Option Pricing Model comparing Louis Bachelier with Black-Scholes Merton

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Part One: Introduction

1.1 Outline

Louis Bachelier[4] published his dissertation on the Theory of Speculation in 1900. The core element of this was his discussion of market price behaviour and development of an option pricing construct and model. The dissertation is famous for application of stochastic analysis in a Brownian Walk, predating Einstein by some five years.

In the early 1973 Black-Scholes[5] publishing their seminal paper documenting the option pricing model written in conjunction with the empirical paper published in 1972[6]; Merton[7] is accredited for contributing to these papers, particularly with regard to the derivation of the portfolio hedge and partial differentiation equation, the Black-Scholes pde. He importantly developed core aspects of the theory including Americanisation in his seminal paper on rational option pricing.

The present paper compares the construct and form of the Black-Scholes Merton (B-SM) and the Louis Bachelier option pricing models in terms of their contemporary markets and contracts, and the underlying pricing construct. To illustrate the comparison the Louis Bachelier model is adapted for features of modern traded option contracts by allowing for the premium's present value form, and incorporating the log-normal and continuous compound assumptions. This demonstrates the B-SM model approximation for the pdf at the boundary point using the differential in the cdf +/- the standard dispersal. This comparison concludes that the Black-Scholes Merton model, and its related precedents and antecedents, are thus approximations and derivative of the Louis Bachelier construct flowing from application of the Fourier heat equation.

As part of the analysis the paper reviews critiques of the Bachelier construct in the financial literature; and similarly critiques the modelling and logical construct behind the Black-Scholes Merton model. In particular,

[1] I would like to acknowledge the University of Saint Joseph, Macao's continued support, and Dr Patrick Harvey's, CUHK, ongoing and long term encouragement for me to work to publication of this and other work that is under development overtime. Any factual, mathematical or other errors or false comment are solely the responsibility of the author.
[2] This paper is intended to be part of a series discussing aspects of derivative pricing models and their application.
[3] This is a revised version of the previous paper (Thomson, 2015) with a restatement of various arguments particularly related to operation of the log-normal distribution in terms of the modern application, and its relationship to market rates. In addition the paper has been extended it to include the B-SM model critique and definition of an appropriate discount rate. The paper now also includes a comparison of the model Greeks.
[4] (Bachelier, Theorie de la Speculation, 1900) (Bachelier, Theory of Speculation, (1900), 1964) (Bachelier, Speculation and the Calculus of Probability, 1938) (Bachelier, Theory of Speculation, 2006)
[5] (Black & Scholes, The Pricing of Options and Corporate Liabilities, 1973)
[6] (Black & Scholes, The Valuation of Option Contracts and a Test of Market Efficiency, 1972)
[7] (Merton, Theory of Rational Option Pricing, 1973)

the core propositions of the risk free or perfect hedge, and the lognormal distribution or limited liability assumption.

The importance of reviewing these constructs and model designs is continued extension of the theory to new applications, and increased model complexity that has allowed application of the base concepts to financial market contracts, accounting and other regulatory pricing rules, derivatives and related more exotic instruments, optimal corporate finance structure, business valuation (real options) and management decisions. In essence it is crucial in finance literature, academic, business and economic, for such a ubiquitous theoretical framework to justify its base principles.

1.2 Paper structure

This comparison paper follows a structure of providing a background for the respective model in their contemporary setting, stating the underlying construct and formal models and providing a critique or review of extant critiques associated with the papers. This provides the framework for a relatively straight-forward comparison of the work to be prepared. From the former critiques and comparative work conclusions are drawn as to an effective pricing model for such contingent contracts and business valuation.

This first part in addition outlines necessary statistical and notational issues. Part Two focuses on the Black-Scholes Merton model outlining its underlying market and contract; development heritage and model construct; and critiques of the core propositions. Part Three undertakes a background of the Louis Bachelier model reviewing the contract form, payoff matrix, the underlying asset price path and issues such as Bachelier's Coefficient of Instability; formulates the Bachelier model in its original form; and then reviews the model's financial literature critiques. Part Four, adapts the Bachelier model for the present value nature of modern option contracts and key B-SM aspects for comparison. That is, the log-normal and continuous compounding formulations. Part Five, provides a brief comparison of the Bachelier and the Black-Scholes Merton models demonstrating that the latter approximates the earlier. Part Six, draws the conclusions and the Appendix provides Greeks for both the standard B-SM model and Bachelier model provided in the paper.

1.3 Notation: Statistical Parameters and Market Returns on Assets

1.3.1 Model Assumptions

The Louis Bachelier and the Black-Scholes Merton models apply asset returns or asset prices in distinctively different manners relating to assumptions on the underlying asset price path and contractual form.

Louis Bachelier's[8] model prices an option contract over a futures contract with all payments at maturity, making a stated discount unnecessary. The statistical analysis applies an arithmetic return method with normally distributed dispersal measure.

The Black-Scholes Merton[9] model prices a contract where the premium is valued at issuance or on purchase, with closing transactions being subject to the option exercise either at maturity or before. Hence this model applies a discounted present value price. The statistical analysis applies a continuously compounded return with the dispersal being described by a lognormal distribution.

[8] (Bachelier, Theory of Speculation, (1900), 1964)
[9] (Black & Scholes, The Pricing of Options and Corporate Liabilities, 1973) (Merton, Theory of Rational Option Pricing, 1973)

1.3.2 Time

In this paper a subscript 't'is used to define the point of valuation for the model and related elements. Where used the subscripts of '0' and 'T' imply respectively the time of issuance or maturity of the derivative contract. Thus, 't' represents any point in the period $0 \leq \mathrm{t} \leq \mathrm{T}$. The subscript, '$\tau$', represents the time to maturity of the contract or investment period implied in the model, that is $\tau = \mathrm{T} - \mathrm{t}$

1.3.3 Measures of Return

For a period Δt measured in annual terms, e.g. for a 6 month term Δt is 0.5, where P_t represents the opening price and $P_{t+\Delta t}$ represents the closing or price at maturity then the returns or yields in annual terms (assuming no dividends or costs) can be measured as:

Arithmetic return, r^a, used by Louis Bachelier is calculated as

$$r^a = \left[\frac{P_{t+\Delta t} - P_t}{P_t}\right]\frac{1}{\Delta t} = \frac{P_{t+\Delta t}}{P_t} - 1; \text{ giving } P_{t+\Delta t} = P_t(1 + r^a \Delta t) \quad (1a)$$

Geometric return, r^g, is given as:

$$r^g = \left(\frac{P_{t+\Delta t}}{P_t} - 1\right)^{\frac{1}{\Delta t}}; \text{ giving } P_{t+\Delta t} = P_t(1 + r^g)^{\Delta t} \quad (1b)$$

Continuously compounded, or exponential return, r^e, as used in the B-SM model is as:

$$r^e = \frac{1}{\Delta t} ln\left(\frac{P_{t+\Delta t}}{P_t}\right); \text{ giving } P_{t+\Delta t} = P_t e^{r^e \Delta t} \quad (1c)$$

A key property arising from the present value discount analysis given the above price paths is that the present value must hold the same value sign as the expected or priced future value of the contract. As in, given $P_{t+\Delta t} > 0$ then $P_t > 0$; or given $P_{t+\Delta t} < 0$ then $P_t < 0$.

1.3.4 Probability Distributions

A key difference in the form of the constructs, as stated, is the shape of the Gaussian probability distributions. Bachelier uses the Normal Distribution; modern constructs use a Lognormal Distribution.

The Normal distribution

$N(.)$ is the cumulative normal probability density function, cdf

$$N(h) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{h} e^{\frac{-z^2}{2}} dz \quad (2a)$$

$z = \frac{x-\mu}{\sigma}$; being the standardised normal

The normal probability density function, pdf, is

$$n(h) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-z^2}{2}} \quad (2b)$$

The Lognormal distribution

This is in essence as above, adapted for the lognormal properties and parameters of the variable

$N^{ln}(.)$ is the cumulative lognormal probability density function, cdf

$$N^{ln}(h) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{h} e^{\frac{-z^2}{2}} dz \quad (2c)$$

$z = \frac{\ln(x)-\mu}{\sigma}$; being the standardised normal where $x > 0$

The normal probability density function, pdf, is

$$n^{ln}(h) = \frac{1}{x\sigma\sqrt{2\pi}} e^{\frac{-z^2}{2}} \qquad (2d)$$

1.3.5 Price Paths

The asset price paths follows a Brownian path allowing for a stochastic or Wiener process, *dz*, relevant to this paper and giving the related market return on the spot asset for the relevant term*, gives:*

Arithmetic terms is

$P_{t+\Delta t} = P_t(1 + \mu^a \Delta t + {}_a\sigma\sqrt{\Delta t}dz)$; by extension $r^a \Delta t = \mu^a \Delta t + \sigma^a \sqrt{\Delta t}dz$ (3a)

For a normal distributed dispersal, where $E(\sigma^a\sqrt{\Delta t}dz) = 0$, then

$E(P_{t+\Delta t}) = P_t(1 + \mu^a \Delta t)$, and the return $E(r^a) = \mu^a$ (3b)

For an arithmetic log-normal distribution, where $E(\sigma^a\sqrt{\Delta t}dz) = \frac{\sigma^{a^2}}{2}\Delta t$, then

$E(P_{t+\Delta t}) = P_t\left[1 + \left(\mu^a + \frac{\sigma^{a^2}}{2}\right)\Delta t\right]$, and the return $E(r^a) = \left(\mu^a + \frac{\sigma^{a^2}}{2}\right)$ (3c)

Exponential terms, with log-normal distribution

$P_{t+\Delta t} = P_t e^{\mu^e \Delta t + \sigma^e \sqrt{\Delta t}dz}$; and by extension $r^e \Delta t = \mu^e \Delta t + \sigma^e \sqrt{\Delta t}dz$ (4a)

$E(P_{t+\Delta t}) = P_t e^{\left(\mu^e + \frac{\sigma^{e^2}}{2}\right)\Delta t}; \; where\ E(r^e) = \mu^e + \frac{\sigma^{e^2}}{2}$ (4b)

It is worth noting the Arithmetic normal mean and Exponential mean statistical relationship, which while similar is not related to the above

$$\mu^e = \mu^a - \frac{\sigma^{a^2}}{2} \qquad (5)$$

1.3.6 Market returns

It is crucial for understanding the modern constructs and then B-SM to recognise the relationship between the market rates and the statistical parameters. At time t the above expected return with the specific characteristics defined equates to the market return over the investment term. This will be subject to various market adjustments as defined in general market analysis – transaction costs, liquidity premia et al. That is, as per above

$r^a = E(r^a) = \mu^a$; for normal distributed arithmetic return; or (6a)

$r^e = E(r^e) = \mu^e + \frac{\sigma^{e^2}}{2}$ for a lognormal distributed exponential return (6b)

An additional notational distinction is between the risk free rate and the underlying asset return. Specifying for illustration the investment term, assumed if not stated, and using either an exponential rate or arithmetic return as appropriate.

$r_{f,\tau}$ risk free discount rate for investment term τ,

$r_{S,\tau}$ underlying spot asset discount rate for investment term τ.

Part Two: Black-Scholes Merton Model

2.1 Outline

The option model construct in Black-Scholes Merton and related papers in general are based on warrant (longer term) or option (short term) contracts being derivatives of underlying equity shares. This model's form draws on the groundwork undertaken by Kruizenga[10] in defining the market behaviour and modelling works by Case Sprenkle[11] and James Boness[12] in the late 1950s and early 1960s. The discussion following is based on their respective papers in the Cootner[13] publication of 1964.

The following reviews and outlines the underlying market contract form, the resultant contract payoffs, and the price path assumed; the B-SM model and behaviour of its value components. This is then reviewed to illustrate alternative forms of the model to facilitate comparison, and to support a critical analysis of key B-SM construct propositions.

2.2 The Contract

In its' simplest form the modern option contract provides the buyer or holder of a Call Option a right but not an obligation to purchase a share on payment at exercise of an agreed price. For Put Options this is a right to sell. To obtain the right the buyer pays on contract issuance or on purchase a premium to the seller or writer, which creates a timing issue for pricing as payments are made at different points in the contracts life.

The exercise terms vary as to when the buyer has the right to exercise. The European form of the option contract is exercisable only at maturity, while the American form may be exercised throughout its life. The Black-Scholes model was for the European form, while Merton generalised this for the American contract[14].

The contracts maybe exchange traded or traded in over the counter option markets, the key modern form which triggered and was supported by the 1970s literature being the equity share and index contracts traded on the CBOT Exchange from 1973. Clearly contractual variations on options had been traded different on exchanges such as the Paris Bourse for an extended time. Exchange traded contracts take a standardised formulation facilitating development of valuation models.

2.3 The Black-Scholes Merton Model

2.3.1 The Option Payoff & Profit

The contract profit and payoff for the modern European call option contract at maturity follow. For simplicity modelling in this paper assumes no rights to disbursements or dividends made to holders, and take the European form.

[10] (Kruizenga, 1964)
[11] (Sprenkle, 1964)
[12] (Boness, 1964). Boness translated Louis Bachelier's dissertation from French in the 1950s (Cootner, 1964)
[13] (Cootner, 1964)
[14] As noted later, it is worth reading the back half of the Bachelier dissertation with respect to Merton's Americanisation arguments

Share Price at maturity, T	Profit at maturity: (1) Call Option Buyer	Issuer	(2) Put Option Buyer	Issuer
$S_T > X$	$S_T - X - C_t e^{\alpha\tau}$	$X - S_T + C_0 e^{\alpha\tau}$	$0 - P_t e^{\alpha\tau}$	$\{S_T\} + P_0 e^{\alpha\tau}$
$S_T \le X$	$0 - C_t e^{\alpha\tau}$	$\{S_T\} + C_0 e^{\alpha\tau}$	$X - S_T - P_t e^{\alpha\tau}$	$S_T - X + P_0 e^{\alpha\tau}$

where

S_T the spot price at maturity

$\{S_T\}$ gives the paper position of the seller at maturity.

X the Exercise price

$C_t e^{\alpha\tau} \,\&\, P_t e^{\alpha\tau}$ respective premiums paid at issuance or on purchase, '*t*' given a maturity, '*T*' for a Call and Put options;

α the investment return applied to align the values at maturity or exercise point with the premium paid taking up the trade. α is typically set at zero for illustrative purposes.

In simple terms the payoff matrices at maturity for the buyer of a call or put option respectively are:

$$C_T = max[S_T - X, 0] \ or \ P_T = max[0, X - S_T] \tag{7}$$

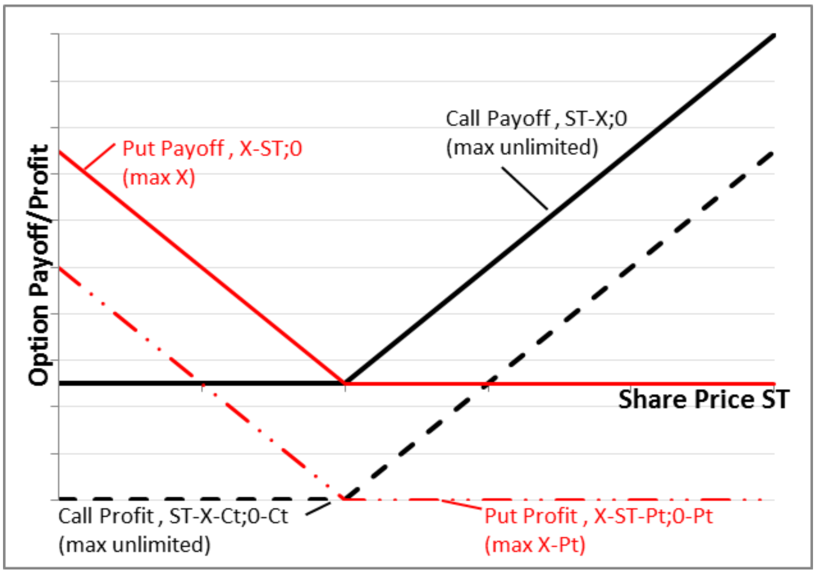

Figure 1 Option Payoff & Profit Diagram given St > 0

Figure 1 illustrates the payoff (solid line) and profit (dashed line) positions on European Calls (Black) and Puts (Red) as the underlying asset (share) price increases. A minimum price of zero is shown as per log-normal model assumptions and illustrates an aspect of outcome difference between Calls, unlimited upside, against Puts with a limited payoff in a log-normal context. This arises as the log-normal is limited at zero, however should a normal Gaussian dispersal be used the Put and call option payoffs equate.

