模擬法在財務工程上的應用

Monte Carlo Methods in Financial Engineering

昀騰金融科技

技術長

董夢雲 博士

dongmy@ms5.hinet.net

大　綱

一、VBA概要

二、期貨與選擇權基礎

三、模擬介紹

四、布朗運動與伊藤補理

五、Black-Scholes模型與選擇權定價

六、產生隨機變數

七、風險管理的應用

八、變異數縮減技巧

九、路徑相關選擇權

十、多資產選擇權

十一、利率模型

十二、馬可夫蒙地卡羅法

第五章、Black-Scholes 模型與選擇權定價

(一)前言

- Black-Scholes(1973), Merton(1973)各自推導了選擇權定價公式。
  - Ito's Lemma 與 No arbitrage 條件。
  - Robert Merton 有專書：Continuous Finance，以連續模型重寫整個財務理論。

- Call Option(歐式)定義：
  - 買方有權利，在到期日(T)，以一定的價格(K)，買入特定資產(S)的權利。期初付權利金。
  - 賣方需配合執行。期初收權利金。

- Put Option(歐式)定義：
  - 買方有權利，在到期日(T)，以一定的價格(K)，賣出特定資產(S)的權利。期初付權利金。
  - 賣方需配合執行。期初收權利金。

- 三種不同執行區間：
  - 歐式
  - 美式
  - 百慕達

Four basic option positions are possible:

1. A long position in a call option. Payoff = $\max(S_T - K, 0)$.
2. A long position in a put option. Payoff = $\max(K - S_T, 0)$.

3. A short position in a call option. Payoff = $-\max(S_T - K, 0)$.
4. A short position in a put option. Payoff = $-\max(K - S_T, 0)$.

(二)單期二項式模型

- 選擇權一期後到期。

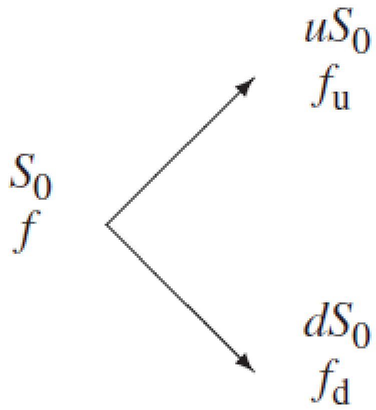

Figure 5.1 One period binomial tree.

- 一期後資產價格只有兩個狀態，up or down。
  - 另 u > 1，d < 1。
  - 期末選擇權價格可知。

$$f_u = \max(Su - K, 0) \text{ and } f_d = \max(Sd - K, 0) \text{ for a call option.}$$

- 期初建立一個避險組合，
  - 買Δ單位股票，放空一單位 Call。

    $\Delta S + f$

- 期末避險組合價值，

$$\begin{aligned} &\Delta u S_0 - f_u, \qquad \text{if stock moves up,} \\ &\Delta d S_0 - f_d, \qquad \text{if stock moves down.} \end{aligned}$$

  - 市場存在無風險利率，r。
  - 適當選擇Δ，此組合無風險，期末報酬皆相等。

$$\Delta u S_0 - f_u = \Delta d S_0 - f_d.$$

  - Δ解得，

$$\Delta = \frac{f_u - f_d}{u S_0 - d S_0}.$$

◆ 期初組合價值，等於期末組合價值折現。

$$S_0\Delta - f = (\Delta u S_0 - f_u)e^{-rT}.$$

➢ 解得期初選擇權價格，

$$\begin{aligned}
f &= S_0\Delta - (\Delta u S_0 - f_u)e^{-rT} \\
&= S_0\Delta(1 - ue^{-rT}) + f_u e^{-rT} \\
&= \frac{f_u - f_d}{u - d}(1 - ue^{-rT}) + f_u e^{-rT} \\
&= e^{-rT}\left[e^{rT}\frac{f_u - f_d}{u - d}(1 - ue^{-rT}) + f_u\right] \\
&= e^{-rT}\left(e^{rT}\frac{f_u - f_d}{u - d} - u\frac{f_u - f_d}{u - d} + f_u\frac{u - d}{u - d}\right) \\
&= e^{-rT}(f_u\frac{e^{rT} - d}{u - d} + f_d\frac{u - e^{rT}}{u - d})
\end{aligned}$$

