MIPT Computational Mathematics 2024

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1 FIRST TASK (1C)

1.1 The order of approximation

Задача 1. Для численного решения уравнения переноса

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

методом неопределенных коэффициентов построить схему **максимального** порядка аппроксмации (порядок указать) по значениями в узлах сетки:

a) (t_{n+1}, x_k), (t_n, x_k), (t_n, x_{k-1}), (t_n, x_{k-2})

b) (t_{n+1}, x_k), (t_n, x_k), (t_{n+1}, x_{k-1}), (t_{n+1}, x_{k-2})

c) (t_{n+1}, x_k), (t_n, x_k), (t_{n+1}, x_{k-1}), (t_{n+1}, x_{k+1})

d) (t_{n+1}, x_k), (t_n, x_k), (t_n, x_{k-1}), (t_{n+1}, x_{k-1})

e) (t_{n+1}, x_k), (t_n, x_k), (t_{n+1}, x_{k-1}), (t_n, x_{k+1})

Исследовать полученный метод на устойчивость.

To investigate the scheme and find the maximum order:

$$\alpha f_k^n + \beta f_{k-1}^{n+1} + \gamma f_k^{n+1} + \theta f_{k+1}^{n+1} = 0$$

using the Taylor series expansion, we obtain:

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^3 u}{\partial t^3} = -c^3\frac{\partial^3 u}{\partial x^3}$$

to obtain the maximum order of approximation, we need to make the coefficients up to 3 orders of magnitude zero:

$$\begin{cases} \alpha + \beta + \gamma + \theta = 0 \\ c^2(\beta + \gamma + \theta)\frac{\partial t^2}{2} + c(\beta - \theta) * \partial x \partial t + (\beta + \theta)\frac{\partial x^2}{2} = 0 \\ c^3(\beta + \gamma + \theta)\frac{\partial t^3}{6} + c^2(\beta - \theta)\frac{\partial x \partial t^2}{2} + c(\beta + \theta)\frac{\partial x^2 \partial t}{2} + (\beta - \theta)\frac{\partial x^3}{6} = 1 \end{cases}$$

then the approximation scheme, which proceeds from the solution of the system of equations given above:

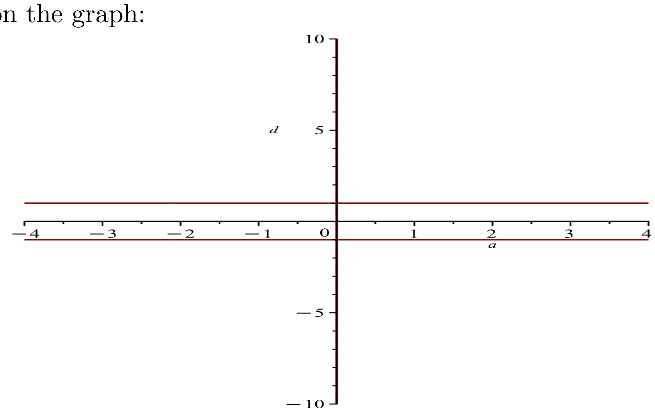
$$\frac{f_k^{n+1} - f_k^n}{dt} + c\frac{f_{k+1}^{n+1} - f_{k-1}^{n+1}}{2dx} + c^2\frac{f_{k-1}^{n+1} - 2f_k^{n+1} + 2f_{k+1}^{n+1}}{2}\frac{dt}{dx^2} = 0(O(x^2+t))$$

1.2 Stability of the difference scheme

Let's denote $f_k^n = \lambda^n e^{iak}$ and we will substitute it into the difference scheme, after that, we will try to express the variable λ using the popular identity $e^{ix} = cos(x) + isin(x)$:

$$\lambda = \frac{1}{cos(a)(\frac{\partial t}{\partial x})^2 + isin(a)(\frac{\partial t}{\partial x}) - (\frac{\partial t}{\partial x})^2 + 1}$$

we can understand that the denominator takes the desired values, which we can see on the graph:



for $|\frac{\partial t}{\partial x}| >= 1$, for any $a|\lambda| < 1$. That is, the scheme is **stable** for $|\frac{\partial t}{\partial x}| >= 1$ **conditionally stable**.