2.3.2 A Price Dichotomy

There is a clear pricing dichotomy implied in options and futures contracts due to the floating and fixed price nature of the underlying asset price and the exercise prices. That is the underlying asset price moves using a stochastic process in time, as per the defined price paths while the Exercise price is fixed or determined on issuance. This dichotomy arguably contributes to justification of risk free rate discount.

Interestingly, this is an example of a relativistic affect in that the dispersal can be seen to be of the underlying asset from the perspective of the exercise price at maturity, or of the exercise price from the perspective of the underlying asset price approaching maturity. The crucial measure in the derivative pricing equations for say options and futures is the relative return necessary or the distance from the exercise price for the underlying asset price and its expected value at maturity.

Mathematically this can be observed in the standard Gaussian cumulative probability function and the density function, refer Part One. Taking the lognormal form of the pdf we have the measure of dispersal

$z = \frac{\ln(x)-\mu}{\sigma}$; which in B-S M modelling translates to

$d = \frac{\ln(\frac{S_t}{Xe^{-t_f\tau}})}{\sigma\sqrt{\tau}}$; which to make it clear equates to $\frac{\ln(S_t)-\ln\left(Xe^{-r_f\tau}\right)}{\sigma\sqrt{\tau}}$

The numerator reflecting the degree of movement in price required for exercise, and the denominator acting as the measure of standard error or dispersal, giving a standardised measure of possible movement at an instant leading to the definition of the probability. That is the critical measure is the degree of distance or price change required for an exercise to occur at the instant of valuation.

2.3.3 The Price Path of the Underlying Asset

The price path, refer section 1.3.5, excluding dividends and other distributions, for the underlying asset in the Black-Scholes Merton construct reflects a continuous price model assuming a lognormal distribution:

$$\ln\left(\frac{S_{t+\Delta t}}{S_t}\right) = \mu^e \Delta t + \sigma^e \sqrt{\Delta t} dz \quad (8a)$$

Or, $S_{t+\Delta t} = S_t e^{\mu^e \Delta t + \sigma^e \sqrt{\Delta t} dz}$ (8b)

Note, at time 't' the expected share price at 'T', where $\tau = T - t$, is given by

$$E\left(S_{t,T}\right) = S_t e^{\mu^e \tau + \frac{\sigma^{e2}}{2}\tau}; \text{ with } E\left(\sigma^e\sqrt{\tau}dz\right) \to \frac{\sigma^{e2}}{2}\tau, \quad (8c)$$

The expected value reflecting properties of the log normal curve

Where

- S_t spot price of the underlying asset at time t.
- μ^e expected yield on the underlying asset ex-ante, or the mean yield of the underlying asset ex-post over the defined period in an exponential setting
- σ^e standard deviation expected for a continuous log-normal return on the price movements for a defined period, giving a dispersal measure expanding with the square root of time $\sqrt{t}$.
- dz Weiner process

2.3.4 The Black-Scholes Merton Model

The Black-Scholes Merton model defines the call option contract premium, C_t^{BS-M}, at the time of issuance or pricing, 't', using continuous compound pricing.

$$C_t^{B-SM} = S_t N^{ln}(d_1) - Xe^{-r_f^e\tau} N^{ln}(d_2) \quad (9)$$

Where,

- X is the exercise price
- r_f^e is the exponential market risk free rate, used to discount the exercise price

$d_1 = \frac{ln\left(\frac{S_t}{X}\right)+\left(r_f^e+\frac{\sigma^{e2}}{2}\right)\tau}{\sigma^e\sqrt{\tau}}$; and $d_2 = d_1 - \sigma^e\sqrt{\tau}$

Note, this model implies a dynamic price path for the options premium based on the dispersal in time of the underlying asset price affected through movement in the spot price, and related movement in the probability of exercise in time.

2.4 Comment

2.4.1 $d_1 \& d_2$

The Black-Scholes Merton formulation of the *boundary* conditions, d_1 & d_2, has created confusion in finance literature over time as neither is a properly stated probability measure, and as the element $r + \frac{\sigma^2}{2}$ is often falsely linked to the arithmetic-exponential return relationship $\mu^a = \mu^e + \frac{\sigma^{a2}}{2}$.

d_1 & d_2 differ due to a statistical approximation technique for the instability coefficient in the price at the boundary point[15] [16], d, or a curvature correction. The technique uses the difference in the cumulative probability density function between d_1 & d_2, that is, $d \pm \frac{\sigma}{2}$ to estimate the differential of the cdf probability which gives probability density function pdf, $n(d)/d$ at the boundary, d ,on the curve. By extension, as shown subsequently, the modern approach represents an approximation of the Bachelier pricing construct which applies the differentiation and thus uses the pdf form.

2.4.2 Sprenkle Approach[17]

Case Sprenkle's applies a statistical solution as opposed to a market discount rate using the return parameters, μ σ, similar to Bachelier; and as with Boness and Bachelier defines the boundary in a standard statistical form , $\frac{X}{S_t}$, such that the cdf equation is the negative of the probability of not exercising. The latter Samuelson and then B-SM approaches invert this achieved in lognormal by switching $ln\left(\frac{X}{kS_t}\right)$ to $-ln\left(\frac{kS_t}{X}\right)$, and in arithmetic by reversing $S_t - X$. Noting in the cdf $N(-h) = 1 - N(h)$

Sprenkle's model uses a single Boundary condition, β, and applies the above approximation as in $\beta_i = \beta \pm \frac{\sigma^e}{2}$. The boundary condition being defined as;

$$\beta = -\frac{ln\left(\frac{X}{kS_t}\right)}{\sigma^e\sqrt{\tau}}, \text{ and} \tag{10a}$$

$$k = e^{\left(\mu^e + \frac{\sigma^{e2}}{2}\right)\tau} \tag{10b}$$

'k' acts either as a discount of the exercise price, X, to the present; or as an expectation return giving the expected future underlying asset price at maturity[18]. In principle, β represents the raw probability

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[15] (Sprenkle, 1964) The form of this approximation appears to have been adopted or developed by Case Sprenkle. Note Sprenkle β is equivalent notation for d. Sprenkle and Boness do not explain the basis of the approximation they have used. The respective PhD dissertations may elaborate on this, which may be drawn on the Feynman Kac work

[16] Bachelier similarly works through a this style of approach, (Bachelier, Theory of Speculation, (1900), 1964), refer p39. In one of several approximations of the option valuation methodology developed in his dissertation applied a related approximation approach using $\pm$25%, or a $\pm 0.6745\sigma$ probability parameter, being the expected dispersal at an instant. Noting Bachelier actually uses 0.4769 being $0.6745/\sqrt{2\pi}$ reflecting the standard application of the Gaussian probability function in his day.
The above, per B-SM, $\pm\frac{\sigma}{2}$ in probability terms is equivalent $\pm$19.15%.

[17] (Sprenkle, 1964)

boundary condition. However, adding to the confusion, due to the form of 'k' we get an apparent shifted result in comparison to latter versions of the model for instance Boness or B-SM. Where:

$$\beta_1 = -\frac{ln\left(\frac{X}{S_t}\right) - \left(\mu^e + \sigma^{e^2}\right)\tau}{\sigma^e\sqrt{\tau}} \qquad (11a)$$

$$\beta_2 = -\frac{ln\left(\frac{X}{S_t}\right) - \mu^e\tau}{\sigma^e\sqrt{\tau}}; \text{ or } \beta_2 = \beta_1 - \sigma^e\sqrt{\tau} \qquad (11b)^{19}$$

That is, β_2 is absent of the volatility measure in the numerator, while for β_1 it is singular note divided by 2. This has implications for the lognormal form of the model as follows in discussing alternative model forms.

2.4.3 Boness[20] Adjustment

James Boness adjusts the boundary condition recognising the underlying asset return, r_s^e, in place of the statistical expected return in determining the premia present value.[21] That is, the definition of k is revised:

$k = e^{r_s^e\tau}$, recognising that $r_s^e = \mu^e + \frac{\sigma^{e2}}{2}$ in the lognormal context

Thus, we find the modern form of the boundary as follows

$$d_1 = -\frac{ln\left(\frac{X}{S_t}\right) - \left(r_s^e + \frac{\sigma^{e2}}{2}\right)\tau}{\sigma^e\sqrt{\tau}}, \text{ and } d_2 = d_1 - \sigma^e\sqrt{\tau} \qquad (12)$$

Black-Scholes Merton later substitute r_f^e for r_s^e

2.4.4 Comments on Sprenkle & Boness

Points and some reiteration on Sprenkle's and Boness's forms of the model

- The boundary condition, as with Bachelier, is inverted relative to Black-Scholes, but thus takes the standard negative Gaussian boundary condition. Samuelson & McKean[22] affect the inversion enabling a positive statement.

$$-\frac{ln\left(\frac{X}{S_t}\right) - \left(r_s^e + \frac{\sigma^{e2}}{2}\right)\tau}{\sigma^e\sqrt{\tau}} \text{ in Boness} \rightarrow +\frac{ln\left(\frac{S_t}{X}\right) + \left(r_s^e + \frac{\sigma^{e2}}{2}\right)\tau}{\sigma^e\sqrt{\tau}} \text{ in Samuelson}$$

- The expected return on the underlying asset, r_s^e or $\mu^e + \frac{\sigma^{e2}}{2}$, is used not the risk free rate, r_f^e
- Sprenkle actually uses the rate to give an expected future share price, i.e. kS_t, in the model. Although as shown by Samuelson and McKean this maybe inverted giving a present value of the exercise price, i.e. is $\frac{X}{k} = Xe^{-r_f^e\tau}$
- The price path is a continuous compounding path, rather than simple interest.
- The lognormal distribution is used to give a zero probability of share prices, S_t, having zero or negative values, justified primarily by the corporate limited liability argument[23].

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[18] As noted previously, this ability to invert the element flows from Samuelson (Samuelson, 1965). This can be applied in affect to both geometric and arithmetic solutions to reverse the sign.
[19] Note this form is later used in defining the log-normal alternative form of B-SM.
[20] James Boness provided the original translation of Bachelier's French dissertation and defers to Bachelier in his paper. (Boness, 1964)
[21] This form of model is then adopted by Samuelson (Samuelson, 1965) and others and then used by Black-Scholes Merton in formulating the classical model. Samuelson try to extend the concept by incorporating CAPM theory.
[22] (Samuelson, 1965), and per (McKean, 1965) in appendix to former

2.5 Alternate formulations of the Black-Scholes Merton Model

To facilitate the comparison the Black-Scholes Merton model is reformulated to emphasise the approximation for the curvature correction at the boundary, and other features of the model. As noted, this is achieved by breaking up the cumulative normal density functions.

2.5.1 *Alternative 1: Using S_t and X in the curvature*

$$C_t^{B-SM} = \left\{\left[S_t - Xe^{-r_f^e\tau}\right] N^{ln}(d)\right\} + \left\{S_t N^{ln}(d_1, d) + Xe^{-r_f^e\tau} N^{ln}(d, d_2)\right\} \quad (13a)$$

where

$\left\{\left[S_t - Xe^{-r_f^e\tau}\right] N^{ln}(d)\right\}$ is the primary probability of exercise in present value terms

$\left\{S_t N^{ln}(d_1, d) + Xe^{-r_f^e T} N^{ln}(d, d_2)\right\}$ is the approximation for the curvature adjustment or instability coefficient at the boundary.

$N^{ln}(d_1, d)$ is for simplicity in writing the net of the cumulative normal probability functions $N^{ln}(d_1) - N^{ln}(d)$

$d = \frac{ln\left(\frac{S_t}{X}\right) + r_f^e\tau}{\sigma\sqrt{\tau}}$, the boundary point

$d_1 = d + \frac{\sigma\sqrt{\tau}}{2}$; and $d_2 = d - \frac{\sigma\sqrt{\tau}}{2}$

2.5.2 *Alternative 2: Using S_t only in the curvature*

$$C_t^{B-SM} = \left\{\left[S_t - Xe^{-r_f^e\tau}\right] N^{ln}(d_2)\right\} + \{S_t N^{ln}(d_1, d_2)\} \quad (13b)$$

This form re-arranges the above model in recognition of a lognormal property. An interesting derivation of the BS-M model reflecting this form is presented by John Hull leading to this form of the model.[24]

Note using the statistical form of the equation as per Sprenkle we have d_2 equating to the Sprenkle's β_2,

$d_2 = -\frac{ln\left(\frac{S_t}{X}\right) + \mu^e\tau}{\sigma^e\sqrt{\tau}}$ in the statistical form of the model.

$d_1 = d_2 + \sigma\sqrt{\tau}$

2.5.3 *Alternative 3: Bachelier (underlying) model*

A further alternative, for reference, is the underlying Louis Bachelier model:

$$C_t^{B-SM} = \{[S_t - Xe^{-r\tau}]N^{ln}(d)\} + \left\{[S_t - Xe^{-r\tau}]\frac{n^{ln}(d)}{d}\right\} \quad (13c)$$

$\left\{[S_t - Xe^{-r\tau}]\frac{n^{ln}(d)}{d}\right\}$ is the element allowing for curvature correction which is most directly approximated by $\left\{S_t N^{ln}(d_1, d) + Xe^{-r_f^e\tau} N^{ln}(d, d_2)\right\}$ in alternative 1.

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[23] This justification for the zero base of the share price, $S_t \geq 0$, using the limited liability argument denies the probability of corporate failure with the underlying equity claim on the business value being negative. Bachelier used arithmetic returns based on analysis of market price changes, and the extreme unlikelihood perceived for failure of the French Government, (Bachelier, Theory of Speculation, 2006) pp40-42.

[24] (Hull, 2009) pp307-309 appendix – Proof of the Black-Scholes Merton Formula

2.6 Black-Scholes Merton in terms of the Bachelier form

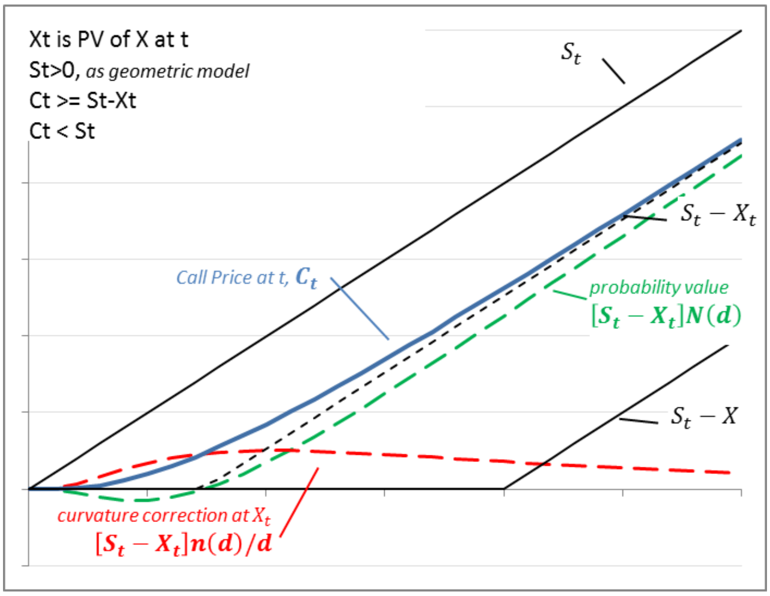

Figure 2 Components of the Black-Scholes Merton model applying the Bachelier approach

Drawing on the final, or Bachelier, alternative above Figure 2 illustrates the model for the call option price at 't', C_t with price constraints consistent with Merton rational pricing - $S_t \geq C_t$; $C_t \geq S_t - X$; and at 't' $C_t \geq S_t - Xe^{-r\tau}$.

