八、三元樹理論介紹

(一)參數的選擇

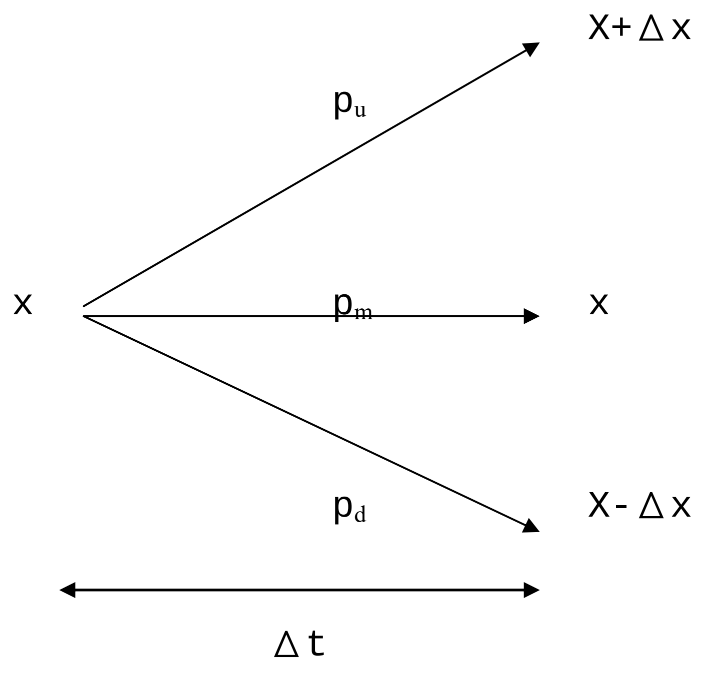
- 發放連續股利收益率之風險中立的幾何布朗運動隨機差分方程式為

$$dS = (r-\delta)Sdt + \sigma SdZ$$

  - 以對數股價方式，$x = \mathrm{Ln}(S)$，表示可改寫為

$$dx = vdt + \sigma dZ, \quad v = r - \delta - \frac{1}{2}\sigma^2$$

- 考慮在單位時間Δt 內，股價可能上漲Δx，維持不變，可能下跌Δx，其機率分別為 p_u, p_m, p_d，之三元樹模型。
  - 股價變動量與時間間距，兩者有一定關聯，$\Delta x = \sigma\sqrt{3\Delta t}$ ，不可分別決定。



