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2 (a) $\|A(x - x^*)\|_2 \le 2\epsilon$ (Derive it)

Proof :

$\|A(x - x^*)\|_2 = \|Ax - Ax^*\|_2$

Given $y = Ax + n$ -(I) & $\|n\|_2 \le \epsilon$ -(II)

$\Rightarrow \|A(x-x^*)\|_2 = \|y - n - Ax^*\|_2$

$\le \|y - Ax^*\|_2 + \|-n\|_2$ (Triangle inequality)

From (I) & (II) $\|-n\|_2 = \|n\|_2$

$\|y - Ax\|_2 = \|n\|_2$

$\Rightarrow \|A(x - x^*)\|_2 \le \|n\|_2 + \|n\|_2$

$< 2\epsilon$

(c) (i) $\|h_{T_j}\|_2 \le s^{1/2}\|h_{T_j}\|_\infty \le s^{-1/2}\|h_{T_{j-1}}\|_1$

Prove the inequality

Given : any element of T_{j-1} is greater than or equal to any element of T_j for any $j \ge 1$ — (1)

$\|h_{T_j}\|_\infty$ = maximum element of h_{T_j}

There are total s non-zero entry in h

$$\underset{\substack{\downarrow \\ \text{sum of squares of} \\ \text{element of } h}}{\|h_{T_j}\|_2^2} \le \underbrace{\|h_{T_j}\|_\infty^2 + \dots \|h_{T_j}\|_\infty^2}_{s\text{ - times}}$$

$\Rightarrow \|h_{T_j}\|_2^2 \le s\|h_{T_j}\|_\infty^2$

$\Rightarrow \|h_{T_j}\|_2 \le s^{1/2}\|h_{T_j}\|_\infty$ -(2)

From (1), we conclude the following

$\|h_{T_j}\|_\infty \le$ least element of T_{j-1}

$\le \dfrac{\|h_{T_{j-1}}\|_1}{s}$ $\quad \|h\|_1 \downarrow$ L_1 norm = sum of elements

$\Rightarrow s^{1/2}\|h_{T_j}\|_\infty < s^{-1/2}\|h_{T_{j-1}}\|_1$ -(3)

Combine (2) & (3), we get

$\|h_{T_j}\|_2 \le s^{1/2}\|h_{T_j}\|_\infty \le s^{-1/2}\|h_{T_{j-1}}\|_1$

(ii) $\sum_{j\ge 2}\|h_{T_j}\|_2 \le s^{-1/2}\|h_{(T_0)^c}\|_1$ Prove this

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Proof: From part (i), we know that

$$\|h_{T_j}\|_2 \le s^{-1/2} \|h_{T_{j-1}}\|_1$$

Sum this inequality for $j \ge 2$

$$\sum_{j\ge 2} \|h_{T_j}\|_2 \le s^{-1/2} \sum_{j\ge 2} \|h_{T_{j-1}}\|_1$$

$$\Rightarrow \sum_{j\ge 2} \|h_{T_j}\|_2 \le s^{-1/2} \sum_{k=1}^{n-1} \|h_{T_k}\|_1$$

$$\le s^{-1/2} \sum_{k=1}^{n} \|h_{T_k}\|_1 \quad \text{(added one more term)}$$

$$= s^{-1/2} \|h_{T_1 \cup T_2 \cup \cdots T_n}\|_1$$

$$= s^{-1/2} \|h_{(T_0)^c}\|_1$$

Hence, we proved $\sum_{j\ge 2} \|h_{T_j}\|_2 \le s^{-1/2} \|h_{(T_0)^c}\|_1$

(iii) Proof: $\|h_{(T_0 \cup T_1)^c}\|_2 = \|\sum_{j\ge 2} h_{T_j}\|_2$

$$\le \sum_{j\ge 2} \|h_{T_j}\|_2 \quad \text{(Using Triangle Inequality)}$$

Using part (ii), $\Rightarrow \sum_{j\ge 2} \|h_{T_j}\|_2 \le s^{-1/2} \|h_{(T_0)^c}\|_1$

Hence proved.

(iv) Proof:

$$\sum_{i\in T_0} |x_i + h_i| + \sum_{i \in (T_0)^c} |x_i + h_i|$$

$$= \sum_{i\in T_0} |x_i - (-h_i)| + \sum_{i\in (T_0)^c} |x_i - (-h_i)|$$

$$\ge \sum_{i\in T_0} |x_i| - \sum_{i\in T_0} |h_i| + \sum_{i\in (T_0)^c} |x_i| - \sum_{i\in T_0} |h_i|$$

(Used Triangle Inequality

$$\Rightarrow \sum_{i\in T_0} |x_i + h_i| + \sum_{i\in (T_0)^c} |x_i + h_i| \ge \|x_{T_0}\|_1 - \|h_{T_0}\|_1 + \|h_{(T_0)^c}\|_1 - \|x_{(T_0)^c}\|_1$$

$B = (b_1, b_2, \dots b_n)$

$\Rightarrow \sqrt{\frac{b_1^2 + b_2^2 \dots b_n^2}{n}} \geq \frac{b_1 + \dots b_n}{n}$

$\Rightarrow \|B\|_2 \geq n^{-1/2} \|B\|_1$

(vi) Proof:

From (iii), we have the following relation

$\|h_{(T_0 \cup T_1)^c}\|_2 \leq s^{-1/2} \|h_{(T_0)^c}\|_1$

From (iv), we have $\|h_{(T_0 \cup T_1)^c}\|_2 \leq s^{-1/2} \|h_{(T_0)}\|_1 + 2s^{-1/2}\|x - x_s\|_1$

We know that $s^{-1/2} \|h_{(T_0)}\|_1 \leq \|h_{(T_0)}\|_2$ (Because RMS > AM)