- 令 $p = \dfrac{e^{rT} - d}{u - d}$ ，
  - 期初價格為，

$$f = e^{-rT}\left[p \cdot f_u + (1-p) \cdot f_d\right]$$

- 若 p 表風險中立下的上漲機率，則 f 可解釋為風險中立下的期望值。

$$f = e^{-rT}\hat{\mathrm{E}}(f) = e^{-rT}(pf_u + (1-p)f_d),$$

  - 適用於標的股票

$$\begin{aligned}
\hat{\mathrm{E}}(S_1) &= puS_0 + (1-p)dS_0 \\
&= pS_0(u-d) + dS_0 \\
&= \frac{e^{rT} - d}{u - d} S_0(u-d) + dS_0 \\
&= e^{rT} S_0.
\end{aligned}$$

Example 5.1 *Suppose the current price of one share of a stock is \$20 and in a period of 3 months, the price will be either \$22 or \$18. Suppose we sold a European call option with a strike price of \$21 in 3 months. Let the annual risk-free rate be 12% and let p denote the probability that the stock moves up in 3 months in the risk-neutral world. Note that the payoff of the option is either $f_u = \$1$ if the stock moves up or $f_d = \$0$ if the stock moves down. How much is the option, f, worth today? To find f, we can use the risk-neutral valuation method. Recall that from the aforementioned discussion,*

$$22p + 18(1-p) = 20e^{0.12/4},$$

so that $p = 0.6523$. *Using the expected payoff of the option, we get*

$$\hat{\mathrm{E}}(f) = pf_u + (1-p)f_d = p + (1-p)0 = p = 0.6523.$$

Therefore, the value of the option for today is

$$f = e^{-rT}\hat{\mathrm{E}}(f) = e^{-0.12/4}(pf_u + (1-p)f_d) = 0.633.$$

Example 5.2 *With the same parameters as in the preceding example, consider solving for Δ. Firstly, as we want a risk-free profit for the hedging portfolio, we want to purchase Δ shares of the stock and short one European call option expiring in 3 months. After 3 months, the value of the portfolio can be either*

$$22\Delta - 1, \text{ if the stock price moves to \$22,}$$

$$\text{or}$$

$$18\Delta, \text{ if the stock price moves to \$18.}$$

This portfolio is risk free if Δ is chosen so that the value of the portfolio remains the same for both alternatives, that is,

$$22\Delta - 1 = 18\Delta \quad \text{which means} \quad \Delta = 0.25.$$

The value of the portfolio in 3 months becomes

$$22 \times 0.25 - 1 = 4.5 = 18 \times 0.25.$$

By the no arbitrage consideration, this risk-free profit must earn the risk-free interest rate. In other words, the value of the portfolio today must equal the present value of \$4.5, that is, $4.5e^{-0.12/4} = 4.367$*. If the value of the option today is denoted by* f*, then the present value of the portfolio equals*

$$20 \times 0.25 - f = 4.5e^{-0.12/4} = 4.367.$$

Solving for f *gives*

$$f = 0.633,$$

which matches with the answer of the preceding example.

(三)BSM Equation

◆ 股價價格行為

$$dS = \mu S\,dt + \sigma S\,dW, \qquad (5.1)$$

◆ 債券價格行為

$$dB = rB\,dt. \qquad (5.2)$$

Theorem 5.1 *Using the notation just defined, and assuming that the price and the bond are described by the geometric Brownian motion* (Eq. 5.1) *and the compound interest rate model* (Eq. 5.2), *respectively, the price of the derivative of this security satisfies*

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf. \qquad (5.3)$$

- 選擇權價格，為 S、t 的函數，

$$f = f(S,t)$$

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial f}{\partial S}\sigma S\, dW.$$

