How Traders Manage Their Risks

Chapter 8

A Trader's Gold Portfolio. How Should Risks Be Hedged? (Table 8.1, page 154)

Position	Value ($)
Spot Gold	9,180,000
Forward Contracts	– 3,060,000
Futures Contracts	2,000
Swaps	180,000
Options	– 6,110,000
Exotics	125,000
Total	317,000

Delta

- Delta of a portfolio is the partial derivative of a portfolio with respect to the price of the underlying asset (gold in this case)
- Suppose that a \$0.1 increase in the price of gold leads to the gold portfolio decreasing in value by \$100
- The delta of the portfolio is −1000
- The portfolio could be hedged against short-term changes in the price of gold by buying 1000 ounces of gold. This is known as making the portfolio delta neutral

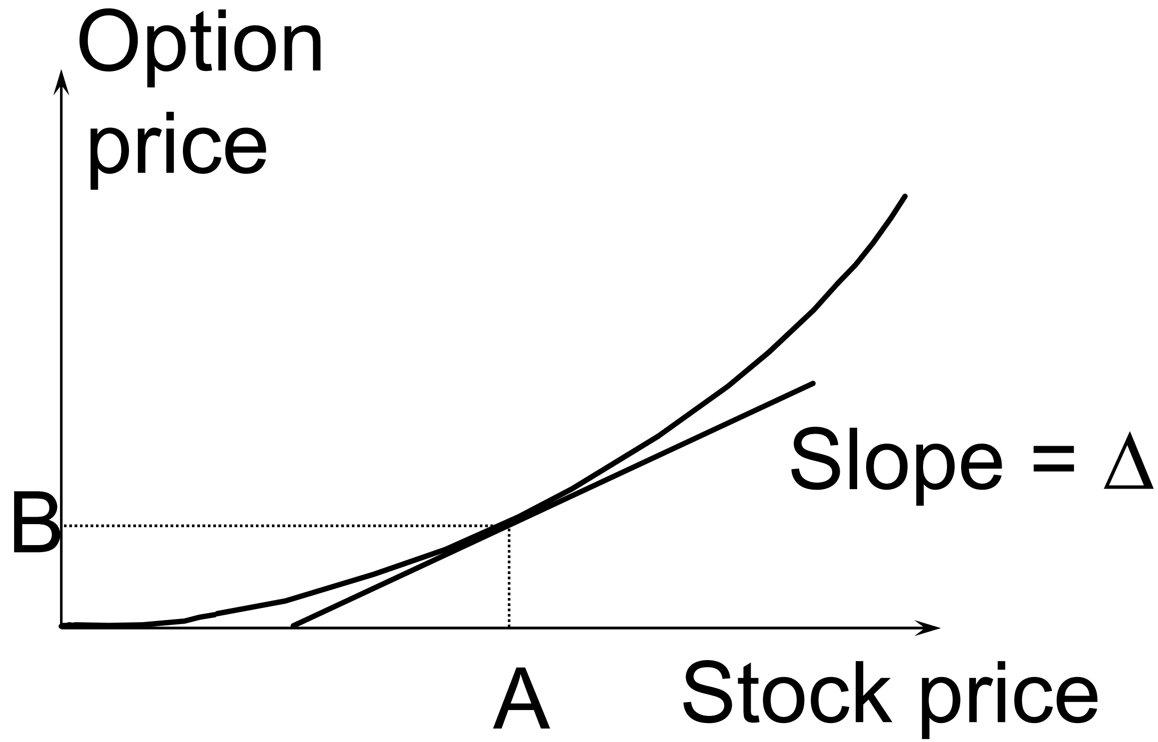
Linear vs Nonlinear Products

- When the price of a product is linearly dependent on the price of an underlying asset a “hedge and forget” strategy can be used
- Non-linear products require the hedge to be rebalanced to preserve delta neutrality

Example of Hedging a Nonlinear Product (pages 156-160)

- A bank has sold for \$300,000 a European call option on 100,000 shares of a nondividend paying stock
- S_0 = 49, K = 50, r = 5%, σ = 20%, T = 20 weeks, μ = 13%
- The Black-Scholes-Merton value of the option is \$240,000
- How does the bank hedge its risk to lock in a \$60,000 profit?

Delta of the Option



Delta Hedging

- Initially the delta of the option is 0.522
- The delta of the position is -52,200
- This means that 52,200 shares must purchased to create a delta neutral position
- But, if a week later delta falls to 0.458, 6,400 shares must be sold to maintain delta neutrality
- Tables 8.2 and 8.3 (pages 158 and 159) provide examples of how delta hedging might work for the option.

Table 8.2: Option closes in the money

Week	Stock Price	Delta	Shares Purchased
0	49.00	0.522	52,200
1	48.12	0.458	(6,400)
2	47.37	0.400	(5,800)
3	50.25	0.596	19,600
….	…..	….	…..
19	55.87	1.000	1,000
20	57.25	1.000	0

Table 8.3: Option closes out of the money

Week	Stock Price	Delta	Shares Purchased
0	49.00	0.522	52,200
1	49.75	0.568	4,600
2	52.00	0.705	13,700
3	50.00	0.579	(12,600)
….	…..	….	…..
19	46.63	0.007	(17,600)
20	48.12	0.000	(700)

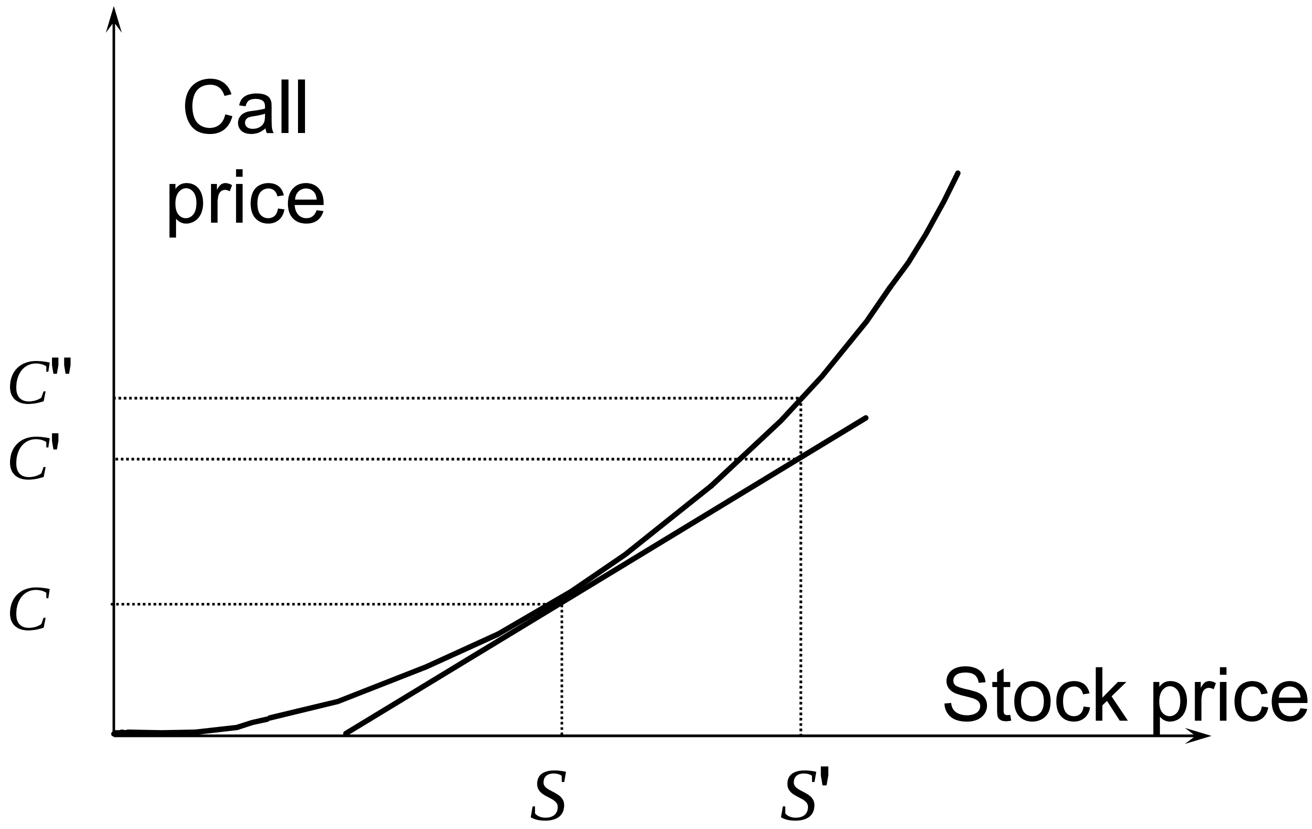
Where the Costs Come From

- Delta hedging a short option position tends to involve selling after a price decline and buying after a price increase
- This is a "sell low, buy high" strategy.
- The total costs incurred are close to the theoretical price of the option

Gamma

- Gamma (Γ) is the rate of change of delta (Δ) with respect to the price of the underlying asset
- Gamma is greatest for options that are close to the money

Gamma Measures the Delta Hedging Errors Caused By Curvature (Figure 8.4, page 161)



Vega

- Vega (ν) is the rate of change of the value of a derivatives portfolio with respect to volatility
- Like gamma, vega tends to be greatest for options that are close to the money

Gamma and Vega Limits

- In practice, a traders must keep gamma and vega within limits set by risk management

Theta

- Theta (Θ) of a derivative (or portfolio of derivatives) is the rate of change of the value with respect to the passage of time
- The theta of a call or put is usually negative. This means that, if time passes with the price of the underlying asset and its volatility remaining the same, the value of the option declines

Rho

- Rho is the partial derivative with respect to a parallel shift in all interest rates in a particular country

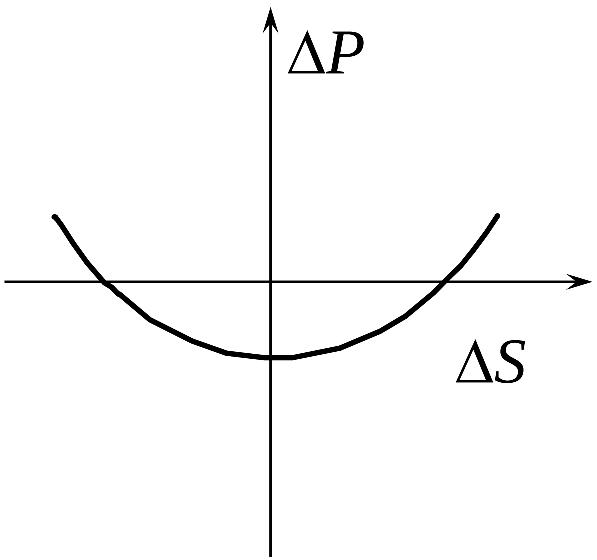
Taylor Series Expansion (Equation 8.1, page 167)

$$\Delta P = \frac{\partial P}{\partial S}\Delta S + \frac{\partial P}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 P}{\partial S^2}(\Delta S)^2 + \frac{1}{2}\frac{\partial^2 P}{\partial t^2}(\Delta t)^2 + \frac{\partial^2 P}{\partial S \partial t}\Delta S \Delta t + \ldots$$

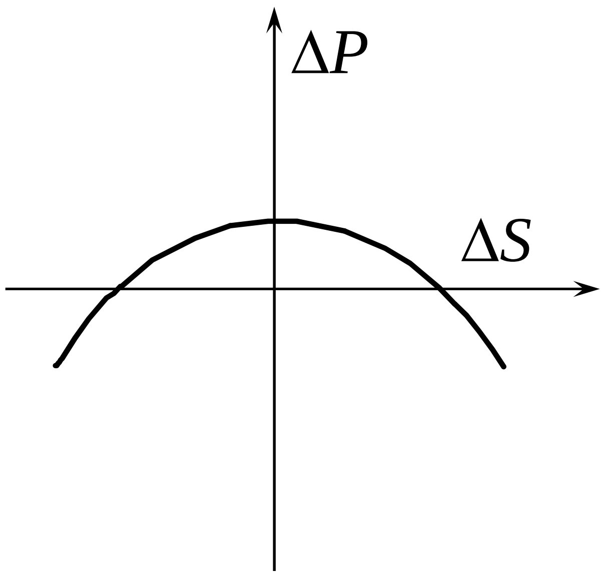
Interpretation of Gamma
(Equation 8.2, page 167)

- For a delta neutral portfolio,

$$\Delta P \approx \Theta\,\Delta t + \tfrac{1}{2}\Gamma \Delta S^{2}$$



Positive Gamma



Negative Gamma

Taylor Series Expansion when Volatility is Uncertain

$$\Delta P = \frac{\partial P}{\partial S}\Delta S + \frac{\partial P}{\partial \sigma}\Delta\sigma + \frac{\partial P}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 P}{\partial S^2}(\Delta S)^2 + \frac{1}{2}\frac{\partial^2 P}{\partial \sigma^2}(\Delta\sigma)^2 + \ldots$$

Managing Delta, Gamma, & Vega

- Δ can be changed by taking a position in the underlying
- To adjust Γ & ν it is necessary to take a position in an option or other derivative

Hedging in Practice

- Traders usually ensure that their portfolios are delta-neutral at least once a day
- Whenever the opportunity arises, they improve gamma and vega
- As portfolio becomes larger hedging becomes less expensive

Static Options Replication
(pages 169-170)

- This involves approximately replicating an exotic option with a portfolio of vanilla options
- Underlying principle: if we match the value of an exotic option on some boundary, we have matched it at all interior points of the boundary
- Static options replication can be contrasted with dynamic options replication where we have to trade continuously to match the option

Scenario Analysis

A scenario analysis involves testing the effect on the value of a portfolio of different assumptions concerning asset prices and their volatilities

Interest Rate Risk

Chapter 9

Management of Net Interest Income
(Table 9.1, page 176)

- Suppose that the market's best guess is that future short term rates will equal today's rates
- What would happen if a bank posted the following rates?

Maturity (yrs)	Deposit Rate	Mortgage Rate
1	3%	6%
5	3%	6%

- How can the bank manage its risks?

