

模擬法在財務工程上的應用

Monte Carlo Methods in Financial Engineering

昀騰金融科技

技術長

董夢雲 博士

dongmy@ms5.hinet.net

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第五章、Black-Scholes 模型與選擇權定價

(一)前言

◆ Black-Scholes(1973), Merton(1973)各自推導了選擇權定價公式。

- Ito's Lemma 與 No arbitrage 條件。
- Robert Merton 有專書：Continuous Finance，以連續模型重寫整個財務理論。

◆ Call Option(歐式)定義：

- 買方有權利，在到期日(T)，以一定的價格(K)，買入特定資產(S)的權利。期初付權利金。
- 賣方需配合執行。期初收權利金。

◆ Put Option(歐式)定義：

- 買方有權利，在到期日(T)，以一定的價格(K)，賣出特定資產(S)的權利。期初付權利金。
- 賣方需配合執行。期初收權利金。

◆ 三種不同執行區間：

- 歐式
- 美式
- 百慕達

Four basic option positions are possible:

1. A long position in a call option. Payoff = $\max(S_T - K, 0)$.
2. A long position in a put option. Payoff = $\max(K - S_T, 0)$.
3. A short position in a call option. Payoff = $-\max(S_T - K, 0)$.
4. A short position in a put option. Payoff = $-\max(K - S_T, 0)$.

(二)單期二項式模型

◆ 選擇權一期後到期。

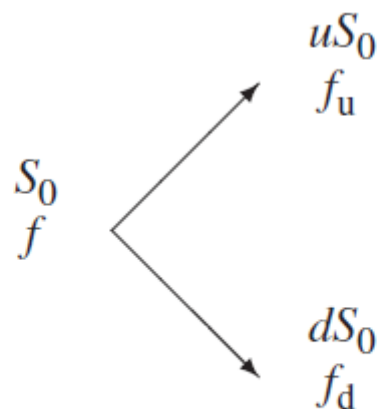


Figure 5.1 One period binomial tree.

◆ 一期後資產價格只有兩個狀態，up or down。

- 另 $u > 1$ ， $d < 1$ 。
- 期末選擇權價格可知。

$$f_u = \max(Su - K, 0) \text{ and } f_d = \max(Sd - K, 0) \text{ for a call option.}$$

◆ 期初建立一個避險組合，

- 買 Δ 單位股票，放空一單位 Call。

$$\Delta S + f$$

◆ 期末避險組合價值，

$$\Delta uS_0 - f_u, \quad \text{if stock moves up,}$$

$$\Delta dS_0 - f_d, \quad \text{if stock moves down.}$$

- 市場存在無風險利率， r 。
- 適當選擇 Δ ，此組合無風險，期末報酬皆相等。

$$\Delta uS_0 - f_u = \Delta dS_0 - f_d.$$

- Δ 解得，

$$\Delta = \frac{f_u - f_d}{uS_0 - dS_0}.$$

◆ 期初組合價值，等於期末組合價值折現。

$$S_0\Delta - f = (\Delta uS_0 - f_u)e^{-rT}.$$

➤ 解得期初選擇權價格，

$$\begin{aligned} f &= S_0\Delta - (\Delta uS_0 - f_u)e^{-rT} \\ &= S_0\Delta(1 - ue^{-rT}) + f_ue^{-rT} \\ &= \frac{f_u - f_d}{u - d}(1 - ue^{-rT}) + f_ue^{-rT} \\ &= e^{-rT} \left[e^{rT} \frac{f_u - f_d}{u - d}(1 - ue^{-rT}) + f_u \right] \\ &= e^{-rT} \left(e^{rT} \frac{f_u - f_d}{u - d} - u \frac{f_u - f_d}{u - d} + f_u \frac{u - d}{u - d} \right) \\ &= e^{-rT} \left(f_u \frac{e^{rT} - d}{u - d} + f_d \frac{u - e^{rT}}{u - d} \right) \end{aligned}$$

◆ 令 $p = \frac{e^{rT} - d}{u - d}$,

➤ 期初價格為 ,

$$f = e^{-rT} [p \cdot f_u + (1 - p) \cdot f_d]$$

◆ 若 p 表風險中立下的上漲機率，則 f 可解釋為風險中立下的期望值。

$$f = e^{-rT} \hat{E}(f) = e^{-rT} (p f_u + (1 - p) f_d),$$

➤ 適用於標的股票

$$\begin{aligned} \hat{E}(S_1) &= p u S_0 + (1 - p) d S_0 \\ &= p S_0 (u - d) + d S_0 \\ &= \frac{e^{rT} - d}{u - d} S_0 (u - d) + d S_0 \\ &= e^{rT} S_0. \end{aligned}$$

