Certificate in Quantitative Finance

CQF

EXAM 2

Presented by:

Andres Girardot Martinez

April 29th, 2019

Table of Contents

Introduction

In today's complex financial world, computer simulation is one of the most relevant topics when assessing and analyzing the different risks that a particular investment decision might have. As such, the financial industry has demonstrated increased interest in the utilization of computational simulation methods such as the Monte Carlo method. One of the many uses of Monte Carlo methods in the quantitative finance field is the valuation of financial derivatives. The purpose of this paper is to show how can these instruments be priced using Monte Carlo simulations, how to measure the accuracy of the results obtained quantitatively and how to improve the accuracy of such results. The project is divided into two main parts: the first one deals with the valuation of a Binary options, the convergence of the numerical valuation to the analytical solution and the methods that can be employed in order to improve the precision of the numerical method. The second part of the project deals with the valuation of Lookback options (with fixed and floating strikes) by also using Monte Carlo simulations. The performance of the numerical valuation is also evaluated through a comparison with the analytical, closed-form solution by analyzing the behavior of the numerical error and the convergence towards the analytical value as the number of simulations increases and as the number of time steps increases as well.

Part I: Binary Option Valuation with Monte Carlo Methods

The very first step of the process is to assume that the underlying asset(stock) follows the following geometric Brownian Motion in order to have a model to simulate the underlying asset's price:

$$dS = rSdt + \sigma SdX \tag{1}$$

Where dX is the increment in the stochastic process and is determined by a random draw from a Normal distribution, with mean of zero and standard deviation of $\sqrt{dt}$. σ is the volatility of the stock, r is the drift rate of the process (which is the risk free rate under the risk neutral measure) and S is the stock price. Consequently:

$$dX = \phi\sqrt{dt} \tag{2}$$

Where ϕ represents a random draw from a Normal distribution. Therefore, the price of the underlying asset over the next time step can be simulated using the following formula:

$$S(t+\delta) = S(t)\exp(\left(r - \tfrac{1}{2}\sigma^2\right)\delta t + \sigma\phi\sqrt{\delta t}) \tag{3}$$

where $S(t+\delta)$ is the realized stock price over the next time step, $S(t)$ is the stock price at time t, r is the risk-free interest rate and δ represents the size of the time step interval.

To approximate the value of an option using Monte Carlo methods once all simulations are generated, the following formula is applied:

$$Option = \tfrac{1}{I}\textstyle\sum_{i=1}^{I} e^{-rT} f(S_T^i) \tag{4}$$

Where I is the number of simulations and $f(S_T^i)$ is the payoff of the Binary option from the i^{th} simulation at option maturity (T). The payoff function for the Binary call is defined as:

$$f(S_T^i)_{Call} = \begin{cases} 1 & if \quad S_T > K \\ 0 & if \quad S_T \leq K \end{cases} \tag{5}$$

While the payoff function for the Binary put is defined as:

$$f(S_T^i)_{Put} = \begin{cases} 1 & if \quad S_T < K \\ 0 & if \quad S_T \geq K \end{cases} \tag{6}$$

