The Mathematics behind Regular WFF 'N Proof

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1 Welcome

WFF 'N Proof is a board game that introduces players to propositional logic. At the Academic Games Leagues of America National Tournament, two different versions of the game are played - Basic and Regular. Many current and former players, including myself, agree that WFF 'N Proof does an excellent job of teaching players the fundamentals of propositional logic. Regular WFF 'N Proof, which is played mostly by high school students at the National Tournament, does a particularly good job of introducing players to the structure of a formal logical proof. However, in my opinion, it is quite unfortunate that Regular WFF 'N Proof does not encourage players to explore the deep mathematics that lies just beyond the game.

The game takes players to the beach. This note is the opportunity to jump in the ocean, an opportunity that the game does not provide. In the context of Regular WFF 'N Proof, this note will explain the basic concepts and prove two deep properties of propositional logic. It will even suggest new strategies that players can use to win games.

2 Truth or Consequences

The beauty of WFF 'N Proof - and the beauty of propositional logic - lies in the fact that the symbols mean something. In Regular WFF 'N Proof, every WFF is composed of base WFFs (p, q, r, s) and connectors (C, A, K, E, N). We think about base WFFs as statements and the connectors as ways of combining statements to get a new statement as follows.

Cpq	if p then q
Apq	either p or q
Kpq	both p and q
Epq	p if and only if q
Np	not p

We can also think about a composite WFF is via a truth table. The truths/falsehoods of Cpq, Apq, Kpq, and Epq depend on the truths/falsehoods of p and q as follows.

p	q	Cpq	Apq	Kpq	Epq
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	F	F	T

Furthermore, the truth/falsehood of Np depends on the truth/falsehood of p as follows.

p	Np
T	F
F	T

Of course, a WFF can have more than one connector. Given a composite WFF α, the statement corresponding to α is found by recursively finding the statements corresponding to its sub-WFFs and the truth/falsehood of α is found by recursively finding the truth/falsehoods of its sub-WFFs. For example, consider the composite WFF $CKprs$. The sub-WFFs of $CKprs$ are Kpr and s. The statement corresponding to Kpr is "both p and r" and the statement corresponding to s is "s," hence the statement corresponding to $CKprs$ is "if both p and r then s." Furthermore, we can construct the truth table for $CKprs$ by using Kpr and s as intermediates as follows.

p	q	s	Kpr	s	$CKprs$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	F	T	T
F	F	F	F	F	T

Given some WFFs $\alpha_1, ..., \alpha_n$ (here n is some natural number) and another WFF β, here is one question that we can ask.

If $\alpha_1, ..., \alpha_n$ are true, is β necessarily true?

To be a bit more precise about this question, we can phrase the question in terms of the base WFFs $p_1, ..., p_m$ (here m is some natural number) that appear in $\alpha_1, ..., \alpha_n$ and β.

If $p_1, ..., p_m$ are assigned T/F values so that $\alpha_1, ..., \alpha_n$ are all true, is β necessarily true?

Let's do an example. Take $n = 2$, $m = 3$, $p_1 = p$, $p_2 = r$, $p_3 = s$, $\alpha_1 = Aps$, $\alpha_2 = Krr$, and $\beta = CKprs$. We need to consider the truths/falsehoods of α_1, α_2, and β for every possible way to assign T/F values of p_1, p_2, and p_3.

$p_1 = p$	$p_2 = r$	$p_3 = s$	$\alpha_1 = Aps$	$\alpha_2 = Krr$	$\beta = CKprs$
T	T	T	T	T	T
T	T	F	T	T	F
T	F	T	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	T	F	F	T	T
F	F	T	T	F	T
F	F	F	F	F	T

The second line shows there is an assignment of T/F values to p_1, p_2, and p_3 so that α_1 and α_2 are true but β is false. Therefore, the answer to our question is no in this case. Our discussion motivates one definition of notion of logical implication.

Definition 1 *Let $\alpha_1, ..., \alpha_n$ and β be WFFs. Let $p_1, ..., p_m$ be the base WFFs that appear in $\alpha_1, ..., \alpha_n$ and β. If β is true for every assignment of T/F values to $p_1, ..., p_m$ that make $\alpha_1, ..., \alpha_n$ true, then we say that $\alpha_1, ..., \alpha_n$ tautologically imply β and we write $\alpha_1, ..., \alpha_n \vDash \beta$. If $\alpha_1, ..., \alpha_n$ do not tautologically imply β, then we write $\alpha_1, ..., \alpha_n \nvDash \beta$.*

Our example demonstrates $Aps, Krr \nvDash CKprs$.

