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Numerical Valuation of High Dimensional Multivariate American Securities

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Abstract

We consider the problem of pricing an American contingent claim whose payoff depends on several sources of uncertainty. Several efficient numerical lattice-based techniques exist for pricing American securities depending on one or few (up to three) risk sources. However, these methods cannot be used for high dimensional problems, since their memory requirement is exponential in the number of risk sources. We present an efficient numerical technique that combines Monte Carlo simulation with a particular partitioning method of the underlying assets space, which we call Stratified State Aggregation (SSA). Using this technique, we can compute accurate approximations of prices of American securities with an arbitrary number of underlying assets. Our numerical experiments show that the method is practical for pricing American claims depending on up to 400 risk sources.

I. Introduction

We consider the problem of pricing an American contingent claim whose payoff depends on several sources of uncertainty. Several efficient numerical lattice-based techniques exist for pricing American securities depending on one or few (up to three) risk sources. However, these methods cannot be used for high dimensional problems, since their memory requirement is exponential in the number of risk sources.

There are several reasons motivating the development of efficient methods for multidimensional contingent claim pricing. In particular, applications exist in the pricing of over the counter (OTC) warrants, path-dependent instruments (Barraquand and Pudet (1995)), multidimensional interest rate term structure contingent claims (Heath et al. (1992)), mortgage-backed securities, life insurance policies (Fabozzi and Pollack (1987)), and futures contracts with quality delivery options (Cheng (1987) and Boyle (1989)).

For pricing purposes, financial assets can be divided into two major classes. The first class is that of assets whose future cash flows cannot be influenced by

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decisions from the holder taken after the purchase date. We will call European instruments all financial assets belonging to this first class. In particular, stocks, bonds, futures contracts, European options, swaps, caps, floors, and mortgage-backed securities are European instruments. The second class is that of assets whose cash flows can be influenced a posteriori by the holder. American options belong to this class We will call American instruments all financial assets belonging to this second class.

Following the general theory of arbitrage pricing, the theoretical price of a European contingent claim is the discounted expected value of its future cash flows under the so-called "risk-neutral" probability distribution of the underlying economic factors (Harrison and Kreps (1979), Harrison and Pliska (1981), Duffie (1988), and Karatzas and Shreve (1988)) Mathematically, computing the arbitrage price reduces to computing an integral (sum) over the space of the underlying economic factors. When the dimension of the space of the underlying economic factors is small, standard techniques for numerical integration can be used. In some cases, the integral can even be computed analytically (e.g., Black-Scholes formula). However, the computational complexity of evaluating the integral is clearly exponential in the dimension of the space. Efficient numerical techniques for pricing high dimensional European claims are presented in Barraquand (1995).

The price of an American claim is the maximum over all possible cash flow monitoring strategies of the associated present values of cash flows. For example, the price of an American option is the maximum over all possible early exercise strategies of the corresponding present values. Since the space of cash flow monitoring strategies is generally huge, direct maximization of the present value is rarely practical (see Bossaerts (1989) for a discussion). However, when the underlying economy is modeled as a Markov process, one can use the Bellman principle of dynamic programming (Bellman (1957)) to compute the optimal monitoring strategy. American options are typically priced using a discrete approximation of the dynamic programming principle. This is the case, in particular, of the CRR model (Cox et al. (1979)) for American stock option pricing. This approach becomes, however, impractical when the underlying economic space has many dimensions, since the dynamic programming algorithm requires a memory space exponential in the number of dimensions. This fact is known as the "curse of dimensionality" problem for dynamic programming.

In this paper, we present a particular state space partitioning technique that attempts to circumvent the curse of dimensionality problem for American security pricing. More precisely, we partition the space of underlying assets (the state space) into a tractable number of cells, and we compute an approximate early exercise strategy that is constant over those cells. The hope is that, if the partition is appropriately chosen, the approximate strategy will be close to the actual optimal strategy. Such a partitioning technique is classically called a *state aggregation* technique.

Among the many possible ways of choosing a partition, one solution is to choose a real-valued function mapping the state (i e., the prices of the underlying assets) that particularly influences the optimal strategy in the problem at hand. We call this function a stratification map Then, the partition chosen is a stratification of the state space into thin layers along this map. In other words, we limit our search to strategies that only depend upon the stratification map, and not upon the entire state itself. We call this technique Stratified State Aggregation.

In the case of American security pricing, an obvious candidate for the stratification map is the *payoff* of the security, i.e., the function representing the future cash flows associated with the security. When the stratification map chosen is the payoff of the American security, we call the technique Stratified State Aggregation along the Payoff (SSAP).

After quantization of the payoff, the SSAP method can be combined with Monte Carlo simulation techniques in order to compute the set of conditional probabilities corresponding to changes in the payoff value over time. Using these conditional probabilities, an approximation of the American price can then be computed backwards in time using techniques reminiscent of those from the classical CRR integration method.

We implemented the SSAP method on American option pricing problems in dimensions ranging from 1 to 400. On all problems for which we could compare the SSAP method with known optimal solutions, the SSAP price was indistinguishable from the optimal theoretical price. In particular, in dimensions 1, 2, and 3, both put and call prices of options on the maximum of the underlying assets were computed accurately by the SSAP method. Also, the SSAP price of an American call on the maximum of n assets paying no dividends was indistinguishable from the European price for n ranging from 1 to 400, in accordance with a well-known theoretical result. (Indeed, since the discounted prices of assets paying no dividends are martingales under the risk-neutral measure, the discounted maximum of such n assets prices is a submartingale. Hence, the corresponding European call price is always higher than the immediate payoff.) In other cases, no other method exists to compare to our results. However, the SSAP price seems to constitute an accurate approximation of the American price in arbitrarily high dimensions.

To the best of our knowledge, the SSAP method is the first capable of computing American prices and exercise strategies in high dimensional cases.

This paper is organized as follows. In Section II, we relate our contribution to previous work in American security pricing, multidimensional asset pricing, and Monte Carlo valuation. In Section III, we recall the usual assumptions on the stochastic processes governing the evolution of securities prices, and the main results of the Arbitrage Pricing Theory. In Section IV, we briefly review the current numerical methods used for American security pricing. In Section V, we present the method of Stratified State Aggregation. In Section VI, we show how SSA prices can be computed through Monte Carlo simulation. In Section VII, we present numerical experiments illustrating the efficiency of the SSAP method.

II. Relation to Other Work

The theoretical analysis of optimal stopping times for early exercise of American options dates back to the work of McKean (1965). This theory has then been further developed by several authors (Merton (1973), Harrison and Kreps (1979), Bensoussan (1984), Karatzas (1988), and Jaillet et al. (1988)). Myneni (1992) surveys the theory of American option pricing.

The most widely used valuation technique for American options with one underlying asset is the binomial lattice approach of Cox et al. (1979). Cox and Rubinstein (1985) outline the principle of the multidimensional extension. Other numerical valuation techniques are presented in Geske and Johnson (1984), Barone-Adesi and Whaley (1987), and Barone-Adesi and Elliott (1991).