Example 5.1 Suppose the current price of one share of a stock is \$20 and in a period of 3 months, the price will be either \$22 or \$18. Suppose we sold a European call option with a strike price of \$21 in 3 months. Let the annual risk-free rate be 12% and let p denote the probability that the stock moves up in 3 months in the risk-neutral world. Note that the payoff of the option is either $f_u = \$1$ if the stock moves up or $f_d = \$0$ if the stock moves down. How much is the option, f , worth today? To find f , we can use the risk-neutral valuation method. Recall that from the aforementioned discussion,

$$22p + 18(1 - p) = 20e^{0.12/4},$$

so that $p = 0.6523$. Using the expected payoff of the option, we get

$$\hat{E}(f) = pf_u + (1 - p)f_d = p + (1 - p)0 = p = 0.6523.$$

Therefore, the value of the option for today is

$$f = e^{-rT}\hat{E}(f) = e^{-0.12/4}(pf_u + (1 - p)f_d) = 0.633.$$

Example 5.2 *With the same parameters as in the preceding example, consider solving for Δ . Firstly, as we want a risk-free profit for the hedging portfolio, we want to purchase Δ shares of the stock and short one European call option expiring in 3 months. After 3 months, the value of the portfolio can be either*

$22\Delta - 1$, if the stock price moves to \$22,

or

18Δ , if the stock price moves to \$18.

This portfolio is risk free if Δ is chosen so that the value of the portfolio remains the same for both alternatives, that is,

$$22\Delta - 1 = 18\Delta \quad \text{which means} \quad \Delta = 0.25.$$

The value of the portfolio in 3 months becomes

$$22 \times 0.25 - 1 = 4.5 = 18 \times 0.25.$$

By the no arbitrage consideration, this risk-free profit must earn the risk-free interest rate. In other words, the value of the portfolio today must equal the present value of \$4.5, that is, $4.5e^{-0.12/4} = 4.367$. If the value of the option today is denoted by f , then the present value of the portfolio equals

$$20 \times 0.25 - f = 4.5e^{-0.12/4} = 4.367.$$

Solving for f gives

$$f = 0.633,$$

which matches with the answer of the preceding example.

(三)BSM Equation

◆ 股價價格行為

$$dS = \mu S dt + \sigma S dW, \quad (5.1)$$

◆ 債券價格行為

$$dB = rB dt. \quad (5.2)$$

Theorem 5.1 *Using the notation just defined, and assuming that the price and the bond are described by the geometric Brownian motion (Eq. 5.1) and the compound interest rate model (Eq. 5.2), respectively, the price of the derivative of this security satisfies*

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 = rf. \quad (5.3)$$

- ◆ 選擇權價格，為 S 、 t 的函數，

$$f = f(S, t)$$

$$df = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dW.$$

- ◆ 形成一個動態組合 G ，來複製選擇權的價格行為，

$$G = x_t S_t + y_t B_t, \quad dG = x_t dS_t + y_t dB_t$$

$$\begin{aligned} dG &= x_t dS + y_t dB \\ &= x_t(\mu S dt + \sigma S dW) + y_t r B dt \\ &= (x_t \mu S + y_t r B) dt + x_t \sigma S dW. \end{aligned}$$

- Match 隨機項，

$$x_t = \frac{\partial f}{\partial S}.$$

◆ 由於， $G = x_t S_t + y_t B_t$ ， $G=f$ 。

$$y_t = \frac{1}{B(t)}(G(t) - x_t S(t)).$$

$$y_t = \frac{1}{B(t)} \left(f(S, t) - \frac{\partial f}{\partial S} S(t) \right).$$

$$\frac{\partial f}{\partial S} \mu S + \frac{1}{B(t)} \left(f(S, t) - \frac{\partial f}{\partial S} S(t) \right) r B(t) = \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2.$$

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} r S + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 = r f.$$

◆ 另一種導法，

5. Alternatively, we can derive Equation 5.3 as follows. Construct a portfolio that consists of shorting one derivative and longing $\frac{\partial f}{\partial S}$ shares of the stock. Let the value of this portfolio be Π and let the value of the derivative be $f(S, t)$. Then

$$\Pi = -f + \frac{\partial f}{\partial S} S. \quad (5.6)$$

The change $\Delta\Pi$ in the value of this portfolio in the time interval Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S. \quad (5.7)$$