Management of Net Interest Income

- Most banks have asset-liability management groups to manage interest rate risk
- When long term loans are funded with short term deposits interest rate swaps can be used to hedge the interest rate risk
- But this does not hedge the liquidity risk

LIBOR Rates and Swap Rates

- LIBOR rates are rates with maturities up to one year for interbank transactions where the borrower has a AA-rating
- Swap Rates are the fixed rates exchanged for floating in an interest rate swap agreement

Understanding Swap Rates

- A bank can
  - Lend to a series AA-rated borrowers for ten successive six month periods
  - Swap the LIBOR interest received for the five-year swap rate
- This shows that the swap rate has the credit risk corresponding to a series of short-term loans to AA-rated borrowers

Extending the LIBOR Curve

- Alternative 1: Create a term structure of interest rates showing the rate of interest at which a AA-rated company can borrow now for 1, 2, 3 … years
- Alternative 2: Use swap rates so that the term structure represents future short term AA borrowing rates
- Alternative 2 is the usual approach. It creates the LIBOR/swap term structure of interest rates

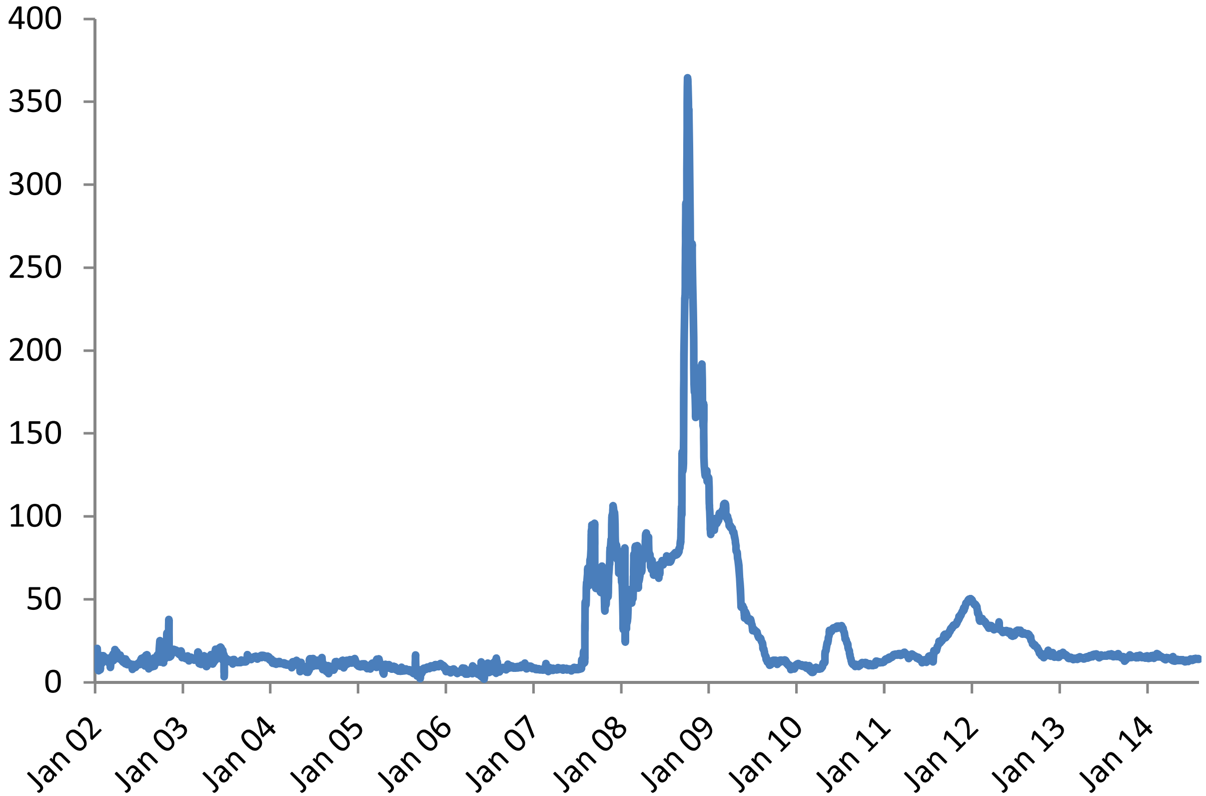
Risk-Free Rate

- Traders has traditionally assumed that the LIBOR/swap zero curve is the risk-free zero curve
- The Treasury curve is about 50 basis points below the LIBOR/swap zero curve
- Treasury rates are considered to be artificially low for a variety of regulatory and tax reasons

OIS Rate

- LIBOR/swap rates were clearly not "risk-free" during the crisis
- As a result there has been a trend toward using overnight indexed swap (OIS) rates as proxies for the risk-free rate instead of LIBOR and swap rates
- The OIS rate is the rate swapped for the geometric average of overnight borrowing rates. (In the U.S. the relevant overnight rate is the fed funds rate)

3-month LIBOR-OIS Spread



Repo Rate

- A financial institution owning securities agrees to sell them today for a certain price and buy them back in the future for a slightly higher price
- It is obtaining a secured loan
- The interest on the loan is the difference between the two prices

Duration (page 182)

- Duration of a bond that provides cash flow c_i at time t_i is

$$\sum_{i=1}^{n} t_i \left(\frac{c_i e^{-yt_i}}{B} \right)$$

where B is its price and y is its yield (continuously compounded)

- This leads to

$$\frac{\Delta B}{B} = -D \Delta y$$

Calculation of Duration for a 3-year bond paying a coupon 10%. Bond yield=12%.
(Table 9.3, page 183)

Time (yrs)	Cash Flow ($)	PV ($)	Weight	Time $\times$ Weight
0.5	5	4.709	0.050	0.025
1.0	5	4.435	0.047	0.047
1.5	5	4.176	0.044	0.066
2.0	5	3.933	0.042	0.083
2.5	5	3.704	0.039	0.098
3.0	105	73.256	0.778	2.333
Total	130	94.213	1.000	2.653

Duration Continued

- When the yield y is expressed with compounding m times per year

$$\Delta B = -\frac{BD\Delta y}{1+y/m}$$

- The expression

$$\frac{D}{1+y/m}$$

is referred to as the "modified duration"

Convexity (Page 185-187)

The convexity of a bond is defined as

$$C=\frac{1}{B}\frac{d^2B}{dy^2}=\frac{\sum_{i=1}^{n}c_i t_i^2 e^{-yt_i}}{B}$$

which leads to

$$\frac{\Delta B}{B}=-D\Delta y+\frac{1}{2}C(\Delta y)^2$$

Portfolios

- Duration and convexity can be defined similarly for portfolios of bonds and other interest-rate dependent securities
- The duration of a portfolio is the weighted average of the durations of the components of the portfolio. Similarly for convexity.

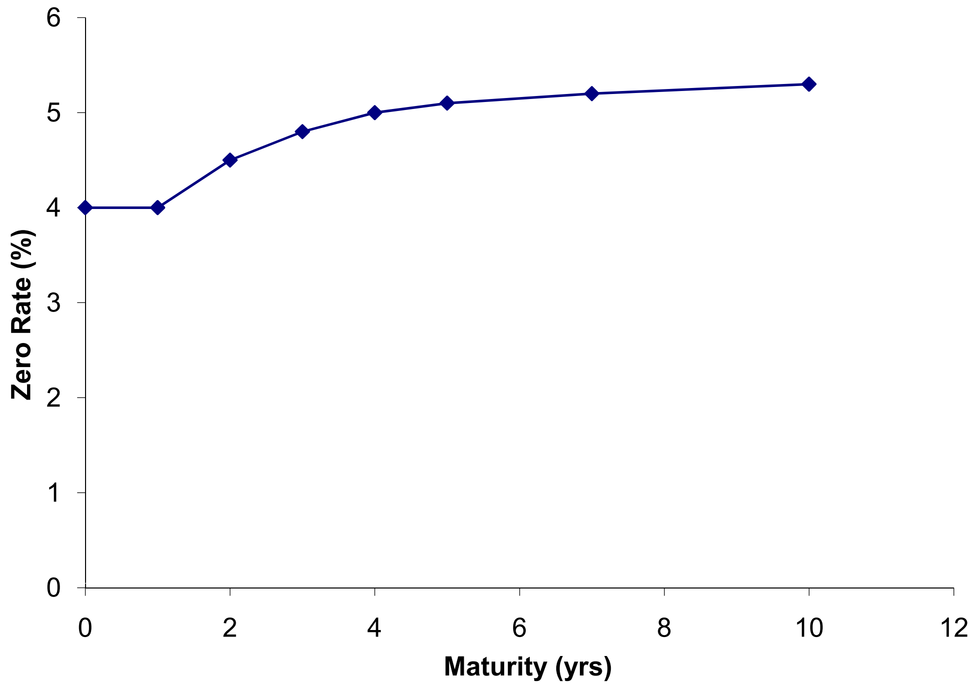
What Duration and Convexity Measure

- Duration measures the effect of a small parallel shift in the yield curve
- Duration plus convexity measure the effect of a larger parallel shift in the yield curve
- However, they do not measure the effect of non-parallel shifts

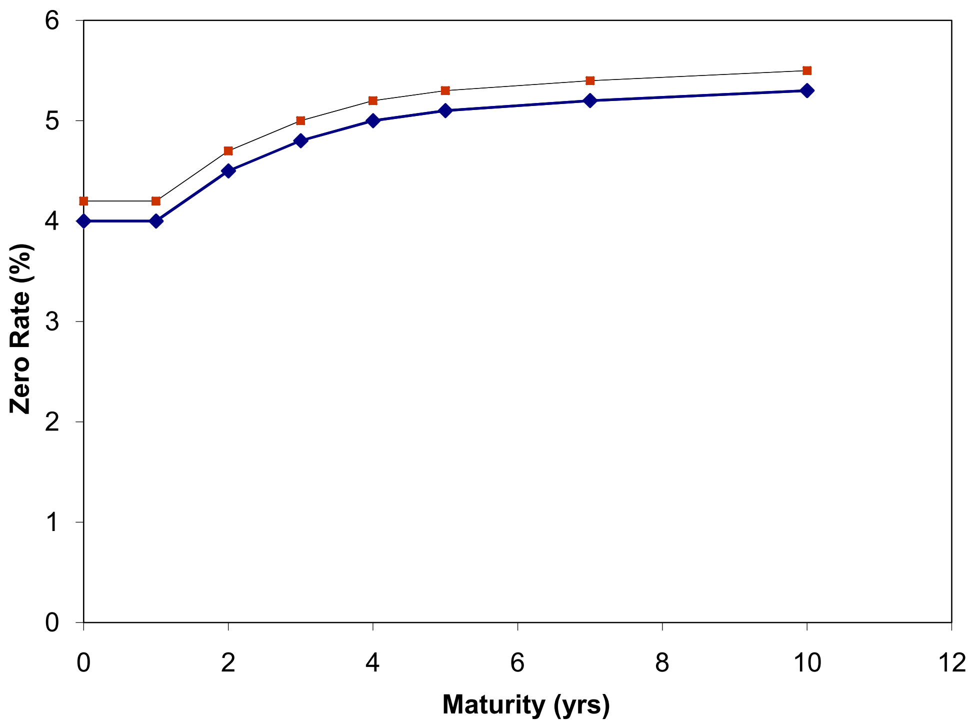
Other Measures

- Dollar Duration: Product of the portfolio value and its duration
- Dollar Convexity: Product of convexity and value of the portfolio

Starting Zero Curve (Figure 9.4, page 190)



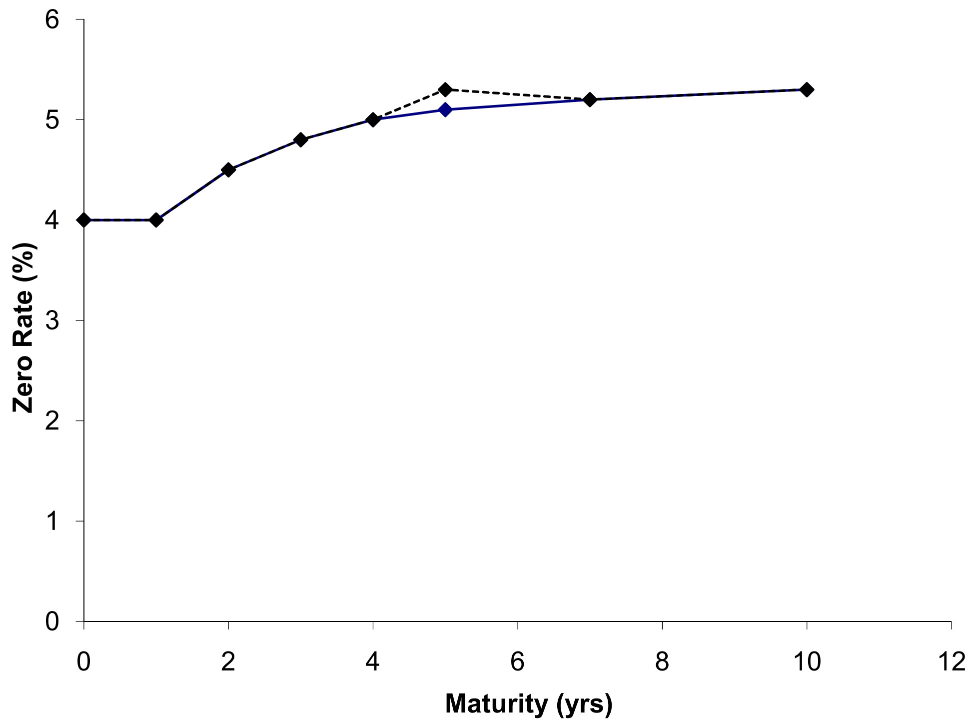
Parallel Shift



Partial Duration

- A partial duration calculates the effect on a portfolio of a change to just one point on the zero curve

Partial Duration continued
(Figure 9.5, page 190)



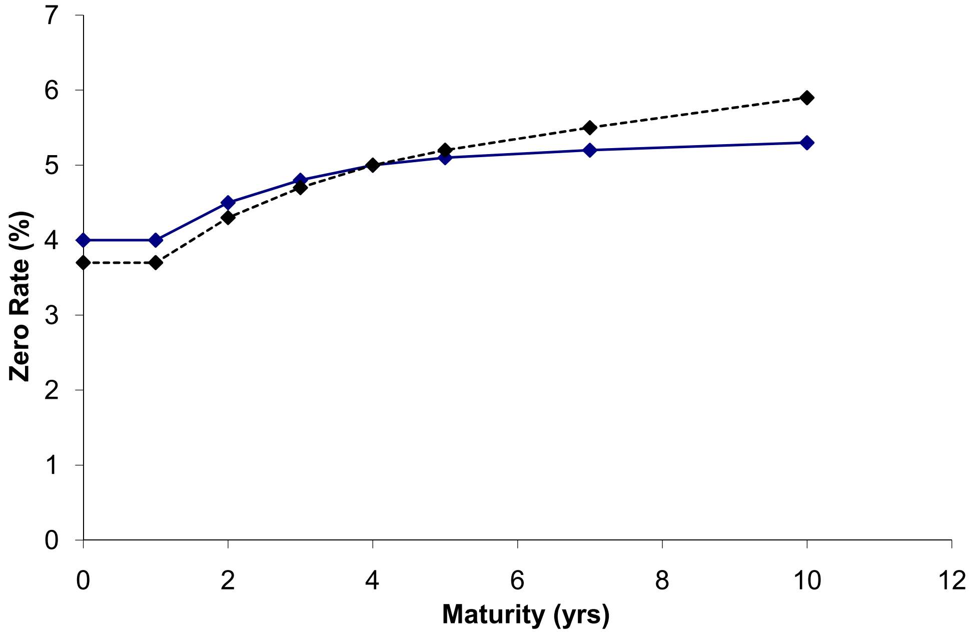
Example (Table 9.5, page 190)

Maturity yrs	1	2	3	4	5	7	10	Total
Partial duration	0.2	0.6	0.9	1.6	2.0	−2.1	−3.0	0.2