The solid blue line is the call price C_t;

the dashed green line is the base cumulative probability component;

the dashed red line is the curvature correction, or measure of instability at the boundary, which follows a lognormal form as $S_t \rightarrow \infty$, with a lower boundary at $S_t = 0$.

This figure identifies the pricing behaviour of the various aspects of the model using the log-normal distribution in determining the option price. The normal distribution form is given later for the Louis Bachelier Model.

2.7 Disputed Points with Black-Scholes Merton Construct

This section discusses justifications for applying the risk free rate and the lognormal dispersal in the model.

2.7.1 A Pricing Conundrum

A pricing conundrum arises under the Black-Scholes Merton option model and other finance derivatives, such as futures pricing models equated through put-call parity, where the risk free rate is applied justified by the instantaneous risk free hedge proposition.

To illustrate, take an option portfolio, $\prod_t(C_t - P_t)$ of long one call and short one put with exercise price, X. Under put-call parity this portfolio equates to a futures contract, f_t, with futures asset price, F, equal to X.

For the underlying asset spot price in time, S_t, price range

$$Xe^{-r_s^e\tau} < S_t \leq Xe^{-r_f^e\tau}$$

S_t, has an expected underlying asset price at maturity, $E(S_T)$, greater than the exercise price:

$$E(S_T) > X; \text{ as } E(S_T) = S_t e^{r_s^e\tau}$$

Thus, at maturity the Call is expected to be exercised, $E(C_T) > 0$, and the Put to lapse, $E(P_T) < 0$.

Hence, the value of the above portfolio, $\prod_t(C_t - P_t)$, will have a positive value at maturity. That is,

$$E[\textstyle\prod_T(C_T - P_T)] = E(C_T) > 0, \text{ as } E(S_T) > X$$

Similarly, the futures contract also has a positive value

$$E[f_T] > 0, \text{ as } E(S_T) > F = X \text{ by definition}$$

But, and this is the conundrum, in the given asset price range, at pricing, i.e. t, both the portfolio value and futures contract price are negative.

$$\prod_t(C_t - P_t) = f_t \leq 0$$

This creates a conundrum as both the B-S M model and related futures contract pricing models run contrary to a basic financial valuation principles. That is, while the portfolio or futures contract values are positive value at maturity, the present value of both has a negative value. That is a positive value can be created in essence from a negative investment which is a simple breach of arbitrage conditions.

This can be extended to show that where the priced 'expected value' or the Futures price for an underlying asset is valued using the risk free rate for any value of the underlying asset an excess portfolio yield to maturity is generated. That is,

For all values of S_t, then at time t, the proposed prices are $S_{t,T}$ and $F_t = S_t e^{r_f^e \tau}$, then

at t, $E\left(S_{t,T}\right) = S_t e^{r_s^e \tau} > S_t e^{r_f^e \tau}$, then

the expected return on the portfolio $\prod_t(C_t - P_t)$, or futures contract, f_t, is $> e^{r_s^e \tau}$ although the underlying asset risk is same.

Both these points question the validity of the proposition in the Black-Scholes Merton construct that discounting the exercise Price by the risk free rate, r_f^e is a valid step. This requires the risk free proposition of B-SM to be tested.

2.7.2 Black-Scholes Merton Risk Free Proposition and pde

2.7.2.1 Standard B-S M pde

This subsection outlines the justification of the risk free rate in the Black-Scholes Merton argument and shows this approach fails to recognise the model's static form in solving a dynamic in time pricing problem. The model is restated using a dynamic underlying asset price path illustrating an arguable flaw in the construct.

The B-S M model uses the instantaneous risk free rate to discount or determine the present value of the exercise price. The primary justification[25] is the contention that by continuously and instantaneously adjusting a hedge portfolio a perfect dynamic hedge can be created, thereby removing the risk exposure arising from the underlying asset's price behaviour. The hedge portfolio is made up proportionally[26] of a long position in a single call option contract offset by $N(d_1)$ or $\frac{\partial C_t}{\partial S_t}$ underlying asset, or shares for example. This ratio, $\frac{\partial C_t}{\partial S_t}$, reflects the instantaneous proportional expected price movement of the call given an instantaneous change in the underlying asset price.

[25] Alternate derivations developed include the expected value approach of for instance Case Sprenkle; and replication arguments developed using risk free debt funding or risk free funds sources to replicate the option payoff. These latter are built on generally unstated justifications essentially reliant on the dichotomy in the price form and the perfect hedge argument. As such these not proofs of the replication construct.

[26] This is argued to be proportional, not absolute, thus overcoming the failure of the hedge portfolio to be self-financing. Self-financing is a requirement stipulated for a perfect hedge by Merton. Refer (Bergman, 1981), (Macdonald, 1997) and related discussion. However, this relationship as shown is stochastic instant to instant, a discussion will be provided separately

Applying this to Ito's Lemma the BSM pde can be derived[27]. Ito's Lemma given the price path and portfolio construct is:

$$dC_t = \left(\frac{\partial C_t}{\partial t} + \mu S \frac{\partial C_t}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_t}{\partial S^2}\right) dt + \sigma S dz \frac{\partial C_t}{\partial S} \tag{14a}$$

With a price path for the underlying asset, as previously defined:

$$S_{t+\Delta t} = S_t e^{\mu^e \Delta t + \sigma^e \sqrt{\Delta t} dz}$$

Then given the above hedge portfolio

$$\Pi_t = C_t - \frac{\partial C_t}{\partial S} S$$

Determining the form for a change in time

$$\frac{d\Pi_t}{dt} = \frac{\partial C_t}{dt} - \frac{\partial C_t}{\partial S}\frac{\partial S}{dt}$$

Substituting the Ito Lemma and underlying asset price paths, then rearranging the Fourier heat equations is given[28]:

$$\frac{d\Pi_t}{dt} = \frac{\partial C_t}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 C_t}{\partial S^2}, \tag{14b}$$

The basic BS-M proposition relies on the above, on basis that as there is no observable stochastic element, i.e. dz is absent, in this equation that substitution into the model of instantaneous risk free rate, r_f^e, is justified. Thus the portfolio return is:

$$\frac{d\Pi_t}{dt} = r_f^e \Pi_t \tag{14c}$$

Restated into the mean movement element to Ito's Lemma gives the Black-Scholes Merton pde

$$\frac{\partial C_t}{\partial t} + r_f^e S \frac{\partial C_t}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_t}{\partial S^2} = r_f^e C_t \tag{14d}$$

That is the Black-Scholes Merton pde is dependent on the capacity to create a perfect hedge through instantaneously adjusting the hedge reflecting market price movements given an apparent absence of stochasticity in the arbitrage portfolio.

2.7.2.2 Dynamic v Static formulation

A key property of this model is its static form, as implicit in Louis Bachelier's dissertation:

> *it is possible to study mathematically the static state of the market at a given instant, that is to say, to establish the probability distribution of the variations in price that the market admits at this instant.*[29]

That is, while the underlying asset price is recognised to follow a defined stochastic price path the option price model and above pde derivation does not incorporate this price path into the construct.

The price path can be incorporated to give a dynamic process for the model and thereby affected in the partial derivative of the option price for a change in time. Putting aside the potential for instability in the price path behavioural parameter's, μ and σ and recognising the price path is stochastic, we have

[27] Refer Black Scholes, (Black & Scholes, The Pricing of Options and Corporate Liabilities, 1973), pp641-643 John Hull for a version of such a derivation (Hull, 2009) pp287-289.
[28] Recognised key to Bachelier's work, refer (Bachelier, Theory of Speculation, 2006) pp 40-42.
[29] (Bachelier, Theory of Speculation, 2006) p15

$S_{t+\Delta t} = S_t e^{\mu^e \Delta t + \sigma^e \sqrt{\Delta t} dz}$; and for notational simplicity

$\lambda_t = e^{\mu^e \Delta t + \sigma^e \sqrt{\Delta t} dz}$; then

$S_{t+\Delta t} = S_t \lambda_{t+\Delta t}$; which is dynamic, or (15a)

$\frac{dS_t}{dt} = S_t \frac{d\lambda_t}{dt} = \left(\mu^e \Delta t + \sigma^e \sqrt{\Delta t} dz\right) S_t$

Noting, at $\Delta t = 0$, then $\lambda_t = 1$ and thus $S_t = S_t.1$, giving the standard or classic model form.

Then substituting into the standard option price model, we have

$$C_t^{BS-M} = S_t \lambda_t N^{ln}(d_1) - X e^{-\alpha^e \tau} N^{ln}(d_2) \quad (15b)$$

$d_1 = \frac{ln\left(\frac{S_t \lambda_t}{X}\right) + \left(\alpha^e + \frac{\sigma^{e2}}{2}\right)\tau}{\sigma^e \sqrt{\tau}}$; and $d_2 = d_1 - \sigma^e \sqrt{\tau}$

α^e is an un defined return factor subject to analysis of the pde.

The sensitivity of the model to time, as in theta, can then be restated as:

$$\frac{dC_t^{BS-M}}{dt} = \widehat{\theta_t} = \left\{\frac{\partial \lambda_t}{\partial t} S_t N^{ln}(d_1)\right\} + S_t \frac{\partial N^{ln}(d_1)}{\partial t} - \frac{\partial e^{-\alpha^e \tau}}{\partial t} X N^{ln}(d_2) - X e^{-\alpha^e \tau} \frac{\partial N^{ln}(d_2)}{\partial t}$$

$$\frac{dC_t^{BS-M}}{dt} = \widehat{\theta_t} = \left\{\left(\mu^e \tau + \sigma^e \sqrt{\tau} dz\right) S_t N^{ln}(d_1)\right\} + \frac{S_t \sigma^e}{2\sqrt{\tau}} n^{ln}(d_1) + \alpha^e \tau X e^{-\alpha^e \tau} N^{ln}(d_2) \quad (15c)$$

The element in brackets {} relates to the dynamic element and remains stochastic.

This revised pricing formulation of the model retains a stochastic element. That is, the standard B-S M model construct does not allow for a dynamic setting where the underlying option price remains stochastic. Hence in the B_SM models non-stochastic price justification arises from the static definition of the model and not absence of the stochastic process. Thus, the hedge is exposed to underlying asset price risk.

2.7.2.3 A Dynamic pde

This is demonstrated by reworking the above pde solution in the dynamic setting. Taking the solution from the heat equation

$$\frac{d\Pi_t}{dt} = \frac{\partial C_t}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_t}{\partial S^2} \quad (14b)$$

Determining the Gamma or sensitivity of the Call to the underlying asset price to second derivative gives

$$\frac{\partial^2 C_t}{\partial S^2} = \frac{n^{ln}(d_1)}{S_t \sigma^e \sqrt{\tau}}$$

And bringing down $\hat{\theta}$ and arranging then we have

$$\frac{d\widehat{\Pi_t}}{dt} = \left\{\left(\mu^e \tau + \sigma^e \sqrt{\tau} dz\right) S_t N^{ln}(d_1)\right\} + \frac{S_t \sigma^e}{2\sqrt{\tau}} n^{ln}(d_1) + \alpha^e \tau X e^{-\alpha^e \tau} N^{ln}(d_2) - \frac{\sigma^{e2}}{2} S_t^2 \frac{n^{ln}(d_1)}{S_t \sigma^e \sqrt{\tau}}$$

$$\frac{d\widehat{\Pi_t}}{dt} = \left\{\left(\mu^e \tau + \sigma^e \sqrt{\tau} dz\right) S_t N^{ln}(d_1)\right\} + \alpha^e \tau X e^{-\alpha^e \tau} N^{ln}(d_2) \quad (15d)$$

Thus, given inclusion of a dynamic price path in the model, the portfolio value is dynamic in time and is directly sensitive to the price path of the underlying asset with an expected yield r_S^e.

The result is that Ito's Lemma remains unaltered and the equation retains the underlying asset price behaviour:

$$\left(\frac{\partial C_t}{\partial t} + \mu S \frac{\partial C_t}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_t}{\partial S^2}\right) dt + \sigma S dz \frac{\partial C_t}{\partial S} = dC_t \quad (15e)$$

Then recognising $dC_t = r_s^e C_t$; and

$r_s^e S = \mu S + \sigma S dz$; we have

$$\frac{\partial C_t}{\partial t} + r_s^e S \frac{\partial C_t}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_t}{\partial S^2} = r_s^e C_t \qquad (15f)$$

Thus it is recommended that the Black-Scholes pde should be restated to incorporate the dynamic underlying asset price path in model. A consequence is that the subjective nature of the market price for the underlying is carried through to the option pricing and is not eliminated.

2.7.2.4 Restated B-SM Model

Adapting the BS-M model to incorporate the above in essence produces the James Boness model

$$C_t^{BS-M} = S_t \lambda_t N^{ln}(d_1) - X e^{-r_s^e \tau} N^{ln}(d_2) \qquad (16)$$

$d_1 = \frac{ln\left(\frac{S_t \lambda_t}{X}\right) + \left(r_s^e + \frac{\sigma^{e2}}{2}\right)\tau}{\sigma^e \sqrt{\tau}}$; and $d_2 = d_1 - \sigma^e \sqrt{\tau}$

r_s^e is expected return on the underlying asset for the specified investment term

λ_t is the dynamic price element & $E(\lambda_t) = e^{\left(r_s^e + \frac{\sigma^{e2}}{2}\right)\tau}$

Where at the instant of pricing, $t, \lambda_t \to 1$, and hence the model at this instant is mathematically static as recognised by Bachelier.

2.7.2.5 Comment[30]

This above illustrates that the B-SM model's reliance on the instantaneous risk free or perfect hedge is satisfied due to the static nature of the option model applied in determining the pde. It is then shown that restatement of the model to recognise the dynamic price path of the underlying asset in time enables Ito's Lemma to be applied without adjustment, and that price behaviour in the model is driven by the underlying asset return.

It is proposed that the appropriate rate for discounting the exercise price in option price models is thus the underlying asset expected return. We can thus make several quick points arising from the prior discussion:

- A model adjusted for the underlying asset return is consistent with the James Boness and the Case Sprenkle versions.
- As the underlying asset return is subjective and dependent on market pricing mechanisms then the option model takes on this character.
- the price conundrum outlined is eliminated as $r_f^e \to r_s^e$
- where $S_t = Xe^{-}r_s^e\tau$ then the option price is driven solely by the instability coefficient and given over time, τ, the movement is expected then the solely price variance is due to the dispersal.

The underlying asset return is used in defining the Bachelier model for the modern contract.

[30] This result is in keeping with Warren Buffet's (Buffet, 2011) idiosyncratic criticism of the Black-Scholes model in terms of pricing long term Put options which by inference implies standard model Call premiums are undervalued and Put premiums are overvalued.
It can also be shown that the CAPM derivation by Black Scholes can be redrawn to reflect the above outcome where the result is similar to above A separate note will be published on this
Also, note the instantaneous rate must be that associated with suitable term contracts or investment period.

2.7.3 Limited Liability justification for the Lognormal distribution

The modern option contract pricing constructs and asset pricing models utilise the lognormal distribution.

$N^{ln}(h)$ exists for $h > 0$

This is justified both by a corporate limited liability proposition and by a no negative price exchange trading rule. Essentially that the underlying asset (an equity share) listed on an exchange cannot take on a negative value. Analogously, adopting the lognormal distribution proposition implies that there is a 'zero probability' of the underlying asset having a price at or below zero. That is,

Application of $N^{ln}(h)$ implies $S_t > 0$ and $E(S_T) > 0$ for all values of t.

However, this proposition appears flawed both for most non-equity share applications where no such limit exists; and as even for equity shares there is a positive probability of default on the underlying equity or business, $\ddot{S}_t$, and by extension delisting[31]. . That is,

$\Phi\big(default, or\ \ddot{S}_t \leq 0\big) > 0$, and hence $\Phi\big(\ddot{S}_t[equity\ share] = 0\big) > 0$

As this breaches the no zero probability condition on lognormal distribution its use in the pricing model should be questioned. Consequently, an appropriate dispersal assumption for the general case should adopt the normal distribution, or form thereof, at least in terms that it gives a measure or proxy for default or delisting of an equity share. That is the probability needs to allow for $\ddot{S}_t$ being in the price range:

$-\infty < \ddot{S}_t < \infty$

This result can then be reasonably adapted in general applications of option pricing models with an underlying asset return used as the discount rate. Such generality can be applied in cases for equity shares traded on exchanges, options on internal business investments say real options, executive compensation corporate finance structuring and by extension futures contracts. Noting, many real option cases are in effect unlimited in risk exposure to the downside.