The valuation of options depending of several underlying assets has been extensively studied. Brennan and Schwartz (1979) addresses the problem of pricing options under two sources of risk by direct finite difference approximation of the generalized Black-Scholes equation. In this example, the two sources of risk are the short-term and the long-term interest rates. The approach is clearly limited to a few assets, since the memory space requirements and the computation time are both exponential in the number of underlying assets. Boyle et al. (1989) developed a multinomial lattice method for pricing multidimensional options, in the spirit of the approach outlined in Cox and Rubinstein (1985). According to the authors, the computation becomes very burdensome for more than two assets. In fact, the multinomial lattice approach can be viewed as a finite difference approximation of the generalized Black-Scholes equation using an explicit Euler scheme and an appropriate change of variables (Brennan and Schwartz (1978)).

Stulz (1982) presents an analytical solution to the problem of pricing a European option on the maximum or minimum of two underlying assets. The analytical solution is generalized in Johnson (1987) to the case of an arbitrary number of assets, taking as given the cumulative multivariate normal distribution function. Boyle (1989) and Boyle and Tse (1990) developed an approximate method for the same problem. Although the problem is solved analytically in Johnson (1987), the approximate method does not require preliminary computation of the cumulative multivariate normal distribution function. To the best of our knowledge, no closed form solutions have ever been obtained for American pricing problems.

The application of the Monte Carlo method to option pricing was first presented in Boyle (1977), in the context of claims contingent to a single underlying asset. It has then been used by several authors for the valuation of path-dependent contingent claims. In particular, the method has been used for pricing mortgage-backed securities (see Schwartz and Torous (1989), Hutchinson and Zenios (1991)). Barraquand (1995) presents the method of Quadratic Resampling for Monte Carlo valuation of European securities with many underlying assets. The method presented in this paper is an extension of this earlier work to the American pricing problem.

III. Arbitrage Pricing Theory

A. Fundamental Results

The arbitrage pricing theory is described in many textbooks. We refer the reader to Duffie (1992) for a comprehensive presentation.

We consider a complete market spanned by n securities with prices $X(t) = (x_1(t), \ldots, x_n(t))$, representing all the information available to investors at time t. We assume that X follows the Itô process,

$$\forall i \in [1,n], \quad \frac{dx_i}{x_i} = \mu_{x_i}(x_1, \ldots, x_n, t)\, dt + \sum_{j=1}^{n} v_{ij}(x_1, \ldots, x_n, t)\, dw_j.$$

The matrix $V = (v_{ij})_{(i,j)\in[1,n]^2}$ is called the volatility matrix, and the covariance of relative returns is the matrix $\mathcal{K} = (k_{ij})_{(i,j)\in[1,n]^2}$,

$$\mathcal{K} = VV^T.$$

We consider the so-called *risk-neutral* information process $\tilde{X}$ for which all market prices for risk are zero,

$$(1) \qquad \forall i \in [1,n], \quad \frac{d\tilde{x}_i}{\tilde{x}_i} = \left(r - d_{x_i}\right) dt + \sum_{j=1}^{n} v_{ij} dw_j.$$

In the above equations, d_{x_i} is the dividend yield generated by the security x_i. A security is called a *European security* if and only if future cash flows cannot be influenced by decisions from the holder taken after the purchase date (besides, of course, selling back the security). Since the market is assumed complete and X is assumed Markov, the cash flows are only functions of time and of the n prices $x_1, \ldots, x_n$. The cash flow generated by a European security C during the time dt can be written,

$$f_C(X,t)dt.$$

A security is called an *American security* if and only if it is not European, i.e., if future cash flows can be influenced by decisions from the holder taken after the purchase date. Then, the cash flows are not only functions of time and information, but also functions of the decisions taken by the security's holder. A cash flow monitoring strategy u is a stochastic process associating with each time and information state a decision $u(X(t),t) \in U$, U being an appropriate *decision space*. We denote by CMS the space of cash flows monitoring strategies. The cash flow generated by an American security C during the time dt can be written,

$$f_C(u,X,t)dt.$$

The price C of an American contingent claim is the maximum over all possible cash flow monitoring strategies of the discounted expected value of future cash flows under the risk-neutral process $\tilde{X}$.

$$C(X(t),t) = \max_{u\in \mathrm{CMS}} \tilde{E}_t \left(\int_t^{\infty} \frac{f_C(u(X(\tau),\tau), X(\tau), \tau)}{L(t,\tau)} d\tau \right),$$

where $L(t,T)$ is the value at time T of one dollar invested at time t in the money market,

$$L(t,T) = \exp\left(\int_t^T r(X,s)ds \right),$$

and $r(X,t)$ is the riskless short interest rate.

B. American Options

By definition, an American call (resp. put) option on an asset S with expiration date T and strike price K gives its owner the right to purchase (resp. sell) *on or before time* T the asset S for the price K. The space of admissible cash flow monitoring strategies CMS is composed of all adapted processes u taking the two possible values,

$$\forall t \in [0,T], \quad u(X,t) \in \{\text{exercise, no-exercise}\},$$

and verifying at all times $t \leq T$,

$$u(X(t),t) \quad = \quad \text{exercise} \quad \Rightarrow \quad \forall \tau \;>\; t, u(X(\tau),\tau) \quad = \quad \text{no-exercise}.$$

That is, exercise cannot occur twice. For the sake of simplicity, we will assume that the payoff of the option at exercise time t^* is distributed during a small time interval $[t^*, t^* + \Delta t]$. We will also assume that exercise decisions $u(X(t),t)$ are piecewise-constant, i.e., change only at the beginning of time intervals of duration Δt. For notational convenience, we will define the "spike" function $\delta(u)$ associating the value $1/\Delta t$ to the decision exercise and the value 0 to the decision no-exercise.

The cash flow function of a call option is

$$f(u,X,t) \quad = \quad (S(X,t) - K)\delta(u).$$

The optimal early exercise strategy can be written,

$$u^*(X,t) \quad = \quad \begin{cases} \text{exercise} & \text{if} \quad C(X,t) \leq S(X,t) - K \\ \text{no-exercise} & \text{otherwise} \end{cases} .$$

We see that the computation of the optimal early exercise strategy requires us to precompute the pricing relationship between the option and the underlying asset, which is what we were trying to compute in the first place.

The price of the American call (resp. put) option can be written,

$$(2) \quad C(X(t),t) \quad = \quad \max_{u(X(\tau),\tau), \tau \geq t, u \in \text{CMS}} \tilde{E}_t \left(\sum_{\tau=t}^{T} \frac{f(X(\tau),\tau)}{L(t,\tau)} \delta(u(X(\tau),\tau))\Delta t \right),$$

with

$$f(X,t) \quad = \quad \max(0, S(X,t) - K) \quad (\text{resp. } \max(0, K - S(X,t))).$$

More generally, any contingent claim entitling its holder to the single cash flow f on or before an expiration date T can be priced according to the above formula.

IV. Numerical Methods for American Security Pricing

A. Stochastic Dynamic Programming

The explicit numerical valuation of an American option using the above formula involves a maximization over the set of of all possible early exercise strategies. The strategy u can be any function associating to each current value of the underlying assets $X = (x_1, \ldots, x_n)$ and to each current time t the binary decision to exercise or not exercise. Since the number of such possible strategies is huge, direct maximization is rarely practical (see Bossaerts (1989) for a discussion).

