模擬法在財務工程上的應用

Monte Carlo Methods in Financial Engineering

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大　綱

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第一章、VBA 概要

(一)使用範例補充

- Pdf_CDF
- Black-Scholes Call
- Black-Scholes Graph
- Black-Scholes Implied Volatility
- QuantLib Addin Call

第二章、期貨與選擇權基礎

(一)簡介

- 衍生商品
  - 如期貨、遠期交易、選擇權
    - 其價格依賴基本證券的價值。
    - 遠期匯率交易
    - 股票選擇權
- 基本證券
  - 股票、債券、外幣、商品
    - 由其收益決定價值
    - 股利、債息、貨幣、黃金

◆ 套利

- 同時進入兩筆交易，賺取無風險利得
- 衍生商品之價格為無套利價格
- 金融市場的均衡是無套利均衡，不是一般的供需均衡

◆ 避險

- 農夫使用遠期交易避險
- 衍生商品重要經濟功能

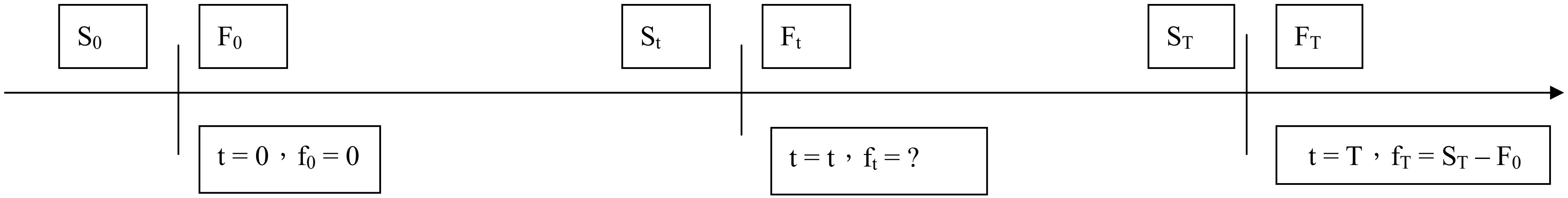
◆ 風險中立定價

$$r_k = r_f + risk\_premium$$

- 投資人為風險趨避的
- 無風險證券收取無風險報酬
- 風險性資產需要風險溢酬

(二)遠期契約

- 美元現貨 $S_0 = 30.0$，進口商三個月後需要美元購入商品，擔心上漲，預購美元遠匯。
  - 三個月遠匯目前報價 $F_0 = 30.2$，買入 USD 100 萬。
  - 期初契約價值 $f_0 = 0$。



  - 令 $K = F_0$，則 $f_t = ?$

- 交易策略一：
  - 買入遠匯 f_0，銀行存入 Ke^{-rT} 。
- 交易策略二：
  - 買入現貨 S_0。
- 兩者代價相同：

$$f_0 + Ke^{-rT} = S_0$$

$$f_0 + F_0 e^{-rT} = S_0$$

$$0 + F_0 e^{-rT} = S_0$$

$$F_0 = S_0 e^{rT}$$

$$F_t = S_t e^{r(T-t)}$$

◆ 另一方式證明：

- 買入現貨 S_0，持有三個月。
- 或直接買入遠匯 F_T，
- 兩者成本要相同，r，台幣融資成本，y，美元資產收益。

$$F_{\mathrm{T}} = S_0 + rS_0(T-0) - yS_0(T-0) = S_0(1+(r-y)T) \approx S_0 e^{(r-y)T}$$

$$F_{\mathrm{T}} = S_t + rS_t(T-t) - yS_t(T-t) = S_t\left[1+(r-y)(T-t)\right] \approx S_t e^{(r-y)\times(T-t)}$$

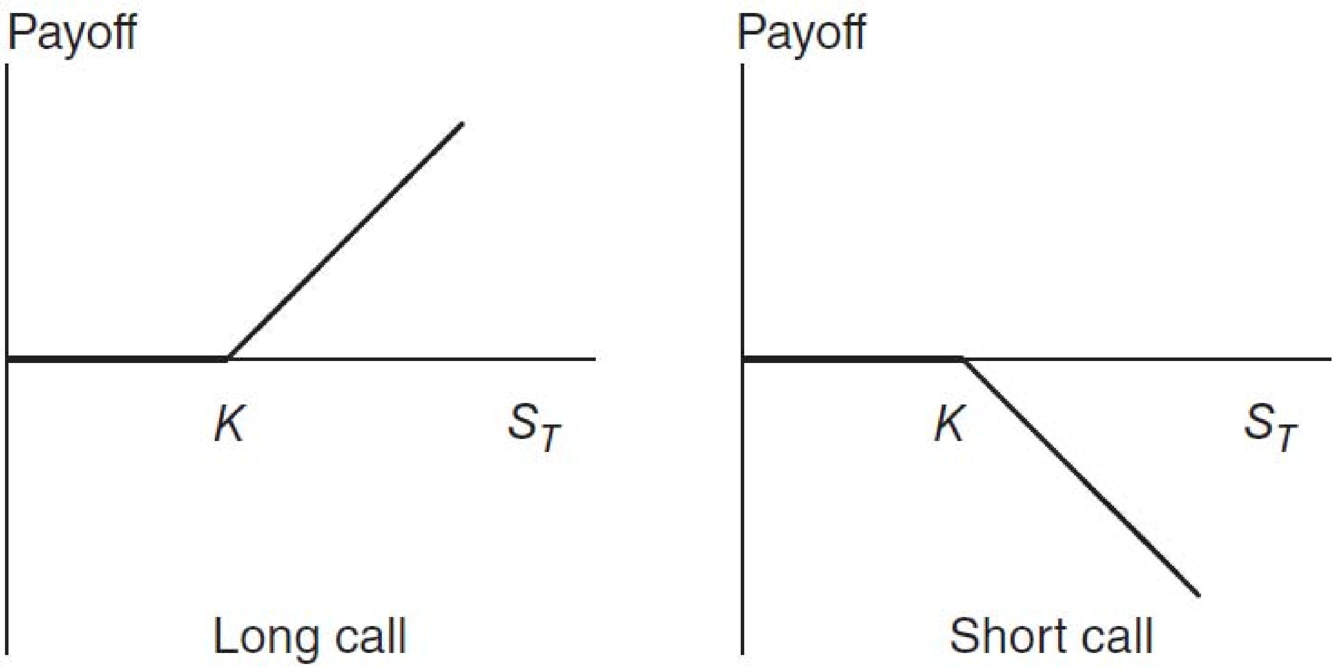
◆ 期中評價

$$f_{\mathrm{t}} = (F_t - F_0)e^{-r(T-t)}$$

(三)選擇權契約

◆ 買權(Call)