[31] An interesting issue for exploration in pricing and corporate finance structure is that the equity (share) value premium is maximised at the point the underlying business equity value is nil, a property of derivative pricing.

Part Three: Louis Bachelier Model

Louis Bachelier's Theory of Speculation PhD dissertation was published in 1900[32] referencing derivatives contracts in place contemporaneously. The statement and description of the underlying contract valued by Bachelier is necessary to understanding aspects of this model and its critiques[33]. This part correctly states the form of contract and market operation allowing Bachelier's model to be properly constructed. The following part then reformulates it for the modern contract.

3.1 The Option Contract

The option contract considered by Louis Bachelier was a contract on exchange traded futures over French Government Rentes with, for instance a 3% coupon. Key features of the contract are:

- the contract is in future values priced at maturity - as the contract was over a futures contract, with the premium forfeited or paid at maturity offset as appropriate against the contract value.
- the exercise price was negotiable as a spread over the market futures price for the Rentes
- the priced spread was for fixed 'forfeits' (premiums) - 50, 25, 10 and 5 centimes per contract. The lower the premium the greater the implied spread against the expected future price.
- the option was exercisable up to the day before delivery on the futures contract, i.e. an American form of contract.

3.2 French Government Rentes

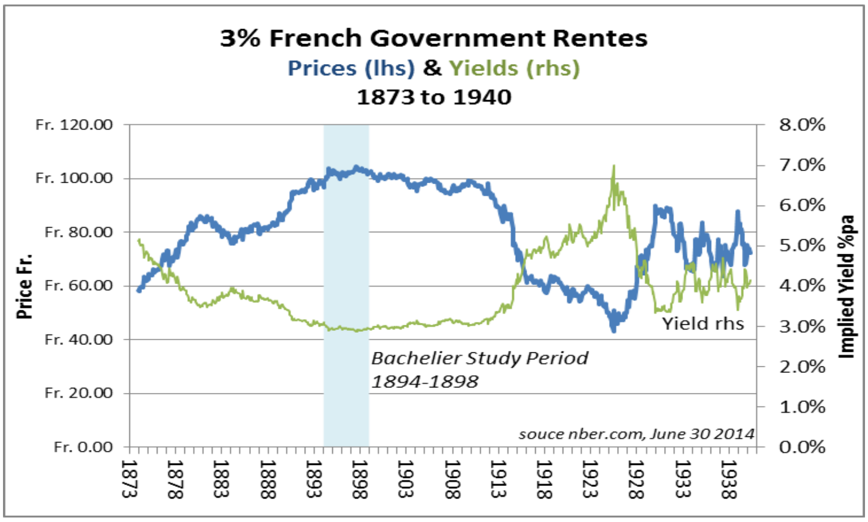

Figure 3 French Rentes Price Fr. 1873-1941, showing period analysed by Bachelier

The underlying asset for this contract was the French Government Rentes, which are perpetuities with a constant coupon, similar to British Consols – i.e. perpetual coupon paying bonds. In stable economic conditions these can be expected to hold a constant price, as experienced when Bachelier conducted his study, 1894 to 1898 – see figure 3.

[32] (Bachelier, Theorie de la Speculation, 1900) (Bachelier, Theory of Speculation, 2006) (Bachelier, Theory of Speculation, (1900), 1964) and rewritten over time, for instance (Bachelier, Speculation and the Calculus of Probability, 1938)

[33] Most modern critiques refer primarily to Bachelier's paper being flawed as it did not address contract issues related the modern options over equity shares. Examples related to his treatment of equity share price behaviour are by Samuelson, Merton and Smith et al. His work and analysis addressed specifically Options on Futures over French Government Rentes hence the aspects of equity shares discussed in modern academic papers are not directly relevant. Although, Bachelier's construct and model are sufficiently general that we can reformulate to give a sound model for options over equity shares.

3.3 Market & Price Behaviour Assumptions by Bachelier

To set up the model Bachelier undertook an extended digression on market price behaviour and mathematical expectation. He recognises certain price precepts and defines a series of propositions on expected behaviour necessary to apply statistical inference in the pricing model. Bachelier argues that suspension of value analysis provides a superior result as it removes risk arising from intrinsic or economic value techniques these being absorbed into the instability factor in his analysis.

3.3.1 Mathematical disinterest

The first principle is mathematical disinterest in aspects of economic valuation.

> *Therefore we are careful not to undertake the analysis of the causes of the fluctuations; such an analysis would be vain and would only lead to errors. …It is precisely because this study appears inextricably complicated that it is in reality, very simple.*[34]

That is, techniques applied take no interest in the underlying drivers of economic value enabling, Bachelier argues, market uncertainties to be packaged together into the statistical probability measures of volatility and market expectation.

3.3.2 Market is in equilibrium and efficient[35]

> *At a given instant the market believes neither in a rise nor a fall of the true price*

Enabling the disinterest to apply facilitates application of fair game principles. The market is assumed to be efficient and to clear at every instant.

3.3.3 Mathematical expectation & a fair game[36]

Bachelier sets down the fundamental economic and mathematical argument enabling the probability and value expectations to be brought together.

This is in effect a state probability approach, where the total mathematical expectation, for a speculator, j, given both the possible states facing that speculator, $s_{j,i}$, and the respective probabilities of those states, $\Phi(s_{j,i})$, can be determined as a weighted average of these elements, at time t. That is, giving the expected value for an investment.

$$E(S_{j,t}) = \sum_{i=1}^{N} \Phi(s_{j,i,t}).\, s_{j,i,t} \qquad (17a)$$

Bachelier notes that this expectation when summed across all participants should imply that for the market to be fair then there can be no net expectation of a price increase:

$$\sum_{j=1}^{M} E(S_{j,t+1}) = \sum_{j=1}^{M} E(S_{j,t}) \qquad \textit{(17b)}$$

$$\sum_{j=1}^{M} E(S_{j,t+1}) = (1 + i_{t+1}).\sum_{j=1}^{M} E(S_{j,t}) \qquad \textit{(17c)}$$

---

[34] (Bachelier, 1938, p. 10) – emphasis added. Note, Bachelier does not disavow the economic principles involved but argues that the statistical approach requires these to be put aside in the pricing process.

[35] P26

[36] Bachelier outlines these arguments in the initial part of his dissertation. The extended backend of the dissertation addresses certain points more formally and extends to timing of exercise and point of value maximisation. This 'backend' is worth exploration and it is arguable that Merton's arguments on American options and on hedging in a continuous pricing framework replicate these discussions.

3.3.4 The random walk & independence through time

Bachelier through the definition of the probability function assumes a random walk and independence of prices from one instant to the next and enabling him to effectively apply a continuous identical distribution argument, i.e. iid[37]

There can be only one probability law

3.3.5 A Sub-Martingale Price mechanism

While Bachelier notes though that a sub-martingale price mechanism is not consistent with the behaviour of prices exhibited on the Bourse[38], he concludes that the market mechanism allows for an expectation of an increase in prices over time, giving a true price around which the probabilities can adjust. Thus, although not explicit in the dissertation and not necessary in the prevailing market for Rentes or given form of the option contract, Bachelier's approach conceptually provides for a sub-martingale process with a drift being defined by the underlying asset in an arithmetic context. This can be stated:[39]

$$\Delta S_t = S_0\left(1 + \mu\Delta t + \sigma\sqrt{\Delta t}dz\right) \qquad \textit{(3a)}$$

The general statistical argument concludes[40]:

that the mathematical expectation of the speculator is zero

From the above, Bachelier allows for prices to move to a true price at maturity, on an option for instance, and by extension accepts that the interim market must likewise behave in this manner.

3.4 Market and price behaviours:

3.4.1 Relationship futures and option contract pricing

Bachelier recognises the pricing relationships between the option contract and the futures contract over the same underlying asset given the same contract term and related payoffs. The dissertation expands on this by discussing various trading strategies. This is tied to the notion of Put-Call Parity with futures.

$$f_t = C_t - P_t \qquad (18a)$$

Where, f_t is the futures contract price, vis-a-vis the futures price, $F_{t,T}$

Bachelier notes the while futures contract leaves the holder exposed to unlimited gains and losses, while the option contract exposes the holder only to the upside. Thus he argues the option holder pays a premium to gain protection against the downside losses based on the spread quoted. That is,

$$C_t = f_t + \rho_{Ct} \qquad (18b)$$

Where, $\rho_{C,t}$ is the premium on a call option representing the benefit from the downside protection by the contract in obtaining access solely to the upside price gain.[41] ρ_{Pt} is for a put option. Giving:

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[37] (Bachelier, 1938, p. 11 & 12)
[38] refer price behaviour 1894-1898, figure 3
[39] This is a key issue in critiques of Bachelier's work as the papers represent he overlooked drift or the Martingale argument; note (Schachermayer & Teichmann, 2007, p. 2) do not. Additionally, authors present that the dissertation did not anticipate Merton's arbitrage arguments although this is implicit in the Put-Call Parity argument above. This is arguable as Bachelier addresses expected dispersal at ∂t, $\pm$25%, and Merton does not adjust for the static formulation in a dynamic context.
[40] P28, and (Bachelier, 1938, p. 11) that the game is fair.
[41] P20

$$f_t = (f_t + \rho_{Ct}) - (f_t + \rho_{Pt}) = \rho_{Ct} - \rho_{Pt} \tag{18c}$$

These premiums by necessity go to zero value at maturity

$$\rho_{Ct}, \rho_{Pt} \rightarrow 0; as\ t \rightarrow T\ or\ \tau \rightarrow 0$$

Tied to this relationship is the concept of price equivalence.

3.4.2 *Equivalence of Prices & Concept of True Prices*

Following on from statements of basic price and contract issues Bachelier outlines the concept of equivalent prices[42] across the respective financial instruments - the underlying asset, i.e. Rentes, the futures contract and the option contract. As shown in figure 4 below the prices for these instruments are necessarily linked and move in relation to each other.

A key pricing concept used by Bachelier and contemporaries is the true price allowing for the futures holder to receive a payout adjustment at maturity for coupons paid when the contract is open.

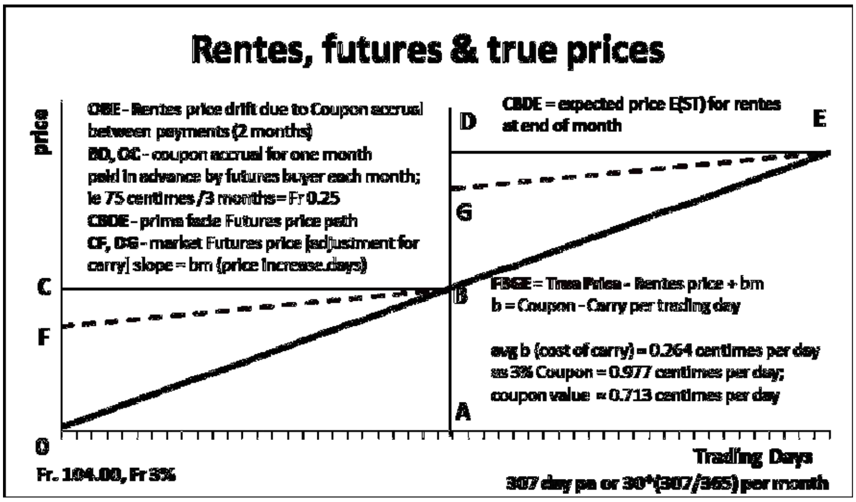

Figure 4 Rentes, futures and True Prices

The true price is determined by a two-step process. First, determine the futures price at time 't', noting $F_{t,T} \equiv E(S_{t,T})$ adjusted for the above coupon entitlement to be paid at contract maturity.

Second, this coupon is discounted at a market rate to give the present value or true Futures Price at any particular time.

This True Price is then used in quoting the option spread, so this spread which is the principle pricing outcome that Bachelier addresses, is what the market uses as the base price.

3.4.3 *The Underlying Asset (Arithmetic) Price Path*

The price for the underlying Rentes, in arithmetic terms, is

$$R_t = \frac{Coupon(Fr.100\ face\ value\ .3\%\ rate)}{market\ yield} + Coupon.(inflator, days\ to\ payment) \tag{19a}$$

$$R_t = \frac{c}{\ddot{r}_t} + c.\frac{m}{91} \tag{19b}$$

Where

R_t the rentes price at time t.

c the coupon payment on the rentes, on a Fr.3 rentes then the quarterly coupon is 75 centimes , or monthly impact is 25 centimes Being the coupon rate pa, i, times face value, $\mathcal{F}$, per quarter. That is: $c = i.\mathcal{F}/4$

$c.\frac{m}{91}$ the arithmetic approach used by Bachelier, geometrically this is $(1 + \ddot{r}_t)^{\frac{m}{91}}$

[42] P17

$\ddot{r}_t$ the market yield or discount price[43]

m days since the last coupon payment on the rentes with 91 days being average period between coupons.[44]

The futures adjusted price, the 'true price', being the expected Futures price for maturity, T, less the discount at time t is:

$$F_{t,T} = E(R_{t,T}) - n\mathscr{b} \tag{19c}$$

Where

$E(R_{t,T})$ expected rentes price, at t, at end of month when next contango payment is due, T, on one month contract. When a contango is paid then

$E(R_{t,T}) = R_t + \frac{c}{3}$, that is for Fr. 3 rentes pa paid quarterly, i.e. $(R_{t,T}) = R_t + 25\ centimes$

$n\mathscr{b}$ the, arithmetic price adjustment for net funding cost of contango.

n number of days remaining on the contract; i.e. $\tau\ or\ (T-t)$

$\mathscr{b}$ the net market discount, 0.264 centimes per day[45].

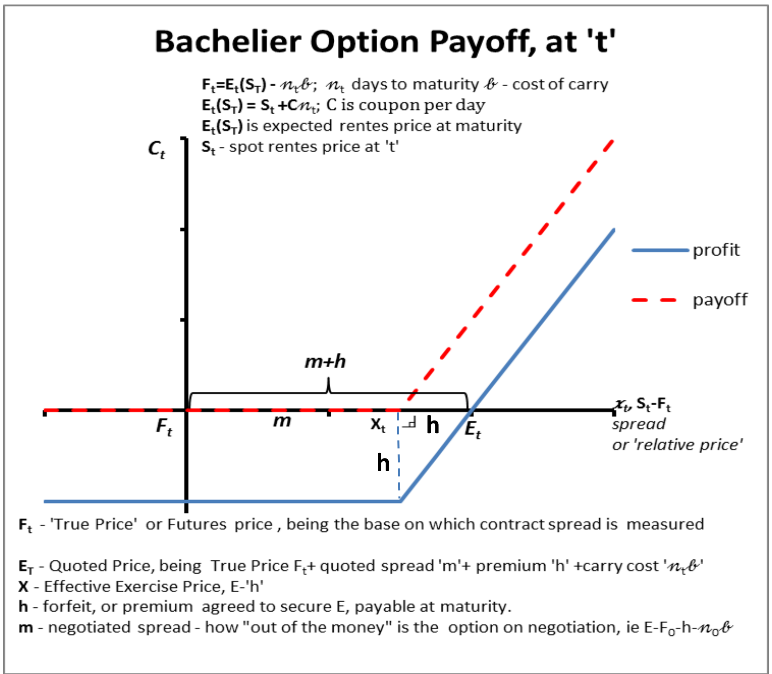

Figure 5 The Rentes futures Option Payoff Diagram

3.5 Option Payoff Diagram[46][47]

In the payoff diagram, figure 5, the horizontal axis represents the spread or differential between the underlying asset price, i.e. the rentes price, at t, and the futures price ('true price' in prevailing parlance) of the contract on issuance, i.e. t=0, signified by x[48]. The Bachelier price is a spread on the option, $x = F_{t,T} - F_{0,T}$.

The vertical axis, unlabelled by Bachelier, is the payoff or option intrinsic value at maturity.