The only practical technique to date consists in using the Bellman principle of Dynamic Programming. This principle can be applied since the information vector X is assumed to be a Markov process and, therefore, the optimal early exercise strategy $u(X(t), t)$ only depends upon time and the current vector $X(t)$.

Assuming that exercise decisions can only be taken at discrete time intervals of constant duration Δt, the maximization problem (2) can be rewritten using the law of iterated expectations and the property $L(t, t^*) = L(t, t + \Delta t)L(t + \Delta t, t^*)$,

$$\begin{aligned} C(X(t), t) \quad &= \max_{u(X(t),t), u \in \mathrm{CMS}} \tilde{E}_t \left[\frac{f(X(t))}{L(t, t+\Delta t)} \delta(u(X(t), t))\Delta t + \frac{1}{L(t, t + \Delta t)} \right. \\ &\quad \times \left. \max_{u(X(\tau),\tau), \tau \geq t+\Delta t, u \in \mathrm{CMS}} \tilde{E}_{t+\Delta t} \left(\sum_{\tau = t+\Delta t}^{T} \frac{f(X(\tau))}{L(t + \Delta t, \tau)} \delta(u(X(\tau), \tau))\Delta t \right) \right] \end{aligned}$$

Examining successively the two cases $u(X(t), t)$ = exercise and $u(X(t), t)$ = no-exercise, and using the expression of $C(X(t + \Delta t), t + \Delta t)$ from equation (2), we obtain

$$\text{(3)} \qquad C(X(t), t) \quad = \quad \exp(-r\Delta t) \max \left(f(X(t)), \tilde{E}_t(C(X(t + \Delta t), t + \Delta t)) \right),$$

assuming that the interest rate r is piecewise constant on intervals of duration Δt. The above recursive expression, called the Bellman equation, allows us to compute the price C of an American option by proceeding backward in time from the expiration date T.

B. Lattice Methods and the Cox-Ross-Rubinstein Approach

The general method for solving the above Bellman equation is to quantize it using a finite difference or lattice method (see e.g., Duffie (1992), chapter 10). This is the paradigm underlying the original Cox-Ross-Rubinstein approach (Cox et al. (1979)).

We will illustrate this approach using a number of simplifying assumptions We will assume that the process of the underlying securities is jointly lognormal, i.e., that the mean and covariance matrix of relative returns are constant. We will also assume that the interest rate r and the dividend yields of the underlying securities are constant. Then, it can be shown that the solution of the Itô process (1) is

$$\text{(4)} \qquad \forall i \in [1, n], \quad x_i(t) \quad = \quad x_i(0) \exp \left(\left(r - d_{x_i} - \frac{1}{2} k_{ii} \right) t + \sum_{j=1}^{n} v_{ij} w_k(t) \right).$$

The Brownian motion W is approximated by an n-dimensional binomial process $\tilde{W}^{\Delta t}$ defined as follows,

$$\forall \epsilon \quad = \quad (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n,$$

$$\text{Prob}\left(\tilde{W}^{\Delta t}(t+\Delta t) = \tilde{W}^{\Delta t}(t) + \epsilon\sqrt{\Delta t}\right) = \frac{1}{2^n}.$$

The quantized Bellman equation is written as

$$\text{(5)} \quad C(W,t) = \exp(-r\Delta t)\max\left(f(X(t)), \frac{1}{2^n}\sum_{\epsilon\in\{-1,1\}^n} C(W+\epsilon\sqrt{\Delta t}, t+\Delta t)\right).$$

We implemented a simple variant of the above numerical scheme for an arbitrary number n underlying assets. The method yields accurate results for $n \leq 3$, but its memory requirement is intrinsically exponential in n. Hence, it cannot be used for $n > 3$ with a quantization step Δt small enough to yield accurate results (see Section VII).

V. State Aggregation

A. State Aggregation Price

Since classical numerical methods are unable to deal with high dimensional American valuation problems, one must resort to alternate approximation schemes. State aggregation is a classical approximation technique for the numerical solution of stochastic optimal control problems (see e.g., Bertsekas (1987), and Kushner and Dupuis (1992)).

For the problem of American security pricing, the relevant state space in the Bellman equation (3) is the n-dimensional space of the underlying assets values $X = (x_1, \ldots, x_n)$. State aggregation consists of partitioning the state space into a tractable number of cells, and of computing an approximate early exercise strategy $u(X,t)$ that is constant over those cells. The hope is that, if the partition is appropriately chosen, the approximate strategy will be close to the actual optimal strategy.

$\forall t \in \{0, \Delta t, \ldots, T\}$, let us consider a finite partition $P(t) = (P_1(t), \ldots, P_{k(t)}(t))$ of the state space R_+^n, i.e., a set of $k(t)$ subsets of R_+^n verifying,

$$\bigcup_{i\in[1,k(t)]} P_i(t) = R_+^n \quad \text{and} \quad \forall i \neq j, \quad P_i(t)\bigcap P_j(t) = \emptyset.$$

We assume that the partition $P(0)$ only has two cells,

$$P_1(0) = \{X(0)\}, \quad \text{and} \quad P_2(0) = R_+^n\backslash\{X(0)\}.$$

Among the set CMS of all possible early exercise strategies, we consider the subset $\mathcal{U}(P)$ of piecewise constant strategies, i.e., of strategies $u(X,t)$ that are *constant* over each cell $P_i(t)$ of the partition.

Then, we define the *state aggregation price* $C^*(i,t)$ as the maximum over all possible piecewise constant strategies in $\mathcal{U}(P)$ of the expected risk-neutral discounted future cash flow conditional on the event $X(t) \in P_i(t)$,

$$\text{(6)} \quad C^*(i,t) =$$

$$\max_{u(X(\tau),\tau),\tau\geq t,\ u\in\mathcal{U}(P)} \tilde{E}_0\left(\sum_{\tau=t}^{T}\frac{f(X(\tau),\tau)}{L(t,\tau)}\delta(u(X(\tau),\tau))\Delta t \mid X(t)\in P_i(t)\right).$$

Since $P_1(0) = \{X(0)\}$, and since $\mathcal{U}(P) \subset$ CMS, the state aggregation price at initial time $C^*(1,0)$ is obviously upper bounded by the true American price $C(X(0), 0)$. Furthermore, since the strategy u_{Euro} corresponding to the European price consists of never exercising before the expiration date, it is clearly constant over the cells of any partition before expiration. Hence, by definition of the state aggregation price, the European price is upper bounded by the state aggregation price at the initial time $C^*(1,0)$. We can state

> For any family of finite partitions P, the state aggregation price $C^* = C^*(1,0)$ is lower bounded by the European price and upper bounded by the American price,
>
> $$C^{\text{Euro}} \quad \leq \quad C^* \quad \leq \quad C^{\text{Amer}}.$$

We will now derive a recursive backward valuation formula for the state aggregation price, under an additional Markovian assumption.