Partial Durations Can Be Used to Investigate the Impact of Any Yield Curve Change

- Any yield curve change can be defined in terms of changes to individual points on the yield curve
- For example, to define a rotation we could change the 1-, 2-, 3-, 4-, 5-, 7, and 10-year maturities by $-3e$, $-2e$, $-e$, 0, e, $3e$, $6e$

Combining Partial Durations to Create Rotation in the Yield Curve
(Figure 9.6, page 191)



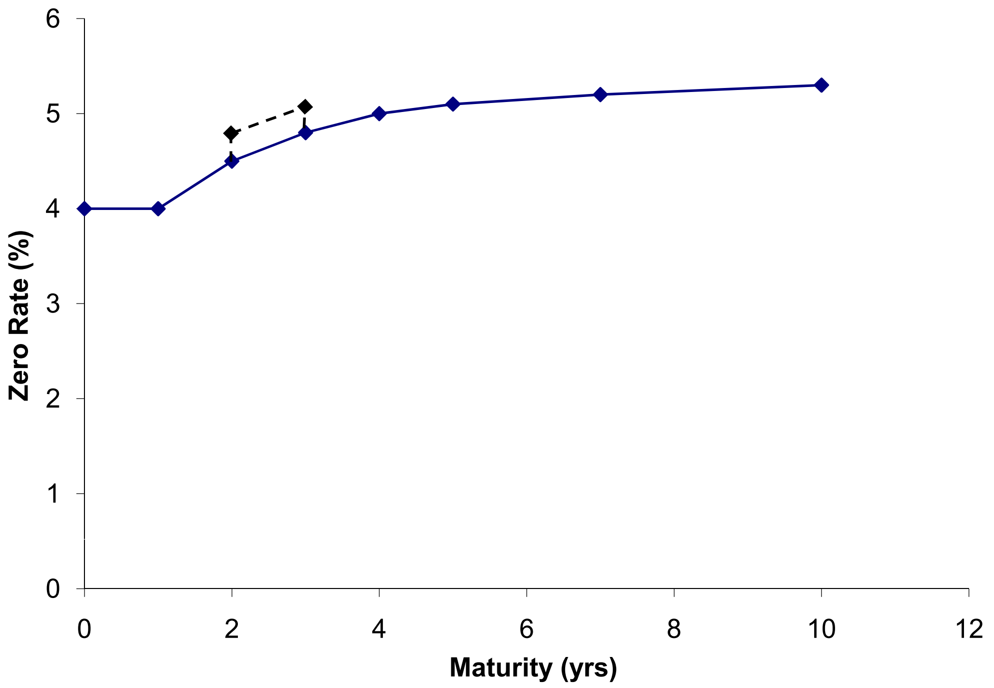
Impact of Rotation

- The impact of the rotation on the proportional change in the value of the portfolio in the example is

$$-[0.2\times(-3e)+0.6\times(-2e)\ldots+(-3.0)\times(+6e)]=25.0e$$

Alternative approach (Figure 9.7, page 192)

Bucket the yield curve and investigate the effect of a small change to each bucket



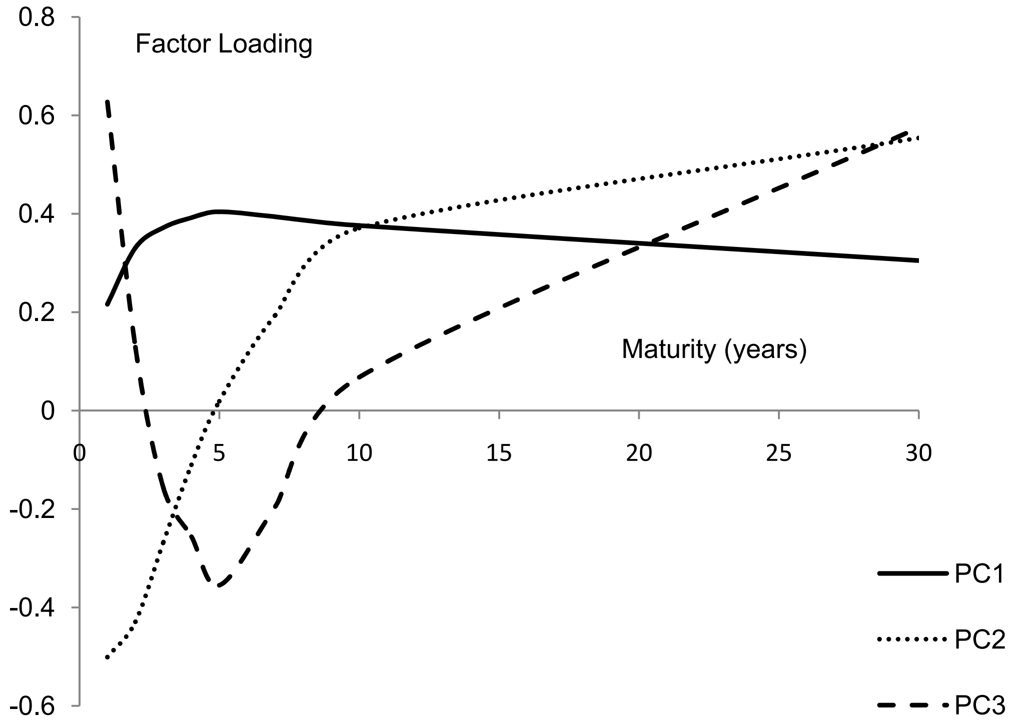
Principal Components Analysis

- Attempts to identify standard shifts (or factors) for the yield curve so that most of the movements that are observed in practice are combinations of the standard shifts

Results (Tables 9.7 and 9.8)

- The first factor is a roughly parallel shift (90.9% of variance explained)
- The second factor is a twist 6.8% of variance explained)
- The third factor is a bowing (1.3% of variance explained)

The Three Factors (Figure 9.8 page 195)



Alternatives for Calculating Multiple Deltas to Reflect Non-Parallel Shifts in Yield Curve

- Shift individual points on the yield curve by one basis point (the partial duration approach)
- Shift segments of the yield curve by one basis point (the bucketing approach)
- Shift quotes on instruments used to calculate the yield curve
- Calculate deltas with respect to the shifts given by a principal components analysis.

Gamma for Interest Rates

- Gamma has the form

$$\frac{\partial^2 P}{\partial x_i \partial x_j}$$

  where x_i and x_j are yield curve shifts considered for delta
- To avoid information overload one possibility is consider only $i = j$
- Another is to consider only parallel shifts in the yield curve and calculate convexity
- Another is to consider the first two or three types of shift given by a principal components analysis

Vega for Interest Rates

- One possibility is to make the same change to all interest rate implied volatilities. (However implied volatilities for long-dated options change by less than those for short-dated options.)
- Another is to do a principal components analysis on implied volatility changes for the instruments that are traded

Volatility

Chapter 10

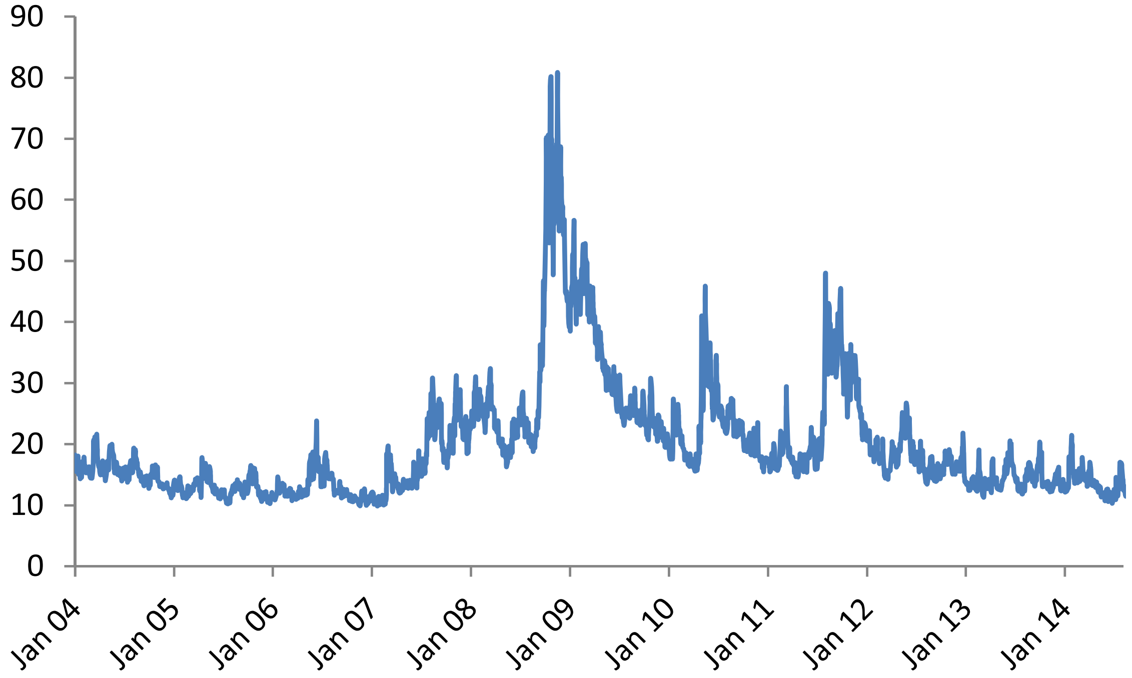
Definition of Volatility

- Suppose that S_i is the value of a variable on day i. The volatility per day is the standard deviation of $\ln(S_i / S_{i\text{-}1})$
- Normally days when markets are closed are ignored in volatility calculations (see Business Snapshot 10.1, page 203)
- The volatility per year is $\sqrt{252}$ times the daily volatility
- Variance rate is the square of volatility

Implied Volatilities

- Of the variables needed to price an option the one that cannot be observed directly is volatility
- We can therefore imply volatilities from market prices and vice versa

VIX Index: A Measure of the Implied Volatility of the S&P 500 (Figure 10.1, page 204)



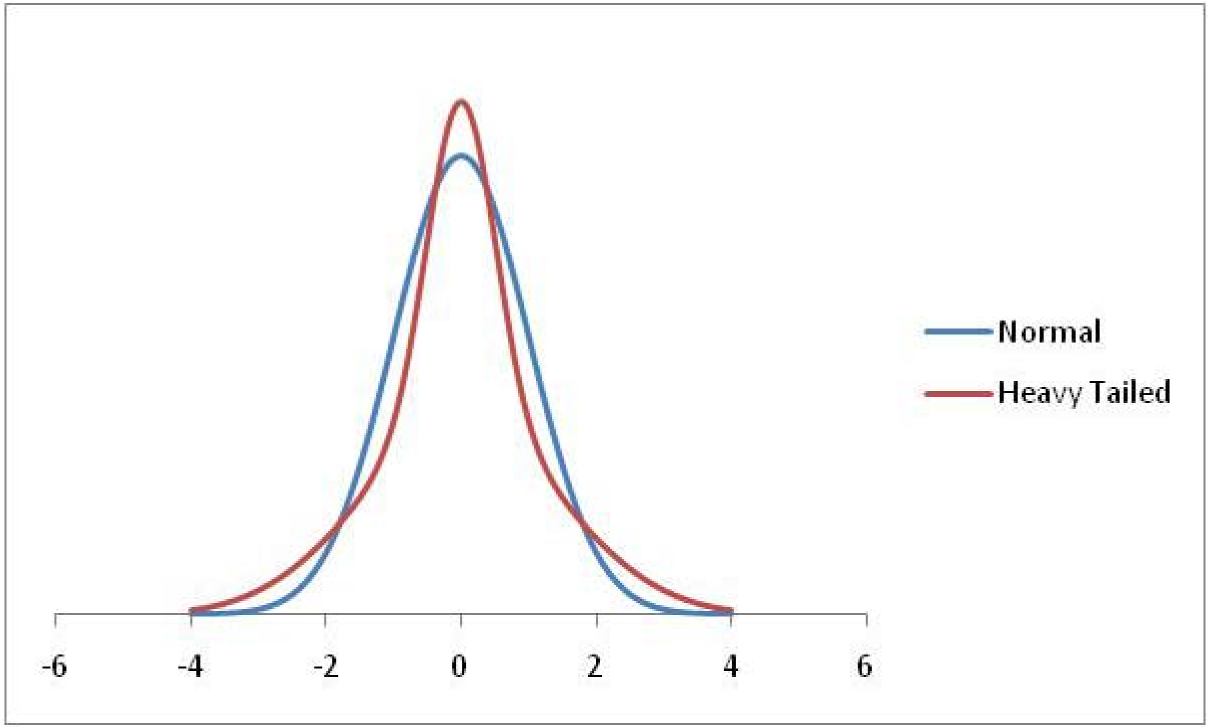
Are Daily Changes in Exchange Rates Normally Distributed? Table 10.1, page 205

	Real World (%)	Normal Model (%)
>1 SD	25.04	31.73
>2SD	5.27	4.55
>3SD	1.34	0.27
>4SD	0.29	0.01
>5SD	0.08	0.00
>6SD	0.03	0.00

Heavy Tails

- Daily exchange rate changes are not normally distributed
  - The distribution has heavier tails than the normal distribution
  - It is more peaked than the normal distribution
- This means that small changes and large changes are more likely than the normal distribution would suggest
- Many market variables have this property, known as excess kurtosis

Normal and Heavy-Tailed Distribution

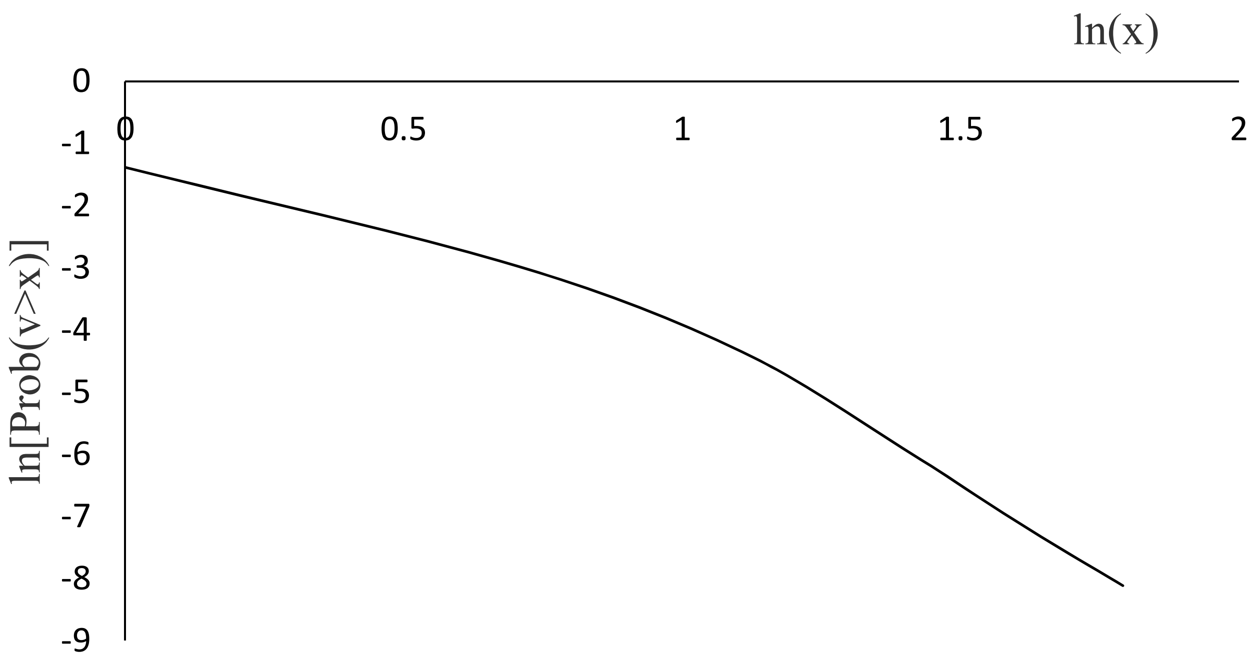


Alternatives to Normal Distributions: The Power Law (See page 207)

$$\text{Prob}(v > x) = Kx^{-\alpha}$$

This seems to fit the behavior of the returns on many market variables better than the normal distribution