The origin of the diagram is set at the 'true price' on issuance of the contract,

[43] for clarity this is distinctive from the instantaneous risk free rate used by Black-Scholes
[44] Bachelier uses similar notation for different purposes.
[45] P20 and pp50-51, it is difficult to be explicit about the nature of this discount without a full data set being available. However, reviewing empirical data for short term French Government yields, r_f, and the Rentes yields, r_R, implies $\mathscr{b}$=r_R-r_f i.e. 2.9% pa (est. average Rentes yield '94-'98) –est. 2.1% pa (est. average ST yield is 1.79%). That is, $\mathscr{b} = \frac{2.9\%}{307\ days} - \frac{2.1\%}{307\ days} \approx 0.264\ centimes\ per\ day\ [0.92\%\ pa]$ if applied for trading days pa with 6 day week giving 307 trading days then equates to 313 centimes per day.
[46] This diagram represents an amalgam of diagrams on pp 24, 43 and 44 and related text
[47] Where possible this paper uses Bachelier's notation and then converts this to modern notation for clarity.
[48] Do not confuse this with X also used to represent the what in modern contracts is the exercise price

$F_{0,T}$, which differs from modern pricing in that it has an adjustment due to the interest accrual and offsetting contango.

This notation is difficult, as the Bachelier quoted price is the spread over the futures price at issuance, and that the effective exercise price at issuance must be a positive add-on given the fixed nature of the premium. This form is used as the prices quoted on options are based on spreads, although quoted in terms of the price e.g. on a rentes with price Fr. 104.00 today an option was quoted at Fr. 104.34/50, the spread being Fr. 0.34, or 34 centimes and 50 the forfeit to be paid.

In the diagram:

- the red dashed line is the payoff of the option which is nil below the exercise price 'X'.
- The blue line represents the profit on the contract which is $(-h)$ below 'X', i.e. the cost of the premium, and becomes a profit at E.

On issuance, three prices must hold to the equivalence rules:

S_0 the rentes price at issuance, $E(S_{0,T})$ the expected rentes price at maturity on issuance, i.e. $S_0 + 25\ centimes$[49], and $F_{0,T}$ which equals $E(S_{0,T}) - '\mathcal{nb}'$.

We can then define the exercise price, spread and premium as follows

X is the effective exercise price. Where, $\mathrm{X} = F_0 + E - h$, in the graphic the equivalent of the present value of the modern exercise price is given as X.

E is the Bachelier Exercise Price, as a spread over the true price:[50]

$$E = m + h + \mathcal{nb} \tag{20a}$$

Where:

h is the option premium or forfeit paid at maturity whether the option is exercised or not. This is, netted off the maturity settlement should an option be exercised, $x > m + \mathcal{nb}$.[51] Forfeits are specified prices - 50, 25, 10 and 5 centimes per contract.

m is the spread on issuance. Bachelier modelled and prices this element with worked examples in the dissertation valuing this component which is netted out of the value proposition.

m is approximated by Bachelier, using a quadratic solution, as:

$$m = \pi\varphi \pm \sqrt{\pi^2\varphi^2 - 4\pi\varphi(\varphi - h)} \tag{20b}$$

Where, φ the coefficient of instability, can be approximated

$$\varphi = \frac{\pi(2h+m) \pm \sqrt{\pi^2(2h+m)^2 - 4\pi m^2}}{4\pi} \tag{20c}$$

$\mathcal{nb}$ is the arithmetic present value adjustment for the entitlement to the coupon payment, based on the futures price mechanism.

[49] Fr.3 option contract for one month, i.e. Fr. 100 .3% / 4qtrs / 3months
[50] Refer discussion pp 54-57 for a numerical determination of the spread and effectively formulation of the completed model
[51] This is shown twice in the diagram to emphasise equality of the lines.

3.6 The Bachelier price path[52]

3.6.1 Background

The underlying asset price path in Bachelier is essentially the same as per modern literature although arithmetic rather than geometric. The Rentes price is in essence stable due to the fixed coupon effect, i.e. a zero growth factor in Gordon dividend model terms, unlike dividends on equity shares, the sole determinant of the Rentes or Consol prices are the term to fixed coupon and the market yield. In a stable economic and market setting there is no expected drift or sub-martingale effect in the pricing, for instance Bachelier's empirical study 1894-98. Although it can be shown that in a climate of expected changing parameters the pricing will adjust – witness rentes prices 1870s to 1890 for instance, figure 3.

For the period analysed the underlying asset price has a zero drift, $\mu^a t = 0$. While Bachelier did not directly address this issue[53] he did recognise there could be a drift in prices and made necessary adjustments, as discussed.

3.6.2 Extended Price path

The rentes price path set-out by Bachelier allowing for a sub-martingale process, can be stated as

$$S_{t+\Delta t} = S_t\left(1 + \mu^a \Delta t + c.\frac{m-\Delta t}{91} + \sigma^a\sqrt{\Delta t}dz\right) \tag{21}$$

Where:

S_t the spot price of the underlying asset, Rentes in this case, at time '*t*'

μ^a the arithmetic drift or mean return on the underlying asset, in this case μ^a *= 0,* as the asset is a perpetuity with a fixed coupon in a stable market.

$c.\frac{m}{91}$ the price adjustment for the coupon

σ^a the arithmetic standard deviation of returns on the underlying asset, refer below for an outline of Bachelier's coefficient of instability[54]

dz *the standard Weiner process, describing a Brownian motion, applying a normal distribution with mean 0.*

This can be shown to be the standard Bachelier price path by applying the mean outcome, $\mu^a t = 0$, and putting aside the coupon oscillation, discussed previously, we find

$$S_{t+\Delta t} = S_t\left(1 + \sigma^a\sqrt{\Delta t}dz\right); given\ \mu^a = 0$$

And, recognising that dz is centralised on zero then in the context above for Rentes

$$E(S_{t+1}) = S_t$$

Thus we have the standard representation of Bachelier. The drift element needs to be incorporated in further models where this is applicable to the underlying asset.

[52] It is clear that Bachelier offered a single solution to these questions, however he did for instance on normal v log-normal question imply by description of the price range a limit to his use of a normal distribution, p29. Bachelier is thus aware of the potential issue but considered it 'a priori as effectively negligible'. And note in 1941 the price series finished due to the fall of the French government so a lognormal approach would fail in this circumstance.

[53] Bachelier approached the issue of price equivalence and the pricing mechanisms of equilibrium and statistical analysis – his pricing mechanism effectively therefore triangulates the price

[54] p47 and other extensive comment

3.6.3 The Coefficient of Instability, k - measure of risk

Bachelier, develops the concept of the *instability function,* $\varphi(t)$*,* to determine the likelihood, p, that the price in terms of the spread, x, is quoted in the range $x + dx$ at time t based in the Gaussian density function. He does this by defining the coefficient, k, in absolute terms for which his approach is criticised. However, as shown below this absolute form is in effect a relative form tied to the fixed exercise or Futures price underlying the agreement. Bachelier's solution is not relative to the underlying asset spot price as erroneously stated or reiterated in the finance literature on Bachelier.

The instability function is as follows:

$$\frac{1}{\sqrt{\pi}\sqrt{\varphi(t)}} e^{\frac{-x_t{}^2}{\varphi(t)}} dx \text{ }^{55} \qquad (22)$$

Where

x_t the spread of the price in current terms, $\left[F_{t,T} - F_{0,T}\right]$

$\varphi(t)$ $= 4\pi k^2 t$, is the instability function *a priori* a positive and increasing function of t[56]

Where

k is the coefficient of instability, given movement of Futures prices in time.

For completeness:

$$k = \sqrt{\sum_{t=\Delta t}^{T} \left[\left(F_{t,T} - F_{0,T}\right) - \overline{\left(F_{t,T} - F_{0,T}\right)}\right]^2}$$

As such, 'k' is an absolute measure of variance of the spread[57] given 'x'. That is the boundary condition becomes in simple terms a measure of standard statistical error:

$$\frac{x_t}{k} = \frac{\left(F_{t,T} - F_{0,T}\right)}{\sqrt{\sum_{t=\Delta t}^{T}\left[\left(F_{t,T} - F_{0,T}\right) - \overline{\left(F_{t,T} - F_{0,T}\right)}\right]^2}\sqrt{\tau}}$$

This is changed to a relative measure by dividing both terms by the central point, $F_{0,T}$ giving

$$\frac{x_t}{\sigma_x} = \frac{\left(F_{t,T} - F_{0,T}\right)/F_{0,T}}{\left(\sqrt{\sum_{t=\Delta t}^{T}\left[\left(F_{t,T} - F_{0,T}\right) - \overline{\left(F_{t,T} - F_{0,T}\right)}\right]^2} \Big/ F_{0,T}\right)\sqrt{\tau}} = \frac{\left(F_{t,T} - F_{0,T}\right)/F_{0,T}}{\sigma_x^a\sqrt{\tau}} = \frac{\left(F_{t,T} - F_{0,T}\right)}{F_{0,T}\sigma_x^a\sqrt{\tau}}$$

This gives the variance or a measure of standard error at the boundary in relative terms. In modern applications the relevant probability and pricing are around the behaviour at the maturity of the contract at the Exercise price then

$\mathrm{k} = \frac{X\sigma^a}{\sqrt{2\pi}}$, hence

$\varphi(t) = 2X^2{\sigma^a}^2 t$

Where

[55] Adapted from p32, (Bachelier, 1938, p. 16)

[56] P31 he recognises the functionality of time. Bachelier credits Laplace with the concept of small causes acting independently in various directions leading to a single law, he notes that in this context there is an exact solution (Bachelier, 1938, p. 12).

[57] P50-51, where $k = 5$; and given $\tilde{X}, or\ F_{0,T} = Fr\ 100$ then $\ddot{\sigma}_x = 5\%$
Note $\mathscr{b}$=0.264 centimes or 1% pa discount on the coupon payments up to 30 to 60 days hence.

$X\sigma^a$ or $F_{0,T}\sigma^a$, is equivalent Bachelier's absolute measure of standard deviation, 'k', in arithmetic terms at the boundary.

As noted, this is applied in modern forms using a relative standard deviation against the exercise price, reflected in his discussion of this factor's instability.[58]

Moving forward, the above allows us to restate the probability function:

$$\frac{1}{\sqrt{\pi}\sqrt{2F_{0,T}{}^2\sigma^{a^2}t}}e^{\frac{-x^2}{2F_{0,T}{}^2\sigma^{a^2}t}}dx = \frac{1}{\sqrt{2\pi}F_{0,T}\sigma^a\sqrt{t}}e^{\frac{-x^2}{2F_{0,T}{}^2\sigma^{a^2}t}}dx$$

Recognise that $\frac{1}{\sqrt{2\pi}}$ is the Gaussian normalisation quotient

Now creating this as a probability, we get

$$p = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}F_{0,T}\sigma^a\sqrt{t}}e^{\frac{-y^2}{2F_{0,T}{}^2\sigma^{a^2}t}}dy$$

Adjusting the boundaries

$$p = \int_{-\infty}^{\frac{x}{F_{0,T}\sigma^a\sqrt{t}}}\frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}dy$$

Recognising that $x = F_{t,T} - F_{0,T}$ by definition then[59]

$$p = \int_{-\infty}^{\frac{F_{t,T}-F_{0,T}}{F_{0,T}\sigma^a\sqrt{t}}}\frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}dy \text{ ; or}$$

$$p = \int_{-\infty}^{d}\frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}dy \qquad (23a)$$

Where

$d = \frac{F_{t,T}-F_{0,T}}{F_{0,T}\sigma^a\sqrt{t}}$ which is effectively the arithmetic version of the boundary condition

That is, p is the Gaussian cumulative normal density function, $N(d)$, and the normal density function

$$n(d) = \frac{1}{\sqrt{2\pi}}e^{\frac{-d^2}{2}} \qquad (23b)$$

3.6.4 Bachelier Model as Presented

We now have the tools to determine the probability relevant to the option, being the sum of the probability that the underlying asset price will be greater than the exercise price plus the probability that the underlying asset price will equal the exercise price at maturity, being the instability coefficient or the curvature adjustment. That is[60] the model can be stated as being tied to the forfeit price given the

[58] Bachelier legitimately uses a stable and absolute measure of stability in his model. This is contrary to modern commentary. The Black-Scholes Merton model is actually consistent in this application.

[59] In this section and paper the boundary is stated in positive terms. Bachelier stated these boundaries in the negative. This is achieved by reversing the measure of spread. That is $-x = F_{0,T} - F_{t,T}$; $+x = F_{t,T} - F_{0,T}$

[60] Bringing together the numerical analysis and Bachelier's representation of the model, p44, and (Bachelier, 1938, p. 23)paragraph 38 Law of spread of options.

$$h + m\int_m^\infty \tilde{\omega}\,dy = \int_m^\infty \tilde{\omega}\,xdy$$

the rearranging, recognising h as the option premium and combining the integrals we have as above

$$h = (x-m)\int_{x-m}^\infty \tilde{\omega}\,dy$$

difference in spot and exercise price plus the probability of exercise adjusted for the curvature correction being the first differential of the probability at the boundary.

In general terms this is:

$$h = (\mathrm{x}-\mathrm{m})\left[p_{(x-m)}+\frac{dp_{(x-m)}}{d(x-m)}\right] \text{ for the Call Option; and} \tag{24a}$$

$$h = (\mathrm{x}-\mathrm{m})\left[p_{(x+m)}+\frac{dp_{(x+m)}}{d(x+m)}\right] \text{ for the Put Option} \tag{24b}$$

3.7 The Louis Bachelier Model for Options on Futures over Rentes[61]

Reiterating, the Bachelier model[62], published in 1900, was constructed to price an option on a futures contract over French Government Rentes where the contract was in future terms. The variable being priced was the spread around the exercise price or agreed Futures price on issuance. Despite the variations the basic construct can be readily transformed to comply with the modern contract as it was based on market assumptions and statistical analysis directly applicable.

This section outlines the original model formulation.

3.7.1 Bachelier Model for the Rentes Option Contract

Initially, noting that he recognises that he is applying the "principle of mathematical expectation to the buyer of the option"[63], that is an efficient market in equilibrium, given operation of price equivalence or arbitrage processes. Then, using his notation we have, as per the previous section and refer footnote for 1938 version:

$$h + m\int_m^\infty p\,dy = \int_m^\infty px\,dy \tag{25}$$

Noting this model is static, and all factors determined as at, and are paid at maturity. Where:

h is the forfeit, or call option premium, C_t, or in modern terms that would be paid at maturity and set at fixed prices 50, 25, 10 or 5 centimes,

x is the current spread or net price of the underlying asset being the spot price, S_t *or the true price* $F_{t,T}$(the futures price at time t), less the exercise price adjusted for the next coupon present value, $F_{t,T}-[F_{0,T}-n_t b]$. 'n_t' being the days to maturity at 't', and 'b' being net cost of contango; that is the True Price at 't'.

m is the initial priced spread, X-$[F_{0,T}-n_0 b]$ on the option contract at maturity.

X is the quoted Futures Price giving the spread on issuance

m is effectively a measure of the degree an option needs to be out of the money in order to comply with the set forfeit, and in effect is the key factor Bachelier is modelling, as the premium or forfeit on exercise is certain.

[61] This section builds on Bachelier's various papers and by reference to (Schachermayer & Teichmann, 2007) and (Haug E. G., 2007).
Bachelier's papers are a fascinating although a difficult read. In particular it is interesting to observe how he anticipates later works. The workout, for instance, of a fully developed binomial pricing solution, pp 33-36, an application of de Moivre anticipates the works of Cox-Ross and Rubinstein. Strangely Bachelier is referenced in Cox Ross 1975 working paper (Cox & Ross, 1975) but not in the published 1976 work (Cox & Ross J. &., 1976).

[62] pp 44 & 45

[63] p 43

m is approximated by Bachelier, using a quadratic solution, as per 20(b) and 20(c)

p is the probability function that the asset price 'x' is greater than 'm'.