B. Markovian Approximation

We will now assume that the partition P is such that the process $I(t)$ defined by $X(t) \in P_{I(t)}(t)$ is approximately Markov under the risk-neutral measure,

$$\begin{aligned} &\forall i, j, \phi, \quad \tilde{E}_0 \left(\phi(X(t+2\Delta t)) \mid X(t) \in P_i(t), \;\; X(t+\Delta t) \in P_j(t+\Delta t) \right) \\ &\approx \quad \tilde{E}_0 \left(\phi(X(t+2\Delta t)) \mid X(t+\Delta t) \in P_j(t+\Delta t) \right). \end{aligned}$$

Applying again the law of iterated expectations to the definition of the state aggregation price in equation (6), we obtain

$$\begin{aligned} C^*(I(t), t) \quad = \quad & \max_{u(X(t),t), u \in \mathcal{U}(P)} \tilde{E}_0 \left[\frac{f(X(t))}{L(t, t+\Delta t)} \delta(u(X(t), t)) \Delta t \right. \\ & + \frac{1}{L(t, t+\Delta t)} \max_{u(X(\tau),\tau), \tau \geq t+\Delta t, u \in \mathcal{U}(P)} \\ & \tilde{E}_0 \left(\sum_{\tau = t+\Delta t}^{T} \frac{f(X(\tau))}{L(t+\Delta t, \tau)} \delta(u(X(\tau), \tau)) \Delta t \right. \\ & \left. \left. \mid X(t) \in P_{I(t)}(t), \;\; X(t+\Delta t) \in P_{I(t+\Delta t)}(t+\Delta t) \right) \mid X(t) \in P_{I(t)}(t) \right]. \end{aligned}$$

By examining successively the two cases $u(X(t), t) =$ exercise and $u(X(t), t) =$ no-exercise, and applying formula (6) at time $t + \Delta t$,

$$\begin{aligned} & C^*(I(t), t) \quad \simeq \\ & \quad \max \left(\tilde{E}_0 \left(e^{-r\Delta t} f(X(t)) \mid X(t) \in P_{I(t)}(t) \right), \right. \\ & \quad \left. \tilde{E}_0 \left(e^{-r\Delta t} C^* \left(I(t+\Delta t), t+\Delta t \right) \mid X(t) \in P_{I(t)}(t) \right) \right), \end{aligned}$$

where the substitution of formula (6) at time $t + \Delta t$ is justified by the above Markovian approximation.

By definition of a partition, we have

$$1 = \sum_{j=1}^{k(t+\Delta t)} 1_{X(t+\Delta t)\in P_j(t+\Delta t)},$$

where 1_A is the indicator variable of the event A (i.e., takes the value 1 if and only if A is true, and 0 otherwise). Combining this with the above recursive equation, we obtain

$$\forall i \in [1, k(t)], \quad C^*(i,t) \approx \max\left(f_i(t), \sum_{j=1}^{k(t+\Delta t)} \pi_{ij}(t) C^*(j, t+\Delta t)\right),$$

where we have defined for notational convenience,

$$f_i(t) = \tilde{E}_0\left(e^{-r\Delta t} f(X(t)) \mid X(t) \in P_i(t)\right),$$

and

$$\pi_{ij}(t) = \tilde{E}_0\left(e^{-r\Delta t} 1_{X(t+\Delta t)\in P_j(t+\Delta t)} \mid X(t) \in P_i(t)\right).$$

Furthermore, the value at expiration date is determined by the terminal condition,

$$C^*(i,T) = f_i(T).$$

C. Recursive State Aggregation

We define the recursive state aggregation price C_{SA} as the solution of the following program,

$$C_{\text{SA}}(i,T) = f_i(T),$$

and

$$C_{\text{SA}}(i,t) = \max\left(f_i(t), \sum_{j=1}^{k(t+\Delta t)} \pi_{ij}(t) C_{\text{SA}}(j, t+\Delta t)\right).$$

When the partitions $P(t)$ are chosen in such a way that the process $I(t)$ is actually Markovian, the recursive state aggregation price C_{SA} is exactly the true state aggregation price C^*. The implementation of a recursive state aggregation program proceeds in two steps: i) definition of an appropriate family of partitions, and ii) computation of the expected payoffs $f_i(t)$ and the conditional probability matrices $p_{ij}(t)$. Then, the approximate price of an American contingent claim with terminal payoff f can be computed backward in time using the above recursive equation

Finite difference or lattice-based methods can be viewed as particular instances of the recursive state aggregation method. Indeed, each grid point or lattice point W^ι can be viewed as the center of the hypercube Hyper(W^ι) defined by

$$\text{Hyper}\left(W^\iota\right) = \left\{ Z \in R^n, \forall j \in [1,n], \quad w_j^\iota - \frac{1}{2}\Delta w_j \leq z_j < w_j^\iota + \frac{1}{2}\Delta w_j \right\}.$$

Although the process $I(t)$ such that $W(t) \in \text{Hyper}(W_{I(t)})$ is not actually Markovian, it can be approximated by a Markov process with reasonable accuracy for a fine enough lattice grid. In the lattice approach, this corresponds to approximating the Brownian motion W by the binomial process $W^{\Delta t}$. Then, the expected value over the hypercube $f_\iota(t)$ is approximated by the value in the center $f(X(W^\iota))$. Similarly, the conditional probability is simply taken from that of the binomial approximation, which yields the recursive formula (5). As explained in Section IV.B, the limitation of this approach is the exponential growth of the number of lattice hypercubes in the number n of underlying assets.

D. Stratified State Aggregation (SSA)

One solution for choosing a partition with a tractable number of cells is to fix a real-valued function mapping the state that particularly influences the optimal strategy in the problem at hand. We call such a function a stratification map. Then, the partition chosen is a stratification of the state space into thin layers along this map.

In order words, stratification consists in limiting the search to strategies that only depend upon the stratification map, and not upon the entire state itself. We call this technique Stratified State Aggregation.

In general, we can consider a vector-valued l-dimensional stratification map ($l < n$):

$$\begin{array}{rrcl} h: & R^n \times R & \longrightarrow & R^l \\ & (X,t) & \longrightarrow & h(X,t), \end{array} \tag{7}$$

and a family Q of partitions of R^l. From the family Q and the map h, we can build the reciprocal image partition $P = h^{-1}(Q)$,

$$P_\iota(t) = \left\{X \in R_+^n, h(X,t) \in Q_\iota(t)\right\}.$$

In the case of American security pricing, an obvious candidate for the stratification map is the *payoff* of the security. When the stratification map chosen is the payoff of the American security, we call the technique Stratified State Aggregation along the Payoff (SSAP).

