MIPT Computational Mathematics 2024

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1 FIRST TASK (1C)

1.1 The order of approximation

Задача 1. Для численного решения уравнения переноса

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

методом неопределенных коэффициентов построить схему **максимального** порядка аппроксмации (порядок указать) по значениями в узлах сетки:

a) $(t_{n+1}, x_k), (t_n, x_k), (t_n, x_{k-1}), (t_n, x_{k-2})$

b) $(t_{n+1}, x_k), (t_n, x_k), (t_{n+1}, x_{k-1}), (t_{n+1}, x_{k-2})$

c) $(t_{n+1}, x_k), (t_n, x_k), (t_{n+1}, x_{k-1}), (t_{n+1}, x_{k+1})$

d) $(t_{n+1}, x_k), (t_n, x_k), (t_n, x_{k-1}), (t_{n+1}, x_{k-1})$

e) $(t_{n+1}, x_k), (t_n, x_k), (t_{n+1}, x_{k-1}), (t_n, x_{k+1})$

Исследовать полученный метод на устойчивость.

To investigate the scheme and find the maximum order:

$$\alpha f_k^n + \beta f_{k-1}^{n+1} + \gamma f_k^{n+1} + \theta f_{k+1}^{n+1} = 0$$

using the Taylor series expansion, we obtain:

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^3 u}{\partial t^3} = -c^3\frac{\partial^3 u}{\partial x^3}$$

to obtain the maximum order of approximation, we need to make the coefficients up to 3 orders of magnitude zero:

$$\begin{cases} \alpha + \beta + \gamma + \theta = 0 \\ c^2(\beta+\gamma+\theta)\frac{\partial t^2}{2} + c(\beta-\theta) * \partial x \partial t + (\beta+\theta)\frac{\partial x^2}{2} = 0 \\ c^3(\beta+\gamma+\theta)\frac{\partial t^3}{6} + c^2(\beta-\theta)\frac{\partial x \partial t^2}{2} + c(\beta+\theta)\frac{\partial x^2 \partial t}{2} + (\beta-\theta)\frac{\partial x^3}{6} = 1 \end{cases}$$

then the approximation scheme, which proceeds from the solution of the system of equations given above:

$$\frac{f_k^{n+1} - f_k^n}{dt} + c\frac{f_{k+1}^{n+1} - f_{k-1}^{n+1}}{2dx} + c^2\frac{f_{k-1}^{n+1} - 2f_k^{n+1} + 2f_{k+1}^{n+1}}{2}\frac{dt}{dx^2} = 0$$

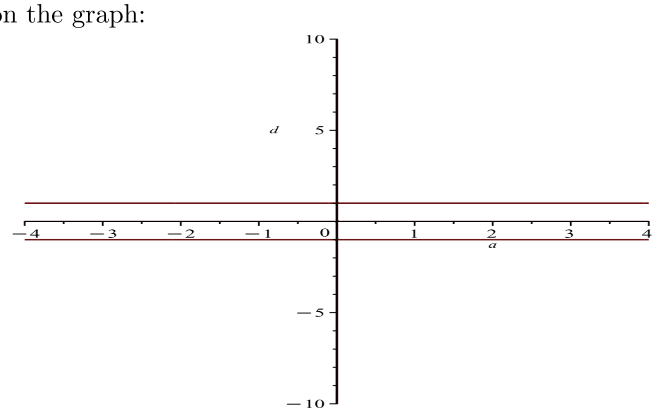
where can we get the order of our approximation from $\rightarrow O(x+t)$

1.2 Stability of the difference scheme

Let's denote $f_k^n = \lambda^n e^{iak}$ and we will substitute it into the difference scheme, after that, we will try to express the variable λ using the popular identity $e^{ix} = cos(x) + isin(x)$:

$$\lambda = \frac{1}{cos(a)(\frac{\partial t}{\partial x})^2 + isin(a)(\frac{\partial t}{\partial x}) - (\frac{\partial t}{\partial x})^2 + 1}$$

we can understand that the denominator takes the desired values, which we can see on the graph:



for $|\frac{\partial t}{\partial x}| >= 1$, for any $a|\lambda| < 1$. That is, the scheme is **stable** for $|\frac{\partial t}{\partial x}| >= 1$ **conditionally stable**.