If we think about $\alpha_1, ..., \alpha_n$ as premises and β as the conclusion, then the game of Regular WFF 'N Proof gives another notion of logical implication.

Definition 2 *Let $\alpha_1, ..., \alpha_n$ and β be WFFs. If it is possible in Regular WFF 'N Proof to prove the conclusion β from the premises $\alpha_1, ..., \alpha_n$ using any rules, then we say that $\alpha_1, ..., \alpha_n$ deductively imply β and we write $\alpha_1, ..., \alpha_n \vdash \beta$. If $\alpha_1, ..., \alpha_n$ do not deductively imply β, then we write $\alpha_1, ..., \alpha_n \nvdash \beta$.*

There are 12 rules that can be used in a proof in Regular WFF 'N Proof. Using parentheses to denote a sub-proof, we can list the rules as follows.

Ci: $(\alpha \vdash \beta) \vdash C\alpha\beta$
Co: $C\alpha\beta,\ \alpha \vdash \beta$
Ai: $\alpha \vdash A\alpha\beta,\ A\beta\alpha$
Ao: $A\alpha\beta,\ (\alpha \vdash \gamma),\ (\beta \vdash \gamma) \vdash \gamma$
Ki: $\alpha,\beta \vdash K\alpha\beta$
Ko: $K\alpha\beta \vdash \alpha, \beta$
Ei: $C\alpha\beta,\ C\beta\alpha \vdash E\alpha\beta$
Eo: $E\alpha\beta \vdash C\alpha\beta,\ C\beta\alpha$
Ni: $(\alpha \vdash \beta, N\beta) \vdash N\alpha$
No: $(N\alpha \vdash \beta, N\beta) \vdash \alpha$
Rp: $\alpha \vdash \alpha$
R: $\alpha \vdash (...(\alpha)...)$

As an example, the following proof demonstrates $Epq, q \vdash p$.

	$Epq, q \to Kpq$	
1	Epq	s
2	q	s
3	Cpq	Eo 1
4	p	Co 2, 3
5	Kpq	Ki 4, 2

We have a method for determining whether $\alpha_1, ..., \alpha_n \vDash \beta$ or $\alpha_1, ..., \alpha_n \nvDash \beta$. Specifically, we create a truth table in which each row corresponds to an assignment of T/F values to the base WFFs $p_1, ..., p_m$ that appear in $\alpha_1, ..., \alpha_n$ and β. If β is true in every row in which $\alpha_1, ..., \alpha_n$ are all true, then we can conclude $\alpha_1, ..., \alpha_n \vDash \beta$. Otherwise, we can conclude $\alpha_1, ..., \alpha_n \nvDash \beta$.

If you can produce a proof of the conclusion β from the premises $\alpha_1, ..., \alpha_n$, then you can conclude $\alpha_1, ..., \alpha_n \vdash \beta$. However, it is very difficult to determine in general if $\alpha_1, ..., \alpha_n \vdash \beta$ or $\alpha_1, ..., \alpha_n \nvdash \beta$. Just because you failed to come up with a proof does not mean that a proof does not exist. Indeed, Regular WFF 'N Proof players often struggle with this problem during their games - "I wasn't able to come up with a proof with these premises, but I maybe one exists. Should I change my premises altogether or should I continue looking for a proof with these premises? I wish there were an easy way to tell because I'm running out of time." Many players confront this problem by searching for a proof. If they can't find a proof, then they assume they would have been clever enough to find one if one existed and hence conclude a proof does not exist. This method requires a lot of time, persistence, and intuition, three things that not all players have all the time. Furthermore, this method often fails. Shakes are often decided because one player continued looking for a solution where the other two players gave up. If these issues have come up in your games, then you will want to continue reading.

