Polar Coordinate Plot

Use PolarPlot[] to plot the following functions from $\phi \in [0, 2\pi]$ and θ ranging in $15°$ steps (ie $\theta = 15°, \theta = 30°$, etc):

$$r_1(\theta,\phi) = \frac{1}{2}\left(3 + \cos(2\theta)\right)\cos(2\phi)$$
$$r_2(\theta,\phi) = -4\cos\theta\sin\phi$$

Plots are given in Figs. 3 and 4.

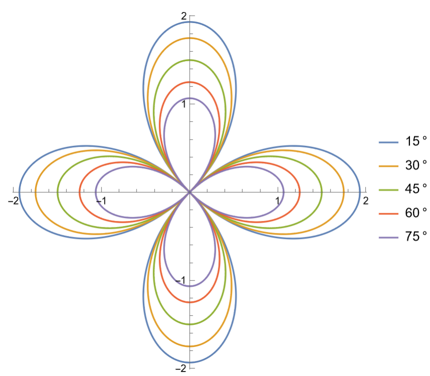

Figure 3: First polar plot

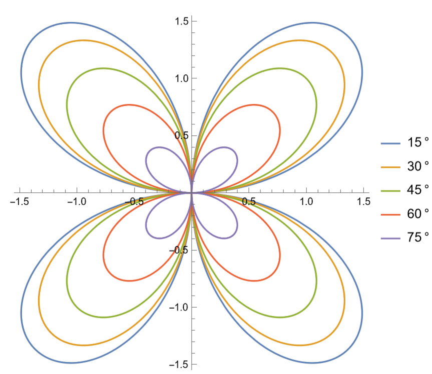

Figure 4: Second polar plot

6.2 3D Spherical Plot

Now take the same 2 functions from before and instead of picking specific θ, we will plot in 3D for continuous $\theta \in [0, \pi]$. Use the function SphericalPlot3D[]. This gives us the following plots Figs. 5 and 6. Note: these are fun and you can drag around the perspective.

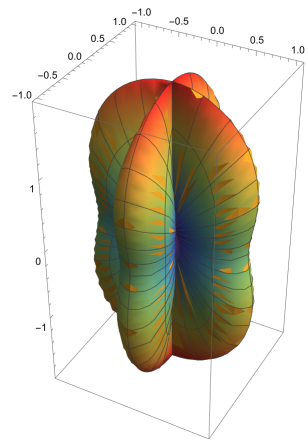

Figure 5: Spherical plot for r_1

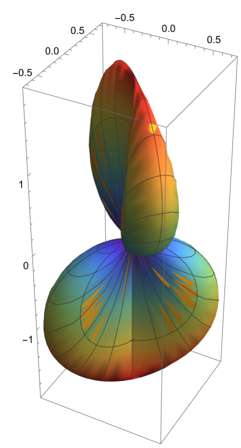

Figure 6: Spherical plot for r_2