第五章 Heston 模型與解析解

第一節 Heston 模型介紹

一、資產價格行為

Steven Heston(1993)提出下面模型，

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^1 \quad \text{(3.1)}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^2 \quad \text{(3.2)}$$

$$dW_t^1 dW_t^2 = \rho \cdot dt \quad \text{(3.3)}$$

其中 $\{S_t\}_{t\geq 0}$ 表價格過程，$\{V_t\}_{t\geq 0}$ 表波動性過程。

以 P 測度表示此真實世界下的機率測量。

$\{W_t^1\}_{t\geq 0}$ 與 $\{W_t^2\}_{t\geq 0}$ 表真實世界中兩相關的布朗運動過程，相關係數為 ρ 。

$\{V_t\}_{t\geq 0}$ 為一平方根均數回覆過程，長期平均為 θ，回覆速率為 κ，σ 稱之為波動性之波動性。

μ 、ρ 、θ 、κ 、σ 均為常數。

第二節 價格函數

一、封閉解公式

對不發放股利的歐式買權，Heston 模型的封閉解為，

$$C(S_t,V_t,t,T)=S_tP_1-Ke^{-r(T-t)}P_2 \qquad (4.1)$$

$$P_j(x_t,V_t,T,K)=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left(\frac{e^{i\phi\ln(K)}f_j(x_t,V_t,T,\phi)}{i\phi}\right)d\phi \qquad (4.2)$$

$x_t=\ln(S_t)$，$\tau=T-t$，

$$f_j(x_t,V_t,\tau,\phi)=\exp\{C(\tau,\phi)+D(\tau,\phi)V_t+i\phi x_t\} \qquad (4.3)$$

$$C(\tau,\phi)=r\phi i\tau+\frac{a}{\sigma^2}\left[(b_j-\rho\sigma\phi i+d)\tau-2\ln\left(\frac{1-ge^{d\tau}}{1-g}\right)\right] \qquad (4.4)$$

$$D(\tau,\phi)=\frac{b_j-\rho\sigma\phi i}{\sigma^2}\left(\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right) \qquad (4.5)$$

$$g=\frac{b_j-\rho\sigma\phi i+d}{b_j-\rho\sigma\phi i-d} \qquad (4.6)$$

$$d=\sqrt{(\rho\sigma\phi i-b_j)-\sigma^2(2u_j\phi i-\phi^2)} \qquad (4.7)$$

$j=1,2$，其中

$u_1=\frac{1}{2}$，$u_2=-\frac{1}{2}$，$a=k^*\theta^*$，$b_1=k^*-\rho\sigma$，$b_2=k^*$

第三節 複數運算

前面(4.2)~(4.7)式中，涉及複數的運算，下面簡單摘要其規則。

$z = x + iy$ ，$i = \sqrt{-1}$ ，$\mathrm{Re}(z) = x$ ，$\mathrm{Im}(z) = y$ 。

$z = (x, y)$

$z_1 = x_1 + iy_1 = (x_1, y_1)$ ，$z_2 = x_2 + iy_2 = (x_2, y_2)$

一、四則運算

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) = (x_1 + x_2, y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) = (x_1 - x_2, y_1 - y_2)$$

$$z_1 \times z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$$

$$z_1 / z_2 = \frac{(x_1 + iy_1)}{(x_2 + iy_2)} \times \frac{(x_2 - iy_2)}{(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} - i\frac{(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

二、極座標、冪次與根

$z = x + iy = r(\cos\theta + i\sin\theta)$, $r = \sqrt{x^2 + y^2}$, $\theta = \arctan\dfrac{y}{x} = \arg z$,

$x = r\cos\theta$ ，$y = r\sin\theta$ ，

$\bar{z} = x - iy$ ，$|z| = \sqrt{z\bar{z}} = r$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\dfrac{\theta + 2k\pi}{n}\right) + i\sin\left(\dfrac{\theta + 2k\pi}{n}\right)\right)$ ，$k = 0,1,\ldots,n-1$