Let us consider an American security with a single cash flow $f(X)$ on or before an expiration date T. In particular, in the case of an American call option, the cash flow is $f(X) = \max(0, S(X) - K)$, where K is the strike price. In order to illustrate the SSAP method, we set $l = 1$ and choose $h(X,t) = f(X)$ for the stratification map. We also take $k(t) = k$ constant for all times $t \in [0,T]$. In the numerical examples developed in Section VII, we assume that $S(X) = \max_{\iota\in[1,n]} x_\iota$, and that the process X is jointly lognormal, of the form described in Section IV.B. The partitions $Q(t)$

of the image space $R^l = R$ are chosen logarithmic in all these examples, i.e., the interval $Q_i(t), t > 0$ is of the form,

$$\forall i \in [2, k-1], \quad Q_i(t) \;=\; \left(A(t)e^{B(t)(i-2)}, A(t)e^{B(t)(i-1)}\right],$$

and

$$Q_1(t) \;=\; (-\infty, A(t)], \quad Q_k(t) \;=\; \left(A(t)e^{B(t)(k-2)}, +\infty\right),$$

for adequate parameters $A(t)$ and $B(t)$. The cell $P_i(t)$ is then, by definition,

$$\forall i \in [2, k-1], \quad P_i(t) \;=\; \left\{X \in R^n_+,\, A(t)e^{B(t)(i-2)} \;<\; f(X) \;\leq\; A(t)e^{B(t)(i-1)}\right\},$$

and

$$\begin{aligned} P_1(t) &= \{X \in R^n_+,\, f(X) \leq A(t)\}, \\ P_k(t) &= \left\{X \in R^n_+,\, f(X) \;>\; A(t)e^{B(t)(k-2)}\right\}. \end{aligned}$$

In our experiments, the number k of cells is set to 100. The numbers $A(t)$ and $B(t)$ are automatically adjusted so as to ensure

$$\text{Prob}\,(X(t) \in P_1(t)) \;\approx\; \text{Prob}\,(X(t) \in P_k(t)) \;\approx\; 0.1\%.$$

The numerical results obtained with the SSAP method, presented in Section VII, show that these empirical parameters are adequate for a broad range of American security pricing problems.

VI. Monte Carlo Estimation of American Price

A. Generation of Sample Paths

Once the family of partitions P has been chosen, for example, using the SSAP method, it remains to compute numerically the expected payoffs $f_i(t)$ and discounted conditional probabilities (Arrow-Debreu prices) $\pi_{ij}(t)$. These numbers can be expressed as integrals over the state space. In general, they must be computed numerically. The only general tractable method for computing such high dimensional integrals is the Monte Carlo method.

It consists in generating a given number M of sample paths for the underlying assets price process $X(t)$. In general, this can be done through direct numerical integration of the Itô equation (1). A simple explicit Euler scheme is given by

$$x_i(t+\Delta t) \;=\; x_i(t)\exp\left(\left(r - d_{x_i} - \frac{1}{2}k_{ii}\right)(X(t), t)\Delta t + \sum_{j=1}^{n} v_{ij}(X(t), t)\sqrt{\Delta t}\;\, z_j^t\right),$$

where z_j^t follows independent standard normal distributions for all j and t. Since $d = T/\Delta t$ is the number of time steps in $[0, T]$, we must draw a total of $M \times d \times n$ standard normal variates in order to generate M n-dimensional sample paths $X^1(t), \ldots, X^M(t)$ for all $t > 0$.

In general, Δt must be chosen small enough so as to reach a reasonable accuracy. In practice, a number of time steps $d = 100$ is sufficient in most asset pricing applications. However, when the joint process $X(t)$ is assumed lognormal as in Section IV.B, d can be chosen much smaller. Indeed, the underlying assets price process X can then be obtained by formula (4) from a standard Brownian motion W. In our experiments, we found that a number of time steps $d = 10$ is sufficient for American security pricing with lognormal underlying assets price processes.

B. Conditional Probabilities and Payoff Expectations

Once the M sample paths $X^1(t), \ldots, X^M(t)$ are computed, the number $a_i(t)$ of samples crossing $P_i(t)$ and the number $b_{ij}(t)$ of samples moving from $P_i(t)$ to $P_j(t + \Delta t)$ are easily computed,

$$\begin{aligned} a_i(t) &= \operatorname{Card}\left\{p \in [1, M], X^p(t) \in P_i(t)\right\}, \\ b_{ij}(t) &= \sum_{\left\{p \in [1,M], X^p(t) \in P_i(t), X^p(t+\Delta t) \in P_j(t+\Delta t)\right\}} e^{-r\Delta t}. \end{aligned}$$

Similarly, the sum $c_i(t)$ over of samples X^k of payoff values $f(X^k(t))$ is computed from

$$c_i(t) = \sum_{\{p \in [1,M], X^p(t) \in P_i(t)\}} e^{-r\Delta t} f(X^p(t)).$$

By the law of large numbers, we have the following identities,

$$\pi_{ij}(t) = \lim_{M \longrightarrow \infty} \frac{b_{ij}(t)}{a_i(t)}, \quad \text{and} \quad f_i(t) = \lim_{M \longrightarrow \infty} \frac{c_i(t)}{a_i(t)}.$$

C. Backward Integration Algorithm

Using the above Monte Carlo estimates of the conditional probabilities and payoff expectations, an approximation of the American price can then be computed backward in time using the simple algorithm described below.

i) At time T, the approximate SSAP price is initialized at

$$C(i, T) = \frac{c_i(T)}{a_i(T)}.$$

ii) At time $T - \Delta t$, we can compute for all $i \in [1, k]$,

$$C(i, T - \Delta t) = \max\left(\frac{c_i(T - \Delta t)}{a_i(T - \Delta t)}, \sum_{j=1}^{k} C(j, T) \frac{b_{ij}(T - \Delta t)}{a_i(T - \Delta t)}\right).$$

iii) The above procedure is then applied recursively, backward in time, to compute all the prices $C(i, T - 2\Delta t), C(i, T - 3\Delta t), \ldots, C(1, 0) = C_{\text{SSAP}}$.

The memory required in the SSAP method is proportional to $k^2 \times d$, corresponding to the storage of the conditional probabilities $p_{ij}(t)$. The computation time is proportional to $M \times n^2 \times d + k^2 \times d$, the first term corresponding to the drawing of the M Monte Carlo sample paths, and the second to the backward integration. Hence, the memory and time complexities of the SSAP method are polynomial in n. This is to be contrasted with classical lattice methods, which are exponential in n.

VII. Experimental Results

A. A Test Case

We implemented the method of Stratified State Aggregation along the Payoff function (SSAP). We present below some numerical results for several European and American option pricing problems ranging from 1 to 400 underlying assets. In all the experiments, we assumed that the American options can only be exercised at $d = 10$ different dates during the life of the option. This corresponds to choosing a time step $\Delta t = T/d = T/10$. We experimented with several different payoff functions, in particular with payoffs corresponding to the maximum, the minimum, or the average of the n underlying assets. We obtained similar results for all these different payoff functions. We only present below the case of an option on the maximum of the underlying assets. Its payoff function is defined by

$$f(x_1, \ldots, x_n) = \max(0, \max(x_1, \ldots, x_n) - K),$$

where K is the strike price of the option.

In all the experiments, we assumed that the underlying assets price process $X(t)$ is lognormal, with a covariance matrix of relative returns $\mathcal{K}$ of the form,

$$\forall i \in [1,n], \quad k_{ii} = \sigma_i^2,$$

and

$$\forall (i,j) \in [1,n]^2, \quad i \neq j, \quad k_{ij} = \rho \sigma_i \sigma_j,$$

for $n+1$ numbers $\sigma_1 > 0, \ldots, \sigma_n > 0$ and $-1/(n-1) \leq \rho \leq 1$.

Volatilities (σ_i), correlations (ρ), and interest rate (r) are counted in percent per year. The time to expiration T is counted in months, with the convention one month = 30 days. All asset and strike prices are counted in dollars.