- 形成一個動態組合 G，來複製選擇權的價格行為，

$G = x_t S_t + y_t B_t$, $dG = x_t dS_t + y_t dB_t$

$$\begin{aligned} dG &= x_t\, dS + y_t\, dB \\ &= x_t(\mu S\, dt + \sigma S\, dW) + y_t rB\, dt \\ &= (x_t \mu S + y_t rB)\, dt + x_t \sigma S\, dW. \end{aligned}$$

  - Match 隨機項，

$$x_t = \frac{\partial f}{\partial S}.$$

◆ 由於，$G = x_t S_t + y_t B_t$，G=f。

$$y_t = \frac{1}{B(t)}(G(t) - x_t S(t)).$$

$$y_t = \frac{1}{B(t)}\left(f(S,t) - \frac{\partial f}{\partial S} S(t)\right).$$

$$\frac{\partial f}{\partial S}\mu S + \frac{1}{B(t)}\left(f(S,t) - \frac{\partial f}{\partial S} S(t)\right) rB(t) = \frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2.$$

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf.$$

◆ 另一種導法，

5. Alternatively, we can derive Equation 5.3 as follows. Construct a portfolio that consists of shorting one derivative and longing $\frac{\partial f}{\partial S}$ shares of the stock. Let the value of this portfolio be Π and let the value of the derivative be $f(S, t)$. Then

$$\Pi = -f + \frac{\partial f}{\partial S} S. \tag{5.6}$$

The change $\Delta\Pi$ in the value of this portfolio in the time interval Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S. \tag{5.7}$$

Recall that S follows a geometric Brownian motion so that

$$\Delta S = \mu S \,\Delta t + \sigma S \,\Delta W.$$

In addition, from Equation 5.4, the discrete version of df is

$$\Delta f = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \,\Delta W.$$

Substituting these two expressions into Equation 5.7, we get

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\,\sigma^2 S^2 \right) \Delta t. \tag{5.8}$$

Note that by holding such a portfolio, the random component ΔW has been eliminated completely. Because this equation does not involve ΔW, this portfolio must equal to the risk-free rate during the time Δt. Consequently,

$$\Delta\Pi = r\Pi\,\Delta t,$$

where r is the risk-free rate. In other words, using Equation 5.8 and Equation 5.6, we obtain

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\,\sigma^2 S^2 \right) \Delta t = r(f - \frac{\partial f}{\partial S} S)\,\Delta t.$$

Therefore,

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\,rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\,\sigma^2 S^2 = rf.$$

Example 5.3 *Let f denote the price of a forward contract on a non-dividend-paying stock with delivery price K and delivery date T. Its price at time t is given by*

$$f(S,t) = S - Ke^{-r(T-t)}. \tag{5.9}$$

Hence,

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}, \quad \frac{\partial f}{\partial S} = 1, \quad \textit{and} \quad \frac{\partial^2 f}{\partial S^2} = 0.$$

Substituting these into Equation 5.3, *we get*

$$-rKe^{-r(T-t)} + rS = rf.$$

Thus, the price formula of f given by Equation 5.9 *is a solution of the Black–Scholes equation, indicating that* Equation 5.9 *is the correct formula.*

◆ Risk-Neutral 模擬法的合理性

Theorem 5.2 *Consider a European option with payoff $F(S)$ and expiration time T. Suppose that the continuous compounding interest rate is r. Then, the current European option price is determined by*

$$f(S,0) = e^{-rT}\widehat{\mathrm{E}}[F(S_T)], \tag{5.10}$$

where $\widehat{\mathrm{E}}$ *denotes the expectation under the risk-neutral probability that is derived from the risk-neutral process*

$$\frac{dS}{S} = r\,dt + \sigma\,dW(t). \tag{5.11}$$

◆ 證明：

➢ 有另一隨機過程$\{X_t\}$，有如下條件，

$$X_0 = S \quad \text{and} \quad \frac{dX_t}{X_t} = r\,dt + \sigma\,dW(t).$$

✓ 注意，選擇權的當前價格 f(S, t)為股價 S 與時間 t 的確定函數。故 f(S, 0) = f(X, 0)