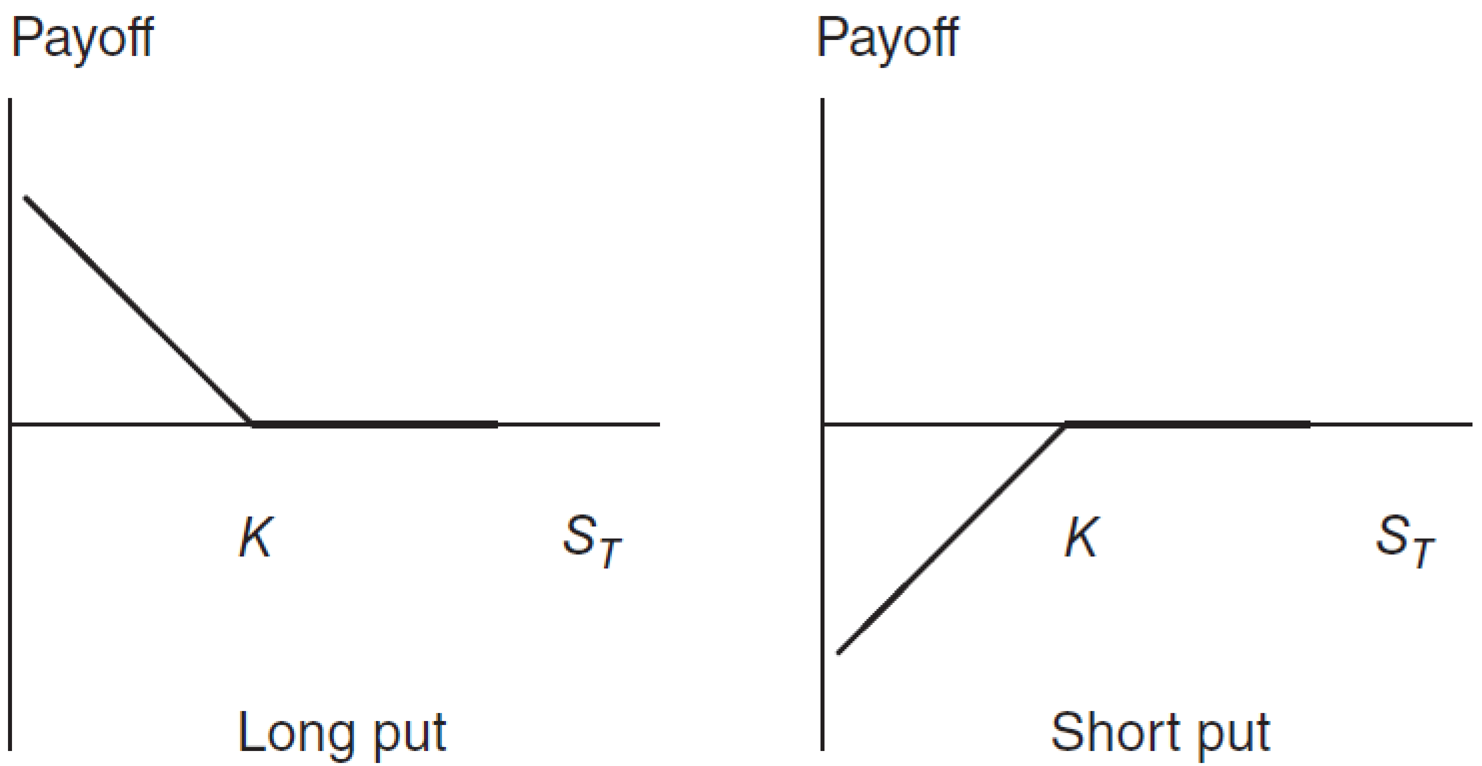
- 買方：台積電股價 S_0 = 400，三個月後可以用 K = 420，買入 1000 股。期初付權利金 C_0。
- 賣方：買方執行權利，交付股票，收 420。期初收權利金 C_0。



- $C_T = \max[S_T - K, 0]$

◆ 賣權(Put)

- 買方：台積電股價 $S_0 = 400$，三個月後可以用 $K = 380$，賣出 1000 股。期初付權利金 P_0。
- 賣方：買方執行權利，收取股票，付 380。期初收權利金 P_0。



- $P_T = \max[K - S_T, 0]$

◆ 期初 C_0、P_0 如何決定?

- 1900，Louis Bachelier，使用常態分配，二次效用函數，推導價格。
- https://www.macroption.com/black-scholes-history/#bachelier-1900。

On this page:

Option Trading and Pricing Before 1900

Louis Bachelier (1900)

Option Pricing Research in the 1960’s

Black, Scholes, and Merton before 1973

The Original Black-Scholes Paper (1973)

Merton’s Extension (1973)

Futures Options: Black Model (1976)

Currency Options: Garman-Kohlhagen (1983)

Nobel Prize (1997)

References

◆ Bachelier Model

We assume that, under the risk-neutral measure, the stock process $\{S_t, t \geq 0\}$ satisfies an SDE of the form

$$dS_t = rS_t dt + \sigma dW_t,$$

where r is the constant interest rate, σ is the constant volatility, and $\{W_t, t \geq 0\}$ is standard Brownian motion. For $0 \leq t \leq T$,

$$S_T = S_t e^{r(T-t)} + \sigma \int_t^T e^{r(T-s)} dW_s.$$

That is,

$$\begin{aligned} S_T \mid S_t &\sim N\left(S_t e^{r(T-t)}, \frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)\right) \\ &\sim S_t e^{r(T-t)} + \sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}\, \xi, \end{aligned}$$

where ξ is standard normal random variable. Then

$$
\begin{aligned}
C_t &= e^{-r(T-t)} E\left((S_T - K)^+ \mid \mathcal{F}_t\right) \\
&= e^{-r(T-t)} E\left(\left(S_t e^{r(T-t)} + \sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}\,\xi - K\right)^+ \mid \mathcal{F}_t\right) \\
&= e^{-r(T-t)} \sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}\, E\left(\left(\xi - \frac{K - S_t e^{r(T-t)}}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}}\right)^+ \mid \mathcal{F}_t\right) \\
&= e^{-r(T-t)} \left(S_t e^{r(T-t)} - K\right) \Phi\left(\frac{S_t e^{r(T-t)} - K}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}}\right) \\
&\quad + e^{-r(T-t)} \sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}\, \phi\left(\frac{S_t e^{r(T-t)} - K}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}}\right),
\end{aligned}
$$

where Φ is the cumulative distribution function of a standard normal random variable, and ϕ is the corresponding density function.