Since SSAP uses Monte Carlo simulation, we report confidence intervals together with all results. These confidence intervals are computed from the central limit theorem, i.e., we assume that at a confidence level of 99.95 percent, the error must be less than four times the observed standard deviation of the result. The confidence interval reported is $4 \times$ stdev.

All the simulations were run on a DEC 3000 model 500X workstation, with an ALPHA AXP processor running at a clock rate of 200 Mhz, and one gigabyte of main memory.

B. One Underlying Asset

We first study the one-dimensional case. In this case, the SSAP price should converge toward the theoretical arbitrage price when both the number of time steps d and the number of cells k converge towards infinity. Both European calls and European puts can be priced according to the original Black-Scholes formula. These prices are reported in the columns European C_{BS} and European P_{BS} of Table 1. The American call can also be priced according to the same formula, since we assume the underlying asset pays no dividends. The price is reported in the column American C_{BS}. For the American put, we computed the price using a variant of the lattice method presented in Section IV.B. We call this method PDE, since it consists in solving a partial differential equation. In dimension 1, it is essentially equivalent to the Cox-Ross-Rubinstein binomial lattice method. We used 120 time steps for T (time to expiration) ranging from one to four months, and 210 time steps for $T = 7$ months. The corresponding price is reported in column American C_{PDE}. The SSAP prices were computed using $M = 100,000$ samples, and $k = 100$ buckets. The number of time steps was set to $d = 10$ in all the experiments.

The observed differences between the SSAP prices and the reference prices are below 0.7 percent. The confidence interval values are below 1 percent of the reference prices. American put prices given by the SSAP method are very accurate even when the difference with the European put prices are important (up to 30 cents). The computation time of a price using the SSAP method is about 21 seconds, compared with less than one second with a classical integration method (PDE). In dimension 1, classical finite difference of binomial lattice methods should be preferred to the SSAP method.

C. Three Underlying Assets

In this case, the SSAP method only finds an approximation of the optimal price. However, numerical experiments show that the SSAP price always remains within a few cents of the optimal theoretical price.

The European and American call and put option prices can be computed by the PDE method. This integration requires 120 time steps for $T = 1$ and $T = 4$ months and 210 time steps for $T = 7$ months. These results are reported in the columns European C_{PDE}, P_{PDE} and American C_{PDE}, P_{PDE} in Tables 2 and 3. The SSAP method was run using $M = 100,000$ samples, and $k = 100$ buckets. The number of time steps was set to $d = 10$ in all the experiments, and the SSAP method. Results are presented in Tables 2 and 3.

The observed differences between the SSAP prices and the reference prices are below 1 percent. The confidence interval value is below 1 percent of the reference prices, except for very low prices where it remains under 1 cent. American put prices given by the SSAP method are very accurate. The computation time of a price using the SSAP method is about 32 seconds, compared with 202 seconds for the classical integration method (PDE). In dimension 3, the SSAP method is as accurate and about six times faster than the classical integration method PDE.

TABLE 1

Results of the SSAP Method with One Underlying Asset

Panel A Call Option Prices $x_1(0)$ = \$40, r = 5%

Parameters			European			American		
σ_1	T	K	C_{BS}	C_{SSAP}	4 stdev	C_{BS}	C_{SSAP}	4 stdev
		35	5 15	5 15	0 001	5 15	5 15	0 002
20 %	1	40	1 00	1 00	0 006	1 00	1 00	0 008
		45	0 02	0 02	0 001	0 02	0 02	0 001
		35	5 76	5 76	0 003	5 76	5 76	0 005
20 %	4	40	2 16	2 16	0 006	2 16	2 16	0 008
		45	0 50	0 50	0 004	0 50	0 50	0 004
		35	6 40	6 40	0 004	6 40	6 40	0 006
20 %	7	40	3 00	2 99	0 010	3 00	3 00	0 010
		45	1 09	1 09	0 006	1 09	1 09	0 006
		35	5 38	5 38	0 003	5 38	5 40	0 007
40 %	1	40	1 91	1 91	0 010	1 91	1 92	0 010
		45	0 41	0 41	0 003	0 41	0 41	0 003
		35	6 88	6 88	0 010	6 88	6 90	0 020
40 %	4	40	3 96	3 96	0 020	3 96	3 97	0 020
		45	2 08	2 08	0 006	2 08	2 09	0 010
		35	8 07	8 08	0 010	8 07	8 10	0 020
40 %	7	40	5 35	5 34	0 024	5 35	5 36	0 040
		45	3 40	3 40	0 012	3 40	3 42	0 020

Panel B Put Option Prices $x_1(0)$ = \$40, r = 5%

Parameters			European			American		
σ_1	T	K	P_{BS}	P_{SSAP}	4 stdev	P_{PDE}	P_{SSAP}	4 stdev
		35	0 00	0 00	0 001	0 00	0 00	0 001
20 %	1	40	0 83	0 83	0 008	0 84	0 84	0 008
		45	4 84	4 84	0 001	5 00	5 00	0 000
		35	0 19	0 19	0 003	0 19	0 19	0 004
20 %	4	40	1 50	1 50	0 006	1 56	1 56	0 010
		45	4 77	4 77	0 004	5 06	5 07	0 010
		35	0 41	0 41	0 003	0 42	0 42	0 006
20 %	7	40	1 86	1 86	0 010	1 96	1 96	0 012
		45	4 82	4 82	0 006	5 24	5 23	0 016
		35	0 24	0 24	0 003	0 24	0 24	0 003
40 %	1	40	1 74	1 74	0 010	1 75	1 76	0 010
		45	5 23	5 23	0 003	5 27	5 29	0 015
		35	1 31	1 31	0 010	1 32	1 33	0 020
40 %	4	40	3 31	3 31	0 020	3 36	3 37	0 020
		45	6 35	6 35	0 006	6 47	6 49	0 015
		35	2 08	2 09	0 012	2 12	2 13	0 015
40 %	7	40	4 21	4 21	0 020	4 31	4 31	0 016
		45	7 12	7 13	0 012	7 36	7 34	0 030

D. Ten Underlying Assets

In the previous subsections VII.B and VII.C, we compared the efficiency and accuracy of the SSAP method with that of the classical integration method (PDE). In this subsection, we report results obtained with the SSAP method on American