Binary Option Valuation with 100,000 Simulations

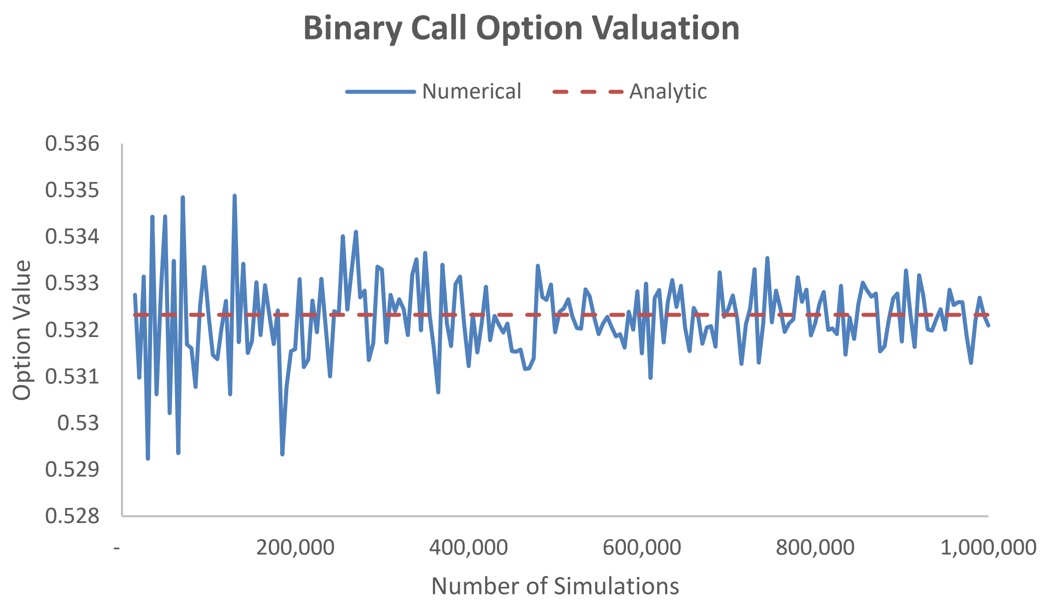
The Binary options that are priced consist on a call and a put with the same pricing parameters. The pricing parameters for the options are the following: stock price (S) of 100, strike *(K)* of 100, time to maturity (T) of 1 year, volatility of the underlying asset (σ) of 20% and risk-free rate (r) of 5%. For the very first valuation, 100,000 simulations are generated in order to calculate the value of both options. Additionally, the time steps considered for this first valuation are 252, which assumes one time-step per trading day over a year. With the purpose of comparing the numerical valuation with the analytical solution, both options are valued using the closed form solution formulae (Wilmott, 2006). Details of such formulae can be found in Appendix I. The results of all four valuations are:

	Numerical	Analytical	Error	Relative Error
Binary Call	0.535456555	0.532324815	0.003131740	0.5883%
Binary Put	0.417304349	0.418904609	0.001600261	0.3820%

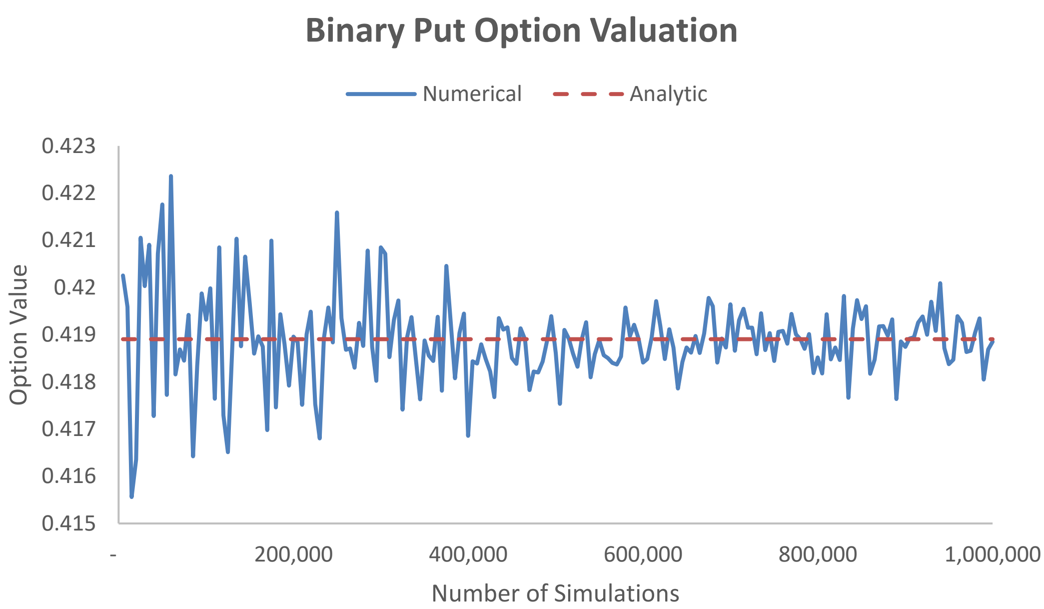
Table 1.1: Analytical Solution vs. Numerical Solution with 100,000 Simulations

Binary Option Valuation with Different Number of Simulations

It is well known that the accuracy of Monte Carlo simulation methods increases as the number of simulations (I) increase, consequently the performance (accuracy) of the option valuation is evaluated using different levels of I. Each option is priced using a range of I from 5,000 to 1,000,000, with increments of 5,000. The results produced can be observed below.