Comments

Let $K^* = e^{-r(T-t)}K$, and

$$v^2(t,T) = \frac{\sigma^2}{2r}\left(1 - e^{-2r(T-t)}\right).$$

Then, we can re-express the price as

$$C_t = (S_t - K^*)\,\Phi\left(\frac{S_t - K^*}{v(t,T)}\right) + v(t,T)\,\phi\left(\frac{S_t - K^*}{v(t,T)}\right).$$

甲、無套利的價格關係

- ◆ Put-Call parity：不支付股利情況
  - ➢ 命題 I ：$C - P = S - K(1+r)^{-T}$
    - ✓ 我們的證明方向為指出 C-P 既不會大於 $S-K(1+r)^{-T}$，也不會小於 $S-K(1+r)^{-T}$。
    - ✓ 因為如果 C-P 大於或小於 $S-K(1+r)^{-T}$，則市場上便存在套利機會。這違背我們的基本假設—市場為效率的。
    - ✓ 依據三一律，C-P 只可能等於 $S-K(1+r)^{-T}$。

證明：(1)　若 $C-P>S-K(1+r)^{-T}$，則 $C-P-S+K(1+r)^{-T}>0$

		到期日	
	今日	$S_T \leqq K$	$S_T \geqq K$
賣出買權	+C	0	$-(S_T-K)$
買入賣權	-P	$+(K-S_T)$	0
買入股票	-S	$+S_T$	$+S_T$
借入款項	$K(1+r)^{-T}$	-K	-K
	>0	0	0

證明：(2)　若 $C-P<S-K(1+r)^{-T}$ 則 $C-P-S+K(1+r)^{-T}<0$，或 $-C+P+S-K(1+r)^{-T}>0$

		到　期　日	
	今　日	$S_T \leqq K$	$S_T \geqq K$
買入買權	$-C$	0	$+(S_T-K)$
賣出賣權	$+P$	$-(K-S_T)$	0
賣出股票	$+S$	$-S_T$	$-S_T$
借出款項	$-K(1+r)^{-T}$	$+K$	$+K$
	>0	0	0

➢ 由三一律可知，唯有 $C-P=S-K(1+r)^{-T}$ 才是一個均衡的情況，因此證明完畢。

◆ 範　　例

➢ 目前股價\$100，執行價格 98，六個月後到期的歐式買權價格\$8。若市場上無風險利率為 10%，則由命題 I ，可求得相同條件的賣權價格為，

$$P = C-S+K(1+r)^{-T} = 8-100+98(1.1)^{-0.5} = 1.4393$$

◆ Upper Bound

Whatever the price of an underlying asset, the value of a call option (with payoff $\max(S-K,0)$) can never be worth more than the stock, and an American call is always worth more than its European counterpart because it can be exercised at any time, including at maturity, so we have

$$C_E \le C_A \le S.$$

For put options, no matter how low the stock price becomes, the put can never be worth more than the strike price:

$$P_E \le K \qquad \text{and} \qquad P_A \le K.$$

Furthermore, for a European put, the option will not be worth more than K at maturity, so the current value of the put cannot be larger than the present value of the strike:

$$P_E \le Ke^{-r(T-t)}.$$

◆ Lower Bound

The lower bounds for call options can be derived as follows:

$$\max\left(S - Ke^{-r(T-t)}, 0\right) \le C_E \le C_A.$$

Similarly, we can obtain the inequality for put options:

$$\max(Ke^{-r(T-t)} - S, 0) \le P_E \le P_A.$$

◆ Difference between American call and put prices

For a non-dividend-paying asset, we can further deduce the boundaries of the difference between the prices of American call and put options. According to the put-call parity,

$$
\begin{aligned}
P_A &\geq P_E \\
&= C_E + Ke^{-r(T-t)} - S \\
&= C_A + Ke^{-r(T-t)} - S\,,
\end{aligned}
$$

$$\Rightarrow \qquad C_A - P_A \leq S - Ke^{-r(T-t)}\,.$$

第三章、模擬介紹

(一)模擬

- 一個電腦實驗，對於研究的隨機變數，賦予一定的統計分配，根據其行為方程式，查看其可能變化，並估算我們感興趣的統計量。
  - 真實模型太複雜，簡化之。
  - 推導不出解析解。
  - 尤其，如果牽涉多個隨機變量，行為方程式太複雜。

- 放款損失金額，頻平均數、標準差、99.9%的最大可能損失。
  - 借款人的違約事件發生機率。
  - 違約回收率。
  - 標的物的價值。
    - ✓ 市場利率

(二)範例

甲、Quadrature

- ◆ 數值積分

$$I = \int_a^b f(x)dx$$

- ➢ 以多點值加權近似，

$$\hat{I} = \sum_{i=1}^{n} \omega_i f(x_i)$$

- ➢ 作法：
  - ✓ [a, b]均分 n 段，以中間值估算，

$$\hat{I} = \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + \frac{(2i-1)}{2n}(b-a) \right)$$

乙、模擬法

- 使用隨機點，估計函數值，
  - 若 $x_1, x_2, \ldots, x_n$ 為獨立隨機抽取，分配密度為 $g(x)$。

$$I=\int f(x)dx=\int \frac{f(x)}{g(x)}g(x)dx=E_g\left\{\frac{f(x)}{g(x)}\right\}$$

  - ✓ 可以視為對 g 分配的期望值。
  - 以樣本平均數近似之。

$$\hat{I}=\frac{1}{n}\sum_{i=1}^{n}\frac{f(x_i)}{g(x_i)}$$

  - $\hat{I}$ 為 I 的不偏估計值，變異數為

$$Var(\hat{I})=\frac{1}{n}Var_g\left[\frac{f(x)}{g(x)}\right]$$

  - 樣本變異數為

$$\frac{1}{n^2}\sum_{i=1}^{n}\frac{f^2(x_i)}{g^2(x_i)}-\frac{\hat{I}^2}{n}$$

(三)隨機模擬

- 如果研究的主題涉及隨機過程，如布朗運動、幾何布朗運動、或對數常態分配，可能很難推導出想要統計量的解析式。

- 若 $X \sim N(\mu,\sigma^2)$ ，則隨機變數 $Y = e^X$ 為對數常態， $\ln(Y) = X \sim N(\mu,\sigma^2)$ 。

  - Y 的分配如下，

$$
\begin{aligned}
G(y) &= P(Y \le y) = P(X \le \log y) \\
&= P((X-\mu)/\sigma \le (\log y - \mu)/\sigma) \\
&= \Phi((\log y - \mu)/\sigma),
\end{aligned}
$$

  - $\Phi(\cdot)$表標準常態分配。

◆ X 的動差函數為，

$$M_X(t) = \mathrm{E}(e^{tX}) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

➢ 期望值為，

$$\mathrm{E}Y = \mathrm{E}(e^X) = M_X(1) = e^{\mu + \frac{1}{2}\sigma^2}.$$

➢ 由二階動差可以得到變異數，

$$\mathrm{Var}(Y) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$$

若隨機變量X服從一個位置參數為μ、尺度參數為σ的常態分布，記為：

$$X \sim N(\mu, \sigma^2),$$ [3]

則其機率密度函數為 $f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ [3]