Adjusting per the previous section and allowing for the curvature correction plus setting $m\ to\ 0$[64], we get

$$\text{h} = \text{x}\left[p_x + \frac{dp_x}{dx}\right] \tag{26}$$

First, recognise $C_{t,T} = h$ as a dependent variable, the Call Option premium, and noting 'x' equates to $F_{t,T} - F_{0,T}$, then in future value terms we can restate the above as:

$$C_{t,T}^{B} = \left(F_{t,T} - F_{0,T}\right)\left[p_x + \frac{dp_x}{dx}\right] \tag{27}$$

The second element of the pricing model is the probability function, with an integral, as previously defined above

$$p_d = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy = N(d) \tag{28a}$$

Where

$$d = +\frac{F_{t,T} - F_{0,T}}{F_{0,T}\sigma_t^a\sqrt{\tau}}\ ^{65}$$

Turning now to the differentiated element of the probability equation, $\frac{dp_x}{dx}$[66]:

$$\frac{d\text{p}}{dx} = \frac{\partial N(d)}{\partial d} = -\frac{n(d)}{d} \tag{28b}$$

$$-\frac{1}{d}n(d)^{67} = \frac{F_{0,T}\sigma_t^a\sqrt{\tau}}{F_{t,T} - F_{0,T}}\frac{1}{\sqrt{2\pi}}e^{-\left(\frac{F_{t,T}-F_{0,T}}{\sqrt{2}F_{0,T}\sigma_t^a\sqrt{\tau}}\right)^2} \tag{28c}$$

Formalising, we then have the Bachelier model for the Call Option in non-spread terms:

$$C_{t,T}^{B} = \left(F_{t,T} - F_{0,T}\right)\left[\text{N(d)} + \frac{n(d)}{d}\right] \tag{29a}$$

Alternatively, rearranging and recognising the elimination of the denominator for 'd' we have

$$C_{t,T}^{B} = \left(F_{t,T} - F_{0,T}\right)\text{N(d)} + F_{0,T}\sigma_t^a\sqrt{\tau}n(d) \tag{29b}$$

This model as per Bachelier and per the notation is in future value terms.

3.8 Critiques of the Louis Bachelier Model

The Bachelier model has been variously critiqued in the finance literature, generally such critiques being reiterations of former comment. There are basically four areas of criticism price behaviour, the instability

[64] P 45, Bachelier notes that the premium, *'a'*, of such an option termed a *'simple option'* in his parlance is equal to the positive expectation of a forward buyer. The value of the right for which the buyer pays the seller for the advantage over a futures or forwards buyer to have a positive expectation without incurring risk.

[65] This element is positive for reasons given in previously

[66] He initially recognises an integral approximation as $\pm\sqrt{2\pi}k\sqrt{\tau}, i.e. \pm\sigma^a{}_t\sqrt{\tau}$, being the points of inflexion, refer Sprenkle et al.

[67] That is $\frac{dy}{df(x)} = e^{-f(x)} = \frac{1}{e^{f(x)}} = \frac{1}{f(x)e^{f(x)}} = \frac{1}{f(x)}e^{-f(x)}$

As in this case $f(x)$is +ve; then$\frac{\partial N(d)}{\partial d} = \frac{1}{d}n(d)$

measure is seen as absolute rather than relative, the instability factor is seen as being defined relative to the underlying asset price vis-à-vis the exercise price, that as the measure of dispersal trend to infinity the call price exceeds the underlying asset price.

3.8.1 Price Behaviour

In general critiques regarding the nature of the asset price behaviour arise as the Bachelier option contract is not over equity shares, but over a futures contract on French Government Rentes, or government perpetuity bonds. This results in the model not including an explicit sub-martingale price effect. There are several basic issues:

> First, the underlying Rentes prices exhibited a stable and steady market price. For the relevant empirical period of 1894-1898 Bachelier studied the market experienced stable economic conditions leading to Rentes price stability.;
>
> Second, the Rentes being perpetuities tend to a zero drift price behaviour; and
>
> Third, for Bachelier's option contract all payments were at maturity unlike the modern equity share option contracts.

While Bachelier developed the model in the context of an option over a futures contract on Rentes as the underlying asset which has a stable price and is expected to fulfil a zero yield martingale, the model can be adapted for sub-martingale behaviour of equity shares returns. That is, the pricing path can be shown to comply with a sub-martingale with an arithmetic yield as follows:

$$S_{t+\Delta t} = S_t\left(1 + \mu^a \Delta t + \sigma^a \sqrt{\Delta t} dz\right) \qquad \text{(3a)}$$

3.8.2 Absolute Measure of instability or dispersal

This critique relates to the apparent absolute variance, k. It is shown above that using an absolute variance was reasonable as the respective measure of error being standardised was absolute, $x = F_{t,T} - F_{0,T}$. Further, as the model relates to a fixed boundary then the apparent absolute nature of the measure can be transformed to a relative measure driven by a fixed value, which can be clearly exhibited for the modern contract can be stated as $\frac{S_t - Xe^{-\alpha\tau}}{Xe^{-\alpha\tau}}$ with a relative deviation measure σ^a. That is the Bachelier solution can be treated as a relative, not absolute, measure.

3.8.3 Modern use of the Equity Price in the dispersal adjustment, k

The model is critiqued for using a share price for the measure of instability, $k = S_t \sigma^a \sqrt{\tau}$. However, by extension of above, this is shown to be an incorrect form of Bachelier's model which leads to incorrect interpretations of the model. Possibly arising from a lognormal adaptation?

The correct projection is around the Exercise Price thus, $k = Xe^{-\alpha\tau}\sigma^a\sqrt{\tau}$ relating to the derivative of the cumulative distribution function, which is crucial in determining the error correction and price argument. For instance as $T and\ \sigma^a \to \infty$; $C_t \to S_t$. This error in interpretation applies to both the geometric and arithmetic form of the model.

3.8.4 That per definition of the dispersal, k, as $\sigma^a \to \infty$; $C_t \to \infty$

This critique in the literature is an error in statement of the mathematical argument and relates to the definition of k and the operation of $\frac{\partial N(d)}{\partial d}$ or the curvature correction. This as shown gives:

$$\frac{\partial N(d)}{\partial d} = (S_t - Xe^{-\alpha\tau})\frac{n(d)}{d}$$

The rearranging

$$Xe^{-\alpha\tau}\sigma^a\sqrt{\tau}\,n(d)$$; where

$$n(d) = \frac{1}{\sqrt{2\pi}}e^{\frac{-d^2}{2}}$$; and

$$d = \frac{F_{t,T} - F_{0,T}}{F_{0,T}\sigma^a\sqrt{t}}$$

Then analysing the extreme where $\sigma^a\sqrt{\tau} \to \infty$ and noting the above has two elements - $Xe^{-\alpha\tau}\sigma^a\sqrt{\tau}$ and $n(d)$ we can determine the impact of this argument:

as $\sigma^a\sqrt{\tau} \to \infty$ then we have expansion of the first element $Xe^{-\alpha\tau}\sigma^a\sqrt{\tau} \to \infty$; but

as $\sigma^a\sqrt{\tau} \to \infty$ then $d \to 0$ and hence we have contraction of the pdf element $n(d) \to 0$;

Modern critiques fail to incorporate the latter contraction in the analysis. The resulting model value is though determined by the relative expansion and contraction of the two elements. It can be shown that the latter element, $n(d)$, contracts at a higher rate than $Xe^{-\alpha\tau}\sigma^a\sqrt{\tau}$ expands, as a result

as $\sigma^a\sqrt{\tau} \to \infty$ then $Xe^{-\alpha\tau}\sigma^a\sqrt{\tau}\,n(d) \to 0$; and

$\mathrm{C_t} \to S_t - Xe^{-\alpha\tau}$

Which is the result necessary under rational pricing such that $\mathrm{C_t} \leq S_t$

Part Four: Bachelier adapted for the Modern Contract

4.1 Preliminary

This section reformulates the Bachelier model for the modern equity share contract.

4.1.1 Model Characteristics

There are several key characteristics:

- the pricing of the premium is a present value vis-à-vis being at maturity, or a future value
- the premium is the determined pricing element, as compared tom the spread based on fixed premia
- the exercise price is agreed on issuance.
- the price path is a arithmetic on a normal distribution and a sub-martingale

4.1.2 Notation

First a restatement of the notation for model use:

$S_t \sim F_{t,T}$	the Spot Price or Share price at 't'
$E(S_{t,T})$	expected spot price at maturity, 'T', at time 't'
$X \sim F_{0,T}$	the Exercise Price for the option, payable at maturity
r_s^a	is the arithmetic discount rate based on the underlying asset return that is applied to determine the premium's present value

4.2 Present value terms

First, restating the Bachelier model[68] in present value terms. This change needs to be applied both to the price elements and the probability elements, as below.

$$C_t^B = [F_{t,T} - F_{0,T}]\left[N(d) + \tfrac{1}{d}n(d)\right](1 - r_s^a\tau) \tag{30a}$$

And restating the probability

$$C_t^B = [F_{t,T} - F_{0,T}]\mathcal{P}(1 - r_s^a\tau) \tag{30b}$$

$$\mathcal{P} = N(d) + \tfrac{1}{d}n(d)$$

$$d = -\frac{[F_{t,T} - F_{0,T}](1 - r_s^a\tau)}{F_{0,T}(1 - r_s^a\tau)\sigma_t^a\sqrt{\tau}}$$

And, this can be re-arranged to show

$$C_t^B = [F_{t,T} - F_{0,T}](1 - r_s^aT)N(d) + F_{0,T}(1 - r_s^aT)\sigma_t^a n(d) \tag{30c}$$

[68] The form of equation corrects for the standard form in the literature. For instance, (Haug E. G., 2007, p. 13), (Smith, 1976) presents the call as:
$C_t^B = (S - X)N(-d_1) + \sigma\sqrt{\tau}n(d_1)$; with $d_1 = \frac{S-X}{\sigma\sqrt{\tau}}$ and $\sigma = S_t\sigma^a$
While this recognises that Bachelier uses an absolute variance the authors define it in underlying asset terms, e.g. $\sigma = F_{t,T}\sigma^a \; or \; S_t\sigma^a$ rather than as in $\sigma = F_{0,T}\sigma^a \; or \; X\sigma^a$. They also err on this point by neglecting that the numerator in the Bachelier equation is a spread in absolute term. Note, for record, Bachelier also recognises that, σ, this an historical measure which is unstable and changes through time and can be deduced from the market spread at any time, p50-51 – e.g. the modern VIX could be read easily from market prices.

4.2.3 Revising the Notation for Call Option

Incorporating present value components we get below. The equity share price at '*t*' is the present market value for the underlying asset rather than a future value.

$$C_t^B = [S_t - X(1 - r_s^a\tau)]\left[N(d) + \frac{n(d)}{d}\right] \tag{31a}$$

$$d = \frac{S_t - X(1 - r_s^a\tau)}{X(1 - r_s^a\tau)\sigma_t^a\sqrt{\tau}}$$

And rearranging

$$C_t^B = [S_t - X(1 - r_s^a\tau)]N(d) + X(1 - r_s^a\tau)\sigma_t^a\sqrt{\tau}n(d)$$

4.2.4 Put Option

For completeness the put option model becomes

$$P_t^B = [X - E(S_{t,T})](1 - r_s^a\tau)\mathcal{P}^* \tag{31d}$$

$$\mathcal{P}^* = N(-d) + \frac{n(d)}{d}$$

d is as for the call option.

4.3 Model behaviour

Figure 6, presents the behaviour of the pricing elements giving the form of the pricing model.

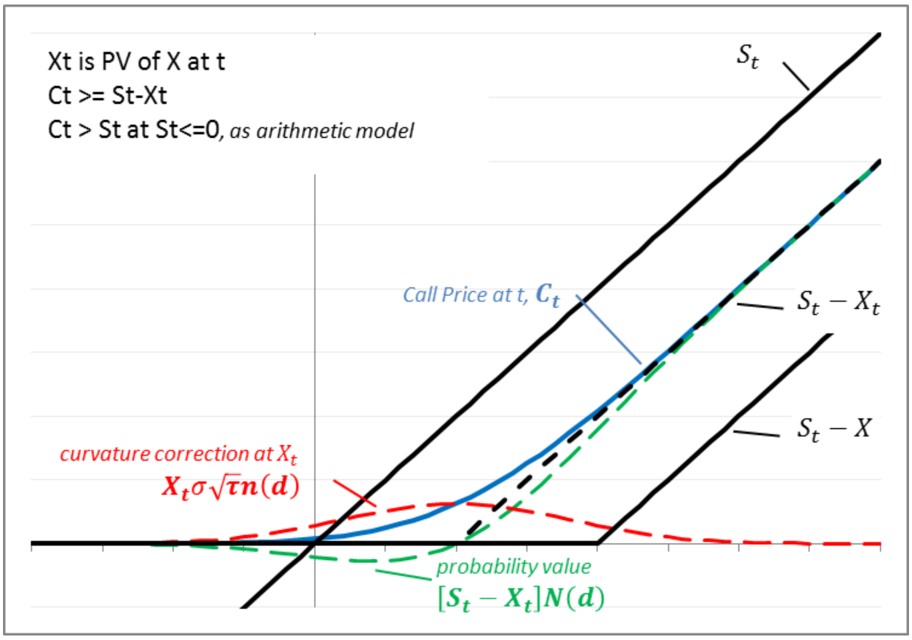

Figure 6 Bachelier Pricing Model showing components for base probability and instability function

The blue line represents the possible call option price path at time '*t*'

$$C_t \geq S_t - Xe^{-r_s^a\tau}\,[X_t]$$

The green dashed line gives the probability of exercise curve against the value of the portfolio. This has negative values which led to certain model criticism as the second element was not correctly defined.

The red dashed line gives the curvature correction around the present value of the exercise price exercise price at time '*t*'. Note if this is related to S_t rather than X_t, then this creates an expanding value leading to certain critiques of the Bachelier model.

Note in above, and as shown below in the properties as $S_t = Xe^{\left(r_s^e + \frac{\sigma^{e2}}{2}\right)\tau}$, or $S_t = [X - n_t b]$, then the option price is driven solely by instability or curvature correction of the model.

4.4 Properties of the Model

Turning to some key features of this model

- As Bachelier notes this model is mathematically exact[69], whereas Black-Scholes Merton use an approximation technique.
- The model is arithmetic in form, with a Gaussian Normal distribution.
- The model is a static formulation[70][71].
- The model is simply the present value of the model in future terms

  $$C_t^B = PV(C_T)$$

  This works given a single discount rate being applied to the respective elements, r_s^a.[72]
- The value of the option $C_t^B \geq 0, \text{for } -\infty < \mathrm{S_t} < \infty$
- In the arithmetic solution the value of the call maybe greater than the underlying asset, for instance $C_t^B \geq \mathrm{S_t}, \text{for } -\infty < \mathrm{S_t} < 0$ and as $\mathrm{S_t} \to 0$.
- In the adapted Bachelier geometric model $C_t^B < \mathrm{S_t}, \text{for } 0 < \mathrm{S_t} < \infty | \mathrm{S_t} \geq 0$
- $as\ \tau \to \infty;\ \frac{n(d)}{d} \to 0$; as while $\sigma\sqrt{\tau} \to \infty, F_{0,T} or\ X(1 - r_s^a\tau) \to 0\ and\ n(d) \to 0.$
- There is a single boundary condition, *d*. $d \pm \frac{\sigma\sqrt{t}}{2}$ is a cdf approximation technique for determining the value of the pdf.
- d represents a standardised measurement of the degree of movement necessary for the underlying asset value to equate with the exercise price. Standardisation is by division by the relative standard

[69] (Bachelier, Speculation and the Calculus of Probability, 1938)p12

[70] This has ramifications in option pricing related to yield and discount rate justification. In his introduction Louis Bachelier states the model is static in that it applies probability distributions and variations in the price that the market admits in that instant, and does not describe the movement of this price instant to instant.