2 SECOND TASK (7.22)

$$\frac{f_k^{n+1} - f_k^n}{dt} = (1-\theta)\frac{f_{k-1}^{n+1} - 2f_k^{n+1} + f_{k+1}^{n+1}}{dx^2} + \theta\frac{f_{k-1}^n - 2f_k^n + f_{k+1}^n}{dx^2}$$

in order to investigate this stability problem, we can apply the same reasoning as in the previous problem, let $f_k^n = exp(iak)\lambda^n$. let's substitute it into the difference scheme, and express our variable and explore the resulting equality:

$$\lambda = \frac{1 + 2\theta\left(\frac{\partial t}{\partial x^2}\right)(cos(a) - 1)}{1 + 2\left(\frac{\partial t}{\partial x^2}\right)(\theta - 1)(cos(a) - 1)}$$

It can be seen that $\lambda \equiv \lambda(a)$ having studied our derivative $\forall a \rightarrow \lambda'(a) > 0$, therefore, we can conclude about the denominator of the first expression $2\left(\frac{\partial t}{\partial x^2}\right)(\theta-1) \neq 1$ where can we get a system for which solutions need to be found $\rightarrow$

$$\begin{cases} -1 > \frac{1+2\left(\frac{\partial t}{\partial x^2}\right)(\theta-1)}{2\left(\frac{\partial t}{\partial x^2}\right)(\theta-1)} \\ 1 < \frac{1+2\left(\frac{\partial t}{\partial x^2}\right)(\theta-1)}{2\left(\frac{\partial t}{\partial x^2}\right)(\theta-1)} \\ \left|\frac{4\theta\left(\frac{\partial t}{\partial x^2}\right)-1}{4\theta\left(\frac{\partial t}{\partial x^2}\right)-4\left(\frac{\partial t}{\partial x^2}\right)-1}\right| < 1 \end{cases}$$

after investigating this system of inequalities, we will get that this system has a solution for $\frac{\partial t}{\partial x^2} > 0 \hookleftarrow$ this scheme is **conditionally stable** at values of $\frac{\partial t}{\partial x^2} > 0$

3 THIRD TASK (7.28)

$$\left(\frac{\partial u}{\partial t}\right) = \left(\frac{\partial^2 u}{\partial x^2}\right) + f(x,t)$$

$$\frac{u_m^{n+1} - u_m^n}{dt} = \frac{u_{m+1}^{n+1} - 2 * u_m^{n+1} + u_{m-1}^{n+1}}{2dx^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{2dx^2} + f_m^{n+1/2}$$

to solve this problem, let's proceed in a simple way, decompose our difference scheme into a Taylor series and see in which approximation it converges

to correctly decompose into a Taylor series, we differentiate our partial differential equation in time:

$$\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial t}\right) = \left(\frac{\partial}{\partial t}\right)\left(\left(\frac{\partial^2 u}{\partial x^2}\right) + f(x,t)\right) \rightarrow \left(\frac{\partial^2 u}{\partial t^2}\right) = \left(\frac{\partial^3 u}{\partial t \partial^2 x}\right) + \left(\frac{\partial f}{\partial t}\right)$$

decompose the Taylor series using Wolfram Alpha

$$\left(\frac{\partial^3 u}{\partial t^3}\right)\frac{dt^2}{6} - \left(\frac{\partial^4 u}{\partial x^4}\right)\frac{dx^2}{12} - \left(\frac{\partial^4 u}{\partial t^2 \partial x^2}\right)\frac{dt^2}{4} + \left(\frac{\partial^4 u}{\partial t^4}\right)\frac{dt^3}{24} - \left(\frac{\partial^4 u}{\partial x^4 \partial t}\right)\frac{dx^2 dt}{24} -$$

$$-\left(\frac{\partial^4 u}{\partial x^2 \partial t^3}\right)\frac{dt^3}{12} - \left(\frac{\partial^2 f}{\partial x^2}\right)\frac{dt^2}{8} - \left(\frac{\partial^3 f}{\partial t^3}\right)\frac{dt^3}{48} = 0(O(x^2+t^2))$$

the difference scheme **converges**