3 If It's True, Then I Can Prove It + If I Can Prove It, Then It's True

The following theorem is a deep theorem of propositional logic and provides a method for checking if $\alpha_1, ..., \alpha_n \vdash \beta$

Theorem 1 *Let $\alpha_1, ..., \alpha_n$ and β be WFFs. Then $\alpha_1, ..., \alpha_n \vDash \beta$ if and only if $\alpha_1, ..., \alpha_n \vdash \beta$.*

This theorem implies that β can be proved from $\alpha_1, ..., \alpha_n$ as long as β is true whenever $\alpha_1, ..., \alpha_n$ are all true. For example, we used a truth table to determine $Aps, Krr \nvDash CKprs$. Therefore, we can conclude that it is impossible to prove $CKprs$ from Aps and Krr. As another example, the following truth table demonstrates $Epq, q \vDash p$, so we can conclude that it is possible to prove p from Epq and q.

p	q	Epq	q	p
T	T	T	T	T
T	F	F	F	T
F	T	F	T	F
F	F	T	F	F

Note that we could conclude a proof was possible without having to think about what such a proof would look like.

As an aside, it should be noted that the theorem is false in Basic WFF 'N Proof. For example, in Basic WFF 'N Proof, you should be able to convince yourself that $p \vDash Cpp$ but $p \nvdash Cpp$.

Now that we understand the theorem and its importance, we should at least think about why its true.

Proof:

We need to show two claims.

- if $\alpha_1, ..., \alpha_n \vdash \beta$ then $\alpha_1, ..., \alpha_n \vDash \beta$ (this is called the soundness property of propositional logic)
- if $\alpha_1, ..., \alpha_n \vDash \beta$ then $\alpha_1, ..., \alpha_n \vdash \beta$ (this is called the completeness property of propositional logic)

We will start with the second claim.

Assume $\alpha_1, ..., \alpha_n \vDash \beta$. We want to show $\alpha_1, ..., \alpha_n \vdash \beta$. We will construct a proof of β from $\alpha_1, ..., \alpha_n$, thereby demonstrating $\alpha_1, ..., \alpha_n \vdash \beta$. Let $p_1, ..., p_m$ be the base WFFs that appear in $\alpha_1, ..., \alpha_n$ and β. The proof consists of four main steps. Step 1 is to list the premises $\alpha_1, ..., \alpha_n$. Step 2 is to prove Ap_iNp_i for $i = 1, ..., m$, which is done as follows.

$$
\begin{array}{l|l}
\quad \begin{array}{|ll}
NAp_iNp_i & \text{s} \\ \hline
\quad \begin{array}{|ll}
p_i & \text{s} \\ \hline
Ap_iNp_i & \text{Ai} \\
NAp_iNp_i & \text{R}
\end{array} & \\
Np_i & \text{Ni} \\
Ap_iNp_i & \text{Ai} \\
NAp_iNp_i & \text{Rp}
\end{array} & \\
Ap_iNp_i & \text{No}
\end{array}
$$

Step 3, which will be detailed later, is to prove $CK...K\alpha_1...\alpha_n\beta$. Step 4 is to prove β, which is done as follows.

$K\alpha_1\alpha_2$	Step 1 (α_1, α_2) + Ki
$KK\alpha_1\alpha_2\alpha 3$	Step 1 (α_3) + Ki
$\vdots$	$\vdots$
$K...K\alpha_1...\alpha_n$	Step 1 (α_n) + Ki
β	Step 3 $(CK...K\alpha_1...\alpha_n\beta)$ + Co

We need to detail Step 3. Using Ao and Step 2, we can prove $CK...K\alpha_1...\alpha_n\beta$ as follows.

$$
\begin{array}{l|l}
\begin{array}{|ll}
p_1 & \text{s} \\ \hline
? & ? \\
CK...K\alpha_1...\alpha_n\beta & ?
\end{array} & \\
& \\
\begin{array}{|ll}
Np_1 & \text{s} \\ \hline
? & ? \\
CK...K\alpha_1...\alpha_n\beta & ?
\end{array} & \\
CK...K\alpha_1...\alpha_n\beta & \text{Step 2 } (Ap_1Np_1) + \text{Ao}
\end{array}
$$

To fill in the proof, we can again use Ao and Step 2.

p_1		s	
	p_2	s	
	?	?	
	$CK...K\alpha_1...\alpha_n\beta$	?	
	Np_2	s	
	?	?	
	$CK...K\alpha_1...\alpha_n\beta$	?	
Ap_2Np_2		Step 2 (Ap_2Np_2) + R	
$CK...K\alpha_1...\alpha_n\beta$		Ao	
Np_1		s	
	p_2	s	
	?	?	
	$CK...K\alpha_1...\alpha_n\beta$	?	
	Np_2	s	
	?	?	
	$CK...K\alpha_1...\alpha_n\beta$	?	
Ap_2Np_2		Step 2 (Ap_2Np_2) + R	
$CK...K\alpha_1...\alpha_n\beta$		Ao	
$CK...K\alpha_1...\alpha_n\beta$			Step 2 (Ap_1Np_1) + Ao