Plot 1.1: Binary Call Value with 252 Time-steps

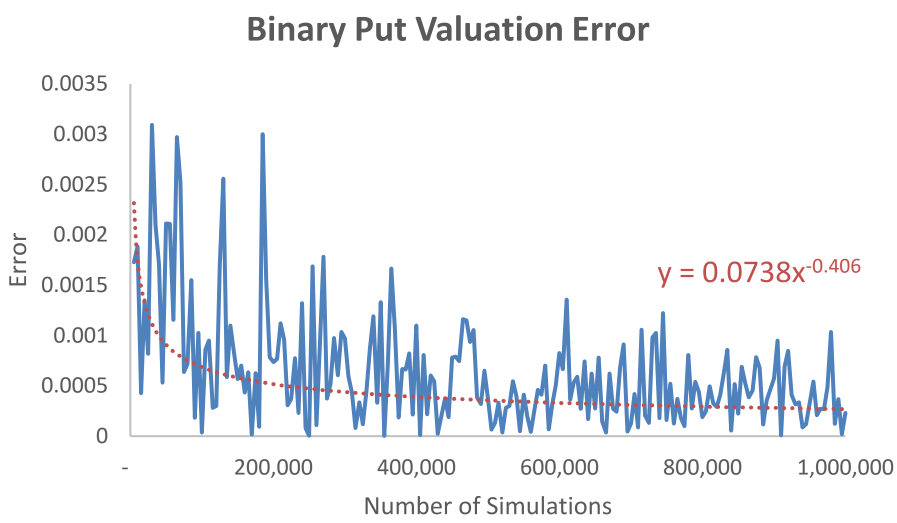


Plot 1.2: Binary Put Value with 252 Time-steps

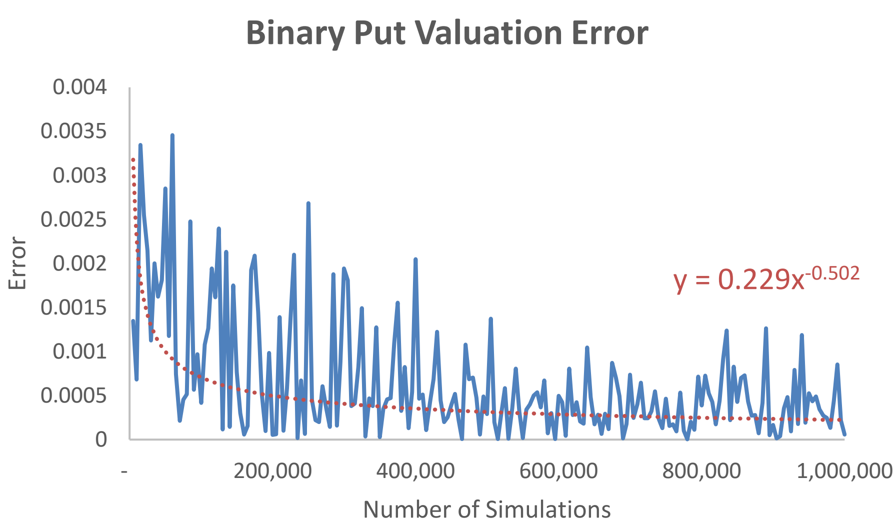
As it is clearly evidenced by both plots, the numerical solution converges to the analytical solution as the number of simulations increases. However, one must always bear in mind that there is an inverse relationship between the error (accuracy) of the calculation and the time required by the machine to execute all required computations. As such, practitioners must always bear in mind this relationship considering that markets move quite rapidly nowadays and the speed at which valuations are executed is of extreme relevance due to the fact that these will automatically change as the underlying parameters vary.

Error Behavior

With the intention of showing further evidence of the fact that the accuracy of the numerical approximation improves as the number of simulations increases, the size of the error is plotted using various number of simulations. The results can be observed in the plots below:



Plot 1.3: Error Size Binary Call Valuation with 252 Time-steps



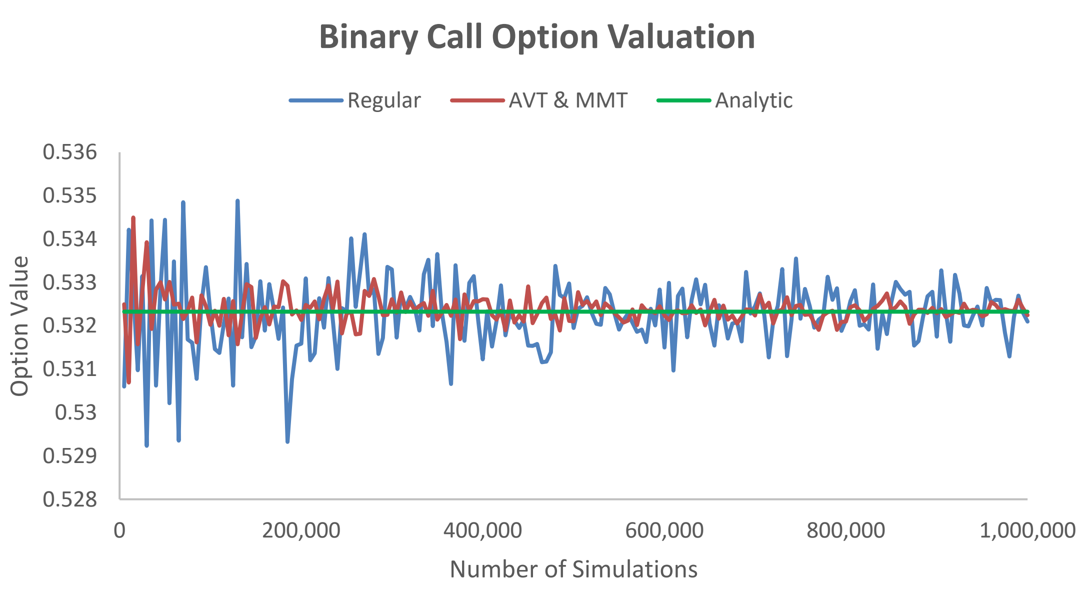
Plot 1.4: Error Size Binary Put Valuation with 252 Time-steps

By observing the behavior of the error displayed in Plots 1.3 and 1.4 it is evident that the error decreases as the number of simulations increases. Furthermore, the errors decrease at a rate of roughly $N^{-0.52}$, which is evidenced by the function describing the behavior of the curve (on the plot). This is a highly relevant result, due to the fact that according to the Central Limit Theorem, the convergence rate of the Monte Carlo numerical solution to the analytical solution is defined by $N^{-0.5}$ (more details on Appendix II), which is relatively close to the result obtained. In addition to that, Plots 1.3 and 1.4 reinforce the evidence that an increase in the number of simulations is directly proportional to an increase in the accuracy of the results. However, this is not a linear relationship as it is clearly evidenced by the curvature of the function.

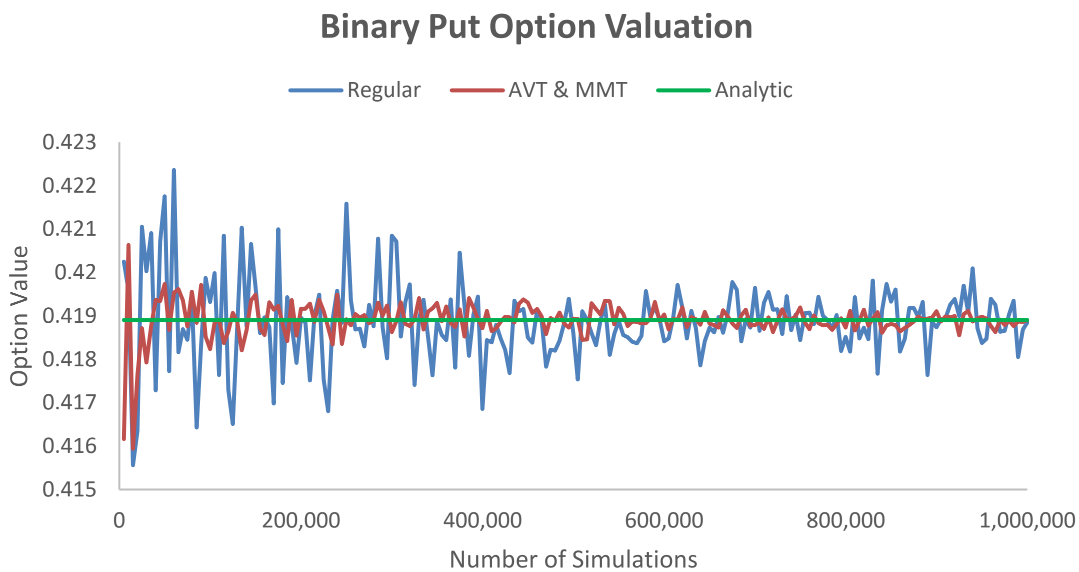
The charts in Appendix III display the numerical valuation values with different number of time steps, ranging from 20 to 2,000, with increments of 20. However, the convergence doesn't really improve as the number of time steps increases, which is exactly what is shown in the plots. The discussion of an improvement in the convergence behavior due to an increase in the number of time steps becomes more relevant with path dependent options, this is explored and discussed in Part II.