- 機率的選擇數值為

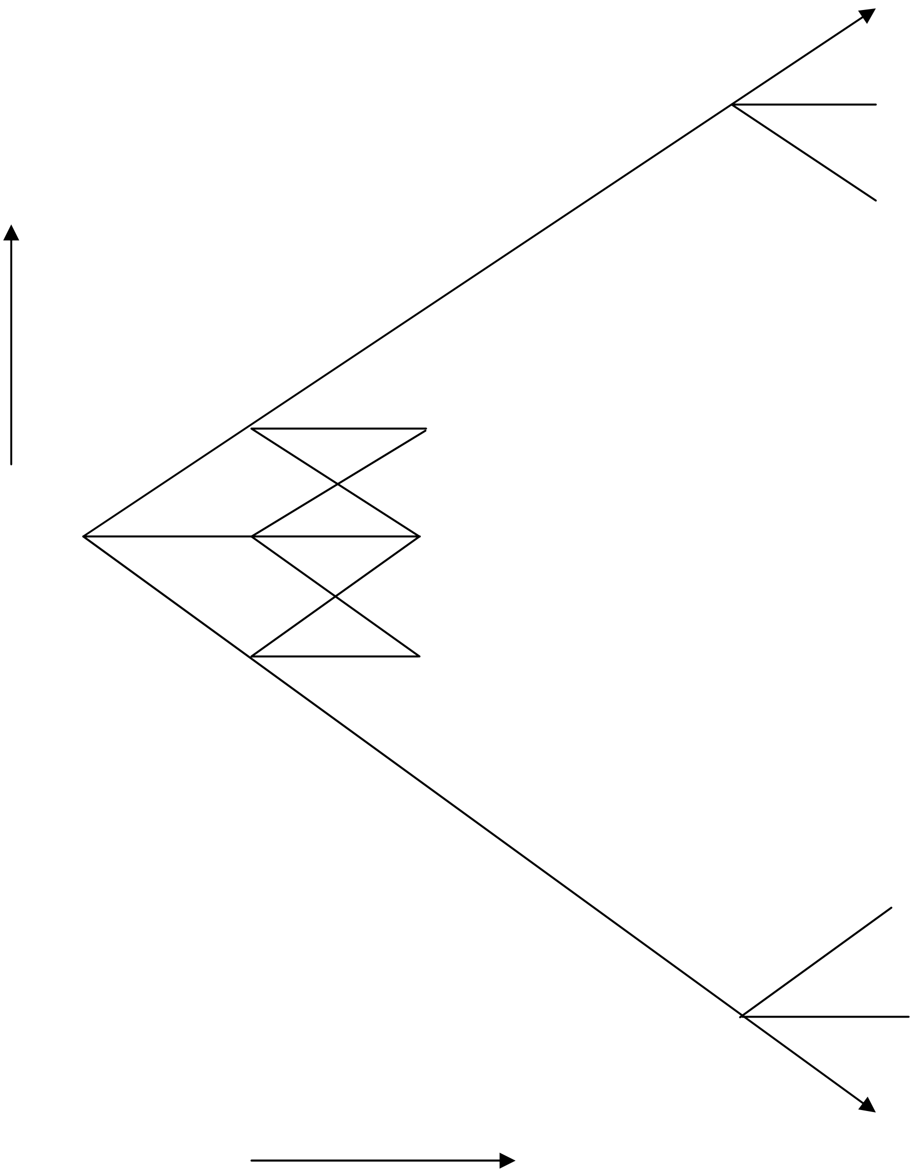
$$p_u = \frac{1}{2}\left(\frac{\sigma^2\Delta t + v^2\Delta t^2}{\Delta x^2} + \frac{v\Delta t}{\Delta x}\right)$$

董夢雲 dongmy@ms5.hinet.net

$$p_m = 1 - \frac{\sigma^2 \Delta t + v^2 \Delta t^2}{\Delta x^2}$$

$$p_d = \frac{1}{2}\left(\frac{\sigma^2 \Delta t + v^2 \Delta t^2}{\Delta x^2} - \frac{v \Delta t}{\Delta x} \right)$$