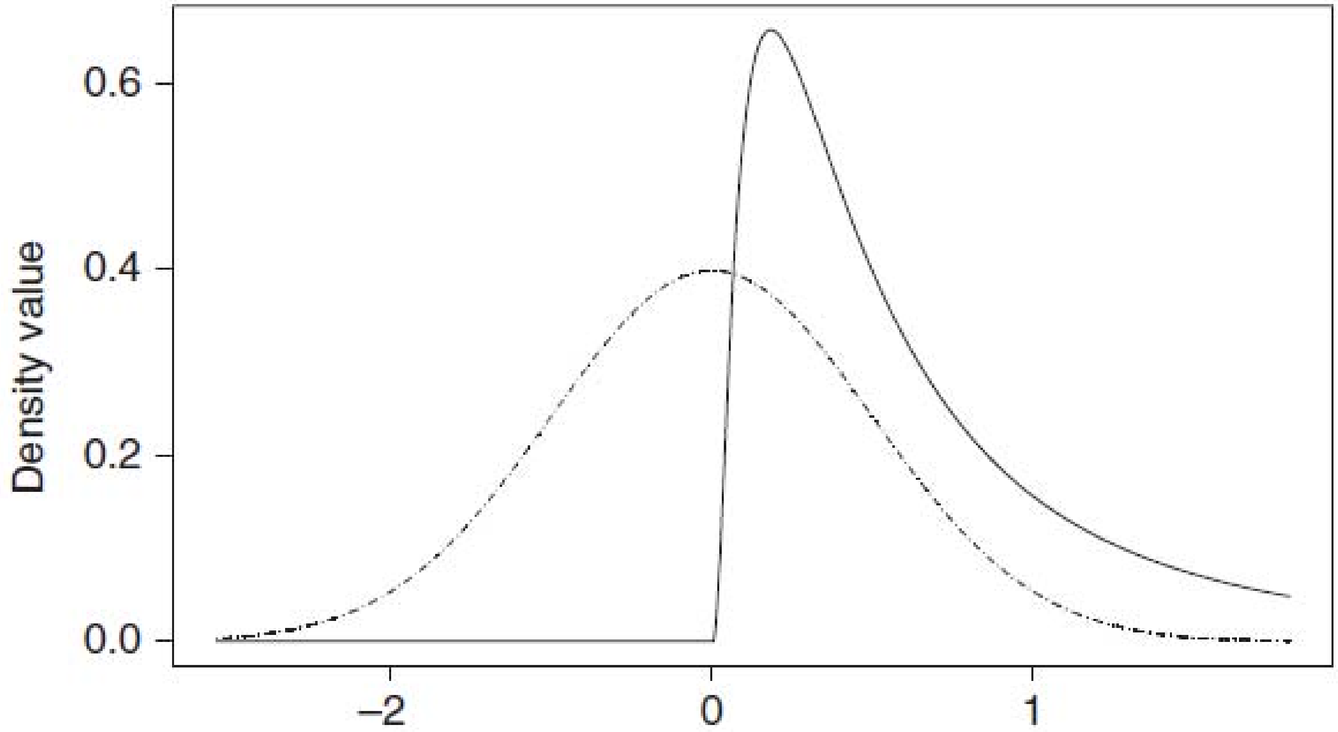
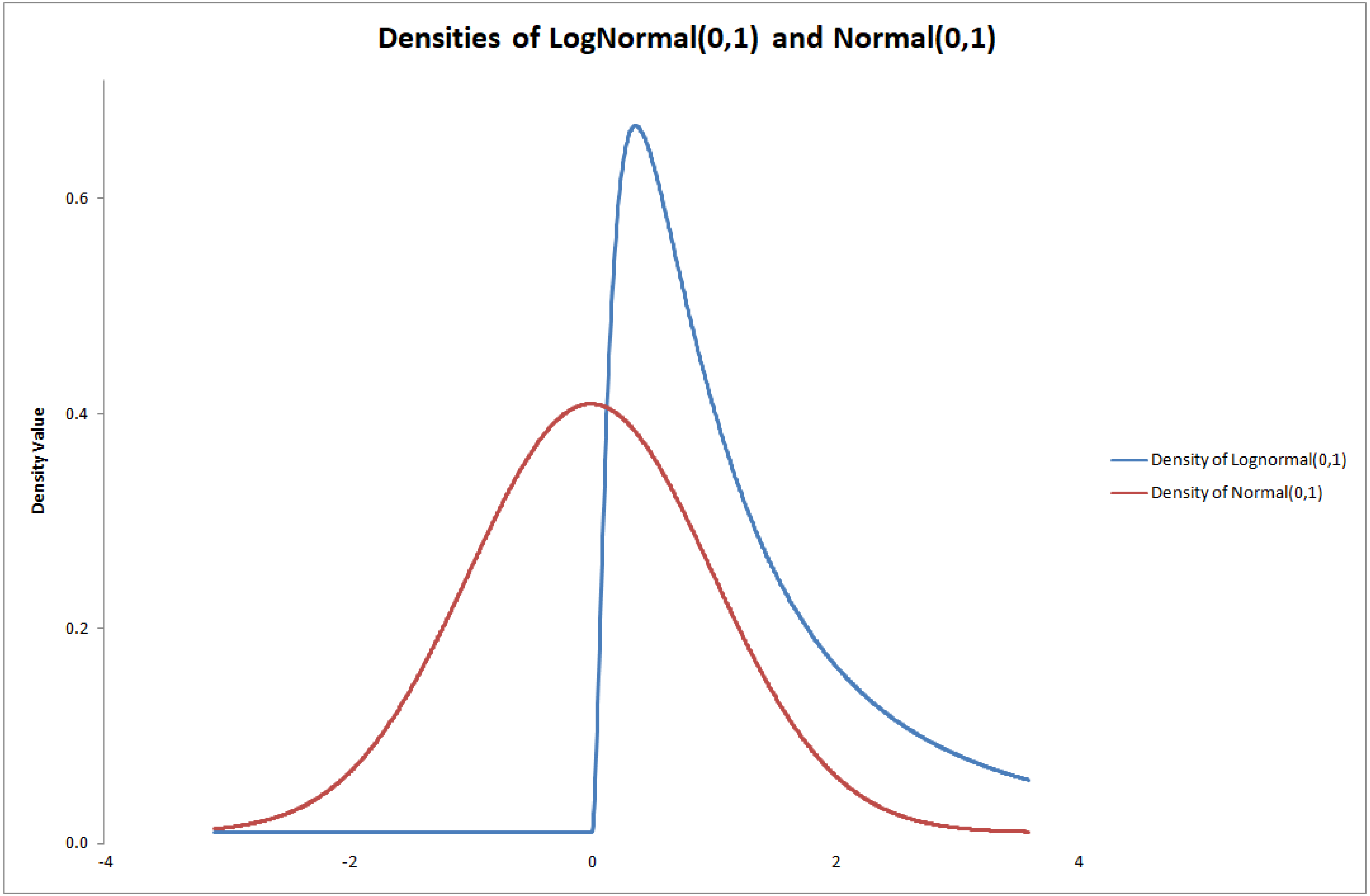


Figure 3.1 Densities of a lognormal distribution with mean $e^{0.5}$ and variance $e(e-1)$, that is, $\mu = 0$ and $\sigma^2 = 1$ and a standard normal distribution.


