Polytropic model of non-relativistic White Dwarfs

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ABSTRACT

Aims. Developing a model for non-relativistic White Dwarfs following the numerical solution of the Lane-Emden equation. Then, comparing the results with analytical solutions and finally obtaining the mass and radius of these Dwarfs.
Methods. First, we will solve numerically the Lane-Emden equation using the Runge-Kutta method. Once solved we will compare with the analytical solution for n=0 and n=1. Finally, to apply these results to non-relativistic White Dwarfs we will be assuming their central density and compositions.
Results. As results, we present the mass and radius of different central mass white dwarfs, for $\rho_{c,1} = 10^9 kgm^{-3}$ we obtained $R_1 = 0.0169 R_\odot$; $M_1 = 0.493 M_\odot$ and for $\rho_{c,2} = 5\times10^9 kgm^{-3}$ we obtained $R_2 = 0.0129 R_\odot$; $M_2 = 1.103 M_\odot$

Key words. Lane-Emden – Runge-Kutta – White Dwarfs

1. The Lane-Emden equation

Even the most simple stellar models have to describe correctly the internal structure of the star, knowing its density, pressure and temperature profile, as well as its emission. All these can be achieved by solving a system of equations which has no analytical solution.

We will be focusing in the case of a star which pressure doesn't depend on the temperature (electron degenerated gas) and a constant median mass per particle. Doing these approximations we can isolate two of these system equations and solve them separately. These two are the hydrostatic equilibrium and the continuity equations, presented below:

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2}$$

$$\frac{dm}{dr} = 4\pi r^2 \cdot \rho$$

By using some changes of variable and applying some equivalences we can transform this equation to the known equation of Lane-Emden (expression 18 in the notes):

$$\frac{2}{\xi}\cdot\frac{d\theta}{d\xi} + \frac{d^2\theta}{d\xi^2} = -\theta^n$$

Where $\theta = \left(\frac{\rho}{\rho_c}\right)^{1/n}$ is the quotient between the density and the central density and $\xi = r/\alpha$ is a dimensionless radial variable that only depends on the radius and α, which follows the expression:

$$\alpha^2 = \frac{(n+1)K}{4\pi G\cdot\rho_c^{\frac{n-1}{n}}}$$

Being K and n both positive real numbers that come from the polytrope equation of state (n is also known as the polytropic index).

1.1. Numerical solution of the Lane-Emden equation

To solve this equation, as it is only possible to solve it analytically for n=0 n=1 and n=5, we will have to use a numerical solving method. In our case, we will be using Runge-Kutta in his fourth order of approximation. To do this we need the change of variable: $f_1 = \frac{d\theta}{d\xi}$. Doing this, our equation system stays as:

$$\begin{cases} \frac{df_1}{d\xi} = -\frac{2}{\xi}\cdot f_1 - \theta^n \\ \\ \frac{d\theta}{d\xi} = f_1 \end{cases}$$

We will use $\theta = 1$ and $y = 0$ in $\xi = 0$ as initial conditions so we can apply the Runge-Kutta method to the system.

1.2. Plotting Lane-Emden equation

Using a polytropic index from 1 to 5 we can graph the solution for the equation at different values by plotting θ against ξ. This is shown in figure 1.

As we can see in the figure, having in mind that θ is a density indicator, the different stars tend to be more diffuse as their polytropic index n grows, arriving to a limit in $n = 5$, which gives us an asymptotic solution that clearly can't be identified as a physical solution. We can also see that this figure corresponds perfectly with the one we saw in the notes.

We can also plot the dependency of ($\theta^n = \rho/\rho_c$) with the radius. To do this we will express the radius in the horizontal axis as ξ/ξ_1, where ξ_1 is the value of ξ when θ tends to 0 for each polytropic index n. This plot is shown in figure 2.