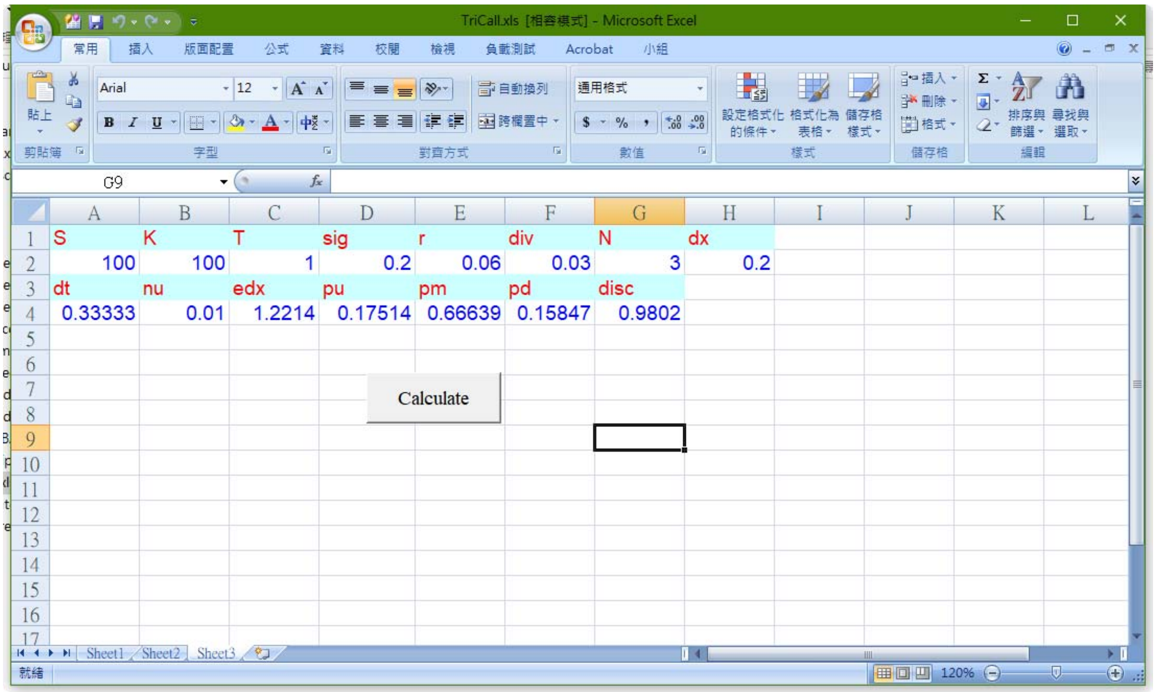
✦ 拓展後的三元樹結構如下



✦ 由到期節點，以下式倒算到期初，美式選擇權需考慮提前執行之時機。

$$C_{N,j} = \max(0, S_{N,j} - K)$$

$$C_{i,j} = e^{-r \bullet \Delta t}(p_u C_{i+1,j+1} + p_m C_{i+1,j} + p_d C_{i+1,j-1})$$



TriCall.xls

(二)Pseudo Code (European)

```
input: S, sig, r, n, T, div, dx
real: dt, nu, dxu, dxd, pu, pd, St[-N..N],C[0..N,-N..N]
      disc
integer: i, j

dt = T/n
nu = r-div-0.5*sig^2
dx = sig*(3*dt)^0.5
edx = exp(dx)
pu = 0.5*((sig^2*dt+nu^2*dt^2)/dx^2+nu*dt/dx)
pm = 1.0-(sig^2*dt+nu^2*dt^2)/dx^2
pd = 0.5*((sig^2*dt+nu^2*dt^2)/dx^2-nu*dt/dx)
disc = exp(-r*dt)

St[-N] = S*exp(-N*dx)
for(j=-N+1 to N)
  St[j] = St[j-1]*edx

For j=-N to N
  C[N,j] = max(0.0, St[j]-K)

For(i = N-1 downto 0)
  for(j=-i to i)
    C[i,j] = disc*(pu*C[i+1,j+1]+pm*C[i+1,j]+
             pd*C[i+1,j-1])

return C[0,0]
```

(三)數值範例

S	K	T	sig	r	div	N	dx
100	100	1	0.2	0.06	0.03	3	0.2

dt	nu	edx	pu	pm	pd
0.3333	0.01	1.2214	0.1751	0.6664	0.1585

disc
0.9802

Key: **St**
C

	i: 0	1	2	3
j \ t	0.0	0.3333	0.6667	1.0
3				182.21 82.21
2			149.18 49.6782	149.18 49.18
1		122.14 24.0802	122.14 22.9051	122.14 22.14
0	100.00 **8.4253**	100.00 4.6546	100.00 3.8008	100.00 0.00
-1		81.87 0.6525	81.87 0.00	81.87 0.00
-2			67.03 0.00	67.03 0.00
-3				54.88 0.00

董夢雲 dongmy@ms5.hinet.net

(四)範例程式

➢ 參數選擇如下

$u = \exp(\sigma\sqrt{2\Delta t})$，$d = \exp(-\sigma\sqrt{2\Delta t})$，

$$p_u = \left(\frac{e^{b\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2, \quad p_d = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{b\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2,$$

$p_m = 1 - p_u - p_d$。

➢ 數字實例

✦ 美式賣權，S = 100，K = 110，T = 0.5，b = r = 10%，σ = 27%，n = 30，使用下面函數與參數

```
11.6493 = TrinomialTree(“a”, “p”, 100, 110, 0.5, 0.1, 0.1, 0.27, 30)
```