2 SECOND TASK (7.22)

$$\frac{f_k^{n+1} - f_k^n}{dt} = (1-\theta)\frac{f_{k-1}^{n+1} - 2f_k^{n+1} + f_{k+1}^{n+1}}{dx^2} + \theta\frac{f_{k-1}^n - 2f_k^n + f_{k+1}^n}{dx^2}$$

in order to investigate this stability problem, we can apply the same reasoning as in the previous problem, let $f_k^n = exp(iak)\lambda^n$. let's substitute it into the difference scheme, and express our variable and explore the resulting equality:

$$\lambda = \frac{1 + 2\theta\left(\frac{\partial t}{\partial x^2}\right)(cos(a) - 1)}{1 + 2\left(\frac{\partial t}{\partial x^2}\right)(\theta - 1)(cos(a) - 1)}$$

having studied our derivative, and having understood how the function behaves at certain intervals, we can understand that the system shown below does not have any solutions (Wolfram Alpha did not give adequate results)

$$\begin{cases} -1 < \frac{1+2\left(\frac{\partial t}{\partial x^2}\right)(\theta-1a0}{2\left(\frac{\partial t}{\partial x^2}\right)(\theta-1)} < 1 \\ |\frac{4\theta\left(\frac{\partial t}{\partial x^2}\right)-1}{4\theta\left(\frac{\partial t}{\partial x^2}\right)-4\left(\frac{\partial t}{\partial x^2}\right)-1}| < 1 \end{cases}$$

from these arguments and some calculations, we can conclude that the difference scheme given in the condition is **unstable**

3 THIRD TASK (7.28)

$$\left(\frac{\partial u}{\partial t}\right) = \left(\frac{\partial^2 u}{\partial x^2}\right) + f(x,t)$$

$$\frac{u_m^{n+1} - u_m^n}{dt} = \frac{u_{m+1}^{n+1} - 2 * u_m^{n+1} + u_{m-1}^{n+1}}{2dx^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{2dx^2} + f_m^{n+1/2}$$

to solve this problem, let's proceed in a simple way, decompose our difference scheme into a Taylor series and see in which approximation it converges

to correctly decompose into a Taylor series, we differentiate our partial differential equation in time:

$$\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial t}\right) = \left(\frac{\partial}{\partial t}\right)\left(\left(\frac{\partial^2 u}{\partial x^2}\right) + f(x,t)\right) \to \left(\frac{\partial^2 u}{\partial t^2}\right) = \left(\frac{\partial^3 u}{\partial t \partial^2 x}\right) + \left(\frac{\partial f}{\partial t}\right)$$

decompose the Taylor series using Wolfram Alpha

$$\left(\frac{\partial^3 u}{\partial t^3}\right)\frac{dt^2}{6} - \left(\frac{\partial^4 u}{\partial x^4}\right)\frac{dx^2}{12} - \left(\frac{\partial^4 u}{\partial t^2 \partial x^2}\right)\frac{dt^2}{4} + \left(\frac{\partial^4 u}{\partial t^4}\right)\frac{dt^3}{24} - \left(\frac{\partial^4 u}{\partial x^4 \partial t}\right)\frac{dx^2 dt}{24} -$$

$$-\left(\frac{\partial^4 u}{\partial x^2 \partial t^3}\right)\frac{dt^3}{12} - \left(\frac{\partial^2 f}{\partial x^2}\right)\frac{dt^2}{8} - \left(\frac{\partial^3 f}{\partial t^3}\right)\frac{dt^3}{48} = 0(O(x^2 + t^2))$$

the difference scheme **converges**