The model can be made dynamic by inclusion of the underlying asset of share price path

$$\gamma_t = 1 + \mu^a dt + c.\frac{m\Delta t}{91} + \sigma^a\sqrt{\Delta t}dz$$

Noting, as $\Delta t \to 0, \gamma_t \to 1$

Recognising this, the Bachelier option model can be restated

$$C_t^B = \left(S_t\gamma_t - X(1 - \alpha\tau)\right).\mathcal{P}$$

And, the Black Scholes Merton model can be restated70

$$C_t^{BSM} = S_t\lambda_t N(d_1) - Xe^{-r\tau}N(d_2)$$

$$\lambda_t = e^{\mu t + \sigma\sqrt{t}dz}$$

[71] In a Black-Scholes Merton dynamic hedge portfolio and Bergman equivalence argument sense the portfolio weights are equal, but negative. That is, $\beta = -\alpha$. Hence the portfolio is a simple funding relationship dependent on value of the spot price of the underlying asset, $\mathrm{S_t}$. This is crucial for specification of the portfolio drift.

[72] This leads to conflict with the logical construct underlying the Black-Scholes Merton model as the B-SM model implicitly uses r_s^a to discount the underlying asset price, and 'r_f' is discount the exercise price.

deviation, $\sigma^a\sqrt{\tau}$, recognising that this is not expected to be stable or constant through time. The relative required movement is given by:

$$\frac{S_t - X(1 - r_s^a\tau)}{X(1 - r_s^a\tau)} \tag{32}$$

$at\ S_t = X(1 - r_s^a\tau)$, then

$S_t - X(1 - \alpha^a\tau) \to 0;\ d \to 0\ and\ \frac{1}{d} \to \infty$; hence the function becomes discontinuous, replaced by

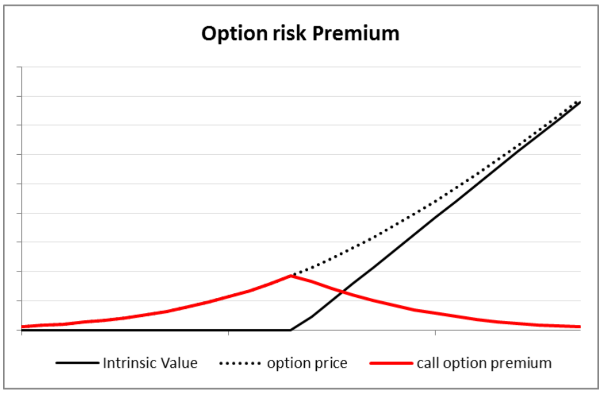

Figure 7 The Option Risk Premium relative to the Asset Price

$X(1 - r_s^a\tau)\sigma^a\sqrt{\tau}n(d)$

$N(d) \to 0.5$, but as $S_t - X(1 - r_s^a\tau) \to 0$ is eliminated.

$n(d)/d \to X(1 - r_s^a\tau)\sigma^a\sqrt{\tau}n(d), as\ S_t - X(1 - r_s^a\tau)\ cancels; i.e. \frac{0}{0} = 1$

Therefore

$C_t \to S_t\sigma^a\sqrt{\tau}n(d) = \frac{1}{\sqrt{2\pi}}X(1 - r_s^a\tau)\sigma^a\sqrt{\tau}$

$as\ n(d) = \frac{1}{\sqrt{2\pi}}e^{-d^2}$

$at\ S_t = X(1 - r_s^a\tau)\ then\ d = 0$

$hence\ e^{-d^2} \to 1; and\ \ n(0) = \frac{1}{\sqrt{2\pi}}$[73] $\cong 0.39894$

That the risk premia over the underlying value maximises at $S_t = [X - n_t\mathcal{b}]$ as shown in the figures, red line

4.5 Bachelier adapted for Log-normal & continuous compounding

This section converts the Bachelier model for lognormal and continuous compounding.

The process of converting the model for continuous compounding and lognormal distribution is conceptually simple two-step process.

4.5.1 Step 1: Conversion from arithmetic to continuous compounding

First we need to change the arithmetic discount $(1 - \mathrm{r}_s^a\tau)$ to $e^{-\mathrm{r}_s^e\tau}$, and substituting the variability measure from $\sigma_t^a\ to\ \sigma_t^e$ for a continuous compounding formula. Note, the present value of the asset price is implied in the model in S_t.

$$S_t = S_0e^{\mu^e t + \sigma^e\sqrt{t}dz} \tag{8a}$$

Note, at time 't' the expected share price at 'T', recognising $\tau = T - t$, is given by

$$E(S_{t,T}) = S_0e^{\mu^e\tau + \frac{\sigma^{e2}\sqrt{\tau}}{2}} = S_0e^{\mathrm{r}_s^e\tau};\ \text{as}\ E(\sigma^e\sqrt{\tau}dz) \to \frac{\sigma^{e2}\sqrt{\tau}}{2} \tag{8c}$$

This gives:

$$\mathrm{C}_t = (S_t - Xe^{-\mathrm{r}_s^e\tau})\left[N(d) + \frac{n(d)}{d}\right]^{74} \tag{33a}$$

---

[73] P42

Simplifying, the statement

$$\mathrm{C}_t = \left(S_t \; - Xe^{-\mathrm{r}_\mathrm{s}^\mathrm{e}T}\right).\mathcal{P} \tag{33b}$$

Where:

$$\mathcal{P} = N(d) + \frac{n(d)}{d}$$

$$d = -\frac{\frac{\left(S_t - Xe^{-\mathrm{r}_\mathrm{s}^\mathrm{e}\tau}\right)}{Xe^{-\mathrm{r}_\mathrm{s}^\mathrm{e}\tau}}}{\sigma^e\sqrt{\tau}} = -\frac{S_t - Xe^{-\mathrm{r}_\mathrm{s}^\mathrm{e}\tau}}{Xe^{-\mathrm{r}_\mathrm{s}^\mathrm{e}\tau}\sigma^e\sqrt{\tau}},$$

Allowing a normal distribution assumed by Bachelier.

4.5.2 Step 2: Conversion from normal to a lognormal model

As the lognormal movement in asset price is normally distributed the model can again be readily converted to a lognormal form, adjusting the numerator in d. Hence as per Sprenkle and Boness:

$$d = -\frac{ln\left(\frac{Xe^{-\mathrm{r}_\mathrm{s}^\mathrm{e}\tau}}{S_t}\right)}{\sigma^e\sqrt{\tau}} = -\frac{ln\left(\frac{X}{S_t}\right) - \mathrm{r}_\mathrm{s}^\mathrm{e}\tau}{\sigma^e\sqrt{\tau}} \tag{34a}$$

Samuelson later recognised that the boundary could be positive by inverting the numerator when using a lognormal model. For the arithmetic form this is achieved by reversing the elements in the numerator, i.e. $(S_t - X) = -(X - S_t)$. Giving

$$d = +\frac{ln\left(\frac{S_t}{X}\right) + \mathrm{r}_\mathrm{s}^\mathrm{e}\tau}{\sigma^e\sqrt{\tau}} \tag{34b}$$

Noting for completeness $ln\left(\frac{S_t}{X}\right) + \mathrm{r}_\mathrm{s}^\mathrm{e}\tau$ can be read in either future value $ln\left(\frac{S_t e^{\mathrm{r}_\mathrm{s}^\mathrm{e}\tau}}{X}\right)$ or present value terms $ln\left(\frac{S_t}{Xe^{-\mathrm{r}_\mathrm{s}^\mathrm{e}\tau}}\right)$. Such an inversion creates a conflict in the Black-Scholes Merton model construct as clearly $r_f \neq \mathrm{r}_\mathrm{s}^\mathrm{e}$.

4.5.3 General interpretation

In general probability terms the pricing models are a relatively simple present value of the probability driven expected payoff[75]. Where the probability element is the basic probability that $S_T > X$. This can be applied in various forms of model, for example binomial models, plus a correction for curvature of the probability curve at the boundary in more sophisticated models. That is the model is the present value of the future expected payoff given its' probability as follows:

$$C_t = PV[(E(S_T) - X)\Phi(E(S_T) > X)] \tag{35a}$$

$$C_t = PV[(E(S_T) - X)_+] \tag{35b}$$

Bachelier addressed this issue through his equivalence discussion recognising the appropriate values of both X and $E(S_T)$ in the discount applied to the underlying asset.

[74] Noting, $Xe^{-b\tau}\sigma\sqrt{T} = (S - Xe^{-b\tau})\left[\frac{Xe^{-bT}\sigma\sqrt{\tau}}{S - Xe^{-b\tau}}\right]$; also at $S = Xe^{-b\tau}$, this cancels to 1, so the above model rather going to zero results in $Xe^{-b\tau}\sigma\sqrt{\tau}\,n(d)$. For the log-normal solution this specific issue is more problematic.

[75] This reflects Bachelier's point that the complexity of the mathematics hides the simplicity of the concept.

Part Five: Model Comparison

Following are statements of the Louis Bachelier model in arithmetic and geometric forms in the modern context; and the Black-Scholes Merton model restated to include the underlying asset return as the discount rate; with a comparison showing the latter to be an approximation.

A measure of pricing error is shown graphically below.

5.1 Bachelier Model - arithmetic

The model construct is based on a simple premise as to the probability and price behaviour at the boundary or Exercise price, as

$$h_t = \mathrm{x}_t\left[p_x + \frac{dp_x}{dx}\right] = \left(F_{t,T} - F_{0,T}\right)\left[\Phi\left(F_{t,T} > F_{0,T}\right) + \frac{d\Phi\left(F_{t,T}>F_{0,T}\right)}{d\left(F_{t,T}>F_{0,T}\right)}\right] \tag{24a}$$

Giving, in arithmetic terms for Bachelier's contemporary contract for the call option

$$C_t^B = \left(F_{t,T} - F_{0,T}\right)\mathcal{P} \tag{36}$$

For the put option

$$P_t^B = \left(F_{0,T} - F_{t,T}\right)\mathcal{P}^*$$

Where

$$\mathcal{P} = N(d) + \frac{n(d)}{d}$$

$$\mathcal{P}^* = N(-d) + \frac{n(d)}{d}$$

$$d = \frac{F_{t,T} - F_{0,T}}{k} = \frac{F_{t,T} - F_{0,T}}{F_{0,T}\sigma^a\sqrt{\tau}}$$

$\mathcal{P}$ is the probability of exercise, $N(d)$, adjusted for a curvature correction, $\frac{n(d)}{d}$, in the probability curve at the boundary

In modern notation with a present value element due to the contractual form then we have

$$C_t^B = [S_t - X(1 - r_s^a\tau)]\mathcal{P} \tag{31c}$$

$$d = -\frac{S_t - X(1 - r_s^a\tau)}{X(1 - r_s^a\tau)\sigma^a\sqrt{\tau}}$$

Thus, expanding

$$C_t^B = [S_t - X(1 - r_s^a\tau)]N(d) + X(1 - r_s^a\tau)\sigma^a\sqrt{\tau}n(d)$$

5.2 Bachelier Model – geometric

Continuing in modern terms

$$C_t^{Be} = \left(S_t - Xe^{-r_s^e\tau}\right)\mathcal{P} \tag{37}$$

For the put option

$$P_t^B = \left(Xe^{-r_s^e\tau} - S_t\right)\mathcal{P}^*$$

$\mathcal{P}$ & $\mathcal{P}^*$; are as above but for a lognormal model.

$$d = \frac{ln\left(\frac{S_t}{X}\right)+r_s^e\tau}{\sigma^e\sqrt{\tau}}$$

5.3 Black-Scholes Merton reconciliation

The classic formulation for the Black-Scholes Merton model is:

$$C_t^{B-SM} = S_t N^{ln}(d_1) - Xe^{-r_f^e\tau}N^{ln}(d_2) \qquad (9)$$

Where

$N^{ln}(.)$ is the cumulative lognormal distribution

$d_1 = \frac{ln\left(\frac{S_t}{X}\right)+r_f^e\tau+\frac{\sigma^{e2}}{2}\tau}{\sigma^e\sqrt{\tau}}$; and $d_2 = d_1 - \sigma^e\sqrt{\tau}$

r_f^e is the instantaneous risk free rate

τ is the time to maturity

In the discussion it has been shown that the above construct in terms of applying the instantaneous risk free rate in discounting the exercise price needs revision. This arises as the proposition is dependent on a static price for the underlying asset value. Thus, the model should be restated along lines proposed by Boness or Sprenkle models of the early 60s using the underlying asset return, r_s^e, used below. Note:

$$r_s^e = \mu^e + \frac{\sigma^{e2}}{2}.$$

Taking this into account the model can then be reformulated to reflect the underlying nature of the pricing proposition, being to determine the present value of a future payout. That is:

$$C_t^{B-SM} = \left\{E\left(S_{t,T}\right)N^{ln}(d_1) - XN^{ln}(d_2)\right\}e^{-r_s^e\tau} \qquad (38)$$

$$E\left(S_{t,T}\right) = S_t e^{r_s^e\tau}$$

$d_1$and d_2 in essence are unchanged, $d_1 = \frac{ln\left(\frac{S_t}{X}\right)+r_s^e\tau+\frac{\sigma^{e2}}{2}\tau}{\sigma^e\sqrt{\tau}}$; and $d_2 = d_1 - \sigma^e\sqrt{\tau}$

We can further break the classic Black-Scholes Merton model into component parts reflecting the Bachelier form of a probability plus instability coefficient reflecting the Fourier heat equation. In this case:

$$C_t^{B-SM} = \left\{\left[S_t - Xe^{-r_s^e\tau}\right]N(d)\right\} + \left\{S_t N(d_1, d) + Xe^{-r_s^e\tau}N(d, d_2)\right\} \qquad (13a)$$

Where

$d = \frac{ln\left(\frac{S_t}{X}\right)+r_s^e\tau}{\sigma^e\sqrt{\tau}}$; and

$d_1 = d + \frac{\sigma^e\sqrt{\tau}}{2}$, $d_2 = d - \frac{\sigma^e\sqrt{\tau}}{2}$

For clarity, $N(d_i, d) = N(d_i) - N(d)$

We can see the Bachelier form more explicitly given

$$C_t^{B-SM} = \left[S_t - Xe^{-r_s^e\tau}\right]N(d) + \left\{S_t N(d_1, d) - Xe^{-r_s^e\tau}N(d_2, d)\right\} \qquad (39)$$

where

$[S_t - Xr_s^e]N(d)$ is the primary probability of exercise in present value terms – i.e. $N(d)$

$S_t N(d_1, d) + Xe^{-r_s^e T} N(d, d_2)$ is an approximation or curvature adjustment for the integral recognising the nature of the above probability at exercise. That is, $\frac{1}{d} n(d)$

5.3 Approximation

The key element here is the approximation element around the boundary condition.

Noting that Bachelier incorporated a solution in his modelling where

$$d_1 = d + 0.6745\sigma\sqrt{\tau}\,,\, d_2 = d - 0.6745\sigma\sqrt{\tau}$$

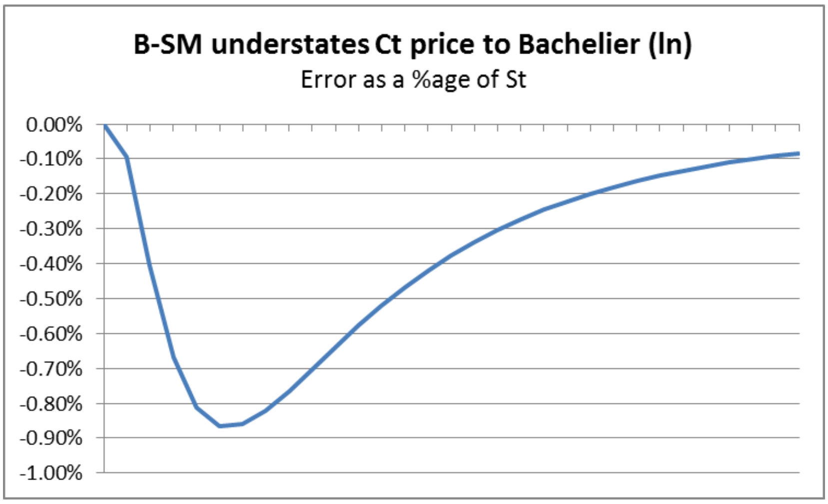

Figure 8 B-SM understatement of the Call Price

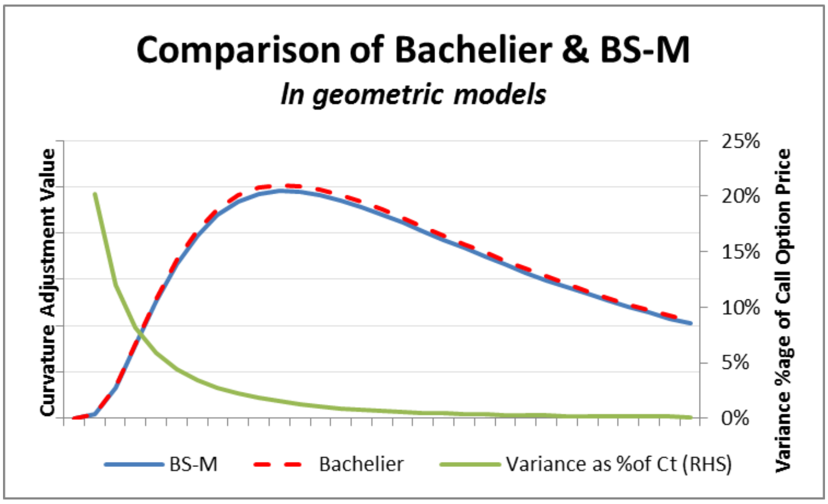

Figure 9 A Comparison of Model Pricing

0.6745 is a probability outcome of $\pm$25%, giving the expected dispersal at a point in time.

