B-S Model 之 Fourier's 解法

NO. 1.
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由(6)式

$$\omega_2 = r\omega - rx\omega_1 - \frac{1}{2}v^2x^2\omega_{11}$$

$$\omega(x, t^*) = x - c \qquad x \geq c$$
$$= 0 \qquad x < c$$

$x(t)$: share price $\quad$ $\omega(x,t)$: call price
t^* : maturity date $\quad$ c : strike price

Transform (6)

$$\omega(x,t) = e^{r(t-t^*)}\, y(u,s)$$

$$u = \left(\frac{2}{v^2}\right)\left(r - \frac{1}{2}v^2\right)\left[\ln\left(\frac{x}{c}\right) - \left(r - \frac{1}{2}v^2\right)(t - t^*)\right]$$

$$s = -\left(\frac{2}{v^2}\right)\left(r - \frac{1}{2}v^2\right)^2 (t - t^*)$$

$$\omega_1 = \frac{\partial \omega}{\partial u}\frac{\partial u}{\partial t} = \left[e^{r(t-t^*)} y_1\right]\left[\frac{\left(\frac{2}{v^2}\right)\left(r - \frac{v^2}{2}\right)}{x}\right]$$

$$\omega_2 = \frac{\partial \omega}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial \omega}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \omega}{\partial t}$$

$$= \left[e^{r(t-t^*)} y_2 \frac{\partial s}{\partial t}\right] + \left[e^{r(t-t^*)} y_1 \frac{\partial u}{\partial t}\right] + \left[r e^{r(t-t^*)} y\right]$$

$$= e^{r(t-t^*)}\left[-y_1\left(\frac{2}{v^2}\right)\left(r - \frac{v^2}{2}\right)^2 - y_2\left(\frac{2}{v^2}\right)\left(r - \frac{v^2}{2}\right)^2 + ry\right]$$

$$\omega_{11} = e^{r(t-t^*)}\left[\frac{-y_1\left(\frac{2}{v^2}\right)\left(r-\frac{v^2}{2}\right)}{x^2} + \frac{y_{11}\left(\frac{2}{v^2}\right)^2\left(r-\frac{v^2}{2}\right)^2}{x^2}\right]$$

代入 (6) 式

$$y_2 = y_{11}$$

with $c = 1$

initial condition : $y(u,0) = \omega(x, t^*) = x - c \quad x \geq c$

$= 0 \quad x < c$

Chose $y(u,0) = c\left[e^{u\left(\frac{1}{2}v^2\right)/\left(r-\frac{1}{2}v^2\right)} - 1\right] \quad u \geq 0$

$= 0 \quad u < 0$

at $t = t^*$

$y(u^*, 0) = c\left[e^{\ln\left(\frac{x}{c}\right)} - 1\right] = x - c \quad u \geq 0$

$= 0 \quad u < 0.$

$f(u) = y(u,0) = c\left[e^{u\left(\frac{1}{2}v^2\right)/\left(r-\frac{1}{2}v^2\right)} - 1\right] \quad u \geq 0$

$= 0 \quad u < 0.$

$$y(u,s) = \frac{1}{\sqrt{\pi}}\int_{-\frac{u}{2\sqrt{s}}}^{\infty} c\left[e^{(u+2\eta\sqrt{s})\frac{1}{2}v^2/\left(r-\frac{1}{2}v^2\right)} - 1\right] e^{-\eta^2}\, d\eta$$

$\because \; u + 2\eta\sqrt{s} \geq 0 \Rightarrow \eta \geq -\frac{u}{2\sqrt{s}}$

NO. 2.1

DATE

$$W_2 = e^{r(t-t^*)}\left[-Y_1\left(\frac{2}{v^2}\right)\left(r-\frac{v^2}{2}\right)^2 - Y_2\left(\frac{2}{v^2}\right)\left(r-\frac{v^2}{2}\right)^2 + rY\right]$$

$$= r \cdot e^{r(t-t^*)} Y - rX\, e^{-r(t-t^*)} \frac{\left(\frac{2}{v^2}\right)\left(r-\frac{v^2}{2}\right)}{X} Y_1$$

$$- \frac{1}{2} v^2 X^2 e^{r(t-t^*)} \left[\frac{Y_1\left(\frac{2}{v^2}\right)\left(r-\frac{v^2}{2}\right)}{X^2} + \frac{Y_{11}\left(\frac{2}{v^2}\right)^2\left(r-\frac{v^2}{2}\right)^2}{X^2}\right]$$

$$-Y_1\left(\frac{2}{v^2}\right)\left(r-\frac{v^2}{2}\right)^2 = -rX \cdot \frac{\left(\frac{2}{v^2}\right)\left(r-\frac{v^2}{2}\right)}{X} Y_1$$