We repeat this process until we have many (2^m) subproofs nested many (m) levels deep. Each subproof corresponds to an assignment of T/F values to $p_1, ..., p_m$. For $i = 1, ..., m$, any subproof nested inside a subproof starting with p_i corresponds to an assignment of T/F values to $p_1, ..., p_m$ in which p_i is assigned T, whereas any subproof nested inside a subproof starting with Np_i will correspond to an assignment of T/F values to $p_1, ..., p_m$ in which p_i is assigned F.

It may seem that introducing all these subproofs will make proving $CK...K\alpha_1...\alpha_n\beta$ more difficult. Instead of proving $CK...K\alpha_1...\alpha_n\beta$ once, we have to prove $CK...K\alpha_1...\alpha_n\beta$ in each subproof. However, we have in fact made the problem easier by breaking the problem down into many easier subproblems. Instead of proving $CK...K\alpha_1...\alpha_n\beta$ outside a subproof from the suppositions $\alpha_1, ..., \alpha_n$, we can prove $CK...K\alpha_1...\alpha_n\beta$ inside a subproof from the suppositions $\alpha_1, ..., \alpha_n$ (after reiterating) plus the suppositions $p_1/Np_1, ..., p_m/Np_m$.

Let's pick a subproof. We need to prove $CK...K\alpha_1...\alpha_n\beta$ in this subproof. Recall that this subproof corresponds to some assignment of T/F values to $p_1, ..., p_m$. I claim that this assignment makes $CK...K\alpha_1...\alpha_n\beta$ true. Indeed, if $CK...K\alpha_1...\alpha_n\beta$ were false, then $\alpha_1, ..., \alpha_n$ would all have to be true and β would have to be false. However, we assumed $\alpha_1, ..., \alpha_n \vDash \beta$, which means that if $\alpha_1, ..., \alpha_n$ are all true then β is true. Therefore, $CK...K\alpha_1...\alpha_n\beta$ is true.

This is where the magic happens. The idea is to accomplish the following goal for every sub-WFF γ of $CK...K\alpha_1...\alpha_n\beta$.

prove either γ, if γ is true, or $N\gamma$, if γ is false, inside our subproof

Before we accomplish this goal for a sub-WFF γ, we will first accomplish this goal for every sub-WFF of γ. This means that we must start with the base WFFs $p_1, ..., p_m$ and end with $CK...K\alpha_1...\alpha_n\beta$. Here are two important observations.

- potentially after reiterating/repeating, we have already accomplished the goal with $p_1, ..., p_m$
- $CK...K\alpha_1...\alpha_n\beta$ is true so this process will result in a proof of $CK...K\alpha_1...\alpha_n\beta$, which is what we wanted

Let's pick some sub-WFF γ of $CK...K\alpha_1...\alpha_n\beta$ and let's assume that we have already accomplished our goal with every sub-WFF of γ. Note that γ must either have the form $C\delta\epsilon$, $A\delta\epsilon$, $K\delta\epsilon$, or $E\delta\epsilon$ for some WFFs δ and ϵ or have the form $N\delta$ for some WFF δ. The truths/falsehoods of $C\delta\epsilon$, $A\delta\epsilon$, $K\delta\epsilon$, or $E\delta\epsilon$ depend on the truths/falsehoods of δ and ϵ as follows.

δ	ϵ	$C\delta\epsilon$	$A\delta\epsilon$	$K\delta\epsilon$	$E\delta\epsilon$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	F	F	T

The truth/falsehood of $N\delta$ depends on the truth/falsehood of δ as follows.