Log-Log Test for Exchange Rate Data (v is number of standard deviations which the exchange rate moves)



Standard Approach to Estimating Volatility

- Define σ_n as the volatility per day between day $n-1$ and day n, as estimated at end of day $n-1$
- Define S_i as the value of market variable at end of day i
- Define $u_i = \ln(S_i/S_{i-1})$

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \bar{u})^2$$

$$\bar{u} = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}$$

Simplifications Usually Made in Risk Management

- Define u_i as $(S_i - S_{i-1})/S_{i-1}$
- Assume that the mean value of u_i is zero
- Replace $m-1$ by m

This gives

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2$$

Weighting Scheme

Instead of assigning equal weights to the observations we can set

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where

$$\sum_{i=1}^{m} \alpha_i = 1$$

ARCH(m) Model

In an ARCH(m) model we also assign some weight to the long-run variance rate, V_L:

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$

EWMA Model (page 212)

- In an exponentially weighted moving average model, the weights assigned to the u^2 decline exponentially as we move back through time
- This leads to

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$$

Attractions of EWMA

- Relatively little data needs to be stored
- We need only remember the current estimate of the variance rate and the most recent observation on the market variable
- Tracks volatility changes
- $\lambda = 0.94$ has been found to be a good choice across a wide range of market variables

GARCH (1,1), page 214

In GARCH (1,1) we assign some weight to the long-run average variance rate

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

Since weights must sum to 1

$$\gamma + \alpha + \beta = 1$$

GARCH (1,1) continued

Setting $\omega = \gamma V_L$ the GARCH (1,1) model is

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

and

$$V_L = \frac{\omega}{1 - \alpha - \beta}$$

Example

- Suppose

$$\sigma_n^2 = 0.000002 + 0.13u_{n-1}^2 + 0.86\sigma_{n-1}^2$$

- The long-run variance rate is 0.0002 so that the long-run volatility per day is 1.4%

Example continued

- Suppose that the current estimate of the volatility is 1.6% per day and the most recent percentage change in the market variable is 1%.
- The new variance rate is

$$0.000002 + 0.13 \times 0.0001 + 0.86 \times 0.000256 = 0.00023336$$

The new volatility is 1.53% per day

GARCH (p,q)

$$\sigma_n^2 = \omega + \sum_{i=1}^{p} \alpha_i u_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2$$

Other Models

- Many other GARCH models have been proposed
- For example, we can design a GARCH models so that the weight given to u_i^2 depends on whether u_i is positive or negative

Maximum Likelihood Methods

- In maximum likelihood methods we choose parameters that maximize the likelihood of the observations occurring

Case 1 (page 216)

- We observe that a certain event happens one time in ten trials. What is our estimate of the proportion of the time, p, that it happens?
- The probability of the outcome is

$$p(1-p)^9$$

- We maximize this to obtain a maximum likelihood estimate: $p = 0.1$

Case 2 (page 216-217)

Estimate the variance of observations from a normal distribution with mean zero

$$\text{Maximize} \quad : \quad \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi v}} \exp\left(\frac{-u_i^2}{2v} \right) \right]$$

$$\text{Same as maximizing} \quad : \quad \sum_{i=1}^{n} \left[-\ln(v) - \frac{u_i^2}{v} \right]$$

$$\text{Maximum value when} \quad v = \frac{1}{n} \sum_{i=1}^{n} u_i^2$$

Application to EWMA and GARCH

We choose parameters that maximize

$$\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi v_i}} \exp\left(-\frac{u_i^2}{2v_i} \right)$$

or

$$\sum_{i=1}^{m} \left[-\ln(v_i) - \frac{u_i^2}{v_i} \right]$$

S&P 500 Excel Application

- Start with trial values of parameters (λ for EWMA and ω, α, and β for GARCH(1,1)
- Update variances
- Calculate

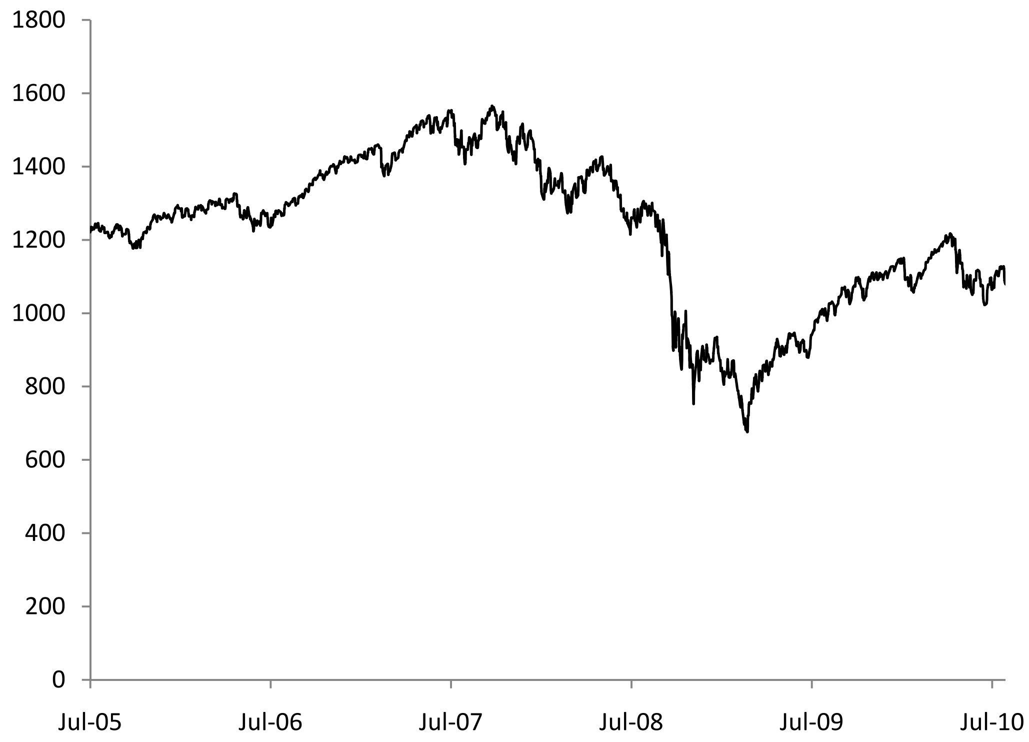
$$\sum_{i=1}^{m}\left[-\ln(v_i)-\frac{u_i^2}{v_i}\right]$$

- Use solver to search for values of parameters that maximize this objective function
- For efficient operation of Solver: set up spreadsheet so that ensure that search is over parameters that are of same order of magnitude and test alternative starting conditions

S&P 500 Excel Application (Table 10.4)

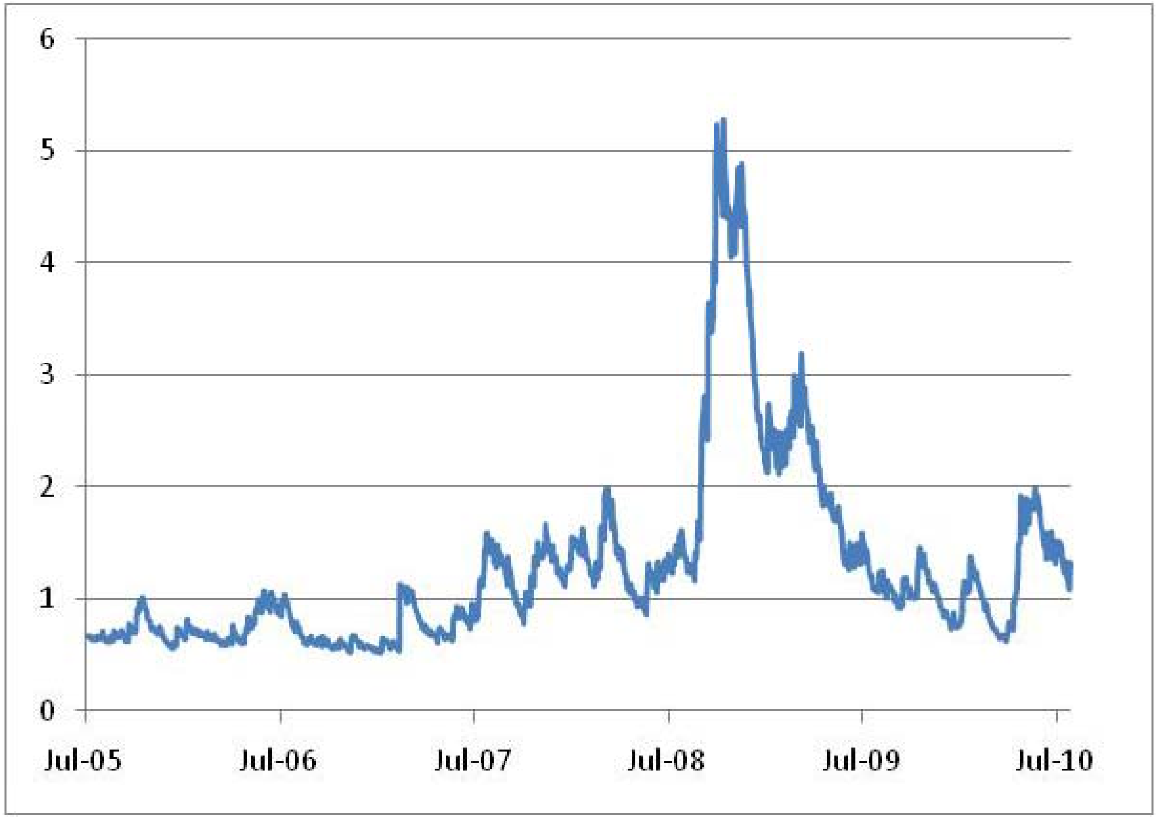
Date	*Day*	S_i	$u_i=(S_i-S_{i-1})/S_{i-1}$	$v_i=\sigma_i^2$	$-\ln(v_i)-u_i^2/v_i$
18-Jul-2005	1	1221.13			
19-Jul-2005	2	1229.35	0.006731		
20-Jul-2005	3	1235.20	0.004759	0.00004531	9.5022
21-Jul-2005	4	1227.04	−0.006606	0.00004447	9.0393
…….	…..	…….	………..	………….	…………
13-Aug-2010	1279	1079.25	−0.004024	0.00016327	8.6209
Total					10,228.2349

The S&P 500 (Figure 10.4)



The GARCH Estimate of Volatility of the S&P 500 (Figure 10.5)

$\omega=0.0000013465$, $\alpha=0.083394$, $\beta=0.910116$



Variance Targeting

- One way of implementing GARCH(1,1) that increases stability is by using variance targeting
- The long-run average variance equal to the sample variance
- Only two other parameters then have to be estimated

How Good is the Model?

- The Ljung-Box statistic tests for autocorrelation
- We compare the autocorrelation of the u_i^2 with the autocorrelation of the u_i^2/σ_i^2

Forecasting Future Volatility
(equation 10.14, page 223)

A few lines of algebra shows that

$$E[\sigma^2_{n+k}] = V_L + (\alpha + \beta)^k (\sigma^2_n - V_L)$$

The variance rate for an option expiring on day m is

$$\frac{1}{m}\sum_{k=0}^{m-1} E\left[\sigma^2_{n+k}\right]$$

Forecasting Future Volatility
continued (equation 10.15, page 224)

Define

$$a = \ln \frac{1}{\alpha + \beta}$$

The estimated volatility per annum for an option lasting T days is

$$\sqrt{252\left(V_L + \frac{1-e^{-aT}}{aT}\left[V(0) - V_L\right]\right)}$$

S&P Example

- ω=0.0000013465, α=0.083394, β=0.910116

$$a = \ln \frac{1}{0.083394 + 0.910116} = 0.006511$$

Option Life (days)	10	30	50	100	500
Est. Volatility (% per annum)	27.36	27.10	26.87	26.35	24.32

Volatility Term Structures

- GARCH (1,1) suggests that, when calculating vega, we should shift the long maturity volatilities less than the short maturity volatilities
- When instantaneous volatility changes by $\Delta\sigma(0)$, volatility for T-day option changes by

$$\frac{1-e^{-aT}}{aT}\frac{\sigma(0)}{\sigma(T)}\Delta\sigma(0)$$

Results for S&P 500 (Table 10.7)

- When instantaneous volatility changes by 1%

Option Life (days)	10	30	50	100	500
Volatility increase (%)	0.97	0.92	0.87	0.77	0.33