We can clearly see how the different values of the polytropic index change the density profile of the stars. This figure confirms that for lower polytropic index values the star has a wider profile, where as for higher values the star has a much denser

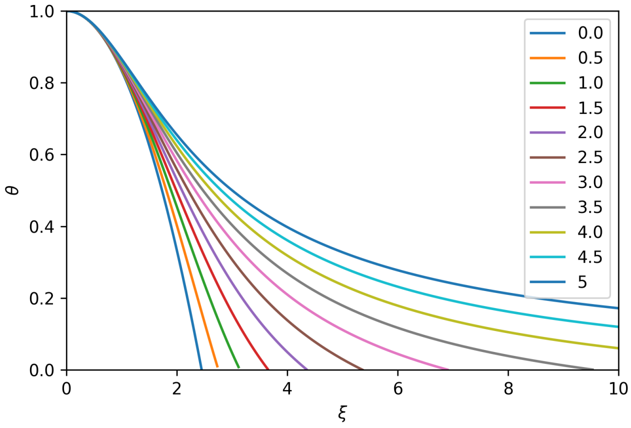

Fig. 1. Lane-Emden equation numerical solutions for different polytropic index values.

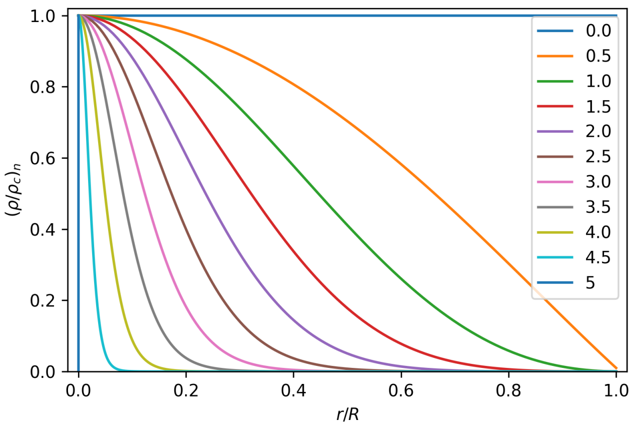

Fig. 2. Density to central density ratio behavior with the radius depending on the polytropic index.

central region which shows a much sharper profile.

2. Comparing analytical and numerical results

As we pointed out on the first section of the work, the Lane-Emden equation can be analytically solved for the polytropic indexes $n = 1, 2, 5$. To do this we will first re-write the expression for the Lane-Emden equation like so:

$$\frac{2}{\xi} \cdot \frac{d\theta}{d\xi} + \frac{d^2\theta}{d\xi^2} = -\theta^n \quad \rightarrow \quad \frac{1}{\xi^2} \cdot \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n$$

Once we have this expression we can focus on each of the cases. We can start with $n = 0$. For this case the solution can be achieved by integration:

$$\frac{1}{\xi^2} \cdot \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -1 \quad \rightarrow \quad d\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\xi^2 d\xi$$

$$\xi^2 \frac{d\theta}{d\xi} = -\int \xi^2 d\xi = -\frac{\xi^3}{3} \quad \rightarrow \quad \boxed{\theta = -\frac{\xi^2}{6}}$$

For n=1 and n=5 the process is more difficult, so we will be giving the solutions directly:

$$n = 1: \qquad \theta = \frac{sin(\xi)}{\xi}$$

$$n = 5: \qquad \theta = \frac{1}{\sqrt{1 + \xi^2/3}}$$

Now that we have the analytical expressions we can plot them side to side to the numerical solutions we obtained in last section. This plots are presented in figure 3.