$\Rightarrow s^{-1/2} (\|h_{(T_0)}\|_1 + 2\|x - x_s\|_1) \leq \|h_{(T_0)}\|_2 + 2s^{-1/2}\|x - x_s\|_1$

Hence Proved

(d) (ii) From the restricted isometry property of A we get

$\|Ah_{T_0 \cup T_1}\|_2 \leq \sqrt{1+\delta_{2s}} \|h_{T_0 \cup T_1}\|_2$ -① (all are (+)ve quantities)

We know from part (a)

$\|Ah\|_2 \leq 2\epsilon$ -② (all are (+)ve quantities)

Multiply ① & ②

$\|Ah_{T_0 \cup T_1}\|_2 \|Ah\|_2 < 2\epsilon \sqrt{1+\delta_{2s}} \|h_{T_0 \cup T_1}\|_2$ -③

We will cauchy-schwartz inequality

$\therefore |\langle Ah_{T_0 \cup T_1}, Ah \rangle| \leq \|Ah_{T_0 \cup T_1}\|_2 \|Ah\|_2$ -④

Combining ③ & ④ we get the final result

(iv) Proof: To prove this part we will use result of part d(i), (ii) & (iii)

$(1 - \delta_{2s}) \|h_{T_0 \cup T_1}\|_2^2 \leq \|Ah_{T_0 \cup T_1}\|_2^2$ (Because of RIP of A)

From d(iii), we have

$|\langle Ah_{T_i}, Ah_{T_j} \rangle| \leq \delta_{2s} \|h_{T_i}\|_2 \|h_{T_j}\|_2$ -①

From d(i), we have $\|Ah_{T_0 \cup T_1}\|_2^2 = \langle Ah_{T_0 \cup T_1}, Ah \rangle - \langle Ah_{T_0 \cup T_1}, \sum_{j \geq 2} Ah_{T_j} \rangle$ -②

From d(ii) we have

$|\langle Ah_{T_0 \cup T_1}, Ah \rangle| \leq 2\epsilon \sqrt{1+\delta_{2s}} \|h_{T_0 \cup T_1}\|_2$ -③

$\Rightarrow \langle Ah_{T_0 \cup T_1}, \sum_{j \geq 2} Ah_{T_j} \rangle \leq \delta_{2s} \|h_{T_0 \cup T_1}\|_2 \|\sum_{j \geq 2} h_{T_j}\|_2$ using ①

$\leq \delta_{2s} \|h_{T_0 \cup T_1}\|_2 \sum_{j \geq 2} \|h_{T_j}\|_2$

$\Rightarrow \|Ah_{T_0 \cup T_1}\|_2^2 \leq 2\epsilon\sqrt{1+\delta_{2s}} \|h_{T_0 \cup T_1}\|_2 + \delta_{2s}(\|h_{T_0}\|_2 + \|h_{T_1}\|_2) \sum_{j \geq 2} \|h_{T_j}\|_2$

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(e) $\|h_{T_0}\|_2 \le \|h_{T_0 \cup T_1} + h_{(T_0 \cup T_1)^c}\|_2$. Apply triangle Inequality

$(T_0 \cup T_1)$ & $(T_0 \cup T_1)^c$ are disjoint

$\Rightarrow \|h\|_2 \le \|h_{T_0 \cup T_1}\|_2 + \|h_{(T_0 \cup T_1)^c}\|_2$

We already have proved the following in c(vi)

$\|h_{(T_0 \cup T_1)^c}\|_2 \le \|h_{(T_0)}\|_2 + 2s^{-1/2}\|x - x_s\|_1$

So, we can write the following

$\|h\|_2 \le \|h_{T_0 \cup T_1}\|_2 + \|h_{(T_0 \cup T_1)^c}\|_2$

$\le \|h_{T_0 \cup T_1}\|_2 + \|h_{(T_0)}\|_2 + 2s^{-1/2}\|x - x_s\|_1$ — (3)

We know that $\|h_{(T_0)}\|_2 \le \|h_{T_0 \cup T_1}\|_2$

Thus eq^n (3) can be written as

$\le 2\|h_{T_0 \cup T_1}\|_2 + 2s^{-1/2}\|x - x_s\|_1$ — (4)

In d(vi), we have seen that

$\Rightarrow \|h_{T_0 \cup T_1}\|_2 \le C' \epsilon + C'' s^{-1/2}\|x - x_s\|_1$

Put it in ~~Rewrite~~ the eq^n (4) with the inequality

$\Rightarrow 2\|h_{T_0 \cup T_1}\|_2 + 2s^{-1/2}\|x - x_s\|_1$

$\le 2(C'\epsilon + C'' s^{-1/2}\|x - x_s\|_1) + 2s^{-1/2}\|x - x_s\|_1$

$= C_0 s^{-1/2}\|x - x_s\|_1 + C_1 \epsilon$

where $C_0 = \dfrac{2(1-\delta_{2s})}{1-(\sqrt{2}+1)\delta_{2s}}$ & $C_1 = \dfrac{4\sqrt{1+\delta_{2s}}}{1-(\sqrt{2}+1)\delta_{2s}}$

C_1 should always be positive because

$$\underbrace{\|h\|_2}_{\text{always +ve}} \le C_0 s^{-1/2} \underbrace{\|x - x_s\|_1}_{\text{This may be zero}} + C_1 \epsilon$$

→ should be +ve

$\Rightarrow 1 - (\sqrt{2}+1)\delta_{2s} \ge 0$

$\Rightarrow \delta_{2s} \le \dfrac{1}{\sqrt{2}+1}$

$\Rightarrow \delta_{2s} \le \sqrt{2} - 1$