Correlations and Copulas

Chapter 11

Correlation and Covariance

- The coefficient of correlation between two variables V_1 and V_2 is defined as

$$\frac{E(V_1V_2)-E(V_1)E(V_2)}{SD(V_1)SD(V_2)}$$

- The covariance is

$$E(V_1V_2)-E(V_1)E(V_2)$$

Independence

- V_1 and V_2 are independent if the knowledge of one does not affect the probability distribution for the other

$$f(V_2 | V_1 = x) = f(V_2)$$

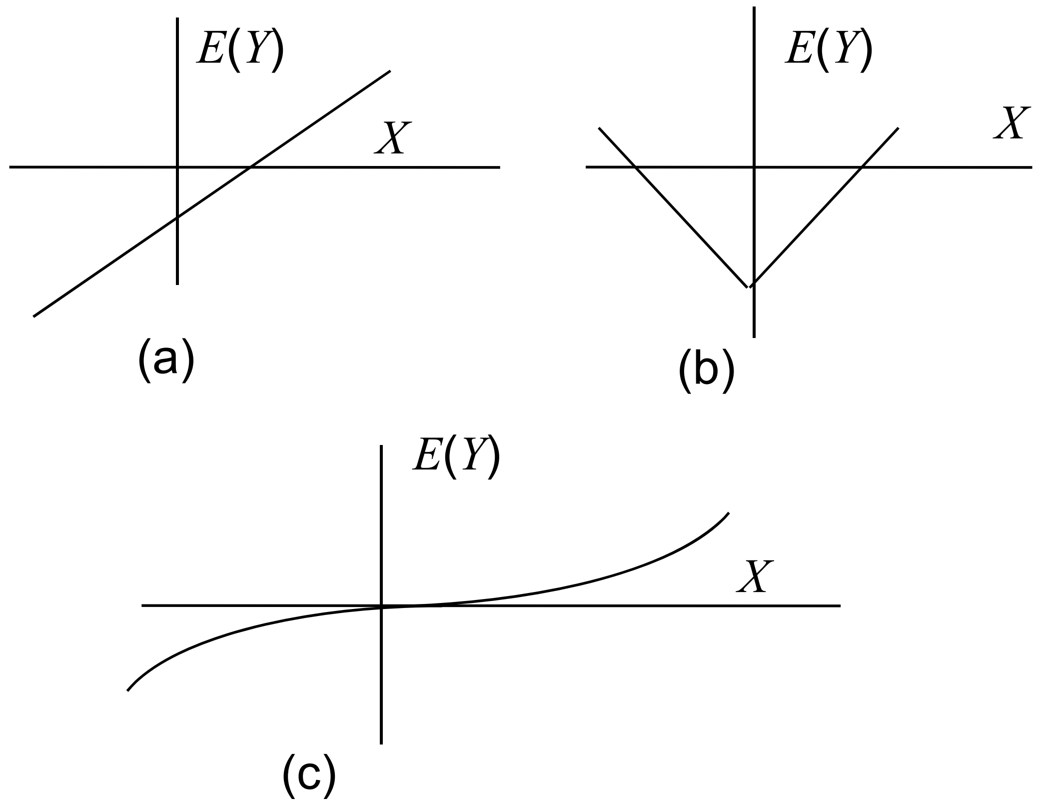
where $f(.)$ denotes the probability density function

Independence is Not the Same as Zero Correlation

- Suppose V_1 = –1, 0, or +1 (equally likely)
- If V_1 = -1 or V_1 = +1 then V_2 = 1
- If V_1 = 0 then $V_2 = 0$

V_2 is clearly dependent on V_1 (and vice versa) but the coefficient of correlation is zero

Types of Dependence (Figure 11.1, page 233)



Monitoring Correlation Between Two Variables *X* and *Y*

Define $x_i=(X_i-X_{i\text{-}1})/X_{i\text{-}1}$ and $y_i=(Y_i-Y_{i\text{-}1})/Y_{i\text{-}1}$

Also

$\text{var}_{x,n}$: daily variance of X calculated on day n-1

$\text{var}_{y,n}$: daily variance of Y calculated on day n-1

cov_n: covariance calculated on day n-1

The correlation is

$$\frac{\text{cov}_n}{\sqrt{\text{var}_{x,n}\ \text{var}_{y,n}}}$$

Covariance

- The covariance on day n is

$$E(x_n y_n) - E(x_n)E(y_n)$$

- It is usually approximated as $E(x_n y_n)$

Monitoring Correlation continued

EWMA:

$$\text{cov}_n = \lambda\,\text{cov}_{n-1} + (1-\lambda)x_{n-1}y_{n-1}$$

GARCH(1,1)

$$\text{cov}_n = \omega + \alpha x_{n-1}y_{n-1} + \beta\,\text{cov}_{n-1}$$

Positive Finite Definite Condition

A variance-covariance matrix, Ω, is internally consistent if the positive semi-definite condition

$$\mathbf{w}^{\mathsf{T}}\Omega\mathbf{w} \geq 0$$

holds for all vectors $\mathbf{w}$

Example

The variance covariance matrix

$$\begin{pmatrix} 1 & 0 & 0.9 \\ 0 & 1 & 0.9 \\ 0.9 & 0.9 & 1 \end{pmatrix}$$

is not internally consistent

Factor Models (page 238)

- When there are N variables, V_i (i = 1, 2,..N), in a multivariate normal distribution there are $N(N-1)/2$ correlations
- We can reduce the number of correlation parameters that have to be estimated with a factor model

One-Factor Model continued

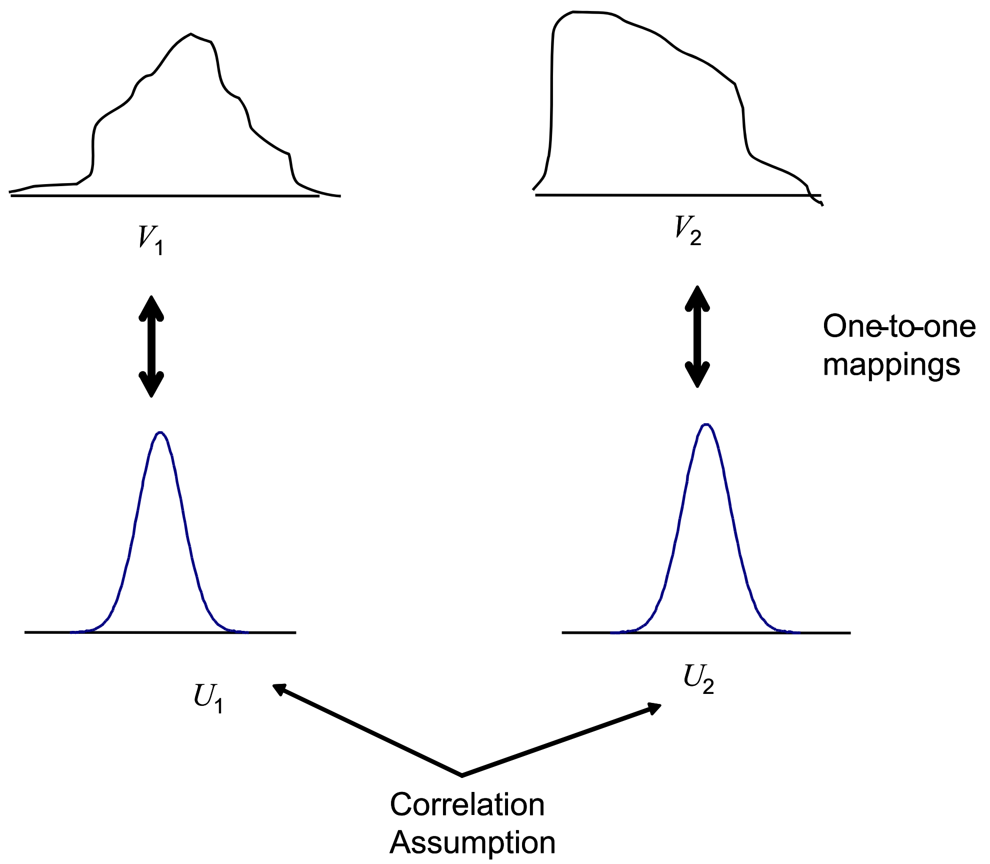
- If U_i have standard normal distributions we can set
$$U_i = a_i F + \sqrt{1 - a_i^2} Z_i$$
where the common factor F and the idiosyncratic component Z_i have independent standard normal distributions
- Correlation between U_i and U_j is $a_i \, a_j$

Gaussian Copula Models:

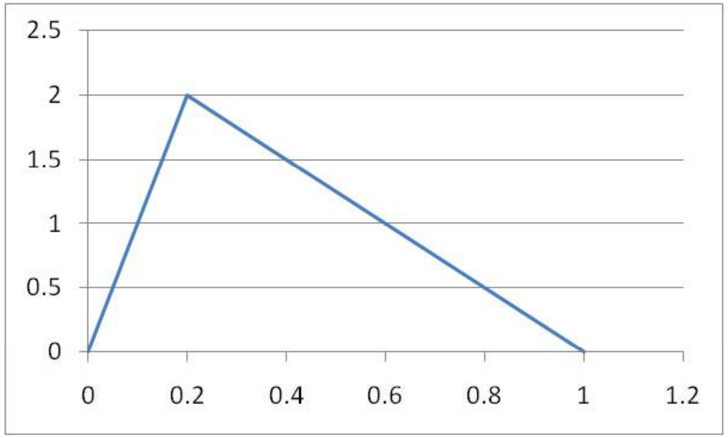
Creating a correlation structure for variables that are not normally distributed

- Suppose we wish to define a correlation structure between two variable V_1 and V_2 that do not have normal distributions
- We transform the variable V_1 to a new variable U_1 that has a standard normal distribution on a "percentile-to-percentile" basis.
- We transform the variable V_2 to a new variable U_2 that has a standard normal distribution on a "percentile-to-percentile" basis.
- U_1 and U_2 are assumed to have a bivariate normal distribution

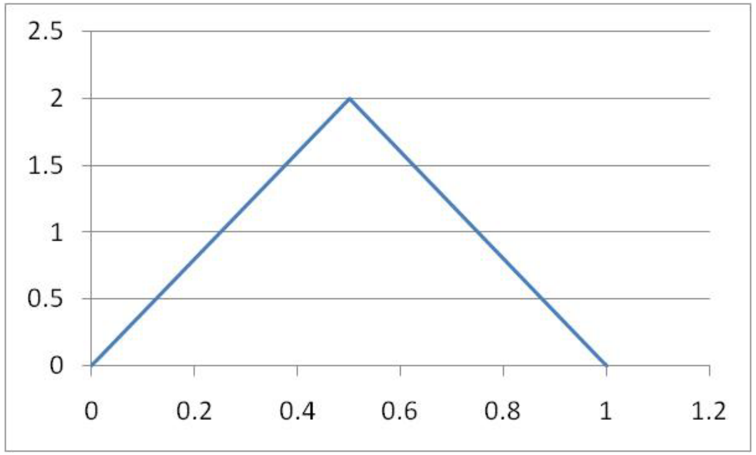
The Correlation Structure Between the V's is Defined by that Between the U's



Example (page 239)



V_1



V_2

V_1 Mapping to U_1

V_1	Percentile	U_1
0.2	20	-0.84
0.4	55	0.13
0.6	80	0.84
0.8	95	1.64

V_2 Mapping to U_2

V_2	Percentile	U_2
0.2	8	−1.41
0.4	32	−0.47
0.6	68	0.47
0.8	92	1.41

Example of Calculation of Joint Cumulative Distribution

- Probability that V_1 and V_2 are both less than 0.2 is the probability that $U_1 < -0.84$ and $U_2 < -1.41$
- When copula correlation is 0.5 this is

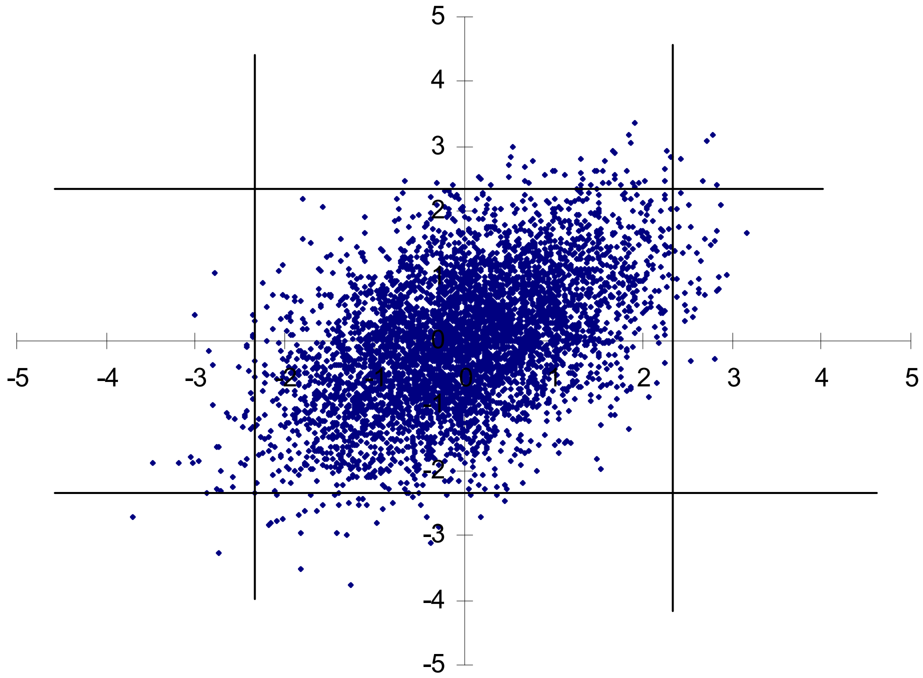
  $M(-0.84, -1.41, 0.5) = 0.043$

  where M is the cumulative distribution function for the bivariate normal distribution

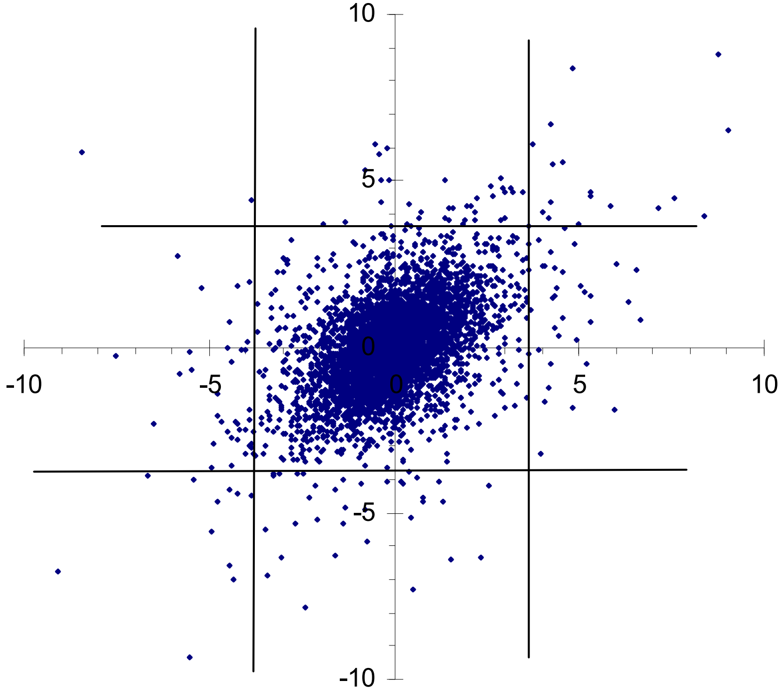
Other Copulas

- Instead of a bivariate normal distribution for U_1 and U_2 we can assume any other joint distribution
- One possibility is the bivariate Student t distribution

5000 Random Samples from the Bivariate Normal (Figure 11.4, page 243)



5000 Random Samples from the Bivariate Student t (Figure 11.5, page 243)



Multivariate Gaussian Copula

- We can similarly define a correlation structure between V_1, V_2,… V_n
- We transform each variable V_i to a new variable U_i that has a standard normal distribution on a “percentile-to-percentile” basis.
- The U’s are assumed to have a multivariate normal distribution

Factor Copula Model

In a factor copula model the correlation structure between the U's is generated by assuming one or more factors.