TABLE 2

Prices for a Call Option with Three Underlying Assets

Parameters			Call Option Prices $x_1(0) = x_2(0) = x_3(0) = \$40$, $\sigma_1 = 20\%$, $\sigma_2 = 30\%$, $\sigma_3 = 50\%$, $r = 5\%$					
			European			American		
ρ	T	K	C_{PDE}	C_{SSAP}	4 stdev	C_{PDE}	C_{SSAP}	4 stdev
		35	8 59	8 58	0 008	8 59	8 59	0 010
0 %	1	40	3 84	3 83	0 010	3 84	3 84	0 012
		45	0 89	0 89	0 006	0 89	0 90	0 007
		35	12 55	12 53	0 020	12 55	12 55	0 016
0 %	4	40	7 87	7 85	0 014	7 87	7 87	0 020
		45	4 26	4 25	0 014	4 26	4 27	0 014
		35	15 29	15 27	0 020	15 29	15 30	0 030
0 %	7	40	10 72	10 70	0 016	10 72	10 73	0 035
		45	6 96	6 95	0 020	6 96	6 98	0 020
		35	7 78	7 77	0 010	7 78	7 78	0 012
50 %	1	40	3 18	3 17	0 010	3 18	3 18	0 012
		45	0 82	0 82	0 004	0 82	0 83	0 004
		35	10 97	10 95	0 010	10 97	10 96	0 015
50 %	4	40	6 69	6 67	0 016	6 69	6 69	0 020
		45	3 70	3 69	0 012	3 70	3 71	0 025
		35	13 23	13 21	0 016	13 23	13 24	0 030
50 %	7	40	9 11	9 09	0 020	9 11	9 12	0 040
		45	5 98	5 98	0 016	5 98	5 99	0 020
		35	6 53	6 52	0 006	6 53	6 54	0 010
100 %	1	40	2 38	2 37	0 010	2 38	2 38	0 012
		45	0 74	0 74	0 002	0 74	0 74	0 003
		35	8 51	8 50	0 008	8 51	8 53	0 016
100 %	4	40	4 92	4 90	0 012	4 92	4 93	0 020
		45	2 97	2 96	0 008	2 97	2 99	0 020
		35	10 04	10 03	0 010	10 04	10 07	0 016
100 %	7	40	6 64	6 63	0 016	6 64	6 67	0 030
		45	4 61	4 60	0 016	4 61	4 64	0 040

option pricing problems with 10 underlying assets (Tables 4 and 5). Since no other method exists to compare to our results, and since the SSAP method only provides an approximation of the optimal price, we cannot guarantee the accuracy of the American premiums reported below. However, both the observed confidence intervals and the fact that the SSAP prices for the American calls without dividends equal the European prices lead us to believe that the SSAP method is reliable in general on 10-dimensional American pricing problems. The parameters of the SSAP method are again $M = 100,000$ and $k = 100$.

The differences between European and American call prices are below 0.5 percent. Since the payoff is the maximum of n underlying assets prices without dividends, these two prices should indeed be identical. Confidence intervals are around 1 percent in the worst case. The computation time using the SSAP method is about 82 seconds.

TABLE 3

Prices for a Put Option with Three Underlying Assets

Parameters			Put Option Prices $x_1(0) = x_2(0) = x_3(0) = \$40$, $\sigma_1 = 20\%$, $\sigma_2 = 30\%$, $\sigma_3 = 50\%$, $r = 5\%$					
			European			American		
ρ	T	K	P_{PDE}	P_{SSAP}	4 stdev	P_{PDE}	P_{SSAP}	4 stdev
		35	0.00	0.00	0.000	0.00	0.00	0.000
0 %	1	40	0.13	0.13	0.003	0.23	0.23	0.003
		45	2.26	2.27	0.012	5.00	5.00	0.000
		35	0.01	0.01	0.002	0.01	0.01	0.001
0 %	4	40	0.25	0.25	0.006	0.44	0.45	0.006
		45	1.55	1.56	0.010	5.00	5.00	0.000
		35	0.03	0.03	0.004	0.04	0.04	0.002
0 %	7	40	0.31	0.32	0.010	0.57	0.58	0.015
		45	1.41	1.42	0.020	5.00	5.00	0.000
		35	0.00	0.00	0.000	0.00	0.00	0.000
50 %	1	40	0.38	0.39	0.003	0.48	0.49	0.006
		45	3.00	3.01	0.010	5.00	5.00	0.000
		35	0.07	0.08	0.006	0.09	0.09	0.004
50 %	4	40	0.72	0.72	0.006	0.93	0.94	0.010
		45	2.65	2.66	0.010	5.00	5.00	0.000
		35	0.17	0.17	0.006	0.20	0.20	0.005
50 %	7	40	0.91	0.91	0.012	1.19	1.21	0.010
		45	2.63	2.65	0.012	5.00	5.00	0.000
		35	0.01	0.01	0.001	0.01	0.01	0.001
100%	1	40	0.84	0.84	0.004	0.84	0.85	0.006
		45	4.18	4.18	0.003	5.00	5.00	0.000
		35	0.19	0.19	0.003	0.19	0.19	0.004
100%	4	40	1.51	1.51	0.006	1.56	1.57	0.008
		45	4.49	4.49	0.005	5.00	5.00	0.008
		35	0.41	0.41	0.004	0.42	0.42	0.004
100%	7	40	1.87	1.86	0.010	1.96	1.97	0.012
		45	4.70	4.70	0.008	5.20	5.20	0.012

E. Experimental Time Complexity

We analyze in this subsection the computation time required by the SSAP method as a function of the number n of underlying assets. We compare these results when possible to those of classical integration methods (PDE). Observed computation times are presented in Table 6. The number of Monte Carlo samples is $M = 100,000$. The number of cells is $k = 100$ as before.

Figure 1 shows that for up to 40 underlying assets, the computation time is linear in n. The dominating term is the computation of the payoff function, which is linear in n ($M \times n \times d$). Figure 2 shows that for larger n, the complexity is quadratic in n ($M \times n^2 \times d$), as expected.

For $n = 2$, the SSAP is comparable to the classical integral method. For $n = 3$, the SSAP method is faster by a factor of six. For $n > 3$, the SSAP method is the only one that can compute the price accurately. The integral (PDE) method is implemented using 60 time steps in the Cox-Rubinstein tree to obtain comparable