三、指數函數、尤拉公式與對數函數

$z = x + iy = r(\cos\theta + i\sin\theta)$, $r = \sqrt{x^2 + y^2}$, $\theta = \arctan\frac{y}{x} = \arg z$,

$$\exp(z) = \exp(x+iy) = \exp(x)\cdot\exp(iy) = \exp(x)\cdot(\cos y + i\sin y)$$

$$\exp(i\theta) = \cos\theta + i\sin\theta$$

$$\ln(z) = \ln(x+iy) = \ln(r(\cos\theta + i\sin\theta)) = \ln(r) + i\theta$$

第四節 數值積分

一、Gauss-Laguerre 求值法

(4.2)式的計算涉及半無限區間的積分，可使用 Gauss-Laguerre 法計算，以加速計算效率，

令積分運算式如下式，

$$G = \int_0^{\infty} f(x)dx$$

令 n 點 Gauss-Laguerre 求值公式為

$$G = \int_0^{\infty} f(x)dx = \sum_{i=0}^{n-1} \lambda_i f(x_i) \qquad \text{.....(4.8)}$$

其中 x_i 為下面 n 階 Laguerre 多項式的 n 個零點，λ_i 為求積係數。

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}) \text{，} 0 \le x \le +\infty \qquad \text{.....(4.9)}$$

當 n=5，5 階 Gauss-Laguerre 求積公式的結點為，

$x_0 = 0.26355990$，$x_1 = 1.41340290$，$x_2 = 3.59642600$，$x_3 = 7.08580990$，$x_4 = 12.64080000$。

相對應的求積系數為，

$\lambda_0 = 0.6790941054$，$\lambda_1 = 1.638487956$，$\lambda_2 = 2.769426772$，$\lambda_3 = 4.315944000$，$\lambda_4 = 7.104896230$。