Recall that S follows a geometric Brownian motion so that

$$\Delta S = \mu S \Delta t + \sigma S \Delta W.$$

In addition, from Equation 5.4, the discrete version of df is

$$\Delta f = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \Delta W.$$

Substituting these two expressions into Equation 5.7, we get

$$\Delta \Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t. \quad (5.8)$$

Note that by holding such a portfolio, the random component ΔW has been eliminated completely. Because this equation does not involve ΔW , this portfolio must equal to the risk-free rate during the time Δt . Consequently,

$$\Delta \Pi = r \Pi \Delta t,$$

where r is the risk-free rate. In other words, using Equation 5.8 and Equation 5.6, we obtain

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t = r \left(f - \frac{\partial f}{\partial S} S \right) \Delta t.$$

Therefore,

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 = rf.$$

Example 5.3 *Let f denote the price of a forward contract on a non-dividend-paying stock with delivery price K and delivery date T . Its price at time t is given by*

$$f(S, t) = S - Ke^{-r(T-t)}. \quad (5.9)$$

Hence,

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}, \quad \frac{\partial f}{\partial S} = 1, \quad \text{and} \quad \frac{\partial^2 f}{\partial S^2} = 0.$$

Substituting these into Equation 5.3, we get

$$-rKe^{-r(T-t)} + rS = rf.$$

Thus, the price formula of f given by Equation 5.9 is a solution of the Black–Scholes equation, indicating that Equation 5.9 is the correct formula.

Theorem 5.2 *Consider a European option with payoff $F(S)$ and expiration time T . Suppose that the continuous compounding interest rate is r . Then, the current European option price is determined by*

$$f(S, 0) = e^{-rT} \hat{\mathbb{E}}[F(S_T)], \quad (5.10)$$

where $\hat{\mathbb{E}}$ denotes the expectation under the risk-neutral probability that is derived from the risk-neutral process

$$\frac{dS}{S} = r dt + \sigma dW(t). \quad (5.11)$$

(四)BS Formula

Lemma 5.1 *Let S be a lognormally distributed random variable such that $\log S \sim N(m, v^2)$ and let $K > 0$ be a given constant. Then*

$$E(\max\{S - K, 0\}) = E(S)\Phi(d_1) - K\Phi(d_2), \quad (5.12)$$

where $\Phi(\cdot)$ denotes the distribution function of a standard normal random variable and

$$d_1 = \frac{1}{v}(-\log K + m + v^2) = \frac{1}{v} \left(\log E \left(\frac{S}{K} \right) + \frac{v^2}{2} \right),$$
$$d_2 = \frac{-\log K + m}{v} = \frac{1}{v} \left(\log E \left(\frac{S}{K} \right) - \frac{v^2}{2} \right).$$

Theorem 5.3 Consider a European call option with strike price K and expiration time T . If the underlying stock pays no dividends during the time $[0, T]$ and if there is a continuously compounded risk-free rate r , then the price of this contract at time 0, $f(S, 0) = C(S, 0)$, is given by

$$C(S, 0) = S \Phi(d_1) - Ke^{-rT} \Phi(d_2), \quad (5.15)$$

where $\Phi(x)$ denotes the cumulative distribution function of a standard normal random variable evaluated at the point x ,

$$\begin{aligned} d_1 &= [\log(S/K) + (r + \sigma^2/2)T] \frac{1}{\sigma\sqrt{T}}, \\ d_2 &= [\log(S/K) + (r - \sigma^2/2)T] \frac{1}{\sigma\sqrt{T}} \\ &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

Example 5.4 Consider a 5-month European call option on an underlying stock with a current price of \$62, strike price \$60, annual risk-free rate 10%, and the volatility of this stock is 20% per year. In this case, $S = 62$, $K = 60$, $r = 0.1$, $\sigma = 0.2$, and $T = \frac{5}{12}$. Applying Equation 5.15, we get

$$\begin{aligned}d_1 &= \frac{1}{0.2\sqrt{5/12}} \left[\log \left(\frac{62}{60} \right) + \left(0.1 + \frac{0.2^2}{2} \right) \frac{5}{12} \right] \\&= 0.641287, \\d_2 &= d_1 - 0.2\sqrt{5/12} = 0.512188.\end{aligned}$$

From the normal table, we get $\Phi(d_1) = 0.739332$ and $\Phi(d_2) = 0.695740$. Consequently,

$$C = (62)(0.739332) - (60)e^{-(0.1)(5/12)}(0.695740) = 5.798.$$