FIGURE 1

Linear Behavior of Computation Time for $0 < n \leq 60$

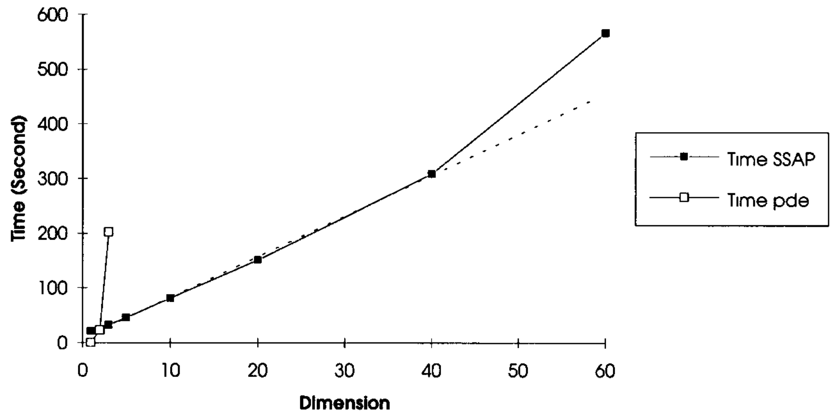

FIGURE 2

Quadratic Behavior of Computation Time for $0 < n \leq 400$

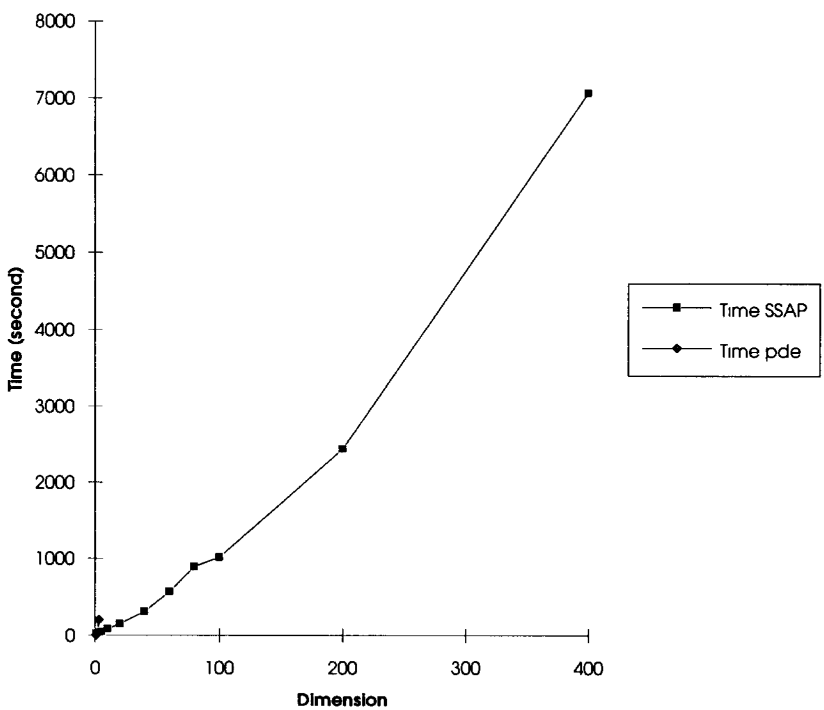

TABLE 4

Prices of a Call Option with Ten Underlying Assets

Parameters			Call option prices $x_1(0) = \cdots = x_{10}(0) = \$40$, $\sigma_1 = \cdots = \sigma_{10} = 40\%$, $r = 5\%$			
			European		American	
ρ	T	K	C_{SSAP}	4 stdev	C_{SSAP}	4 stdev
		35	12.66	0.010	12.66	0.010
0 %	1	40	7.68	0.020	7.68	0.020
		45	2.98	0.020	2.98	0.020
		35	21.53	0.060	21.54	0.050
0 %	4	40	16.62	0.030	16.62	0.030
		45	11.76	0.040	11.76	0.040
		35	27.91	0.060	27.92	0.060
0 %	7	40	23.06	0.060	23.08	0.060
		45	18.24	0.040	18.25	0.040
		35	10.36	0.008	10.36	0.008
50 %	1	40	5.54	0.010	5.54	0.010
		45	1.90	0.010	1.90	0.010
		35	16.52	0.020	16.53	0.020
50 %	4	40	11.87	0.020	11.87	0.020
		45	7.81	0.040	7.81	0.040
		35	20.91	0.060	20.92	0.050
50 %	7	40	16.38	0.040	16.38	0.040
		45	12.28	0.020	12.28	0.020
		35	5.41	0.003	5.42	0.010
100 %	1	40	1.93	0.006	1.93	0.010
		45	0.42	0.004	0.42	0.004
		35	6.93	0.006	6.95	0.010
100 %	4	40	4.00	0.012	4.02	0.020
		45	2.11	0.006	2.11	0.010
		35	8.14	0.010	8.16	0.020
100 %	7	40	5.40	0.016	5.42	0.020
		45	3.44	0.012	3.45	0.020

precision. The exercise condition is also applied only 10 times during the life period of the option.

VIII. Conclusion

In this article, we described a systematic numerical technique for pricing arbitrarily complex American contingent claims, i.e., generalized option contracts with possibilities of early exercise. Besides its obvious applications to trading and hedging in organized and over the counter (OTC) capital markets, American security pricing has many important applications in various areas of risk management such as assets and liabilities management and corporate investment decision making. Using this technique, we were able to compute the prices of complex American instruments in a few tens of seconds on a workstation, and within a few seconds on a network of workstations.

TABLE 5

Prices of a Put Option with Ten Underlying Assets

Parameters			Put Option Prices $x_1(0) = \quad = x_{10}(0) = \$40$, $\sigma_1 = \quad = \sigma_{10} = 40\%$, $r = 5\%$			
			European		American	
ρ	T	K	P_{SSAP}	4 stdev	P_{SSAP}	4 stdev
		35	0 00	0 0000	0 00	0 0000
0 %	1	40	0 00	0 0000	0 00	0 0000
		45	0 28	0 0060	5 00	0 0000
		35	0 00	0 0000	0 00	0 0000
0 %	4	40	0 00	0 0008	0 01	0 0008
		45	0 06	0 0060	5 00	0 0000
		35	0 00	0 0000	0 00	0 0000
0 %	7	40	0 00	0 0008	0 01	0 0010
		45	0 04	0 0030	5 00	0 0000
		35	0 00	0 0008	0 00	0 0006
50 %	1	40	0 16	0 0030	0 26	0 0040
		45	1 49	0 0120	5 00	0 0000
		35	0 05	0 0040	0 07	0 0030
50 %	4	40	0 32	0 0060	0 52	0 0060
		45	1 17	0 0200	5 00	0 0000
		35	0 10	0 0060	0 15	0 0060
50 %	7	40	0 42	0 0120	0 69	0 0080
		45	1 18	0 0200	5 00	0 0000
		35	0 24	0 0020	0 24	0 0020
100 %	1	40	1 73	0 0060	1 75	0 0080
		45	5 20	0 0040	5 27	0 0100
		35	1 30	0 0060	1 32	0 0100
100 %	4	40	3 28	0 0160	3 34	0 0120
		45	6 31	0 0060	6 45	0 0100
		35	2 06	0 0100	2 11	0 0160
100 %	7	40	4 18	0 0160	4 29	0 0160
		45	7 08	0 0150	7 30	0 0240

TABLE 6

Computation Times as Functions of the Dimension

Dimension (N)	Time SSAP Method (Second)	Time PDE Method (Second)
1	21	0 12
2	25	23
3	32	202
5	46	Out of memory
10	81	Out of memory
20	151	Out of memory
40	309	Out of memory
60	567	Out of memory
80	894	Out of memory
100	1018	Out of memory
200	2438	Out of memory
400	7069	Out of memory

Our approach essentially relies on appropriate state aggregation techniques that circumvent the intractability of the computation of the early exercise boundary, combined with a classical Monte Carlo simulation for the computation of the conditional probabilities in the backwards pricing formula. We call this method Stratified State Aggregation along the Payoff function (SSAP) We have successfully implemented the SSAP method for problems with up to 400 dimensions. To the best of our knowledge, no other method has ever been developed to date for pricing American contingent claims with many (more than three or four) underlying assets.

We feel that the method presented in this paper and the experimental results thus obtained make it possible to realistically envision the use of multidimensional stochastic models for practical real-world quantitative risk management problems. This capability of computing the joint influences of several tens of risk factors such as interest rates of various terms in different currencies, equity and commodities of various kinds, and any other relevant economic variables, may dramatically increase the competitive advantage of quantitative methods over more traditional analysis techniques. An application of particular interest is the pricing and hedging of complex derivatives offered on international OTC markets. We plan to backtest on actual market data the performance of the SSAP method as compared to more classical delta-hedging techniques currently used on capital markets.

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