2. $J(\{(u_{i,j}, v_{i,j})\}) = \sum_{i=1}^{N} \sum_{j=1}^{N} (I_{x;i,j} u_{i,j} + I_{y;i,j} v_{i,j} + I_{t;i,j})^2 + \lambda((u_{i,j+1} - u_{i,j})^2 + (u_{i+1,j} - u_{i,j})^2 + (v_{i,j+1} - v_{i,j})^2 + (v_{i+1,j} - v_{i,j})^2)$

First, let us set partial derivative of J w.r.t $u_{k,l}$ equal to 0.

$$\Rightarrow \frac{\partial J}{\partial u_{k,l}} = 2(I_{x;k,l} u_{k,l} + I_{y;k,l} v_{k,l} + I_{t;k,l})(I_{x;k,l})$$

$$+ \lambda(2(u_{k,l+1} - u_{k,l})(-1) + 2(u_{k+1,l} - u_{k,l})(-1)$$

$$+ 2(u_{k,l} - u_{k,l-1}) + 2(u_{k,l} - u_{k-1,l}))$$

$$\cancel{+ 2(v_{k,l+1} - v_{k,l})(-1) + 2(v_{k+1,l} - v_{k,l})(-1))}$$

$$\Rightarrow \frac{\partial J}{\partial u_{k,l}} \frac{1}{2} = (I^2_{x;k,l} + 4\lambda) u_{k,l} + I_{x;k,l} I_{y;k,l} v_{k,l}$$

$$+ I_{t;k,l} I_{x;k,l} - 4\lambda \bar{u}_{k,l}$$

where $4\bar{u}_{k,l} = (u_{k,l+1} + u_{k+1,l} + u_{k,l-1} + u_{k-1,l})$

Set $\frac{\partial J}{\partial u_{k,l}} = 0$

$$\Rightarrow (I^2_{x;k,l} + 4\lambda) u_{k,l} + I_{x;k,l} I_{y;k,l} v_{k,l}$$

$$= 4\lambda \bar{u}_{k,l} - I_{x;k,l} I_{t;k,l} \quad \text{— (1)}$$

Set $\frac{\partial J}{\partial v_{k,l}} = 0$ (Doing similarly as above)

$$\Rightarrow (I_{x;k,l} u_{k,l} + I_{y;k,l} v_{k,l} + I_{t;k,l})(I_{y;k,l})$$

$$+ \lambda((v_{k,l+1} - v_{k,l})(-1) + (v_{k+1,l} - v_{k,l})(-1)$$

$$+ (v_{k,l} - v_{k,l-1}) + (v_{k,l} - v_{k-1,l}))$$

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On rearranging

$$(I^2_{y;k,l} + 4\lambda)\, v_{k,l} + I_{x;k,l}\, I_{y;k,l}\, u_{k,l} = 4\lambda \bar{v}_{k,l} - I_{y;k,l}\, I_{t;k,l} \quad \text{— (2)}$$

where $\bar{v}_{k,l} = (v_{k,l+1} + v_{k+1,l} + v_{k,l-1} + v_{k-1,l})/4$

(1) & (2) are eq^n in $u_{k,l}$ & $v_{k,l}$

On solving, we get

$$\left(\frac{(I^2_{x;k,l} + 4\lambda)}{I_{x;k,l}\, I_{y;k,l}} - \frac{I_{x;k,l}\, I_{y;k,l}}{(I^2_{y;k,l} + 4\lambda)} \right) u_{k,l}$$

$$= \left(\frac{4\lambda \bar{u}_{k,l} - I_{x;k,l}\, I_{t;k,l}}{I_{x;k,l} \cdot I_{y;k,l}} \right) - \left(\frac{4\lambda \bar{v}_{k,l} - I_{y;k,l} \cdot I_{t;k,l}}{I^2_{y;k,l} + 4\lambda} \right)$$