Credit Default Correlation

- The credit default correlation between two companies is a measure of their tendency to default at about the same time
- Default correlation is important in risk management when analyzing the benefits of credit risk diversification
- It is also important in the valuation of some credit derivatives

Model for Loan Portfolio

- We map the time to default for company i, T_i, to a new variable U_i and assume

$$U_i = aF + \sqrt{1-a^2}\,Z_i$$

- Where F and the Z_i have independent standard normal distributions
- The copula correlation is $\rho = a^2$

Analysis

- To analyze the model we
  - Calculate the probability that, conditional on the value of F, U_i is less than some value U
  - This is the same as the probability that T_i is less that T where T and U are the same percentiles of their distributions

This leads to

$$\text{Prob}(T_i < T | F) = N\left[\frac{N^{-1}[\text{PD}] - \sqrt{\rho}F}{\sqrt{1-\rho}}\right]$$

where PD is the probability of default in time T

The Model continued

- Low values of F give high default probabilities
- The value of F is that has an X% chance of being exceeded is $-N^{-1}(X)$
- The "worst case default rate" that will not be exceeded with probability X during time T is therefore

$$\mathrm{WCDR}(T,X) = N\left(\frac{N^{-1}[\mathrm{PD}] + \sqrt{\rho}N^{-1}(X)}{\sqrt{1-\rho}}\right)$$

Estimating PD and ρ

- Table 11.4 shows that default rates ranged friom 0.087% to 5.422% between 1970 and 2013 for all rated companies.
- We can use this data in conjunction with maximum likelihood methods to estimate PD and ρ

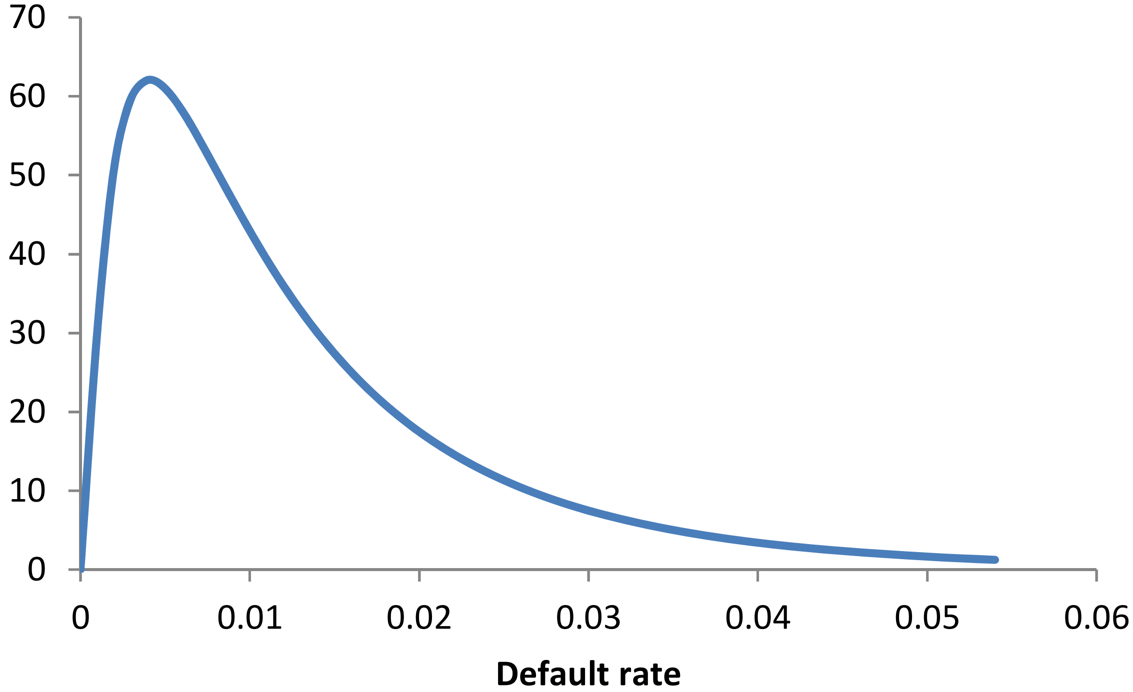
Estimating PD and ρ continued

- The probability density function for the default rate is

$$g(\mathrm{DR})=\sqrt{\frac{1-\rho}{\rho}}\exp\left\{\frac{1}{2}\left[(N^{-1}(\mathrm{DR}))^2-\left(\frac{\sqrt{1-\rho}N^{-1}(\mathrm{DR})-N^{-1}(\mathrm{PD})}{\sqrt{\rho}}\right)^2\right]\right\}$$

- Maximizing the sum of the logarithms of this for the data in Table 11.4 we get PD=1.41% and ρ = 0.108

Probability Distribution for Default Rate (Figure 11.6, page 249)



Alternatives to the Gaussian Factor Copula

- In

$$U_i = aF + \sqrt{1 - a^2} Z_i$$

- We can let the F and Z_i be non-normal distributions with mean zero and standard deviation one.

Value at Risk and Expected Shortfall

Chapter 12

The Question Being Asked in VaR

"What loss level is such that we are X% confident it will not be exceeded in N business days?"

VaR and Regulatory Capital

- Regulators have traditionally used VaR to calculate the capital they require banks to keep
- The market-risk capital has been based on a 10-day VaR estimated where the confidence level is 99%
- Credit risk and operational risk capital are based on a one-year 99.9% VaR

Advantages of VaR

- It captures an important aspect of risk in a single number
- It is easy to understand
- It asks the simple question: “How bad can things get?”

Example 12.1 (page 257)

- The gain from a portfolio during six month is normally distributed with mean \$2 million and standard deviation \$10 million
- The 1% point of the distribution of gains is $2-2.33\times10$ or − \$21.3 million
- The VaR for the portfolio with a six month time horizon and a 99% confidence level is \$21.3 million.

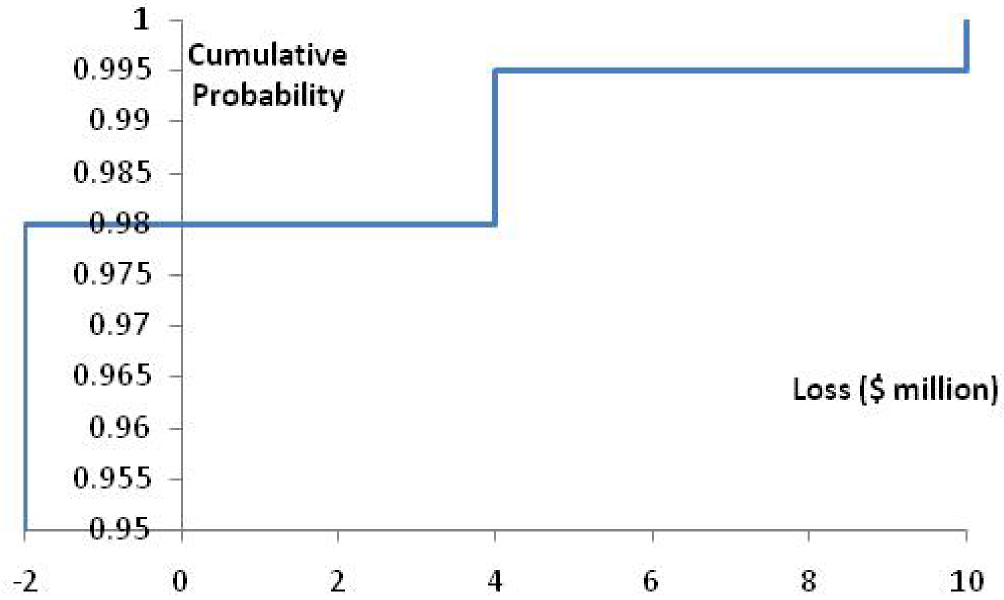
Example 12.2 (page 258)

- All outcomes between a loss of $50 million and a gain of $50 million are equally likely for a one-year project
- The VaR for a one-year time horizon and a 99% confidence level is $49 million

Examples 12.3 and 12.4 (page 258)

- A one-year project has a 98% chance of leading to a gain of $2 million, a 1.5% chance of a loss of $4 million, and a 0.5% chance of a loss of $10 million
- The VaR with a 99% confidence level is $4 million
- What if the confidence level is 99.9%?
- What if it is 99.5%?

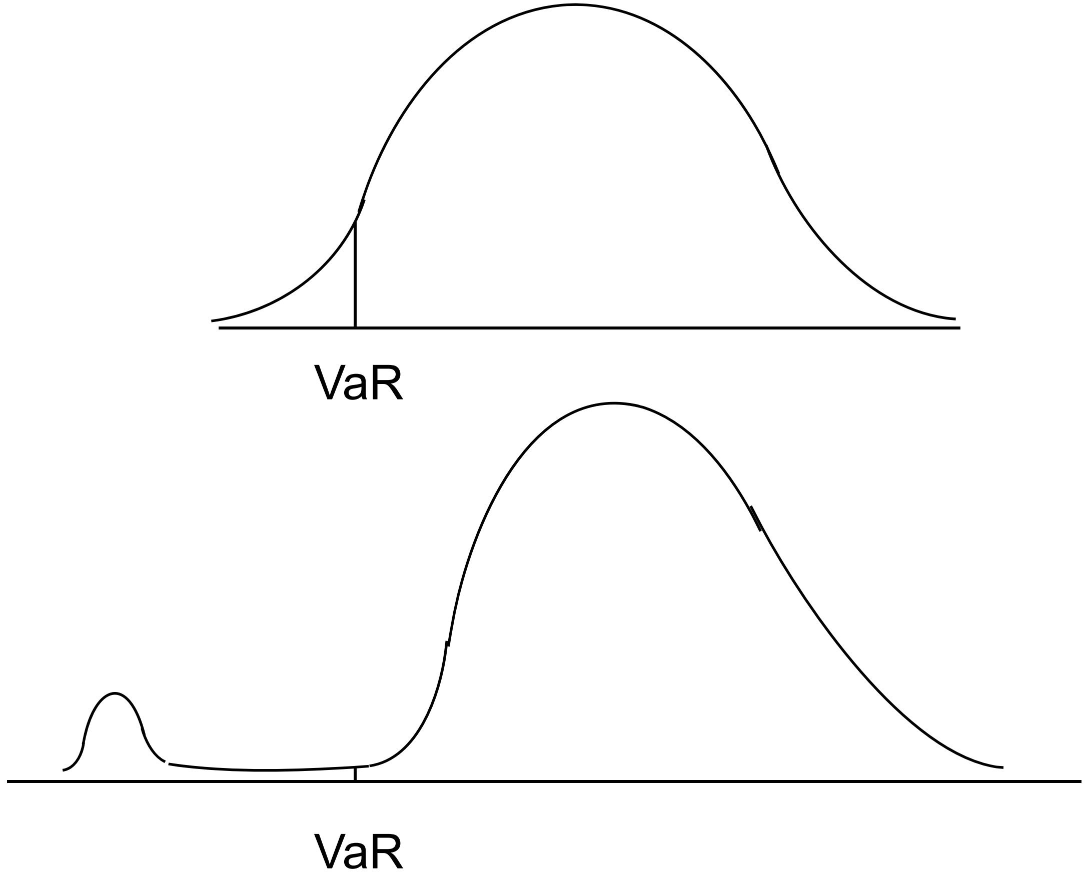
Cumulative Loss Distribution for Examples 12.3 and 12.4 (Figure 12.3, page 258)



VaR vs. Expected Shortfall

- VaR is the loss level that will not be exceeded with a specified probability
- Expected shortfall (ES) is the expected loss given that the loss is greater than the VaR level (also called C-VaR and Tail Loss)
- Regulators have indicated that they plan to move from using VaR to using ES for determining market risk capital
- Two portfolios with the same VaR can have very different expected shortfalls

Distributions with the Same VaR but Different Expected Shortfalls



Coherent Risk Measures (page 260)

- Define a coherent risk measure as the amount of cash that has to be added to a portfolio to make its risk acceptable
- Properties of coherent risk measure
  - If one portfolio always produces a worse outcome than another its risk measure should be greater
  - If we add an amount of cash K to a portfolio its risk measure should go down by K
  - Changing the size of a portfolio by λ should result in the risk measure being multiplied by λ
  - The risk measures for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged

VaR vs Expected Shortfall

- VaR satisfies the first three conditions but not the fourth one
- ES satisfies all four conditions.

Example 12.5 and 12.7

- Each of two independent projects has a probability 0.98 of a loss of \$1 million and 0.02 probability of a loss of \$10 million
- What is the 97.5% VaR for each project?
- What is the 97.5% expected shortfall for each project?
- What is the 97.5% VaR for the portfolio?
- What is the 97.5% expected shortfall for the portfolio?

Examples 12.6 and 12.8

- A bank has two $10 million one-year loans. Possible outcomes are as follows

Outcome	Probability
Neither Loan Defaults	97.5%
Loan 1 defaults, loan 2 does not default	1.25%
Loan 2 defaults, loan 1 does not default	1.25%
Both loans default	0.00%

- If a default occurs, losses between 0% and 100% are equally likely. If a loan does not default, a profit of 0.2 million is made.
- What is the 99% VaR and expected shortfall of each project
- What is the 99% VaR and expected shortfall for the portfolio

Spectral Risk Measures

1. A spectral risk measure assigns weights to quantiles of the loss distribution
2. VaR assigns all weight to Xth percentile of the loss distribution
3. Expected shortfall assigns equal weight to all percentiles greater than the Xth percentile
4. For a coherent risk measure weights must be a non-decreasing function of the percentiles

Normal Distribution Assumption

- When losses (gains) are normally distributed with mean μ and standard deviation σ

$$\text{VaR} = \mu + \sigma N^{-1}(X)$$

$$\text{ES} = \mu + \sigma \frac{e^{-Y^2/2}}{\sqrt{2\pi}(1-X)}$$

Changing the Time Horizon

- If losses in successive days are independent, normally distributed, and have a mean of zero

$$T\text{-day VaR} = 1\text{-day VaR} \times \sqrt{\mathrm{T}}$$

$$T\text{-day ES} = 1\text{-day ES} \times \sqrt{T}$$

Extension

- If there is autocorrelation ρ between the losses (gains) on successive days, we replace $\sqrt{T}$ by

$$\sqrt{T+2(T-1)\rho+2(T-2)\rho^2+2(T-3)\rho^3+\ldots+2\rho^{T-1}}$$

in these equations

Ratio of T-day VaR to 1-day VaR (Table 12.1, page 266)

	T=1	T=2	T=5	T=10	T=50	T=250
ρ=0	1.0	1.41	2.24	3.16	7.07	15.81
ρ=0.05	1.0	1.45	2.33	3.31	7.43	16.62
ρ=0.1	1.0	1.48	2.42	3.46	7.80	17.47
ρ=0.2	1.0	1.55	2.62	3.79	8.62	19.35

Choice of VaR Parameters

- Time horizon should depend on how quickly portfolio can be unwound. Regulators are planning to move toward a system where ES is used and the time horizon depends on liquidity. (See *Fundamental Review of the Trading Book*)
- Confidence level depends on objectives. Regulators use 99% for market risk and 99.9% for credit/operational risk.
- A bank wanting to maintain a AA credit rating might use confidence levels as high as 99.97% for internal calculations.