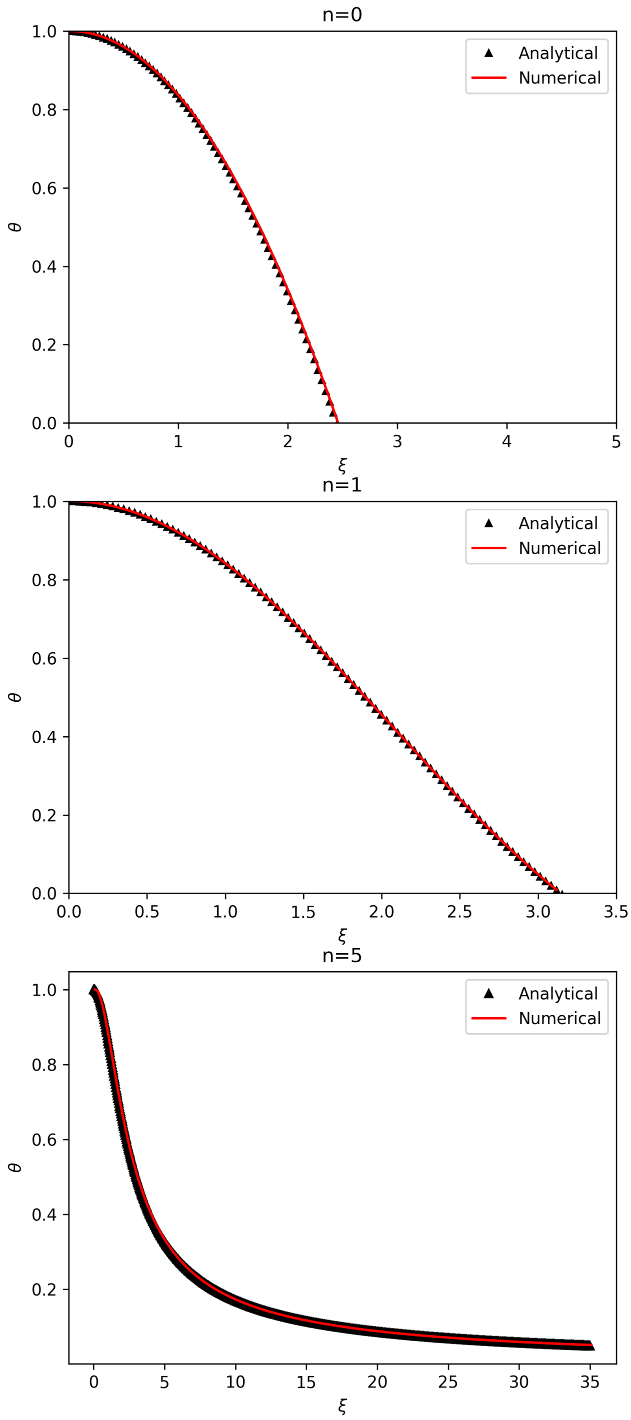

Fig. 3. Comparison between the analytical and numerical solutions of the Lane-Emden equation for different polytropic index values.

We can clearly see how both numerical and analytical curves fit perfectly. This confirms a good analysis.

3. Calculating D_n, M_n, R_n, and B_n

In this section we will calculate the following constants for each value of the polytropic index.

In first place, D_n is the ratio between the central density and the median of the density for all the radial profile. Its expression is the following:

$$D_n = \frac{\rho_c}{\overline{\rho}} = -\left[\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\right]^{-1}$$

M_n is defined as:

$$M_n = -\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}$$

B_n comes up when arranging all the dependencies in n in the relation between the central density and the central pressure:

$$B_n = \frac{(3D_n)^{\frac{3-n}{3n}}}{(n+1)M_n^{\frac{n-1}{n}}R_n^{\frac{3-n}{n}}}$$

Finally, R_n is the re-scalated radius of the star, in which the star density is 0, that is, the radius at which the star ends:

$$R = \alpha R_n \quad \rightarrow \quad R_n = \frac{R}{\alpha}$$

All these values are calculated for every polytropic index. These are presented in Table 1.

Table 1. Different constant values for each polytropic index chosen.

n	D_n	M_n	B_n	R_n
0.0	1.00	4.71	0.345	2.42
0.5	1.80	3.78	0.275	2.73
1.0	3.22	3.14	0.233	3.12
1.5	5.94	2.71	0.206	3.64
2.0	11.36	2.41	0.186	4.34
2.5	23.49	2.19	0.170	5.36
3.0	6.90	2.02	0.157	6.90
3.5	152.66	1.89	0.145	9.53
4.0	630.48	1.80	0.135	15.03
4.5	6685.85	1.73	0.126	32.61
5.0	∞	1.71	0.116	∞

The results in the table for the radius R_n follow the same behaviour as seen in the last section, as the polytropic index increases the stars become more diffuse, which extends for a larger star, increasing its radius. This behaviour is also coherent with D_n, which also increases with n, having a minimum in $D_n = 1$ for $n = 0$, this means for this polytropic index value the star has $\overline{\rho} = \rho_c$.

All this agrees with the fact that B_n is decreasing with n. As we go to higher polytropic indexes, the central pressure decreases, which pairs with a more diffuse and extended star, as we mentioned above.