第五節 特徵函數

(4.2)積分式中 Integrand 對 Phi 的作圖。

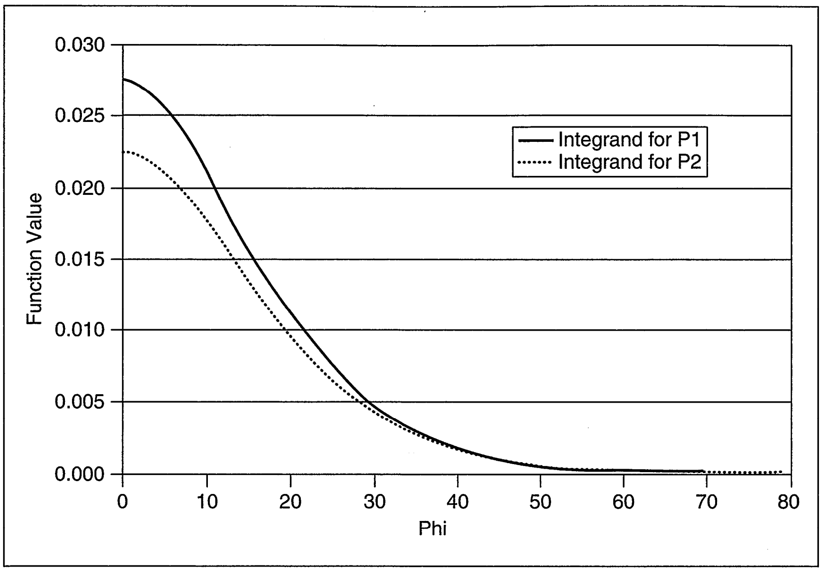

FIGURE 5.4 Convergence of Functions Used in Integration

(4.1)式 Call 價格與 Black-Scholes 計算之 Call 價格的差距，H_C-BS_C。

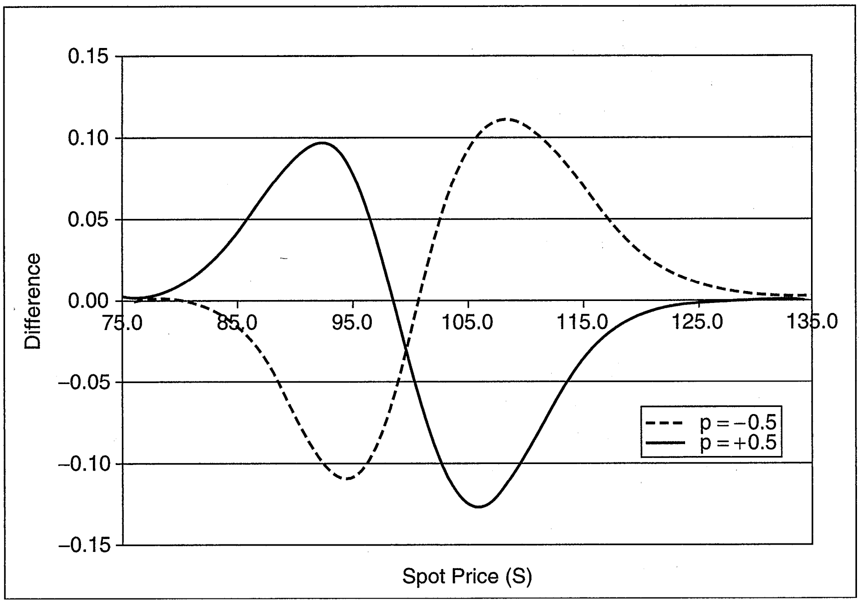

FIGURE 5.8 Plots of Call Price Differences with Varying Correlation

第六節 R 語言與 C#的實作

一、R 語言實作

使用 R 語言內建的函數與功能來撰寫 Heston 模型的解析解相對容易，

```
setwd("D:\\FEMC\\RCode")
source("HestonPrice.R")
source("HestonProb.R")
# Option features
S = 100;          # Spot price
K = 100;         # Strike price
tau = 0.5;       # Maturity
r = 0.03;        # Risk free rate
q = 0.00;        # Dividend yield
kappa = 5;       # Heston parameter : reversion speed
sigma = 0.5;     # Heston parameter : volatility of variance
rho = -0.8;      # Heston parameter : correlation
theta = 0.05;    # Heston parameter : reversion level
v0 = 0.05;       # Heston parameter : initial variance
lambda = 0;      # Heston parameter : risk preference
# Expression for the characteristic function
Trap = 0;        # 0 = Original Heston formulation
               # 1 = Albrecher et al formulation
# Integration range
Lphi = 0.000001; # Lower limit
Uphi = 50;       # Upper limit
num = 100;       # subdivision num

# Obtain the Heston put and call
HPut = HestonPrice('P', kappa, theta, lambda, rho, sigma, tau, K, S, r, q, v0,
    Trap, Lphi, Uphi, num);
HCall = HestonPrice('C', kappa, theta, lambda, rho, sigma, tau, K, S, r, q, v0,
    Trap, Lphi, Uphi, num);

# Output the result
print(HPut);
print(HCall);
```

