Linear Programming: Linear Programming

Linear Programming

Input: An $m \times n$ matrix A and vectors $b \in \mathbb{R}^m, v \in \mathbb{R}^n$

Output: A vector $x \in \mathbb{R}^n$ so that $Ax \geq b$ and $v \cdot x$ is as large (or small) as possible

Network Flow

Variables: f_e for each edge e

Constraints: $0 \leq f_e \leq C_e, \sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e = 0$

Objective: $\sum_{e \text{ out of } s} f_e - \sum_{e \text{ into } s} f_e$

Strange Cases

No solution

No optimum

Linear Programming: Convex Polytopes

Polytope

A polytope is a region in $\mathbb{R}^n$ bounded by finitely many flat surfaces. These surfaces may intersect in lower dimensional facets (like edges), with zero-dimensional facets called vertices.

Convexity

A region $\mathcal{C} \subset \mathbb{R}^n$ is convex, if $\forall x, y \in \mathcal{C}$, the line segment connecting x and y is contained in $\mathcal{C}$.

Lemma

An intersection of halfspaces is convex.

Proof:

- Defined by $Ax \geq b$
- Need for $x, y \in \mathcal{C}$ and $t \in [0, 1]$, $tx + (1-t)y \in \mathcal{C}$
- $A(tx + (1-t)y) = tAx + (1-t)Ay \geq tb + (1-t)b = b$

Theorem

The region defined by a system of linear inequalities is always a convex polytope.

Separation Lemma

Let $\mathcal{C}$ be a convex region and $x \notin \mathcal{C}$ a point. Then, there is a hyperplane H separating x from C.

Extreme Points

A linear function on a polytope takes its minimum/maximum values on vertices

Intuition of Proof:

The corners are the only extreme points. Optima must be there.

Linear function on segment takes extreme values on ends. So, optima can be obtained by pushing linear function towards the corners. By repeatedly pushing to lower dimensional facet, it will eventually end at a vertex.

Linear Programming: Duality

Dual Program

Given the linear program (the primal): $\begin{aligned} &\min v \cdot x \\ &\text{s.t. } Ax \geq b \end{aligned}$

The dual linear program is the linear program: $\begin{aligned} &\max y \cdot b \\ &\text{s.t. } y^T A = v, y \geq 0 \end{aligned}$

Intuition

Suppose we have the following linear program:

$$\begin{aligned} \min v_1 x_1 + v_2 x_2 + \ldots + v_n x_n \text{ s.t.} \\ a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \geq b_1 \\ \ldots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \geq b_m \end{aligned}$$

Let $c_i \geq 0$, we can then combine constraints:

$$\begin{aligned} c_1 \cdot (a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n) \geq c_1 \cdot b_1 \\ \ldots \\ c_m \cdot (a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n) \geq c_m \cdot b_m \end{aligned}$$

Let $w_i = \sum_{j=1}^m c_j a_{ji}, t = \sum_{j=1}^m c_j b_j$. We can write

$$\sum_{i=1}^n w_i x_i \geq t$$

Let $v_i = w_i$, we have

$$\sum_{i=1}^n v_i x_i \geq t$$

By letting $v_i = w_i$, we need to find $c_i \geq 0$ so that $v_i = \sum_{j=1}^m c_j a_{ji} \forall i$, and $t = \sum_{j=1}^m c_j b_j$ as large as possible.

Dual bounds

$\forall x$ such that $Ax \geq b$, $x \cdot v = y^T Ax \geq y^T b = y \cdot b$

Theorem

A linear program and its dual always have the same (numerical) answer.

Theorem (Complementary Slackness)

Consider a primal LP: $\min v \cdot x$
$$\text{s.t. } Ax \geq b$$

and its dual LP: $\max y \cdot b$
$$\text{s.t. } y^T A = v, y \geq 0$$

Then, in the solutions, $y_i > 0$ only if the i-th equation in x is tight.

Linear Programming: Simplex Method

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Simplex:
Start at vertex p
repeat:
    for each equation through p
        relax equation to get edge
        if edge improves objective:
            replace p by other end
            break
    if no improvement: return p
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OtherEndOfEdge:
Vertex p defined by n equations
Relax one, write general solution as p + t*w (Gaussian elimination)
Relaxed inequality requires t >= 0
For each other inequality in system:
    largest t so p + tw satisfies
Let t_0 be the smallest such t
return p + t_0*w
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