δ	$N\delta$
T	F
F	T

Let's assume $\gamma = C\delta\epsilon$ and δ is false and ϵ is true. (The procedure is similar for the other possible choices.) Since δ and ϵ are sub-WFFs of γ, we have already accomplished our goal with δ and ϵ. As δ is false and ϵ is true, this means that we have proven $N\delta$ and ϵ. Since δ is false and ϵ is true, $C\delta\epsilon$ is true. Thus, our goal is to prove $C\delta\epsilon$. Let's do it.

$$
\begin{array}{l|l|lll}
\multicolumn{3}{l}{N\delta} & & \text{earlier goal} \\
\multicolumn{3}{l}{\epsilon} & & \text{earlier goal} \\
 & \multicolumn{2}{l}{\delta} & \text{s} & \\
\cline{1-4}
 & & N\epsilon \quad \text{s} & & \\
\cline{3-3}
 & & \delta \quad \text{R} & & \\
 & & N\delta \quad \text{R} & & \\
 & \multicolumn{2}{l}{\epsilon} & \text{No} & \\
\multicolumn{3}{l}{C\delta\epsilon} & & \text{Ci}
\end{array}
$$

We have just proven the second claim. We not only showed $\alpha_1, ..., \alpha_n \vdash \beta$, but we also devised a proof of β from $\alpha_1, ..., \alpha_n$. In an actual game of Regular WFF 'N Proof, this proof would likely be impractical for a human player to write down on a piece of paper. However, a computer could spit this proof out quickly. In fact, this is what the program MrRegularWff does.

Let's take a breather ... and then we will prove the first claim.

Assume $\alpha_1, ..., \alpha_n \vdash \beta$. We want to show $\alpha_1, ..., \alpha_n \vDash \beta$. Since $\alpha_1, ..., \alpha_n \vdash \beta$, we know that there exists a proof with the suppositions $\alpha_1, ..., \alpha_n$ on the first n lines (in the outer proof) and β on the last line (in the outer proof). The idea is to prove the following statement for every line l of the proof.

for every assignment of T/F values to the base WFFs $p_1, ..., p_M$
appearing in $\alpha_1, ..., \alpha_n$ and the suppositions $s_1, ..., s_N$ of the subproofs in which l is nested and the WFF ω on l,
if $\alpha_1, ..., \alpha_n$ and $s_1, ..., s_N$ are all true, then ω is also true

Before we prove this statement for a line, we will first prove this statement for every previous line. This means that we must start with the first line and end with the last line. Here are two important observations.

- since $\alpha_1, ..., \alpha_n$ occur on the first n lines of the proof, it is obvious that the statement is true for the first n lines
- since β occurs on the last line outside any subproofs, if the statement is true for the last line, then β is true for every assignment of T/F values to the base WFFs $p_1, ..., p_m$ appearing in $\alpha_1, ..., \alpha_n$ and β that make $\alpha_1, ..., \alpha_n$ true i.e. $\alpha_1, ..., \alpha_n \vDash \beta$, which is what we wanted to show

Let's pick a line l of the proof, other than the first n lines, and let's assume the statement is true for every previous line. The WFF ω on this line either is a supposition or is proven from earlier WFFs and subproofs using one of the 12 rules in Regular WFF 'N Proof. If ω is a supposition, then the statement is obviously true, since ω is one of the suppositions of the subproofs in which l is nested. Suppose ω is proven from earlier WFFs and subproofs using a rule. We will deal only with Ki and Ni. (The argument is similar for the other rules.)

Let S denote the subproof containing l. Pick an assignment of T/F values to the base WFFs $p_1, ..., p_M$ that appear in $\alpha_1, ..., \alpha_n$ and the suppositions $s_1, ..., s_N$ of the subproofs in which the line is nested and ω so that $\alpha_1, ..., \alpha_n$ and $s_1, ..., s_N$ are all true. We want to show ω is true.

Suppose that Ki is the rule. This means that $\omega = K\alpha\epsilon$ for some WFF α that occurred on some previous line l_1 in S and some WFF ϵ that occurred on some previous line l_2 in S. Since we assumed that the statement is true for l_1 and l_2 and since l_1 and l_2 are in S and since all base WFFs that occur in α and ϵ also occur in $\omega = K\alpha\epsilon$, we know that α and ϵ are true. Thus $\omega = K\alpha\epsilon$ is true.

Suppose that Ni is the rule. This means two things.

- $\omega = N\alpha$ for some WFF α that was the supposition of a previous subproof T of S
- there is some WFF ϵ such that ϵ occurred on some line l_1 in T and $N\epsilon$ occurred on some line l_2 in T

Note every base WFF that occurs in α occurs in $\omega = N\alpha$. Suppose that α were true. Since we assumed that the statement is true for l_1 and l_2 and since α is the supposition of T, this would mean that ϵ and $N\epsilon$ are true, which would be impossible. Therefore α must be false and thus $\omega = N\alpha$ must be true.

We have now proven both claims. $\square$