Densities of LogNormal(0,1) and Normal(0,1)
Density Value
0.0
0.2
0.4
0.6
-4
-2
0
2
4
Density of Lognormal(0,1)
Density of Normal(0,1)

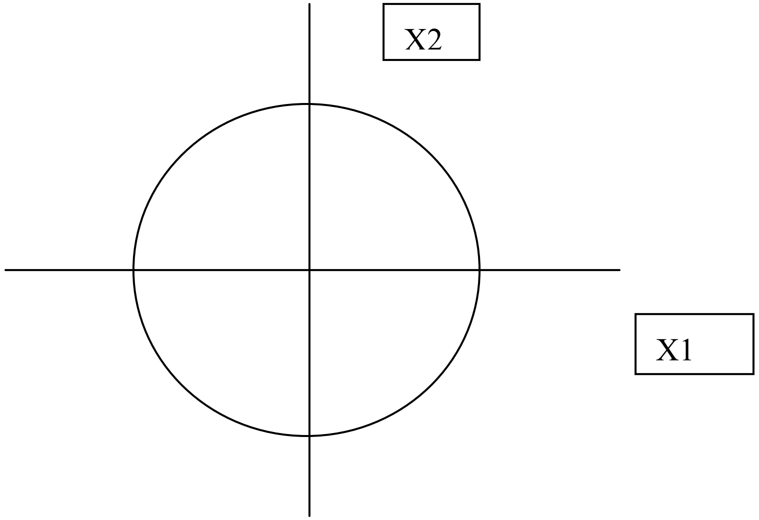
◆ 產生隨機亂數

```
Option Explicit
Sub LogNormGen()

    Sheets("Sheet1").Select
    Columns("H").Select
    Selection.ClearContents

    Dim n As Long, mu As Double, sigma2 As Double, sigma As Double
    Dim i As Long

    n = Cells(1, 2).Value
    mu = Cells(2, 2).Value
    sigma2 = Cells(3, 2).Value
    sigma = Sqr(sigma2)

    For i = 1 To n
        Cells(i, 8).Value = Application.LogInv(Rnd(), mu, sigma)
    Next i

    Cells(5, 2).Select

End Sub
```

	A	B	C	D	E	F	G	H	I
1	Number of Random Variables generated:	1000		Lognormal Random Variables generated:				0.427951639	
2	mu:	0						0.988746007	
3	sigma^2:	1						0.727092588	
4								2.922870128	
5								1.848968823	
6								0.252712478	
7								0.587941631	
8								0.543643234	
9								0.430009295	
10								2.374427803	
11								16.28337025	
12								2.551827352	
13								2.548814685	
14								0.61032746	
15								1.249565912	
16								11.08705049	

Generate Lognormal Random Variables

◆ 單位圓面積(π)模擬估計



- 隨機抽取 2 維亂數(x1, x2)，若 $x_1^2 + x_2^2 \le 1$ ，則點在圓內。
- 抽 10,000 次，若有 m 次落於圓內，則 1/4 圓面積為，

$$\frac{\pi}{4} = \frac{m}{n}$$

$$\pi = 4\frac{m}{n}$$

L3 =4*I3/200

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2			U1	U2		U1^2+U2^2	<=1		M			
3		1	0.010766925	0.12263357		0.015154919	1		158		Pi	3.160
4		2	0.858316571	0.680330633		1.199557105	0					
5		3	0.632162943	0.614575584		0.777333135	1		N			
6		4	0.690280218	0.95183245		1.382471792	0		200			
7		5	0.68357676	0.657896894		0.90010551	1					
8		6	0.614115939	0.926077147		1.234757269	0					
9		7	0.413538224	0.358663582		0.299653427	1					
10		8	0.9288585	0.302394558		0.954220581	1					
11		9	0.221204934	0.069650432		0.053782805	1					
12		10	0.218397243	0.250713814		0.110554772	1					
13		11	0.654107151	0.942818722		1.316763306	0					
14		12	0.466877033	0.739544155		0.764899721	1					
15		13	0.771681394	0.594422487		0.948830267	1					
16		14	0.056737804	0.638827954		0.411320333	1					
17		15	0.936362771	0.049664196		0.879241772	1					

第四章、布朗運動與伊藤補理

(一)Wiener & Ito’s 程序

- 如下模型，

$$W(t_{k+1}) = W(t_k) + \varepsilon_{t_k}\sqrt{\Delta t} \ ,$$

$$\Delta t = t_{k+1}\text{-}t_k$$

$$t_0 = 0, \quad k = 0,\ldots,N$$

$$\varepsilon_{t_k} \sim N(0,1)$$

$$W(t_0) = 0$$

◆ 可以表示為，j<k，

$$W(t_k)\text{-}W(t_j) = \sum_{i=j}^{k-1} \varepsilon_{t_i} \sqrt{\Delta t}$$

- 右邊為常態隨機數的累加
- 統計量為，

$$\mathrm{E}(W(t_k) - W(t_j)) = 0,$$

$$\mathrm{Var}(W(t_k) - W(t_j)) = \mathrm{E}[\sum_{i=j}^{k-1} \epsilon_{t_i} \sqrt{\Delta t}]^2 = (k-j)\Delta t = t_k - t_j.$$

- 對 $t_1 < t_2 \leqq t_3 < t_4$，

$$W(t_4) - W(t_3) \text{ is uncorrelated with } W(t_2) - W(t_1).$$

◆ 布朗運動(Wiener Process)的模擬

- [0, 1]均等分割為 n 區間，$t \subset [0, 1]$
- [nt]表高斯函數，n=10, t=1/3，[nt] = 3。
- 定義如下[0, 1]上的隨機過程，

$$S_{[nt]} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \varepsilon_i$$

- 可得，

$$S_{[nt]} = S_{[nt]\text{-}1} + \varepsilon_{[nt]} \frac{1}{\sqrt{n}} \quad \text{.........................(4.3)}$$

$$S_{[nt]} - S_{[nt]\text{-}1} = \varepsilon_{[nt]} \frac{1}{\sqrt{n}} \sim N(0, \frac{1}{n})$$

- 在 $t = 1$，S_n 為常態分配

$$S_{[nt]} = S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i$$

  - 由 CLT，只要 ε_i 為 i.i.d.，無須常態要求，S_n 也趨向常態分配。
  - 只要 $n \to \infty$，$S_{[nt]}$ 趨近於 Wiener process in distribution。