➢ Source Code

```
'// Trinomial tree
Public Function TrinomialTree(AmeEurFlag As String, CallPutFlag As String, S As Double,
X As Double, T As Double, _
             r As Double, b As Double, v As Double, n As Integer) As Double

   Dim OptionValue() As Double
   Dim dt As Double, u As Double, d As Double
   Dim pu As Double, pd As Double, pm As Double
   Dim i As Integer, j As Integer, z As Integer
   Dim Df As Double

   ReDim OptionValue(n * 2 + 1)

   If CallPutFlag = "c" Then
      z = 1
      ElseIf CallPutFlag = "p" Then
      z = -1
   End If

   dt = T / n
   u = Exp(v * Sqr(2 * dt))
   d = Exp(-v * Sqr(2 * dt))
   pu = ((Exp(b * dt / 2) - Exp(-v * Sqr(dt / 2))) / (Exp(v * Sqr(dt / 2))
        - Exp(-v * Sqr(dt / 2)))) ^ 2

   pd = ((Exp(v * Sqr(dt / 2)) - Exp(b * dt / 2)) / (Exp(v * Sqr(dt / 2))
        - Exp(-v * Sqr(dt / 2)))) ^ 2
   pm = 1 - pu - pd
   Df = Exp(-r * dt)

   For i = 0 To (2 * n)
       OptionValue(i) = Max(0, z * (S * u ^ Max(i - n, 0) * d ^ Max(n * 2
           - n - i, 0) - X))
   Next

   For j = n - 1 To 0 Step -1
      For i = 0 To (j * 2)
         If AmeEurFlag = "e" Then
            OptionValue(i) = (pu * OptionValue(i + 2) + pm * OptionValue(I
                + 1) + pd * OptionValue(i)) * Df
         ElseIf AmeEurFlag = "a" Then
            OptionValue(i) = Max((z * (S * u ^ Max(i - j, 0) * d ^ Max(j
                * 2 - j - i, 0) - X)), (pu * OptionValue(i + 2) + pm *
                 OptionValue(i + 1) + pd * OptionValue(i)) * Df)
         End If
      Next
   Next

   TrinomialTree = OptionValue(0)

End Function
```


董夢雲 dongmy@ms5.hinet.net

九、兩變數之二元樹實作*

(一)參數的選擇

- 若選擇權的期末償付由一個以上的資產價格決定，則我們需同時處理這些資產的模型。
  - 考慮一個兩資產的償付情況，Max[0, $S_{1,T}$-$S_{2,T}$-K]。

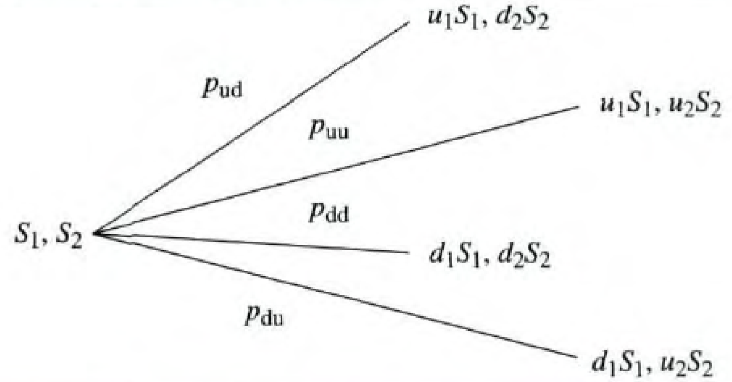
$$dS_1 = (r-\delta_1)S_1dt + \sigma_1 S_1 dZ_1$$

$$dS_2 = (r-\delta_2)S_2dt + \sigma_2 S_2 dZ_2$$

  - 兩資產價格間有相關性 ρ，

$$dZ_1 dZ_2 = \rho dt$$

FIGURE 2.26 Multiplicative Two-Variable Binomial Process



- 對數股價之結構如下

$$dx_1 = v_1 dt + \sigma_1 dZ_1,\quad v_1 = r-\delta_1-\frac{1}{2}\sigma_1^2$$

$$dx_2 = v_2 dt + \sigma_2 dZ_2,\quad v_2 = r-\delta_2-\frac{1}{2}\sigma_2^2$$

✦ 求解可得

$$\Delta x_1 = \sigma_1\sqrt{\Delta t} \text{ ，} \Delta x_2 = \sigma_2\sqrt{\Delta t}$$

$$p_{uu} = \frac{1}{4}\frac{(\Delta x_1\Delta x_2 + \Delta x_2 v_1\Delta t + \Delta x_1 v_2\Delta t + \rho\sigma_1\sigma_2\Delta t)}{\Delta x_1\Delta x_2}$$

$$p_{ud} = \frac{1}{4}\frac{(\Delta x_1\Delta x_2 + \Delta x_2 v_1\Delta t - \Delta x_1 v_2\Delta t - \rho\sigma_1\sigma_2\Delta t)}{\Delta x_1\Delta x_2}$$

$$p_{du} = \frac{1}{4}\frac{(\Delta x_1\Delta x_2 + \Delta x_2 v_1\Delta t + \Delta x_1 v_2\Delta t + \rho\sigma_1\sigma_2\Delta t)}{\Delta x_1\Delta x_2}$$