Aggregating VaRs

An approximate approach that seems to works well is

$$\mathrm{VaR}_{\mathrm{total}} = \sqrt{\sum_i \sum_j \mathrm{VaR}_i \mathrm{VaR}_j \rho_{ij}}$$

where VaR_i is the VaR for the ith segment, $\mathrm{VaR}_{\mathrm{total}}$ is the total VaR, and ρ_{ij} is the coefficient of correlation between losses from the ith and jth segments

VaR Measures for a Portfolio where an amount x_i is invested in the *i*th component of the portfolio (page 268-270)

- Marginal VaR: $$\frac{\partial \text{VaR}}{\partial x_i}$$
- Incremental VaR: Incremental effect of the *i*th component on VaR
- Component VaR: $$x_i \frac{\partial \text{VaR}}{\partial x_i}$$

Properties of Component VaR

- The component VaR is approximately the same as the incremental VaR
- The total VaR is the sum of the component VaR’s (Euler’s theorem)
- The component VaR therefore provides a sensible way of allocating VaR to different activities

Back-testing (page 270-273)

- Back-testing a VaR calculation methodology involves looking at how often exceptions (loss > VaR) occur
- Alternatives: a) compare VaR with actual change in portfolio value and b) compare VaR with change in portfolio value assuming no change in portfolio composition
- Suppose that the theoretical probability of an exception is p (=1−X). The probability of m or more exceptions in n days is

$$\sum_{k=m}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

Bunching

- Bunching occurs when exceptions are not evenly spread throughout the back testing period
- Statistical tests for bunching have been developed by Christoffersen (See page 200)

Historical Simulation and Extreme Value Theory

Chapter 13

Historical Simulation

- Collect data on the daily movements in all market variables.
- The first simulation trial assumes that the percentage changes in all market variables are as on the first day
- The second simulation trial assumes that the percentage changes in all market variables are as on the second day
- and so on

Historical Simulation continued

- Suppose we use n days of historical data with today being day n
- Let v_i be the value of a variable on day i
- There are n-1 simulation trials
- The ith trial assumes that the value of the market variable tomorrow (i.e., on day n+1) is

$$v_n \frac{v_i}{v_{i-1}}$$

Example: Portfolio on Sept 25, 2008 (Table 13.1, page 278)

Index	Amount Invested ($000s)
DJIA	4,000
FTSE 100	3,000
CAC 40	1,000
Nikkei 225	2,000
Total	10,000

U.S. Dollar Equivalent of Stock Indices (Table 13.2, page 279)

Day	Date	DJIA	FTSE	CAC 40	Nikkei
0	Aug 7, 2006	11,219.38	11,131.84	6,373.89	131.77
1	Aug 8, 2006	11,173.59	11,096.28	6,378.16	134.38
2	Aug 9, 2006	11,076.18	11,185.35	6,474.04	135.94
3	Aug 10, 2006	11,124.37	11,016.71	6,357.49	135.44
….	……	……	…..	……	……
499	Sep 24, 2008	10,825.17	9,438.58	6,033.93	114.26
500	Sep 25, 2008	11,022.06	9,599.90	6,200.40	112.82

Scenarios (Table 13.3, page 279)

$$= 11{,}022.06 \times \frac{11{,}173.59}{11{,}219.38}$$

Scenario Number	DJIA	FTSE	CAC	Nikkei	Portfolio Value	Loss
1	10,977.08	9,569.23	6,204.55	115.05	10,014.334	-14.334
2	10,925.97	9,676.96	6,293.60	114.13	10,027.481	-27,481
3	11,070.01	9,455.16	6,088.77	112.40	9,946.736	53,264
….	……	…..	…..	…..	…..	…..
499	10,831.43	9,383.49	6,051.94	113.85	9,857.465	142.535
500	11,222.53	9,763.97	6,371.45	111.40	10,126.439	-126.439

Losses (Table 13.4, page 281)

Scenario Number	Loss ($000s)
494	477.841
339	345.435
349	282.204
329	277.041
487	253.385
227	217.974
131	205.256

One-day 99% VaR=$253,385

One-day 99% ES is
(477,841+345,435+282,204+277,041+253,385)/5=$327,181

Stressed VaR and Stressed ES

- Instead of basing calculations on the movements in market variables over the last n days, we can base calculations on movements during a period in the past that would have been particularly bad for the current portfolio
- This produces measures known as "stressed VaR" and "stressed ES"

Accuracy of VaR (page 282)

Suppose that x is the qth quantile of the loss distribution when it is estimated from n observations. The standard error of x is

$$\frac{1}{f(x)}\sqrt{\frac{q(1-q)}{n}}$$

where $f(x)$ is an estimate of the probability density of the loss at the qth quantile calculated by assuming a probability distribution for the loss

Example 13.1 (page 283)

- We estimate the 0.01-quantile from 500 observations as \$25 million
- We estimate $f(x)$ by approximating the actual empirical distribution with a normal distribution mean zero and standard deviation \$10 million
- The 0.01 quantile of the approximating distribution is NORMINV(0.01,0,10) = 23.26 and the value of $f(x)$ is NORMDIST(23.26,0,10,FALSE)=0.0027
- The estimate of the standard error is therefore

$$\frac{1}{0.0027} \times \sqrt{\frac{0.01 \times 0.99}{500}} = 1.67$$

Weighting of Observations

- Let weights assigned to observations decline exponentially as we go back in time
- Rank observations from worst to best
- Starting at worst observation sum weights until the required quantile is reached

Application to 4-Index Portfolio $\lambda=0.995$ (Table 13.5, page 285)

Scenario Number	Loss ($000s)	Weight	Cumulative Weight
494	477.841	0.00528	0.00528
339	345.435	0.00243	0.00771
349	282.204	0.00255	0.01027
329	277.041	0.00231	0.01258
487	253.385	0.00510	0.01768
227	217.974	0.00139	0.01906

One-day 99% VaR=$282,204

One day 99% ES is
$[0.00528\times477{,}841+0.00243\times345{,}435+(0.01-0.00528-0.00243)\times282{,}204]/0.01=$
$400,914

Volatility Updating

- Use a volatility updating scheme to monitor volatilities of all market variables
- If the current volatility for a market variable is β times the volatility on Day i, multiply the percentage change observed on day i by β
- Value of market variable under ith scenario becomes

$$v_n \frac{v_{i-1} + (v_i - v_{i-1})\sigma_{n+1} / \sigma_i}{v_{i-1}}$$

Volatilities (% per Day) Estimated for Next Day in 4-Index Example (Table 13.6, page 286)

Day	Date	DJIA	FTSE	CAC 40	Nikkei
0	Aug 7, 2006	1.11	1.42	1.40	1.38
1	Aug 8, 2006	1.08	1.38	1.36	1.43
2	Aug 9, 2006	1.07	1.35	1.36	1.41
3	Aug 10, 2006	1.04	1.36	1.39	1.37
….	……	……	……	……	……
499	Sep 24, 2008	2.21	3.28	3.11	1.61
500	Sep 25, 2008	2.19	3.21	3.09	1.59

Volatility Adjusted Losses (Table 13.7, page 287)

Scenario Number	Loss ($000s)
131	1,082.969
494	715.512
227	687.720
98	661.221
329	602.968
339	546.540
74	492.764

One-day 99% VaR = $602,968

One-day 99% ES =
(1,082,969+715,512+687,720+661,221+602,968)/5 = $750,078

A Simpler Approach

- Monitor variance of simulated losses on the portfolio using EWMA
- If current standard deviation of losses is β_P times the standard deviation of simulated losses on Day i, multiply ith loss given by the standard approach by β_P
- This approach gives VaR and ES as $627,916 and $747,619, respectively

The Bootstrap Method to Determine Confidence Intervals

- Suppose there are 500 daily changes
- Calculate a 95% confidence interval for VaR by sampling 500,000 times with replacement from daily changes to obtain 1000 sets of changes over 500 days
- Calcuate VaR for each set and calculate a confidence interval
- This is known as the bootstrap method

Computational Issues

- To avoid revaluing a complete portfolio many times a delta/gamma approximation is sometimes used
- When a derivative depend on only one underlying variable, S

$$\Delta P \approx \delta \Delta S + \frac{1}{2}\gamma(\Delta S)^2$$

- With many variables

$$\Delta P \approx \sum_{i=1}^{n} \delta_i \Delta S_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{2}\gamma_{ij} \Delta S_i \Delta S_j$$

Extreme Value Theory (page 289-295)

- Extreme value theory can be used to investigate the properties of the right tail of the empirical distribution of a variable x. (If we interested in the left tail we consider the variable $-x$.)
- We first choose a level u somewhat in the right tail of the distribution
- We then use Gnedenko's result which shows that for a wide class of distributions as u increases the probability distribution that v lies between u and $u+y$ conditional that it is greater than u tends to a generalized Pareto distribution

Generalized Pareto Distribution

- This has two parameters ξ (the shape parameter) and β (the scale parameter)
- The cumulative distribution is

$$1-\left[1+\frac{\xi}{\beta}y\right]^{-1/\xi}$$

Maximum Likelihood Estimator
(Equation 13.7, page 291)

- The observations, x_i, are sorted in descending order. Suppose that there are n_u observations greater than u
- We choose ξ and β to maximize

$$\sum_{i=1}^{n_u} \ln\left[\frac{1}{\beta}\left(1+\frac{\xi(v_i-u)}{\beta}\right)^{-1/\xi-1}\right]$$

Using Maximum Likelihood for 4-Index Example, u=160 (Table 13.10, page 293)

Scenario	Loss ($000s)	Rank	$\ln\left[\frac{1}{\beta}\left(1+\frac{\xi(v_i-u)}{\beta}\right)^{-1/\xi-1}\right]$
494	477.841	1	-8.97
339	345.435	2	-7.47
349	282.204	3	-6.51
329	277.041	4	-6.42
487	253.385	5	-5.99
304	160.778	22	-3.71
Total			-108.37

Tail Probabilities

Our estimator for the cumulative probability that the variable v is greater than x is

$$\frac{n_u}{n}\left[1+\xi\frac{x-u}{\beta}\right]^{-1/\xi}$$

Setting $u=\beta/\xi$ we see that this corresponds to the power law

$$\text{Prob}(v>x)=Kx^{-\alpha}$$

where

$$K=\frac{n_u}{n}\left(\frac{\xi}{\beta}\right)^{-1/\xi} \qquad \alpha=\frac{1}{\xi}$$

Extreme value theory therefore explains why the power law holds so widely

Estimating VaR and ES Using Extreme Value Theory (page 292)

- Setting the probability that $v>x$ equal to $1-q$ we obtain an estimate of VaR with a confidence level of q

$$\text{VaR} = u + \frac{\beta}{\xi}\left\{\left[\frac{n}{n_u}(1-q)\right]^{-\xi} - 1\right\}$$

- Also

$$\text{ES} = \frac{\text{VaR} + \beta - \xi u}{1-\xi}$$

Model-Building Approach

Chapter 14

The Model-Building Approach

- The main alternative to historical simulation is to make assumptions about the probability distributions of the returns on the market variables
- This is known as the model building approach (or sometimes the variance-covariance approach)

Microsoft Example (page 299-302)

- We have a position worth \$10 million in Microsoft shares
- The volatility of Microsoft is 2% per day (about 32% per year)
- We use N=10 and X=99

Microsoft Example continued

- The standard deviation of the change in the portfolio in 1 day is $200,000
- The standard deviation of the change in 10 days is

$$200{,}000\sqrt{10} = \$632{,}500$$

Microsoft Example continued

- We assume that the expected change in the value of the portfolio is zero (This is OK for short time periods)
- We assume that the change in the value of the portfolio is normally distributed
- Since $N(-2.326)=0.01$, the VaR is

$$2.326 \times 632{,}500 = \$1{,}471{,}300$$

AT&T Example

- Consider a position of $5 million in AT&T
- The daily volatility of AT&T is 1% (approx 16% per year)
- The SD per 10 days is

$$50{,}000\sqrt{10} = \$158{,}144$$

- The VaR is

$$158{,}100 \times 2.326 = \$367{,}800$$

Portfolio (page 301)

- Now consider a portfolio consisting of both Microsoft and AT&T
- Suppose that the correlation between the returns is 0.3

S.D. of Portfolio

- A standard result in statistics states that

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y}$$

- In this case σ_X = 200,000 and σ_Y = 50,000 and ρ = 0.3. The standard deviation of the change in the portfolio value in one day is therefore 220,200

VaR for Portfolio

- The 10-day 99% VaR for the portfolio is

$$220{,}200 \times \sqrt{10} \times 2.326 = \$1{,}620{,}100$$

- The benefits of diversification are

(1,471,300+367,800)–1,620,100=$219,000

- What is the incremental effect of the AT&T holding on VaR?

ES Results (10-day 99%)

- Microsoft shares: $1,687,000
- AT&T shares: $421,700
- Portfolio: $1,857,600

The Linear Model

We assume

- The daily change in the value of a portfolio is linearly related to the daily returns from market variables
- The returns from the market variables are normally distributed

Markowitz Result for Variance of Return on Portfolio

$$\text{Variance of Portfolio Return} = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} w_i w_j \sigma_i \sigma_j$$

w_i is weight of ith asset in portfolio

σ_i^2 is variance of return on ith asset in portfolio

ρ_{ij} is correlation between returns of ith and jth assets

Corresponding Result for Variance of Portfolio Value

$$\Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i$$

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j$$

$$\sigma_P^2 = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + 2 \sum_{i<j} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j$$

σ_i is the daily volatility of the *i*th asset (i.e., SD of daily returns)

σ_P is the SD of the change in the portfolio value per day

$\alpha_i = w_i P$ is amount invested in *i*th asset

Covariance Matrix ($\text{var}_i = \text{cov}_{ii}$)
(page 304)

$$C = \begin{pmatrix} \text{var}_1 & \text{cov}_{12} & \text{cov}_{13} & \cdots & \text{cov}_{1n} \\ \text{cov}_{21} & \text{var}_2 & \text{cov}_{23} & \cdots & \text{cov}_{2n} \\ \text{cov}_{31} & \text{cov}_{32} & \text{var}_3 & \dots & \text{cov}_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \text{cov}_{n1} & \text{cov}_{n2} & \text{cov}_{n3} & \dots & \text{var}_n \end{pmatrix}$$

Alternative Expressions for σ_P^2
page 304

$$\sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\text{cov}_{ij}\,\alpha_i\alpha_j$$

$$\sigma_P^2 = \boldsymbol{\alpha}^{\mathrm{T}} C \boldsymbol{\alpha}$$

where $\boldsymbol{\alpha}$ is the column vector whose ith element is α_i and $\boldsymbol{\alpha}^{\mathrm{T}}$ is its transpose

Four Index Example Using Last 500 Days of Data to Estimate Covariances

	Equal Weight	EWMA : λ=0.94
One-day 99% VaR	$217,757	$471,025
One-day 99% ES	$249,476	$539,637

Volatilities and Correlations Increased in Sept 2008

Volatilities (% per day)