- 只要 $\Delta t \to 0$，可得韋恩程序，

$$dW(t) = \varepsilon(t)\sqrt{dt}$$

  - 只要 $\varepsilon(t)$ 為無相關的標準常態隨機變數。

◆ 迭代(4.3)便可模擬韋恩程序的一個樣本路徑。

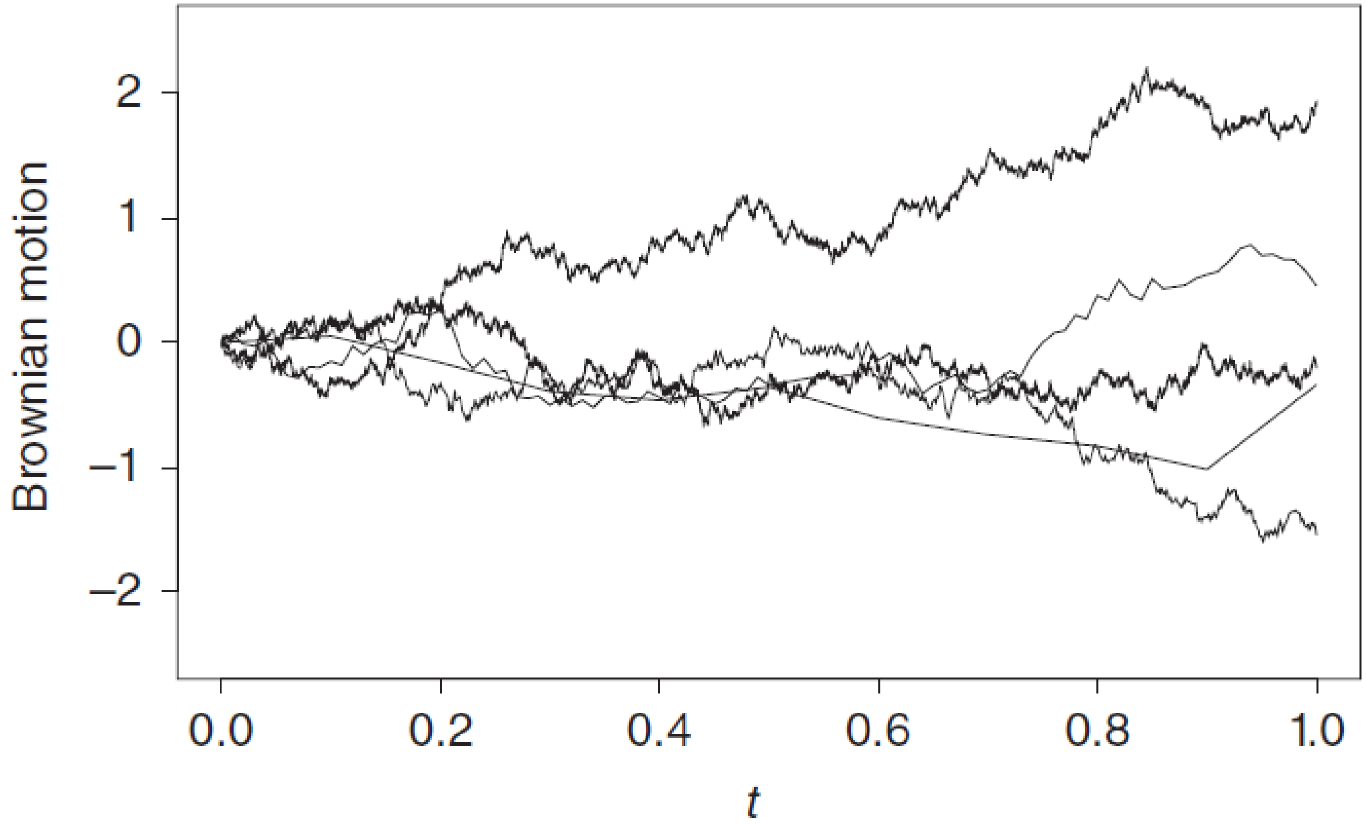


Figure 4.1 Sample paths of the process $S_{[nt]}$ for different n and the same sequence of ϵ_i.

◆ 韋恩程序的定義

Definition 4.1 *A Wiener process $W(t)$ is a stochastic process that satisfies the following properties:*

- *For $s < t$, $W(t) - W(s)$ is a normally distributed random variable with mean 0 and variance $t - s$.*
- *For $0 \leq t_1 < t_2 \leq t_3 < t_4$, $W(t_4) - W(t_3)$ is uncorrelated with $W(t_2) - W(t_1)$. This is known as the independent increment property.*
- *$W(t_0) = 0$ with probability one.*

