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5. Given : $p_{2i} = \alpha R p_{1i} + t + n_i$ — ①

Taking average of the above expression

$\Rightarrow \bar{p}_2 = \alpha R \bar{p}_1 + t$ (assuming n_i ~ 0 mean)

$\Rightarrow t = \bar{p}_2 - \alpha R \bar{p}_1$ — Ⓐ

For all i, eq^n ① can be written as :

$$P_2 = \alpha R P_1 + T + n$$

where $P_2 \rightarrow 2\times N$ $P_1 \rightarrow 2\times N$ $T \rightarrow 2\times N$ every column of T will be "t"

Define : $\tilde{P}_2 = P_2 - \bar{P}_2$ — ②

$\tilde{P}_1 = P_1 - \bar{P}_1$ — ③

Our objective is to minimize: $E = \| P_2 - \alpha R P_1 - T \|^2$

Putting value of P_1 & P_2 from ② & ③ we get

$E = \| \tilde{P}_2 + \bar{P}_2 - \alpha R \tilde{P}_1 - \alpha R \bar{P}_1 - T \|^2$

$\Rightarrow \| \tilde{P}_2 - \alpha R \tilde{P}_1 + \bar{P}_2 - \alpha R \bar{P}_1 - T \|^2$

$\Rightarrow \| \tilde{P}_2 - \alpha R \tilde{P}_1 \|^2$ $(\because T = \bar{P}_2 - \alpha R \bar{P}_1)$

$E = \text{trace}((\tilde{P}_2 - \alpha R \tilde{P}_1)^T (\tilde{P}_2 - \alpha R \tilde{P}_1))$

$= \text{trace}(\tilde{P}_2^T \tilde{P}_2 + \alpha^2 \tilde{P}_1^T R^T R \tilde{P}_1 - \alpha \tilde{P}_2^T R \tilde{P}_1 - \alpha \tilde{P}_1^T R^T \tilde{P}_2)$

$= \text{trace}(\alpha^2 \tilde{P}_1^T \tilde{P}_1 + \tilde{P}_2^T \tilde{P}_2) - 2\alpha\, \text{trace}(\tilde{P}_2^T R \tilde{P}_1)$ — ④

$(\because \text{trace}(A^T) = \text{trace}(A))$

$= x - y$

Now, we have to max. $y = 2\, \text{trace}(\alpha \tilde{P}_2^T R \tilde{P}_1)$

$= 2\, \text{trace}(R \alpha \tilde{P}_1 \tilde{P}_2^T)$

$= 2\alpha\, \text{trace}(R \tilde{P}_1 \tilde{P}_2^T)$

$= 2\alpha\, \text{trace}(R U' S' V'^T)$ $(\because \tilde{P}_1 \tilde{P}_2^T = U' S' V'^T)$

$= 2\alpha\, \text{trace}(S' V'^T R U')$

$= 2\alpha\, \text{trace}(S' X)$ → orthonormal

Since, S'_{ii} are all non-negative and the above expression is max. if $X_{ii} = 1$ $\forall i$. As X is orthonormal, $\therefore X = I \Rightarrow V'^T R U' = I$

$$\boxed{R = V' U'^T}$$

Putting the value of R in ④ we get

$E = \alpha^2 \text{trace}(\tilde{P}_1^T \tilde{P}_1) - 2\alpha\, \text{trace}(\tilde{P}_2^T R \tilde{P}_1) + \text{trace}(\tilde{P}_2^T \tilde{P}_2)$

$= a\alpha^2 - 2\alpha b + c$ $(\text{trace}(\tilde{P}_1^T \tilde{P}_1) > 0)$

this is a quadratic eq^n in α, which will be min. at $\alpha = \frac{b}{a}$ $(\because a > 0)$

Now, $\alpha = \dfrac{trace(\tilde{P}_2^T R \tilde{P}_1)}{trace(\tilde{P}_1^T \tilde{P}_1)}$

Now we have got α & R. We will use this in (A)

$$t = \bar{p}_2 - \alpha R \bar{p}_1$$

Finally, we got $R = V' U'^T$ (where, V' & U' is given by $\tilde{P}_1 \tilde{P}_2^T = U' S' V'^T$)

and $\alpha = \dfrac{trace(\tilde{P}_2^T R \tilde{P}_1)}{trace(\tilde{P}_1^T \tilde{P}_1)}$ & $\tilde{P}_1 = P_1 - \bar{P}_1$

and $t = \bar{p}_2 - \alpha R \bar{p}_1$ $\tilde{P}_2 = P_2 - \bar{P}_2$