程式列表

```
# Heston (1993) price of a European option.
# Uses the original formulation by Heston
# Heston parameters:
#    kappa  = volatility mean reversion speed parameter
#    theta  = volatility mean reversion level parameter
#    lambda = risk parameter
#    rho    = correlation between two Brownian motions
#    sigma  = volatility of variance
#    v0     = initial variance
# Option features.
#    PutCall = 'C'all or 'P'ut
#    K = strike price
#    S = spot price
#    r = risk free rate
#    q = dividend yield
#    T = maturity
# Integration features
#    L = lower limit
#    U = upper limit
#    num = integration increment

HestonPrice = function(PutCall, kappa, theta, lambda, rho, sigma, T, K, S, r,
q, v0, trap, Lphi, Uphi, num)
{
    # The integrals
    I1 = HestonProb(Lphi, Uphi, num, kappa, theta, lambda, rho, sigma,
       T, K, S, r, q, v0, 1, trap);
    I2 = HestonProb(Lphi, Uphi, num, kappa, theta, lambda, rho, sigma,
       T, K, S, r, q, v0, 2, trap);
    # The probabilities P1 and P2
    P1 = 1/2 + I1/pi;
    P2 = 1/2 + I2/pi;
    # The call price
    HestonC = S*exp(-q*T)*P1 - K*exp(-r*T)*P2;
    # Output the option price
    if (PutCall == 'C')
    {
        y = HestonC;
    }
    else
    {
      # The put price by put-call parity
      HestonP = HestonC - S*exp(-q*T) + K*exp(-r*T);
    y = HestonP;
    }
    return(y)
}
```
程式列表

```
# Returns the risk neutral probabilities P1 and P2.
# integrand = integrand of Probability
# phi = integration variable
# Integration features
#    Lphi = lower limit
#    Uphi = upper limit
# Pnum = 1 or 2 (for the probabilities)
# Heston parameters:
#    kappa  = volatility mean reversion speed parameter
#    theta  = volatility mean reversion level parameter
#    lambda = risk parameter
#    rho    = correlation between two Brownian motions
#    sigma  = volatility of variance
#    v0     = initial variance
# Option features.
#    PutCall = 'C'all or 'P'ut
#    K = strike price
#    S = spot price
#    r = risk free rate
#    q = dividend yield
#    Trap = 1 "Little Trap" formulation
#           0  Original Heston formulation

HestonProb = function(Lphi, Uphi, num, kappa, theta, lambda, rho,
  sigma, tau, K, S, r, q,   v0, Pnum, Trap)
{
    x = log(S);
    a = kappa * theta;
    if (Pnum == 1)
    {
        u = 0.5;
        b = kappa + lambda - rho * sigma;
    }
    else
    {
        u = -0.5;
        b = kappa + lambda;
    }

  integrand = function(phi)
  {
    Zi = complex(0, 1);

    d = sqrt((rho*sigma*phi*Zi - b)^2 - sigma^2*(2*u*phi*Zi - phi^2));
    g = (b - rho*sigma*phi*Zi + d) / (b - rho*sigma*phi*Zi - d);

    if (Trap==1)    # "Little Heston Trap" formulation
    {
      c = 1/g;
      D = (b - rho*sigma*Zi*phi - d)/sigma^2*((1-exp(-d*tau))
        /(1-c*exp(-d*tau)));
      G = (1 - c*exp(-d*tau))/(1-c);
      C = (r-q)*Zi*phi*tau + a/sigma^2*((b - rho*sigma*Zi*phi - d)*tau
        - 2*log(G));
    }
```

```
    else
    {
      if (Trap==0)  # Original Heston formulation.
      {
        G = (1 - g*exp(d*tau))/(1-g);
        C = (r-q)*Zi*phi*tau + a/sigma^2*((b - rho*sigma*Zi*phi + d)*tau
          - 2*log(G));
        D = (b - rho*sigma*Zi*phi + d)/sigma^2*((1-exp(d*tau))
          /(1-g*exp(d*tau)));
      }
    }

    # The characteristic function.
    f = exp(C + D*v0 + Zi*phi*x);

    # Return the real part of the integrand.
    integ = Re(exp(-Zi*phi*log(K))*f/Zi/phi);

    return(integ);
  }

  Total = integrate(f=integrand,lower=Lphi, upper=Uphi, subdivisions=num);
  # Get value of the integrate function
  ans = Total$value;

    return(ans);
}
```
程式列表

二、C#語言實作

```
class HestonPriceGaussLaguerre
{
    static void Main(string[] args)
    {
        // 32-point Gauss-Laguerre Abscissas and weights
        double[] x = new Double[32];
        double[] w = new Double[32];
        using(TextReader reader = File.OpenText("../../GaussLaguerre32.txt"))
        {
            for(int k=0;k<=31;k++)
            {
                string text = reader.ReadLine();
                string[] bits = text.Split(' ');
                x[k] = double.Parse(bits[0]);
                w[k] = double.Parse(bits[1]);
            }
        }
        // Heston parameters
        HParam param = new HParam();
        param.kappa = 1.5;
        param.theta = 0.04;
        param.sigma = 0.3;
        param.v0 = 0.05412;
        param.rho = -0.9;
        param.lambda = 0.0;

        // Option settings
        OpSet settings = new OpSet();
        settings.S = 101.52;
        settings.K = 100.0;
        settings.T = 0.15;
```

```
        settings.r = 0.02;
        settings.q = 0.0;
        settings.PutCall = "C";
        settings.trap = 1;

        // The Heston price
        HestonPrice HP = new HestonPrice();
        double Price = HP.HestonPriceGaussLaguerre(param,settings,x,w);
        Console.WriteLine("Heston price using 32-point Gauss Laguerre");
        Console.WriteLine("------------------------------------------ ");
        Console.WriteLine("Option Flavor =  {0,0:F5}",settings.PutCall);
        Console.WriteLine("Strike Price  =  {0,0:0}" ,settings.K);
        Console.WriteLine("Maturity      =  {0,0:F2}",settings.T);
        Console.WriteLine("Price         =  {0,0:F4}",Price);
        Console.WriteLine("------------------------------------------ ");
        Console.WriteLine(" ");
    }
}
```