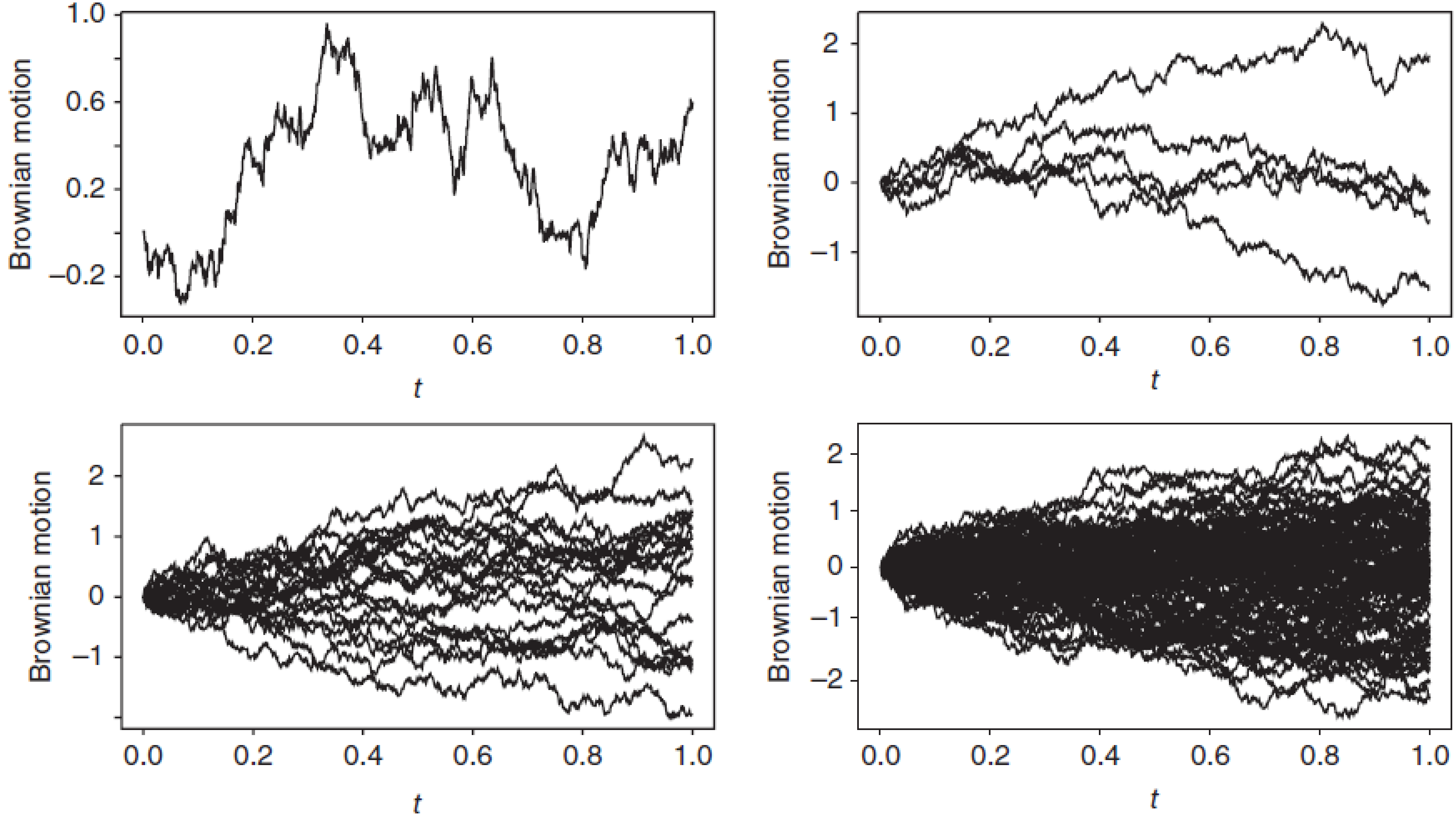


Figure 4.2 Sample paths of Brownian motions on [0,1].

◆ 韋恩程序的特性

From this definition, we can deduce a number of properties.

1. For $t < s$, $\mathrm{E}(W(s)|W(t)) = \mathrm{E}(W(s) - W(t) + W(t)|W(t)) = W(t)$. This is known as the *martingale* property of the Brownian motion.
2. The process $W(t)$ is nowhere differentiable. Consider

$$\mathrm{E}\left(\left(\frac{W(s) - W(t)}{s - t}\right)^2\right) = \frac{1}{s - t}.$$

   This term tends to ∞ as $s - t$ tends to 0. Hence, the process cannot be differentiable, and we cannot give a precise mathematical meaning to the process $dW(t)/dt$.
3. If we formally represent $\xi(t) = \frac{dW(t)}{dt}$ and call it the white noise process, we can use it only as a symbol, and its mathematical meaning has to be interpreted in terms of an integration in the context of a stochastic differential equation.

◆ 一般化的韋恩程序

The idea of Wiener process can be generalized as follows. Consider a process $X(t)$ satisfying the following equation:

$$dX(t) = \mu\, dt + \sigma\, dW(t), \tag{4.4}$$

where μ and σ are constants, and $W(t)$ is a Wiener process defined previously. If we integrate Equation 4.4 over $[0, t]$, we get

$$X(t) = X(0) + \mu t + \sigma W(t),$$

that is, the process $X(t)$ satisfies the integral equation

$$\int dX(t) = \mu \int dt + \sigma \int dW(t).$$

The process $X(t)$ is also known as a diffusion process or a generalized Wiener process. In this case, the solution $X(t)$ can be written down analytically in terms of the parameters μ and σ and the Wiener process $W(t)$.

◆ Ito's 程序的定義

let the parameters μ and σ depend on the process $X(t)$ as well. In that case, we have what is known as a general diffusion process or an Itô's process.

Definition 4.2 *An Itô's process is a stochastic process that is the solution to the following stochastic differential equation (SDE):*

$$dX(t) = \mu(x,t)\,dt + \sigma(x,t)\,dW(t). \tag{4.5}$$

◆ Ito's process 不一定有解，技術問題。

- $\mu(\mathrm{x,t})$ 為 drift function。
- $\sigma(\mathrm{x,t})$ 為 volatility function。
- X(t)需以積分方式來解釋。

(二)股票價格

Definition 4.3 *Let $X(t)$ be a Brownian motion with drift ν and variance σ^2, that is,*

$$dX(t) = \nu\, dt + \sigma\, dW(t).$$

The process $S(t) = e^{X(t)}$ is called a GBM with drift parameter μ, where $\mu = \nu + \frac{1}{2}\sigma^2$. In particular, $S(t)$ satisfies

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dW(t),$$

and

$$d \log S(t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW(t). \tag{4.6}$$

- S(t)為 GBM，起點 S(0) = z，if

$$S(t) = ze^{X(t)} = ze^{\nu t + \sigma W(t)} = ze^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}$$

(三)Ito's Formula

- 前面定義

$$S(t) = ze^{X(t)} = ze^{vt+\sigma W(t)} = ze^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W(t)}$$

$$dX(t) = vdt + \sigma \cdot dW(t)$$

$$S(t) = e^{X(t)}$$

$$d\log S(t) = \frac{dS(t)}{S(t)} = d\log\left(e^{X(t)}\right) \neq dX(t) = vdt + \sigma \cdot dW(t)$$

- X(t)內含 W(t)項，不能以一般微分方式處理，暫且相信下式

$$d\log S(t) = \frac{dS(t)}{S(t)} = d\log\left(e^{X(t)}\right) = \left(v + \frac{1}{2}\sigma^2\right)dt + \sigma \cdot dW(t)$$

$$= \mu \cdot dt + \sigma \cdot dW(t)$$

$$v = (\mu - \frac{1}{2}\sigma^2)$$

◆ 簡單模擬方式

4. Note that Equation 4.7 provides a way to simulate the price process $S(t)$. Suppose we start at t_0 and let $t_k = t_0 + k\Delta t$. According to Equation 4.7, the simulation equation is

$$S(t_{k+1}) - S(t_k) = \mu S(t_k)\Delta t + \sigma S(t_k)\epsilon(t_k)\sqrt{\Delta t},$$

where $\epsilon(t_k)$ are i.i.d. standard normal random variables. Iterating this equation we get

$$S(t_{k+1}) = [1 + \mu\Delta t + \sigma\epsilon t_k\sqrt{\Delta t}]S(t_k), \tag{4.8}$$

which is a multiplicative model, but the coefficient is normal rather than log-normal. So this equation does not generate the lognormal price distribution. However, when Δt is sufficiently small, the differences may be negligible.

◆ 正確模擬方式

5. Instead of using Equation 4.7, we can use Equation 4.6 for the log prices and get

$$\log S(t_{k+1}) - \log S(t_k) = \nu\Delta t + \sigma\epsilon(t_k)\sqrt{\Delta t}.$$

This equation leads to

$$S(t_{k+1}) = e^{\nu\Delta t+\sigma\epsilon(t_k)\sqrt{\Delta t}}S(t_k), \tag{4.9}$$

which is also a multiplicative model, but now the random coefficient is log-normal. In general, we can use either Equation 4.8 or Equation 4.9 to simulate stock prices.