$$+ \frac{1}{2} v^2 X^2 \frac{Y_1\left(\frac{2}{v^2}\right)\left(r-\frac{v^2}{2}\right)}{X^2}$$

$$-\left(r-\frac{v^2}{2}\right)^2 = -r\left(r-\frac{v^2}{2}\right) + \frac{1}{2} v^2 \left(r-\frac{v^2}{2}\right)$$

$$-\left(r-\frac{v^2}{2}\right) = -r + \frac{1}{2} v^2 \quad \#$$

(1) $u_t(x,t) = k\,u_{xx}(x,t) \qquad (-\infty < x < \infty,\ t > 0)$

$u(x,0) = f(x) \qquad (-\infty < x < \infty)$

Separation of Variables:
It applies to the solution of initial and/or boundary value problems for homogeneous versions of the equations and boundary conditions.

$$u(x,t) = M(x)\,N(t) \quad \text{代入 (1) 式}$$

$$M(x)\,N'(t) = k M''(x) N(t)$$

$$\frac{N'(t)}{kN(t)} = \frac{M''(x)}{M(x)} = -\lambda = -r^2$$

$$\begin{cases} N'(t) + k\lambda N(t) = 0 & \text{------- (2)} \\ M''(x) + \lambda M(x) = 0. & \text{------- (3)} \end{cases}$$

For each eigenvalue λ_k, we obtain an equation for N_k.
In (2), (3) 式, assume $\lambda_k > 0$. for all k.

$$N_k(t) = a_k \exp(-\lambda_k k t)$$

$$u_k = M_k \cdot N_k, \qquad (M_k, M_j) = 0 \ \text{for } k \neq j$$

$$*\ (f, g) = \iint_G \rho f \cdot g \, dv.$$

(f, g) : inner product of f and g, defined and integrable over the region G.

Superposition of u_k.

The spectrum (i.e., the set of eigenvalues) is continuous in the unbounded case; whereas in the bounded case the spectrum is discrete.

Discrete form:

$$u(x,t) = \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} M_k N_k.$$

$$u(x,0) = \sum_k M_k(x) N_k(0) = \sum_k a_k M_k(x) = f(x)$$

$$a_k = \frac{(f(x), M_k(x))}{(M_k(x), M_k(x))}$$

$$u(x,t) = \sum_k a_k \exp(-\lambda_k k t) M_k(x)$$

Continuous form:

In the case of a continuous spectrum, the eigenvalues λ range over the set D, which may be the interval $-\infty < \lambda < \infty$, $0 \le \lambda < \infty$, or some other uncountable case.

$$u = \int_D u(\lambda)\, d\lambda = \int_D M(\lambda) N(\lambda)\, d\lambda.$$

Replace λ by γ^2

$$M''(x) + \gamma^2 M(x) = 0 \qquad -\infty < x < \infty$$

$|M(x)|$ bounded as $x \to \infty$

this implies γ is real;

$-\infty < \gamma < \infty$

To

$$M''(x) + \gamma^2 M(x) = 0.$$

general solution

$$M(x;\gamma) = \alpha(\gamma)\, e^{i\gamma x} + \beta(\gamma)\, e^{-i\gamma x}$$

Define :

$$f(x) = \int_{-\infty}^{\infty} [\alpha(\gamma)\, e^{i\gamma x} + \beta(\gamma)\, e^{-i\gamma x}]\, d\gamma$$

$$= \int_{-\infty}^{\infty} \theta(\gamma)\, e^{-i\gamma x}\, d\gamma \text{ ---------- } (4)$$

$$\theta(\gamma) = \beta(\gamma) + \alpha(-\gamma)$$

Fourier Integral Formula.

Define $F(r) = \mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{irx} f(x)\, dx$

$$f(x) = \mathcal{F}^{-1}\{F(r)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-irx} F(r)\, dr. \text{ - - - - - - } (5)$$

By comparing (4) (5), we conclude that

$$\theta(r) = \frac{1}{\sqrt{2\pi}} F(r)$$

Properties of Fourier Transform.

i $\mathcal{F}^{-1}\{F(r) \cdot G(r)\} = f(x) * g(x)$

ii $\mathcal{F}\{f'(x)\} = -ir\, \mathcal{F}\{f(x)\}$

iii $\mathcal{F}\{f^{n}(x)\} = (-ir)^{n}\, \mathcal{F}\{f(x)\}$

iv $\int_{-\infty}^{\infty} |F(r)|^2\, dr = \int_{-\infty}^{\infty} |f(x)|^2\, dx$

Ex. $u_t - c^2 u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0$ ------- (6)

$u(x, 0) = f(x), \quad -\infty < x < \infty$ -------- (7)

$$U(r,t) = \mathcal{F}\{u(x,t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{irx} u(x,t)\, dx$$

(6) $\times \frac{1}{\sqrt{2\pi}} e^{irx}$ and integrate with respect to x from $-\infty$ to ∞