	DJIA	FTSE	CAC	Nikkei
Equal Weights	1.11	1.42	1.40	1.38
EWMA	2.19	3.21	3.09	1.59

Correlations

$$\begin{pmatrix} 1 & 0.489 & 0.496 & -0.062 \\ 0.489 & 1 & 0.918 & 0.201 \\ 0.496 & 0.918 & 1 & 0.211 \\ -0.062 & 0.201 & 0.211 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0.611 & 0.629 & -0.113 \\ 0.611 & 1 & 0.971 & 0.409 \\ 0.629 & 0.971 & 1 & 0.342 \\ -0.113 & 0.409 & 0.342 & 1 \end{pmatrix}$$

Equal weights EWMA

Alternatives for Handling Interest Rates

- Duration approach: Linear relation between ΔP and Δy but assumes parallel shifts)
- Cash flow mapping: Variables are zero-coupon bond prices with about 10 different maturities
- Principal components analysis: 2 or 3 independent shifts with their own volatilities

Handling Interest Rates: Cash Flow Mapping (page 307-309)

- We choose as market variables zero-coupon bond prices with standard maturities (1mm, 3mm, 6mm, 1yr, 2yr, 5yr, 7yr, 10yr, 30yr)
- Suppose that the 5yr rate is 6% and the 7yr rate is 7% and we will receive a cash flow of $10,000 in 6.5 years.
- The volatilities per day of the 5yr and 7yr bonds are 0.50% and 0.58% respectively

Example continued

- We interpolate between the 5yr rate of 6% and the 7yr rate of 7% to get a 6.5yr rate of 6.75%
- The PV of the $10,000 cash flow is

$$\frac{10{,}000}{1.0675^{6.5}} = 6{,}540$$

Example continued

- We interpolate between the 0.5% volatility for the 5yr bond price and the 0.58% volatility for the 7yr bond price to get 0.56% as the volatility for the 6.5yr bond
- We allocate α of the PV to the 5yr bond and (1- α) of the PV to the 7yr bond

Example continued

- Suppose that the correlation between movement in the 5yr and 7yr bond prices is 0.6
- To match variances

$$0.56^2 = 0.5^2\alpha^2 + 0.58^2(1-\alpha)^2 + 2\times 0.6\times 0.5\times 0.58\times \alpha(1-\alpha)$$

- This gives α=0.074

Example continued

The value of 6,540 received in 6.5 years

$$6{,}540 \times 0.074 = \$484$$

in 5 years and by

$$6{,}540 \times 0.926 = \$6{,}056$$

in 7 years.

This cash flow mapping preserves value and variance

Using a PCA to Calculate VaR (page 309-310)

- Suppose we calculate

$$\Delta P = 0.05 f_1 - 3.87 f_2$$

where f_1 is the first factor and f_2 is the second factor

- If the SD of the factor scores are 17.55 and 4.77 the SD of ΔP is

$$\sqrt{0.05^2 \times 17.55^2 + 3.87^2 \times 4.77^2} = 18.48$$

When Linear Model Can be Used

- Portfolio of stocks
- Portfolio of bonds
- Forward contract on foreign currency
- Interest-rate swap

The Linear Model and Options

Consider a portfolio of options dependent on a single stock price, S. Define

$$\delta = \frac{\Delta P}{\Delta S}$$

and

$$\Delta x = \frac{\Delta S}{S}$$

Linear Model and Options continued

- As an approximation

$$\Delta P = \delta\,\Delta S = S\delta\,\Delta x$$

- Similarly when there are many underlying market variables

$$\Delta P = \sum_i S_i \delta_i\,\Delta x_i$$

where δ_i is the delta of the portfolio with respect to the ith asset

Example

- Consider an investment in options on Microsoft and AT&T. Suppose the stock prices are 120 and 30 respectively and the deltas of the portfolio with respect to the two stock prices are 1,000 and 20,000 respectively
- As an approximation

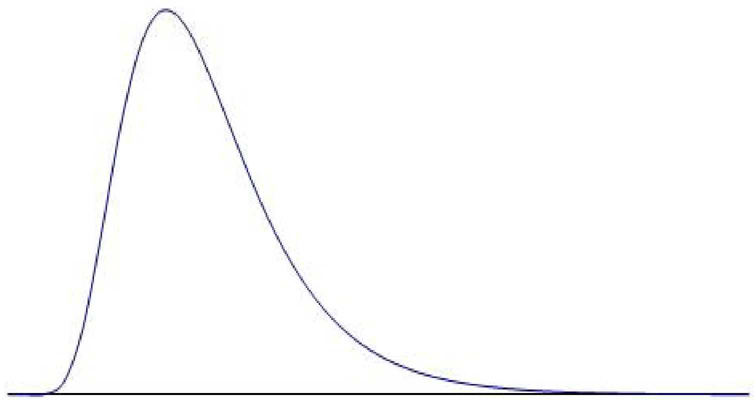
$$\Delta P = 120 \times 1{,}000 \Delta x_1 + 30 \times 20{,}000 \Delta x_2$$

where Δx_1 and Δx_2 are the percentage changes in the two stock prices

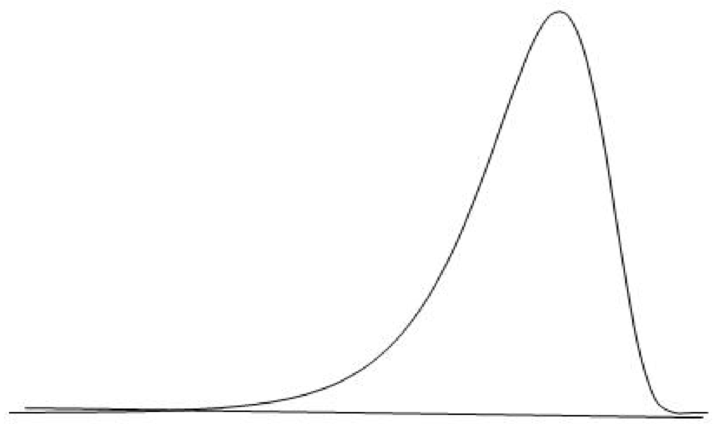
But the Distribution of the Daily Return on an Option is not Normal

The linear model fails to capture skewness in the probability distribution of the portfolio value.

Impact of Gamma
(See Figure 14.1, page 313)



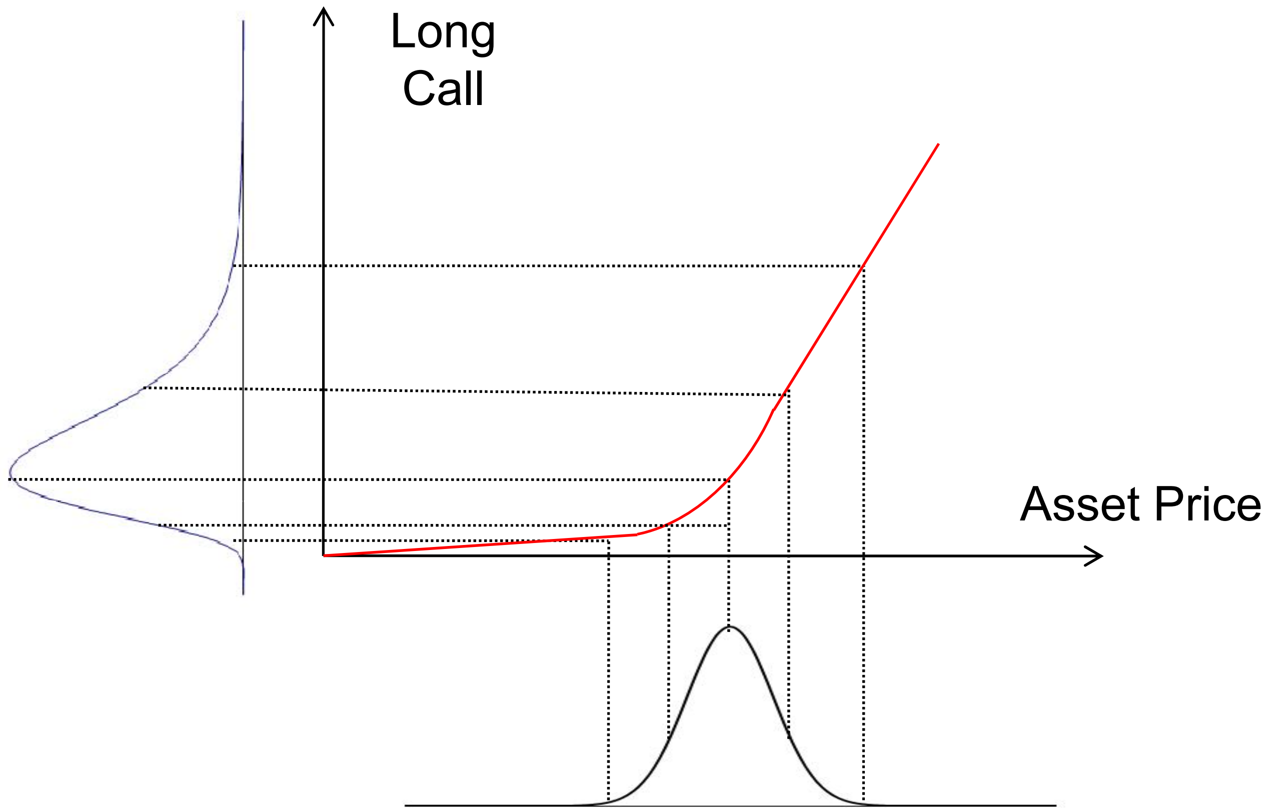
Positive Gamma



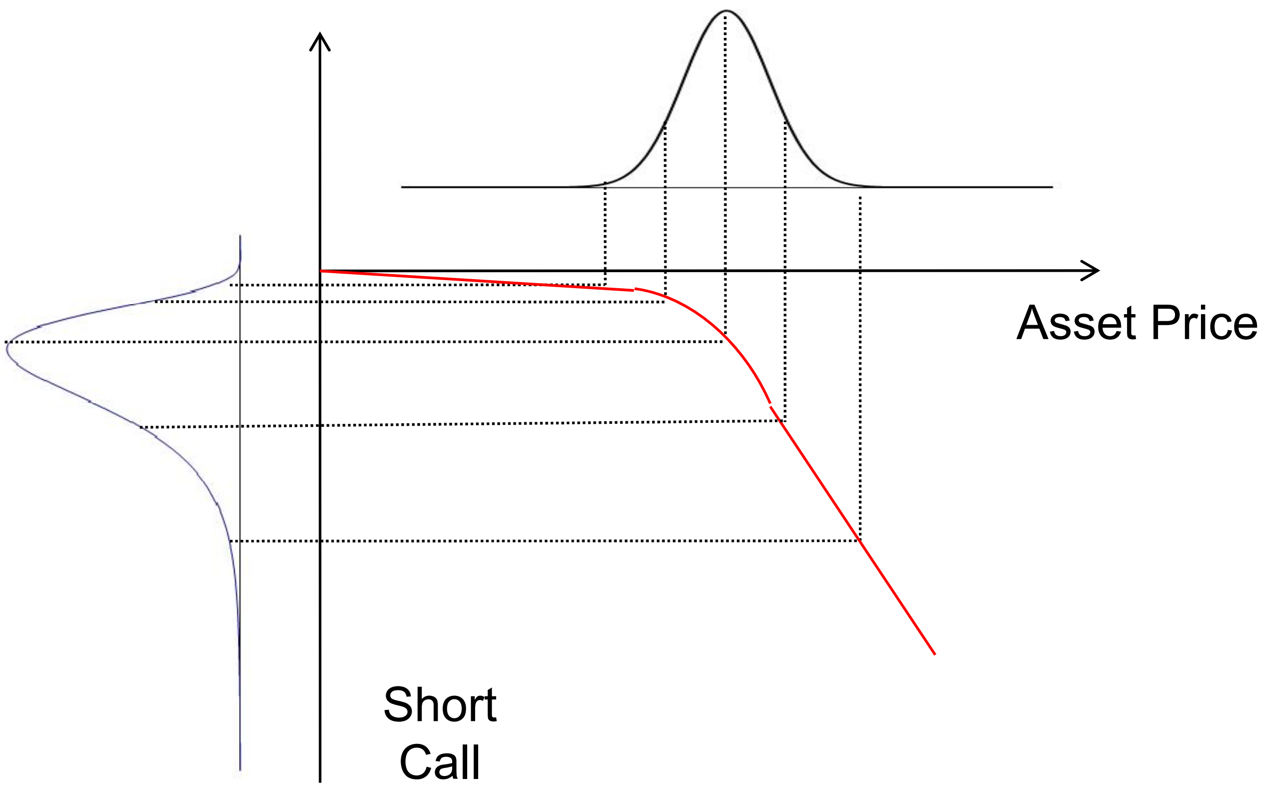
Negative Gamma

Translation of Asset Price Change to Price Change for Long Call

(Figure 14.2, page 313)



Translation of Asset Price Change to Price Change for Short Call
(Figure 14.3, page 314)



Quadratic Model
(page 314-316)

For a portfolio dependent on a single asset price it is approximately true that

$$\Delta P = \delta \Delta S + \frac{1}{2}\gamma(\Delta S)^2$$

so that

$$\Delta P = S\delta\, \Delta x + \frac{1}{2} S^2 \gamma\, (\Delta x)^2$$

Moments are

$$E(\Delta P) = 0.5 S^2 \gamma \sigma^2$$
$$E(\Delta P^2) = S^2 \delta^2 \sigma^2 + 0.75 S^4 \gamma^2 \sigma^4$$
$$E(\Delta P^3) = 4.5 S^4 \delta^2 \gamma \sigma^4 + 1.875 S^6 \gamma^3 \sigma^6$$

Quadratic Model continued

- When there are a small number of underlying market variable moments can be calculated analytically from the delta/gamma approximation
- The Cornish –Fisher expansion can then be used to convert moments to fractiles
- However when the number of market variables becomes large this is no longer feasible

Monte Carlo Simulation (page 316-317)

To calculate VaR using MC simulation we

- Value portfolio today
- Sample once from the multivariate distributions of the Δx_i
- Use the Δx_i to determine market variables at end of one day
- Revalue the portfolio at the end of day

Monte Carlo Simulation continued

- Calculate ΔP
- Repeat many times to build up a probability distribution for ΔP
- VaR is the appropriate fractile of the distribution times square root of N
- For example, with 1,000 trial the 1 percentile is the 10th worst case.

Speeding up Calculations with the Partial Simulation Approach

- Use the approximate delta/gamma relationship between ΔP and the Δx_i to calculate the change in value of the portfolio (as in historical simulation)

Alternative to Normal Distribution Assumption in Monte Carlo

- In a Monte Carlo simulation we can assume non-normal distributions for the x_i (e.g., a multivariate t-distribution)
- Can also use a Gaussian or other copula model in conjunction with empirical distributions

Model Building vs Historical Simulation

Model building approach can be used for investment portfolios where there are no derivatives, but it does not usually work when for portfolios where

- There are derivatives
- Positions are close to delta neutral