However, Bachelier's final model as he noted in 1938 was an exact solution for the valuation model; alternatively the modern Black-Scholes Merton formulation which relies on an approximation method similar to that above is by definition an approximation of Bachelier's construct allowing for a log-normal distribution.

Figures 7 & 8 illustrate the error in the Black-Scholes Merton model arising from the approximation for an increasing underlying asset price given a log-normal distribution assumption.

The second model plots values for the curvature adjustments and gives %age variance impact on the call option value. The probability value is consistent between models.

Part Six: Conclusion

6.1 Summary

This paper reviews both the Black-Scholes Merton and the Bachelier option pricing models in their contemporaneous markets with the purpose of clarifying certain issues in their logical construct and finally comparing the two forms. This enables critiques of Bachelier's model to be seen against his contemporary setting at the end of the 19th century valuing options on futures contracts for French Government perpetual bonds, Rentes; and in terms of misunderstandings of the measures and model used. In addition, the paper critiques the adaptations of the earlier model in general both modern authors and in particular by Black-Scholes Merton – the former in terms of the lognormal assumption and the latter in terms of the risk free rate justification.

Thus Bachelier's model is variously reviewed with regard to its original context as an option contract on futures over Rentes, using an arithmetic formula, and in terms of its underlying mathematical form. This latter model is then adapted to comply with modern market practise both as an arithmetic model for pricing a present value premium, and lognormal continuous compounding geometric form applied in Black-Scholes Merton is shown. This enables the simplest comparison and statement of key properties.

These critiques show that the Bachelier approach is legitimate both in its original and the modern context; and that the B-SM model form should be queried in terms of both the lognormal proposition and the risk free rate justification, noting that the latter issue should have been strictly limited to listed equity share option contracts. Given this the paper recommends that the Louis Bachelier construct and derived models be utilised as the base option pricing model in financial price analysis, with adaptations for the present value premia using the underlying asset discount rate and retaining the normal distribution for a measure of dispersal.

This implication of such a revision would be extensive due to the ubiquitous application of the option pricing framework developed through financial analysis and academic literature.

6.2 Bachelier Model

The core pricing concept in Bachelier's model is defining the probability model with a correction for curvature as the underlying model. Given a spread, x, dispersal around the expected price at maturity using a Gaussian Normal distribution, we get a price premium, h.

$$\mathrm{h} = \mathrm{x}\left[p_x + \frac{dp_x}{dx}\right] \quad (13a)$$

In modern contractual terms applying the arithmetic return with normal dispersal in present value terms we have:

$$C_t^B = [S_t - X(1 - r_s^a \tau)]\mathcal{P} \quad (24a)$$

Where

$\mathcal{P} = p_x + \frac{dp_x}{dx}$, being the probability of exercise with a measure for instability at the point of exercise or boundary condition, d

$p_x = N(d)$ being the measure of probability; and

$\frac{dp_x}{dx} = \frac{n(d)}{d}$ being the curvature correction or measure of instability at the boundary.

$$d = -\frac{S_t - X(1 - r_s^a\tau)}{X(1 - r_s^a\tau)\sigma^a\sqrt{\tau}}$$

and expanding

$$C_t^B = [S_t - X(1 - r_s^a\tau)]N(d) + X(1 - r_s^a\tau)\sigma^a\sqrt{\tau}n(d)$$

6.3 The Black-Scholes Merton Approximation

It is demonstrated in Part Five that the modern form of the option pricing models are approximations of the Bachelier model adjusted for geometric terms, applying a technique Bachelier anticipated, and adjusting for the expected return and price path of the underlying asset reflecting the demonstrated result that the B-S M pde is stochastic in a dynamic price model. This model is restated here including using the return on the underlying asset to discount the exercise price which reflects the risk behaviour of the model in the dynamic context. This change removes a price conundrum identified in using the classical Black-Scholes Merton model arising from the differential implicit in the B-SM model due to the differential between the underlying asset return, r_s^e, and the risk free rate, r_f^e.

$$C_t^{BSM} = \{[S_t - Xe^{-r_s^e\tau}]N(d)\} + \{S_tN(d_1, d) + Xe^{-r_s^e\tau}N(d, d_2)\} \qquad (9)$$

where

$\{[S_t - Xe^{-r_s^e\tau}]N(d)\}$ is the primary probability of exercise in present value terms

$\{S_tN(d_1, d) + Xe^{-r_s^eT}N(d, d_2)\}$ is the approximation for the curvature adjustment.

$d = \frac{ln\left(\frac{S_t}{X}\right) + r_s^e\tau}{\sigma^e\sqrt{\tau}}$, the boundary point

$d_1 = d + \frac{\sigma^e\sqrt{\tau}}{2}$; and $d_2 = d - \frac{\sigma^e\sqrt{\tau}}{2}$

Rather than using $\pm\frac{\sigma^e\sqrt{\tau}}{2}$ reflecting an approximation of the curvature correction or instability coefficient. Bachelier used the expected dispersal, i.e. $\pm 25\%$ probability

6.4 Final Comment

Bachelier's analysis of the problem was based in his contemporaneous market and selection of Rentes as the underlying asset, and he clearly understood the limitations of the model and the work, as is noted in the reports on his dissertation. Bachelier went significantly beyond producing a simple model developing trading rules and also methodologies used by various modern authors to develop their treatise in this area. Although these later authors often did not recognise the model's limitations. It is certainly worth working through the full paper one could make an argument Merton used the logic of these propositions in developing his Americanisation of the Black-Scholes European model.

The nature of this study becomes important in financial literature as the form and structure of the arguments have become so ubiquitous in analysing and structuring pricing and management rules. The critiques of the B-SM form of the model lead to certain misstatements in the literature which should be addressed in the academic discussion of financial management and trading rules.

Ian A. Thomson

March 2016

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Part Eight: Appendix – The Greeks

Note: The following applies a constant yield dividend, ρ^e, as per Merton
X^* represents the present value of the exercise price for simplicity

PRICE PATH		**Black-Scholes (dynamic)**	**Louis Bachelier -Arithmetic**
Price Path	λ^e, λ^a $\tau = T - t$	$e^{\mu^e t + \sigma^e \sqrt{t} dz}$ $as\ t \to 0, \lambda^e \to 1$ $E(\lambda_T^e) = e^{\left(\mu^e + \frac{\sigma^{e2}}{2}\right)\tau};$ given $\sigma^e \sqrt{t} E(dz) = \frac{\sigma^{e2}}{2}$	$1 + \mu^a t + \sigma^a \sqrt{t} dz$ $as\ t \to 0, \lambda^\alpha \to 1$ $E(\lambda_T^\alpha) = 1 + \mu^a \tau;$ given $\sigma^a \sqrt{\tau} E(dz) = 0$
Price Path **Incl. dividend yield, ρ^e**	λ^e, λ^a $\tau = T - t$	$e^{\mu^e t - \rho^e t + \sigma^e \sqrt{t} dz}$ $as\ t \to 0, \lambda^e \to 1$ $E(\lambda_T^e) = e^{\left(\mu^e - \rho^e + \frac{\sigma^{e2}}{2}\right)\tau};$ given $\sigma^e \sqrt{t} E(dz) = \frac{\sigma^{e2}}{2}$	$1 + \mu^a t - \rho^a t + \sigma^a \sqrt{\tau} dz$ $as\ t \to 0, \lambda^e \to 1$ $E(\lambda_T^e) = 1 + \mu^a \tau - \rho^a \tau;$ given $\sigma^a \sqrt{\tau} E(dz) = 0$

MODEL		**Black-Scholes (dynamic)**	**Louis Bachelier -Arithmetic**
Call Option	C_t	$S_t \lambda^e N(d_1) - X e^{-r_f^e \tau} N(d_2)$ $\left[S_t \lambda^e - X e^{-r_f^e \tau}\right] N(\mathrm{d}) +$ $\left\{S_t \lambda^e N(d_1, d) + X e^{-r_f^e \tau} N(d, d_2)\right\}$ $\{curvature\ correction\}$	$(S_t - X^*)\left[N(\mathrm{d}) + \frac{1}{d} n(\mathrm{d})\right]$ $(S_t - X^*) N(\mathrm{d}) + X^* \sigma^a \sqrt{\tau} n(\mathrm{d})$ $\frac{dN(\mathrm{d})}{dd} = \frac{1}{d} n(\mathrm{d})$ for curvature
Boundary		$d_\mathrm{i} = \frac{lnS_t - lnX + \left(r_f^e \pm_i \frac{\sigma^{e2}}{2}\right)\tau}{\sigma^e \sqrt{\tau}}$, with $i \in 1{,}2$	$\frac{S_t - X^*}{X^* \sigma^a \sqrt{\tau}}$ with singular boundary
Differentials	$\frac{\partial N(d_\mathrm{i})}{\partial S_t}$ $\frac{\partial n(d_\mathrm{i})}{\partial S_t}$	$n(d_\mathrm{i}) \frac{\partial d_\mathrm{i}}{\partial S_t}$ with $i \in 1{,}2$ by product rule $-n(d_\mathrm{i}) \frac{\partial d_\mathrm{i}}{\partial S_t}$	$\frac{\partial N(\mathrm{d})}{\partial d} = \frac{1}{d} n(\mathrm{d}) \frac{\partial d}{\partial S_t} = \frac{1}{d} \frac{n(d)}{X^* \sigma^a \sqrt{\tau}}$ by Chain Rule $-n(\mathrm{d}) \frac{\partial d}{\partial S_t} = -\frac{n(d)}{X^* \sigma^a \sqrt{\tau}}$
Relationships **Pdf - BSM**		$n(d_2) = n(d_1) \frac{S_t}{X^*}$	-

MODEL		**Black-Scholes (dynamic)**	**Louis Bachelier -Arithmetic**
Relationships **Negative boundary**		$N(-d_{\mathrm{i}}) = 1 - N(d_{\mathrm{i}})$ $n(-d_{\mathrm{i}}) = n(d_{\mathrm{i}})$	-
Put Option	P_t	$-S_t\lambda^e N(-d_1) + Xe^{-r_f^e\tau}N(-d_2)$ $Xe^{-r_f^e\tau} - S_t\lambda^e + P_t$	$(-S_t + X^*)\left[N(-\mathrm{d}) + \frac{1}{d}n(\mathrm{d})\right]$ $(-S_t + X^*)N(-\mathrm{d}) + X^*\sigma^a\sqrt{\tau}n(\mathrm{d})$
Put-Call Parity		$S_t\lambda^e + P_t = Xe^{-r_f^e\tau} + C_t$	

CALL GREEKS		**Black-Scholes (dynamic)**	**Louis Bachelier -Arithmetic**
Delta, Δ	$\frac{\partial C_t}{\partial S_t}$	$N(d_1)$ As $t \to 0, \lambda^e \to 1$ the dynamic form is the same as the static	$N(d)$
Gamma, Γ	$\frac{\partial^2 C_t}{\partial {S_t}^2}$	$\frac{n(d_1)}{S_t\sigma^{\mathrm{e}}\sqrt{\tau}}$	$\frac{n(d)}{X(1-r_s^a\tau)\sigma^a\sqrt{\tau}}$
Strike, X	$\frac{\partial C_t}{\partial \mathrm{X}}$ $\frac{\partial C_t}{\partial \mathrm{X}e^{-r_f^e\tau}}$	$-N(d_2)e^{-r_f^e\tau}$ $-N(d_2)$	$-\left[\mathrm{N}(d) - \sigma_t\sqrt{\tau}n(d)\right](1 - r_s^a\tau)$ $-\left[\mathrm{N}(d) - \sigma_t\sqrt{\tau}n(d)\right]$
Rho, Ρ	$\frac{\partial C_t}{\partial r_f^e}$	$\tau Xe^{-r_f^e\tau}N(d_2)$	$\tau X\left[N(\mathrm{d}) - \sigma\sqrt{\tau}n(\mathrm{d})\right]$
Vega, $\boldsymbol{\mathcal{V}}$	$\frac{\partial C_t}{\partial \sigma^e}$	$S_t\sqrt{\tau}n(d_1)$	$X(1 - r_s^a\tau)\sqrt{\tau}n(\mathrm{d})$
Theta, Θ **Dynamic Θ**	$\frac{\partial C_t}{\partial \mathrm{t}}$	$S_t\lambda^e n(d_1)\frac{\sigma^{\mathrm{e}}}{2\sqrt{\tau}} + rXe^{-r_f^e\tau}N(d_2)$ $\left(r_f^e + \sigma\frac{1}{2\sqrt{t}}dz\right)S_t\lambda^e N(d_1) + S_t\lambda^e n(d_1)\frac{\sigma^{\mathrm{e}}}{2\sqrt{\tau}} + rXe^{-r_f^e\tau}N(d_2)$	$r_s^a\mathrm{X}N(\mathrm{d}) + \frac{\sigma}{2\sqrt{\tau}}Xn(\mathrm{d}) - r_s^a\sigma\frac{3}{2}\sqrt{\tau}Xn(\mathrm{d})$ $\left[S_t\left(r_s^a + \frac{\sigma}{2\sqrt{t}}dz\right) - r_s^a\mathrm{X}\right]N(\mathrm{d}) + \frac{\sigma}{2\sqrt{\tau}}Xn(\mathrm{d}) - \frac{3}{2}r_s^a\sigma\sqrt{\tau}Xn(\mathrm{d})$

PUT GREEKS		**Black-Scholes (dynamic)**	**Louis Bachelier -Arithmetic**
Delta, Δ Put	$\frac{\partial P_t}{\partial S_t}$	$N(d_1) - 1$ As $t \to 0, \lambda^e \to 1$ the dynamic form is the same as the static	$N(d)$
Theta, Θ Put Dynamic Θ	$\frac{\partial P_t}{\partial \mathrm{t}}$	$S_t n(d_1) \frac{\sigma^{\mathrm{e}}}{2\sqrt{\tau}} + rXe^{-r_f^e \tau}\left(1 - N(d_2)\right)$ $\left(r_f^e + \sigma \frac{1}{2\sqrt{t}} dz\right) S_t \lambda^e \left(1 - N(d_1)\right) + S_t \lambda^e n(d_1) \frac{\sigma^{\mathrm{e}}}{2\sqrt{\tau}} + rXe^{-r_f^e \tau}\left(1 - N(d_2)\right)$	$r_s^a \mathrm{X} N(\mathrm{d}) + \frac{\sigma}{2\sqrt{\tau}} Xn(\mathrm{d}) - r_s^a \sigma \frac{3}{2} \sqrt{\tau} Xn(\mathrm{d})$ $\left[S_t \left(r_s^a + \frac{\sigma}{2\sqrt{t}} dz\right) - \mathrm{rX}\right] N(\mathrm{d}) + \frac{\sigma}{2\sqrt{\tau}} Xn(\mathrm{d}) - \frac{3}{2} r_s^a \sigma \sqrt{\tau} Xn(\mathrm{d})$