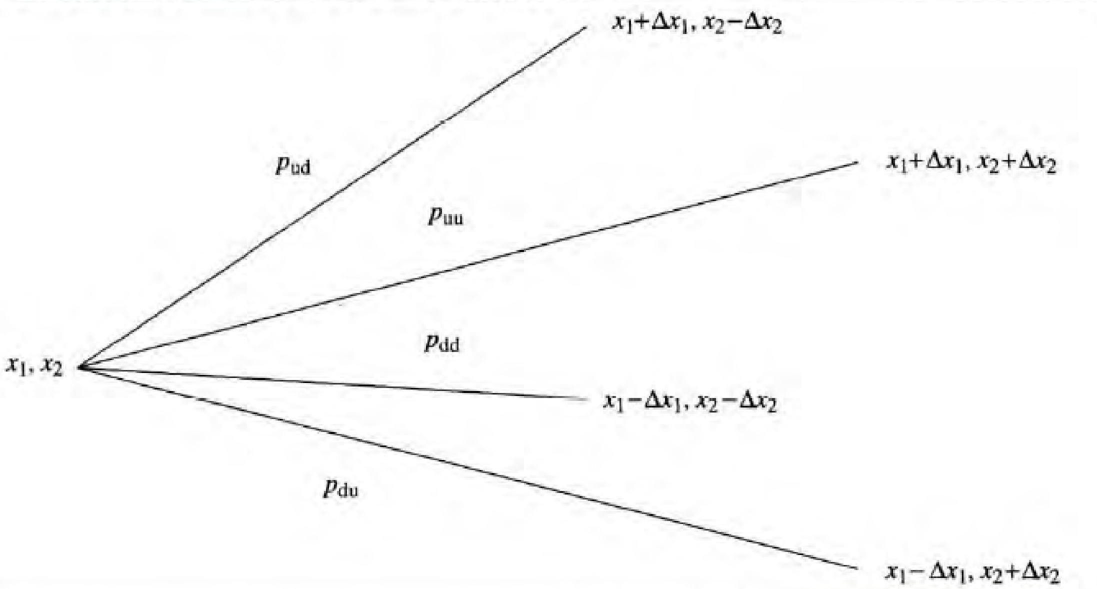
$$p_{dd} = \frac{1}{4}\frac{(\Delta x_1\Delta x_2 - \Delta x_2 v_1\Delta t - \Delta x_1 v_2\Delta t + \rho\sigma_1\sigma_2\Delta t)}{\Delta x_1\Delta x_2}$$