4. Radius and mass of non-relativistic White Dwarfs

For this section of the work we will be calculating the radius and mass of a white dwarf. To do this we will assume the following data:

$$n = 1.5$$

$$\mu_e = 2$$

$$\rho_{c,1} = 10^9 kgm^{-3} \quad ; \quad \rho_{c,2} = 5\times 10^9 kgm^{-3}$$

For the radius we will be using the same expression shown in the last section. The expression for α is the same shown in section 1 of the work, and for R_n we will use the one obtained in last section. First, the values obtained for α are:

$$\alpha^2 = \frac{(n+1)K}{4\pi G\cdot \rho_c^{\frac{n-1}{n}}} \rightarrow \begin{cases}\alpha\,(\rho_{c,1}) = 2.34\times 10^6\\ \alpha\,(\rho_{c,2}) = 3.07\times 10^6\end{cases}$$

It is important to mention that for K we used the expression $K = K_1'/\mu_e^{5/3}$ with $K_1' = 10^7$.

Now with these values we can easily calculate the radius for each central density. To do so we will make a vector for each and select the last value of that vector, corresponding to the largest radius:

$$\boxed{R_1 = 0.0169 R_\odot}$$

$$\boxed{R_2 = 0.0129 R_\odot}$$

For the mass of the stars we will be using the following expression:

$$M = -4\pi\alpha^3\rho_c\,\xi_1^2\left.\frac{d\theta}{d\xi}\right|_{\xi_1}$$

Now that we know the value for the derivate from the first section of the work, we can easily get to the mass values. The solutions are:

$$\boxed{M_1 = 0.493 M_\odot}$$

$$\boxed{M_2 = 1.103 M_\odot}$$

5. Plotting $log\rho$ and m as a function of the radius

As a final analysis of the results we will be plotting these two magnitudes against the radius of the stars for each one of the central densities given. To do this we will use the vectors calculated in the last section of the work. These were the ones we took the last value from to achieve our radius and mass values for each of the stars.

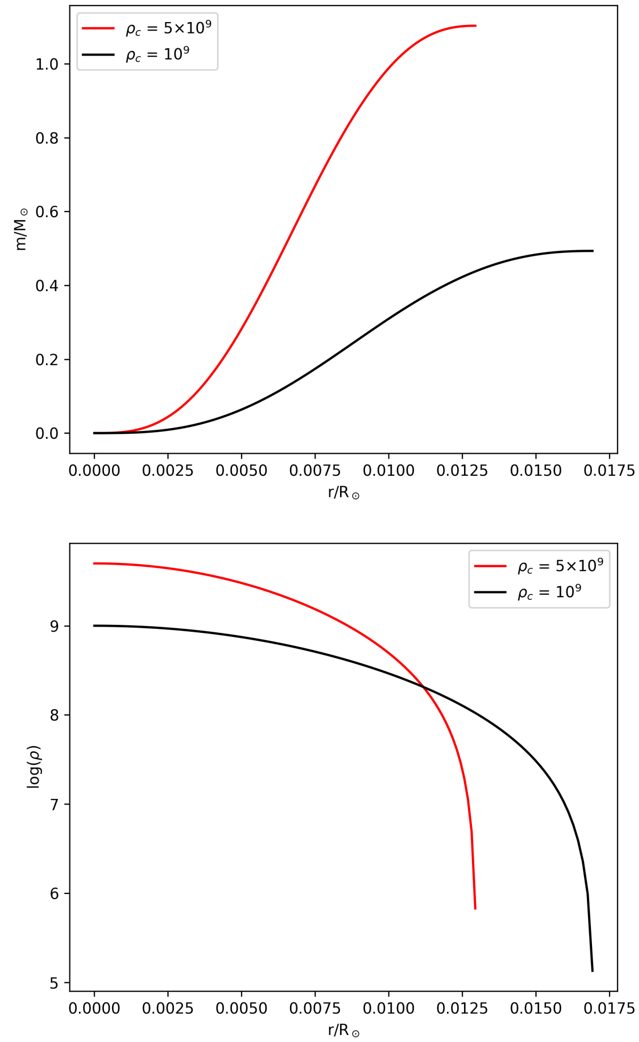

Fig. 4. Mass and *logρ* against the radius for the two different central densities given.

Once we have this functions we can plot them against the radius. These are shown in figure 4.

As we saw in the previous section, the star with higher central density is more massive and also has a smaller radius, this behaviour can be confirmed by figure 4, in which we can see how the higher density star starts decreasing faster than the lower central density star, therefore having a smaller radius.

6. References

[1] Sébastién Comerón. *Subject 5: Simple Stellar Models*, 2021

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