CCE2203 Lab 2: Fourier Series

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Instructions

- This unit of assessment is to be attempted individually. It is essential that the work you submit and present consists only of your own work; use of copied material will be treated as plagiarism. Discussion is only permitted on general issues, and it is absolutely forbidden to discuss specific details with anyone.
- Your lab submission consists of the following deliverables:
  - A report, in the form of a Jupyter Notebook, submitted on the VLE as a single PDF file. The file needs to be less than 20 MiB in size. Be particularly careful with the sizes of any included images, which can easily cause the PDF to be too big.
  - A completed and signed Plagiarism Form for this unit of assessment.

  If any of these submissions is late, the whole unit of assessment will be considered a late submission, even if any part was submitted on time. Other methods of submission will not be considered.
- The report should be paginated on A4 paper, and exported directly as PDF from Jupyter notebook.
- Separate your responses for each question, including a Markdown header to separate the various parts.
- Use a sequence of code blocks, answering each question separately, rather than putting all the code in one big block. Questions need to be answered in sequence.
- Textual answers to any questions must be included in the report, as a Markdown cell. Each answer should appear immediately after the results to which it refers.
- In your submission, include only content that *directly* answers the questions asked. Submission of irrelevant material may lead to a reduction in the grade obtained.
- The deliverables are to be submitted on the VLE by the deadline specified there; late submissions will be rejected and assessed with a grade of zero.
- If there are extenuating circumstances which do not allow you to complete the unit of assessment on time, you are required to follow the procedure specified in the regulations.

Aims and Objectives

The aim of this laboratory session is to explore the use of the Fourier series to represent arbitrary periodic signals. We will see how the representation improves with increasing number of coefficients, both visually and also in numerical terms.

1 Fourier Synthesis

A given periodic function $f(t)$, with period $T = 4$, is represented by the Fourier series with coefficients

$$a_n = \begin{cases} 0, & n \text{ even}, \\ \frac{4}{\pi^2 n^2}, & n \text{ odd}. \end{cases} \tag{1.1}$$

1.1. Write a function `a(n)` that returns the coefficient a_n corresponding to the given index n. **[10 marks]**

1.2. Using this function, plot the coefficients a_n for $n \in [-10, 10]$. Label the axes clearly and make sure that tick marks are shown only for valid values. **[10 marks]**

1.3. The truncated Fourier synthesis is given by

$$\hat{f}(t, \omega_0, n_{\max}) = \sum_{k=-n_{\max}}^{n_{\max}} a_k e^{jk\omega_0 t}. \tag{1.2}$$

Obtain a simplified expression for $\hat{f}(t, \omega_0, n_{\max})$ in terms of trigonometric functions. **[10 marks]**

1.4. Write a function `fhat(t, omega0, nmax)` that returns the truncated Fourier synthesis for any array of time values `t`, fundamental angular frequency `omega0`, and truncation limit `nmax`. **[10 marks]**

1.5. Using this function, plot on a single set of axes the truncated Fourier synthesis for $T = 4$ and $n_{\max} = \{1, 3, 5, 7, 9\}$. Show the range $-2 \leq t \leq 2$. Label the axes clearly. **[10 marks]**

2 Error Analysis

It can be shown that the periodic function is given by

$$f(t) = \begin{cases} 1 - t, 0 \leq t \leq 2, \\ 1 + t, -2 \leq t \leq 0 \end{cases} \tag{2.1}$$

defined on the interval $[-2, 2]$.

2.1. Write a function `f(t)` that returns the periodic function for any array of time values `t`. **[10 marks]**

2.2. Using this function, plot the error of the truncated Fourier synthesis over the range $-2 \leq t \leq 2$, for $T = 4$ and $n_{\max} = \{1, 3, 5, 7, 9\}$. Label the axes clearly. **[10 marks]**

2.3. Write a function `E(nmax)` that returns the energy of the approximation error over one period, defined as

$$E(n_{\max}) = \int_T \left| f(t) - \hat{f}(t, n_{\max}) \right|^2 dt. \tag{2.2}$$

Hint: you may use the `trapz` *function from NumPy.* **[10 marks]**

2.4. Using this function, plot the energy of the approximation error against the number of coefficients, for $1 \leq n_{\max} \leq 20$. Label the axes clearly, using a suitable scale for the y-axis, and make sure that tick marks are shown only for valid values. **[10 marks]**

2.5. Comment on the results obtained. **[10 marks]**