✦ 樹上的(i, j, k)表第 i 步，S_1 資產在 j 狀態，S_2 資產在 k 狀態的情況。

$$S_{1,i,i,k} = S_1 Exp(j\Delta x_1)$$

$$S_{2,i,i,k} = S_2 Exp(k\Delta x_2)$$

FIGURE 2.27 Additive Two-variable Binomial Process



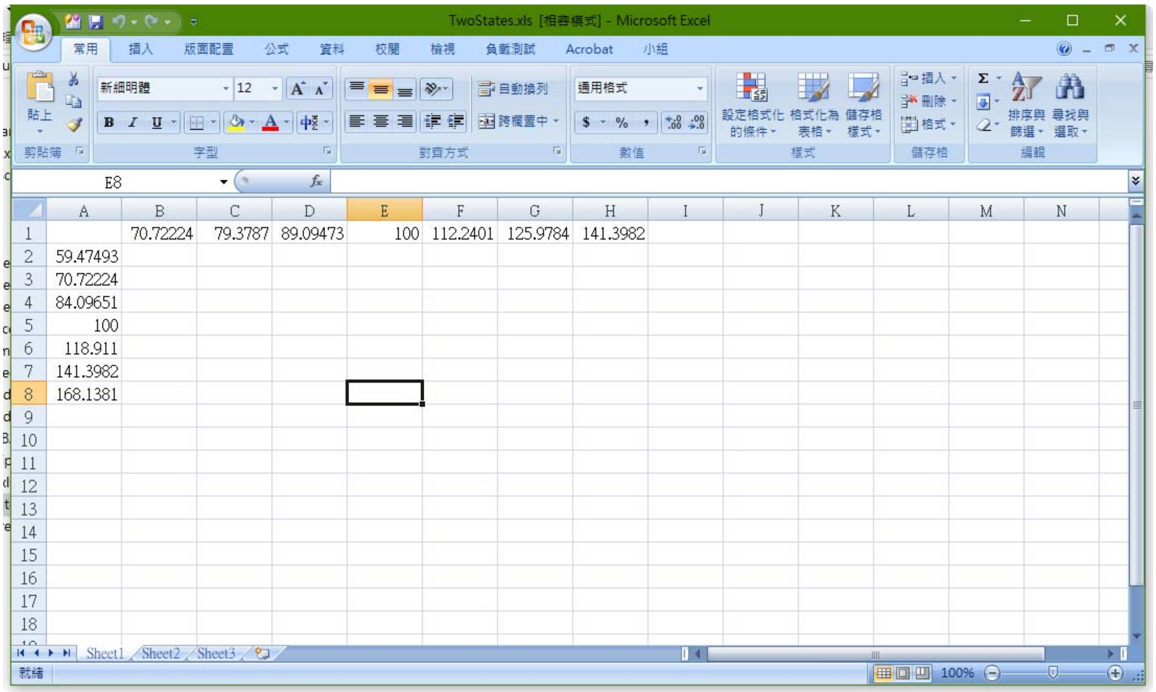
➢ 美式價差選擇權

- ✦ 期末償付為

$$\max\left[0,\quad S_{1,T}-S_{2,T}-K\right]$$

  - Pseudo-code 見下一小節。

- ✦ 兩變數二項式模型收斂情況不理想，使用三元樹或有限差分法來改進。



TwoStates.xls

(二)Pseudo Code

```
input: S1,S2,K,T,sig1,sig2,div1,div2,rho,r,N
real: dt,nu1,nu2,dx1,dx2,disc,puu,pud,pdu,pdd,exd1,
      exd2,S1t[-N..N],S2t[-N..N],C[-N..N,-N..N]
integer: i, j, k

dt = T/N
nu1 = r – div1 – 0.5*sig1^2
nu2 = r – div2 – 0.5*sig2^2
dx1 = sig1 * sqrt(dt)
dx2 = sig2 * sqrt(dt)
disc = exp(-r*dt)

puu = (dx1*dx2 + (dx2*nu1+dx1*nu2+rho*sig1*sig2)*dt)
      / (4*dx1*dx2) * disc
pud = (dx1*dx2 + (dx2*nu1-dx1*nu2-rho*sig1*sig2)*dt)
      / (4*dx1*dx2) * disc
pdu = (dx1*dx2 + (-dx2*nu1+dx1*nu2-rho*sig1*sig2)*dt)
      / (4*dx1*dx2) * disc
pdd = (dx1*dx2 + (-dx2*nu1-dx1*nu2+rho*sig1*sig2)*dt)
      / (4*dx1*dx2) * disc

edx1 = exp(dx1)
edx2 = exp(dx2)

S1t[-N] = S1 * exp( -N*dx1 )
S2t[-N] = S2 * exp( -N*dx2 )
```


董夢雲 dongmy@ms5.hinet.net

```
For j = -N+1 to N do
   S1t[j] = S1t[j-1]*edx1
   S2t[j] = S2t[j-1]*edx2
Next j

For j = -N to N step 2 do
   For k = -N to N step 2 do
      C[j, k] = max(0.0, S1t[j]-S2t[k]-K)

For i = N-1 downto 0 do
   For j = -i to i step 2 do
      For k = -i to i step 2 do
         C[j, k] = pdd*C[j-1,k-1] + pud*C[j+1,k-1]
               +pdu*C[j-1,k+1] + puu*C[j+1,k+1]
         C[j, k] = max(C[j,k], S1t[j] – S2t[k] –K)
      Next k
   Next j
Next i

Return C[0, 0]
```

(三)數值範例

✦ 美式價差買權，$K=1, S_1=100, S_2=100, T=1, \sigma_1=0.20, \sigma_2=0.30, d_1=0.03, d_2=0.04, \rho=0.50, r=0.06, N=3$。