➢ 根據 Ito's Lemma，可知 f 的微分形式如下，

$$df = \left(\frac{\partial f}{\partial t} + rX\frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2X^2\frac{\partial^2 f}{\partial X^2}\right)dt + \sigma X\frac{\partial f}{\partial X}\,dW.$$

➢ 由 Theorem 5.1 可知，dt 的係數為 r · f，因此可得，

$$df = rf\,dt + \sigma X\frac{\partial f}{\partial X}\,dW,$$

✓ 亦即，

$$df - rf\,dt = \sigma X\frac{\partial f}{\partial X}\,dW.$$

- 左式可表示為微分的乘法法則，

$$e^{rt}\, d\left[e^{-rt} f(X,t)\right] = \sigma X \frac{\partial f}{\partial X}\, dW.$$

  - ✓ 對應的積分表達式，

$$e^{-rT} f(X_T, T) - f(X, 0) = \sigma \int_0^T e^{-rt} X \frac{\partial f}{\partial X}\, dW.$$

- 右式為高斯過程之累加，其值為零。取期望值。

$$\hat{\mathrm{E}}\left[e^{-rT} f(X_T, T) - f(X, 0)\right] = 0.$$

$$f(X, 0) = e^{-rT}\, \hat{\mathrm{E}}[f(X_T, T)].$$

- 根據 BS 公式的期末條件，$f(X_T, T) = F(X_T)$，可得。

$$f(S, 0) = e^{-rT}\, \hat{\mathrm{E}}[F(X_T)],$$

(四)BS Formula

Lemma 5.1 *Let S be a lognormally distributed random variable such that $\log S \sim \mathrm{N}(m, v^2)$ and let $K > 0$ be a given constant. Then*

$$\mathrm{E}(\max\{S-K, 0\}) = \mathrm{E}(S)\Phi(d_1) - K\Phi(d_2), \tag{5.12}$$

where $\Phi(\cdot)$ denotes the distribution function of a standard normal random variable and

$$d_1 = \frac{1}{v}(-\log K + m + v^2) = \frac{1}{v}\left(\log \mathrm{E}\left(\frac{S}{K}\right) + \frac{v^2}{2}\right),$$

$$d_2 = \frac{-\log K + m}{v} = \frac{1}{v}\left(\log \mathrm{E}\left(\frac{S}{K}\right) - \frac{v^2}{2}\right).$$

Theorem 5.3 *Consider a European call option with strike price K and expiration time T. If the underlying stock pays no dividends during the time* $[0, T]$ *and if there is a continuously compounded risk-free rate r, then the price of this contract at time* 0, $f(S, 0) = C(S, 0)$, *is given by*

$$C(S, 0) = S\,\Phi(d_1) - Ke^{-rT}\Phi(d_2), \tag{5.15}$$

where $\Phi(x)$ *denotes the cumulative distribution function of a standard normal random variable evaluated at the point x,*

$$d_1 = [\log(S/K) + (r + \sigma^2/2)T]\frac{1}{\sigma\sqrt{T}},$$

$$d_2 = [\log(S/K) + (r - \sigma^2/2)T]\frac{1}{\sigma\sqrt{T}}$$

$$= d_1 - \sigma\sqrt{T}.$$

Example 5.4 *Consider a 5-month European call option on an underlying stock with a current price of* \$62, *strike price* \$60, *annual risk-free rate* 10%, *and the volatility of this stock is* 20% *per year. In this case,* $S = 62$, $K = 60$, $r = 0.1$, $\sigma = 0.2$, *and* $T = \frac{5}{12}$. *Applying* Equation 5.15, *we get*

$$d_1 = \frac{1}{0.2\sqrt{5/12}}\left[\log\left(\frac{62}{60}\right) + \left(0.1 + \frac{0.2^2}{2}\right)\frac{5}{12}\right]$$

$$= 0.641287,$$

$$d_2 = d_1 - 0.2\sqrt{5/12} = 0.512188.$$

From the normal table, we get $\Phi(d_1) = 0.739332$ *and* $\Phi(d_2) = 0.695740$. *Consequently,*

$$C = (62)(0.739332) - (60)e^{-(0.1)(5/12)}(0.695740) = 5.798.$$