➢ 移轉方程式(Transformation Equation)。

◆ Ito's lemma

Theorem 4.1 *Suppose the random process $x(t)$ satisfies the diffusion equation*

$$dx(t) = a(x,t)\,dt + b(x,t)\,dW(t),$$

where $W(t)$ is a standard Brownian motion. Let the process $y(t) = F(x,t)$ for some function F. Then the process $y(t)$ satisfies the Itô's equation

$$dy(t) = \left(\frac{\partial F}{\partial x}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2\right)dt + \frac{\partial F}{\partial x}b\,dW(t). \tag{4.10}$$

◆ 確定過程，一般展開，少一項(x 的二次微分項)

$$dy = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial t}\,dt$$
$$= \frac{\partial F}{\partial x}(a\,dt + b\,dW) + \frac{\partial F}{\partial t}\,dt.$$

◆ 將 y 對 x 展開到二次，

$$
\begin{aligned}
y+\Delta y &= F(x,t)+\frac{\partial F}{\partial x}\Delta x+\frac{\partial F}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\Delta x)^2 \\
&= F(x,t)+\frac{\partial F}{\partial x}(a\Delta t+b\Delta W)+\frac{\partial F}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(a\Delta t+b\Delta W)^2.
\end{aligned}
$$

- 展開二次平方項

$$a^2(\Delta t)^2+2ab(\Delta t)(\Delta W)+b^2(\Delta W)^2.$$

- 第三項要保留，前兩項為 Δt 的高次項，$\Delta t \to 0$ 時，會消失。
- $\Delta W \sim N(0, \Delta t)$，$(\Delta W)^2 \to \Delta t$。

$$dW(t)^2 \cong dt \quad \text{or} \quad dW(t) \cong \sqrt{dt}.$$

甲、範例

Example 4.1 *Suppose $S(t)$ satisfies the geometric Brownian motion equation*

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t).$$

◆ 要研究 log(S(t))的行為，F(S) = log(S)，

$$d\left(\log S(t)\right) = \frac{dF}{dS}dS + \frac{1}{2}\frac{d^2F}{dS^2}(dS)^2 = \frac{1}{S}\left(\mu S\cdot dt + \sigma S\cdot dW\right) + \frac{1}{2}\left(-\frac{1}{S^2}\right)\left(\mu S\cdot dt + \sigma S\cdot dW\right)^2$$

$$= \left(\mu\cdot dt - \frac{1}{2}\sigma^2\cdot dt\right) + \left(\sigma\cdot dW\right)$$

◆ 補充

$$dX(t) = vdt + \sigma \cdot dW(t) \text{ ， } S(t) = e^{X(t)}$$

$$S = F(X)$$

$$dS = \frac{\partial F}{\partial X} dX + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} (dX)^2 = SdX + \frac{1}{2} S(dX)^2$$

$$= S(vdt + \sigma \cdot dW) + \frac{1}{2} S(vdt + \sigma \cdot dW)^2 = S(v + \frac{1}{2}\sigma^2)dt + S(\sigma \cdot dW)$$

$$dS = S(v + \frac{1}{2}\sigma^2)dt + S(\sigma \cdot dW)$$

$$\frac{dS}{S} = (v + \frac{1}{2}\sigma^2)dt + \sigma \cdot dW = \mu \cdot dt + \sigma \cdot dW$$

$$\mu = v + \frac{1}{2}\sigma^2$$

Example 4.2 *Evaluate*

$$\int_0^t s\,dW(s).$$

$\mathrm{X(t)} = W(t)$ ， $\mathrm{dX(t)} = dW(t)$

$\mathrm{Y(t)} = F(t, X(t)) = t \cdot W(t)$

$$\mathrm{d}\big(\mathrm{Y(t)}\big) = \frac{\partial F}{\partial t}dt + \frac{\partial \mathrm{F}}{\partial X}dX + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2 = X(t)dt + tdX(t) = W(t)dt + tdW(t)$$

$$\int \mathrm{d}\big(\mathrm{Y(t)}\big) = \int \big(W(t)dt + tdW(t)\big) = \int W(t)dt + \int tdW(t)$$

$$\mathrm{Y(t)} = \int W(t)dt + \int tdW(t)$$

$$\int tdW(t) = \mathrm{Y(t)} - \int W(t)dt$$

Example 4.3 *Evaluate*

$$\int_0^t W(s)\,dW(s).$$

1. Let $X(t) = W(t)$, then $dX(t) = dW(t)$, and we identify $a = 0$ and $b = 1$ in Equation 4.10.
2. Let $Y(t) = F(W(t)) = W^2(t)/2$. Then $\partial F/\partial W = W$, $\partial^2 F/\partial W^2 = 1$, and $\partial F/\partial t = 0$.
3. Recite Itô's Lemma:

$$dY(t) = \left[\frac{\partial F}{\partial X}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}b^2\right]dt + \frac{\partial F}{\partial X}b\,dW(t),$$

   so that

$$dY(t) = \frac{1}{2}\,dt + W(t)\,dW(t).$$

4. Integrating the preceding equation, we get

$$W^2(t)/2 = Y(t) = \frac{t}{2} + \int_0^t W(s)\,dW(s).$$

In other words,

$$\int_0^t W(s)\,dW(s) = \frac{W^2(t)}{2} - \frac{t}{2}\,!!!$$

Example 4.4 *Let W_t be a standard Brownian motion and let $Y_t = W_t^3$. Evaluate dY_t.*

Let $X_t = W_t$ and $F(X, t) = X_t^3$. Then the diffusion is $dX_t = dW_t$ with $a = 0$ and $b = 1$. Further

$$\frac{\partial F}{\partial X} = 3X^2, \quad \frac{\partial^2 F}{\partial X^2} = 6X, \quad \frac{\partial F}{\partial t} = 0.$$

Using Itô's lemma, we have

$$dY_t = 3W_t\,dt + 3W_t^2\,dW_t.$$

Integrating both sides of this equation, we get

$$\int_0^t dY_s = \int_0^t 3W_s\,ds + \int_0^t 3W_s^2\,dW_s,$$

$$Y_t = W_t^3 = 3\int_0^t W_s\,ds + 3\int_0^t W_s^2\,dW_s,$$

In other words,

$$\int_0^t W_s^2\,dW_s = \frac{W_t^3}{3} - \int_0^t W_s\,ds.$$

In general, one gets

$$\int_0^t W_s^m\,dW_s = \frac{W_t^{m+1}}{m+1} - \frac{m}{2}\int_0^t W_s^{m-1}\,ds, \quad m = 0, 1, 2, \dots\ . \tag{4.11}$$

Example 4.5 *Let*

$$dX_t = \frac{1}{2}X_t\,dt + X_t\,dW_t. \tag{4.12}$$

Evaluate $d\log X_t$.

From the given diffusion, we have $a = \frac{X_t}{2}$ and $b = X_t$. Let $Y_t = F(X, t) = \log X_t$. Then

$$\frac{\partial F}{\partial X} = \frac{1}{X}, \quad \frac{\partial^2 F}{\partial X^2} = -\frac{1}{X^2}, \quad \frac{\partial F}{\partial t} = 0.$$

Using Itô's lemma, we get $dY_t = d\log X_t = dW_t$. That is, $Y_t = W_t$ Therefore, $X_t = e^{W_t}$ is a solution to Equation 4.12.