FIGURE 2.29 American Spread Call Option by Two-variable Binomial

K	T	S_1	S_2	sig1	sig2	div1	div2	rho	r	N	
1	1	100	100	0.2	0.3	0.03	0.04	0.50	0.06	3	
dt	**nu1**	**nu2**	**dx_1**	**dx_2**	**disc**	**puu**	**pud**	**pdu**	**pdd**	**edx1**	**edx2**
0.3333	0.0100	−0.0250	0.1155	0.1732	0.9802	0.3629	0.1414	0.1037	0.3723	1.1224	1.1891

i 0
t 0

S2t							
168.14							
141.40							
118.91							
100.00				10.04479			
84.10							
70.72							
59.47							
S1t	70.72	79.38	89.09	100.00	112.24	125.98	141.40

i 1
t 0.333333

S2t							
168.14							
141.40							
118.91			0.9635		6.7420		
100.00							
84.10			9.4563		28.1353		
70.72							
59.47							
S1t	70.72	79.38	89.09	100.00	112.24	125.98	141.40

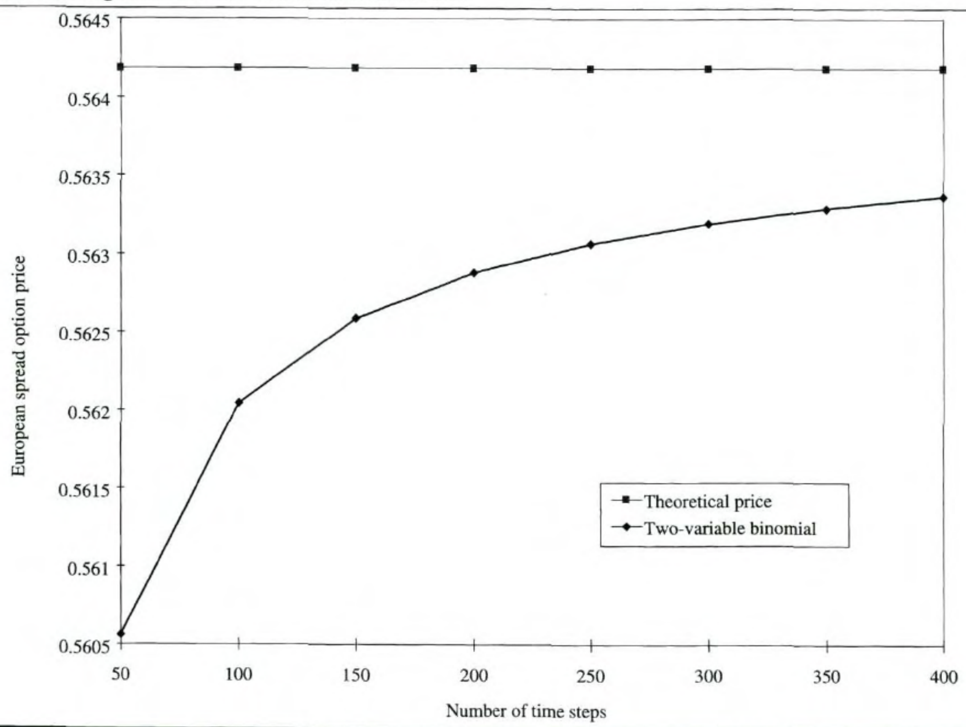
i 2
t 0.666667

S2t							
168.14							
141.40		0.0000		0.0000		3.0381	
118.91							
100.00		0.5653		5.3263		25.8626	
84.10							
70.72		9.3123		28.2778		54.2561	
59.47							
S1t	70.72	79.38	89.09	100.00	112.24	125.98	141.40

i 3
t 1

S2t							
168.14	0.0000		0.0000		0.0000		0.0000
141.40							
118.91	0.0000		0.0000		0.0000		21.4873
100.00							
84.10	0.0000		3.9982		27.1436		56.3017
70.72							
59.47	10.2473		28.6198		51.7652		80.9233
S1t	70.72	79.38	89.09	100.00	112.24	125.98	141.40

FIGURE 2.30 Convergence of the Two-variable Binomial Method



收斂不理想