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t e^{irx}\, dx + (cr)^2 U(r,t)$$

$$= \frac{\partial}{\partial t} U(r,t) + (cr)^2 U(r,t) = 0 \text{ ------- (8)}$$

[By property (iii)]

$$U(r,0) = F(r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{irx} f(x)\, dx \text{ ----- (9)}$$

$$\begin{cases} \frac{\partial}{\partial t} U(r,t) + (cr)^2 U(r,t) = 0 \\ U(r,0) = F(r) \end{cases}$$

$$U(r,t) = F(r) \exp\left[-(cr)^2 t\right]$$

$$u(x,t) = \mathcal{F}^{-1}\{U(r,t)\}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-irx - r^2c^2t}\, F(r)\, dr.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ir(x-s) - r^2c^2t} f(s)\, dr\, ds \quad **$$

$$\int_{-\infty}^{\infty} e^{-ir(x-s)-r^2c^2t}\,dr = \int_{-\infty}^{0} e^{-ir(x-s)-r^2c^2t}\,dr + \int_{0}^{\infty} e^{-ir(x-s)-r^2c^2t}\,dr.$$

$$= 2\int_0^{\infty} e^{-r^2c^2t}\cos[r(x-s)]\,dr$$

define :

$$I(\alpha) = 2\int_0^{\infty} e^{-r^2c^2t}\cos[\alpha r]\,dr.$$

$$\frac{dI(\alpha)}{d\alpha} = -\frac{\alpha}{2c^2t} I(\alpha) \quad \text{------ (10)}$$

$$I(0) = 2\int_0^{\infty} e^{-r^2c^2t}\,dr = \sqrt{\frac{\pi}{c^2t}} \quad \text{------ (11)}$$

from (10) (11), $\displaystyle I(\alpha)\Big|_{\alpha=x-s} = \sqrt{\frac{\pi}{c^2t}}\exp\left[-\frac{(x-s)^2}{4c^2t}\right]$ ---- (12)

Substitute (12) to $**$

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2t}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x-s)^2}{4c^2t}\right] f(s)\,ds.$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2\sigma\sqrt{c^2t})\exp(-\sigma^2)\,d\sigma \quad \#\#$$

$$s = x+2\sigma\sqrt{c^2t}$$

令 $\eta = \frac{q}{\sqrt{2}}$

$$y(u,s) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{u}{\sqrt{2s}}}^{\infty} C\left[e^{(u+2\eta\sqrt{s})(\frac{1}{2}v^2)/(r-\frac{1}{2}v^2)} - 1\right] e^{-\frac{q^2}{2}} \, dq$$

令 $-\frac{u}{\sqrt{2s}} = -d_2$

$$W(x,t) = e^{r(t-t^*)} \, y(u,s)$$

$$W(x,t) = \frac{1}{\sqrt{2\pi}} C e^{r(t-t^*)} \int_{-d_2}^{\infty} e^{\{[(u+q\sqrt{2s})(\frac{1}{2}v^2)/(r-\frac{1}{2}v^2)] - \frac{q^2}{2}\}} \, dq$$

$$- \frac{C e^{r(t-t^*)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{q^2}{2}} \, dq$$

2nd term:

$$\frac{C e^{r(t-t^*)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{q^2}{2}} \, dq = C e^{r(t-t^*)} [1 - N(-d_2)]$$

$$= C e^{r(t-t^*)} N(d_2)$$

1st term:

$$\frac{1}{\sqrt{2\pi}} C e^{r(t-t^*)} \int_{-d_2}^{\infty} e^{[u\cdot(\frac{1}{2}v^2)/(r-\frac{1}{2}v^2)]} \cdot e^{[q\sqrt{2s}(\frac{1}{2}v^2)/(r-\frac{1}{2}v^2)]} \cdot e^{-\frac{q^2}{2}} \, dq$$

$$= e^{r(t-t^*)} C e^{\ln(\frac{x}{C})} \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-r(t-t^*)} e^{[-\frac{q^2}{2} + qv\sqrt{t^*-t} - \frac{1}{2}v^2(t^*-t)]} \, dq$$

$$= x \cdot \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}[q^2 - 2qv\sqrt{t^*-t} + v^2(t^*-t)]} \, dq$$

$$= x \cdot \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}q'^2} \, dq' \qquad q' = q - v\sqrt{t^*-t}$$

$$= x[1 - N(-d_1)]$$

$$= x[1 - 1 + N(d_1)]$$

$$= x\,N(d_1) \qquad d_1 = d_2 + v\sqrt{t^* - t}$$

$$W(x,t) = x\,N(d_1) - c e^{r(t-t^*)} N(d_2) \quad \#$$

reference : 1. Partial Differential Equations of Applied Mathematics

Eric Zauderer 1983.

2. Fourier Series and Boundary Value Problems 4th ed.

Churchill, R. V