$$\Rightarrow (I^2_{y;k,l} + I^2_{x;k,l} + 4\lambda)\, u_{k,l}\, 4\lambda$$

$$= 4\lambda(\bar{u}_{k,l}\, I^2_{y;k,l} + 4\lambda \bar{u}_{k,l} - I_{x;k,l}\, I_{t;k,l})$$

$$- 4\lambda\, \bar{v}_{k,l}\, I_{x;k,l}\, I_{y;k,l}$$

$$\Rightarrow u_{k,l} = \bar{u}_{k,l} - \frac{I_{x;k,l}(I_{x;k,l}\, \bar{u}_{k,l} + I_{y;k,l}\, \bar{u}_{k,l} + I_{t;k,l})}{I^2_{y;k,l} + I^2_{x;k,l} + 4\lambda}$$

Similary solving for $v_{k,l}$ we get

$$v_{k,l} = \bar{v}_{k,l} - \frac{I_{y;k,l}(I_{x;k,l}\, \bar{u}_{k,l} + I_{y;k,l}\, \bar{v}_{k,l} + I_{t;k,l})}{I^2_{y;k,l} + I^2_{y;k,l} + 4\lambda}$$

Now, verifying Jacobi update equation –

from (1) and (2), we can write it in matrix form

$$\underbrace{\begin{bmatrix} I_{x_{k,l}}^2 + 4\lambda & I_{x_{k,l}} I_{y_{k,l}} \\ I_{x_{k,l}} I_{y_{k,l}} & I_{y_{k,l}}^2 + 4\lambda \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_{k,l} \\ v_{k,l} \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 4\lambda \bar{u}_{k,l} - I_{x_{k,l}} I_{t_{k,l}} \\ 4\lambda \bar{v}_{k,l} - I_{y_{k,l}} I_{t_{k,l}} \end{bmatrix}}_{b}$$

$$A \downarrow$$

$$\underbrace{\begin{bmatrix} I_{x_{k,l}}^2 + 4\lambda & 0 \\ 0 & I_{y_{k,l}}^2 + 4\lambda \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & I_{x_{k,l}} I_{y_{k,l}} \\ I_{x_{k,l}} I_{y_{k,l}} & 0 \end{bmatrix}}_{R}$$

A/c Jacobi update, we can write

$$x^{(t+1)} = D^{-1}(b - Rx^{(t)})$$

$$\begin{bmatrix} u_{k,l}^{(t+1)} \\ v_{k,l}^{(t+1)} \end{bmatrix} = \begin{bmatrix} \frac{1}{I_{x_{k,l}}^2 + 4\lambda} & 0 \\ 0 & \frac{1}{I_{y_{k,l}}^2 + 4\lambda} \end{bmatrix} \left(\begin{bmatrix} 4\lambda \bar{u}_{k,l}^{(t)} - I_{x_{k,l}} I_{t_{k,l}} \\ 4\lambda \bar{v}_{k,l}^{(t)} - I_{y_{k,l}} I_{t_{k,l}} \end{bmatrix} - \begin{bmatrix} 0 & I_{x_{k,l}} I_{y_{k,l}} \\ I_{x_{k,l}} I_{y_{k,l}} & 0 \end{bmatrix} \begin{bmatrix} u_{k,l}^{(t)} \\ v_{k,l}^{(t)} \end{bmatrix} \right)$$

$$u_{k,l}^{(t+1)} = \frac{4\lambda \bar{u}_{k,l}^{(t)} - I_{x_{k,l}} I_{t_{k,l}} - I_{x_{k,l}} I_{y_{k,l}} v_{k,l}^{(t)}}{I_{x_{k,l}}^2 + 4\lambda}$$

$$v_{k,l}^{(t+1)} = \frac{4\lambda \bar{v}_{k,l}^{(t)} - I_{y_{k,l}} I_{t_{k,l}} - I_{x_{k,l}} I_{y_{k,l}} u_{k,l}^{(t)}}{I_{y_{k,l}}^2 + 4\lambda}$$