Variance Reduction Techniques: Antithetic Variable and Moment Matching

The Antithetic Variable Technique (AVT) and Moment Matching Technique (MMT) are two different methods that are commonly employed to reduce the variance of the valuation and improve the accuracy of a Monte Carlo numerical approximation. The base of the AVT is to calculate two option values for each random draw (ϕ). The average of these two figures represents the value of one simulation. On the other hand, MMT consists in calculating the arithmetic mean and standard deviation of all the random draws obtained, subtract each random draw by the mean of the sample and divide it by the standard deviation of the sample as well. By doing so, the distribution of the random draws will automatically have a mean of zero and a variance of one (which is in line with a standard Normal distribution). A more detailed description of both techniques is provided in Appendix IV. The same number of random variables generated for the previous section is used to approximate the value of the Binary options, but this time implementing the variance reduction techniques (AVT and MMT). Plots 1.5 and 1.6 compare the regular Monte Carlo valuation with the Monte Carlo valuation that implements AVT and MMT for the Binary call and put, respectively. By observing these plots, it is quite evident that the accuracy of the numerical approximation improves significantly.

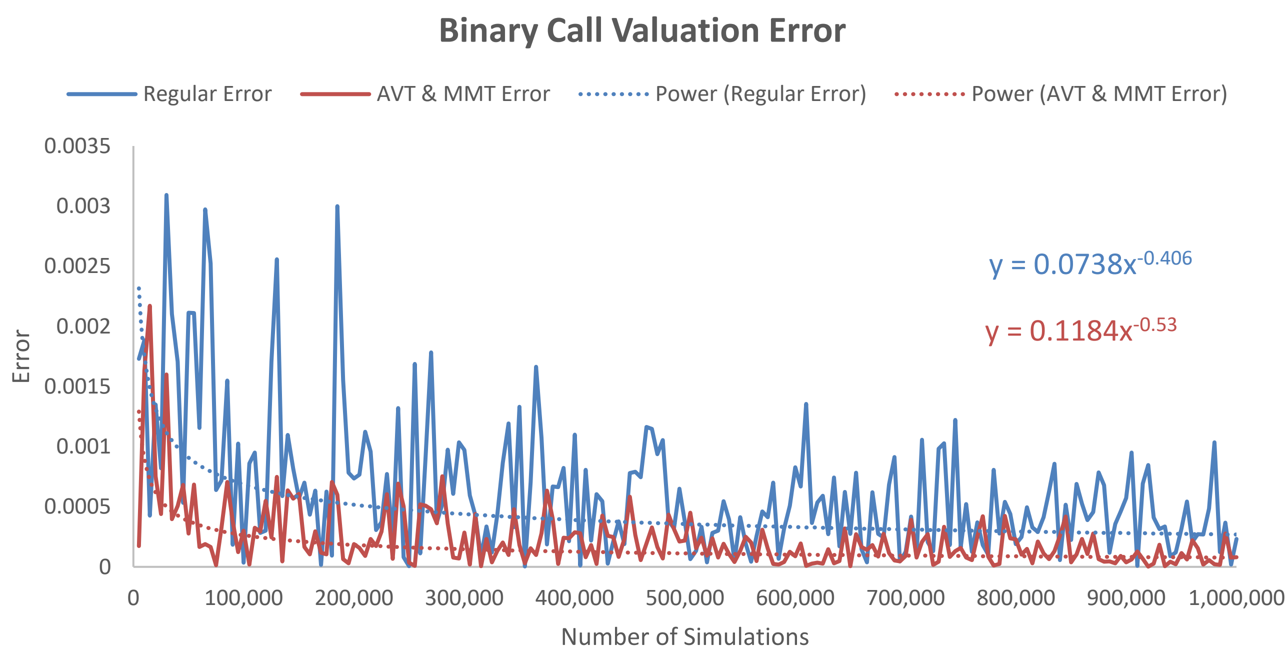


Plot 1.5: Binary Call Value with 252 Time-steps (Regular and AVT with MMT)

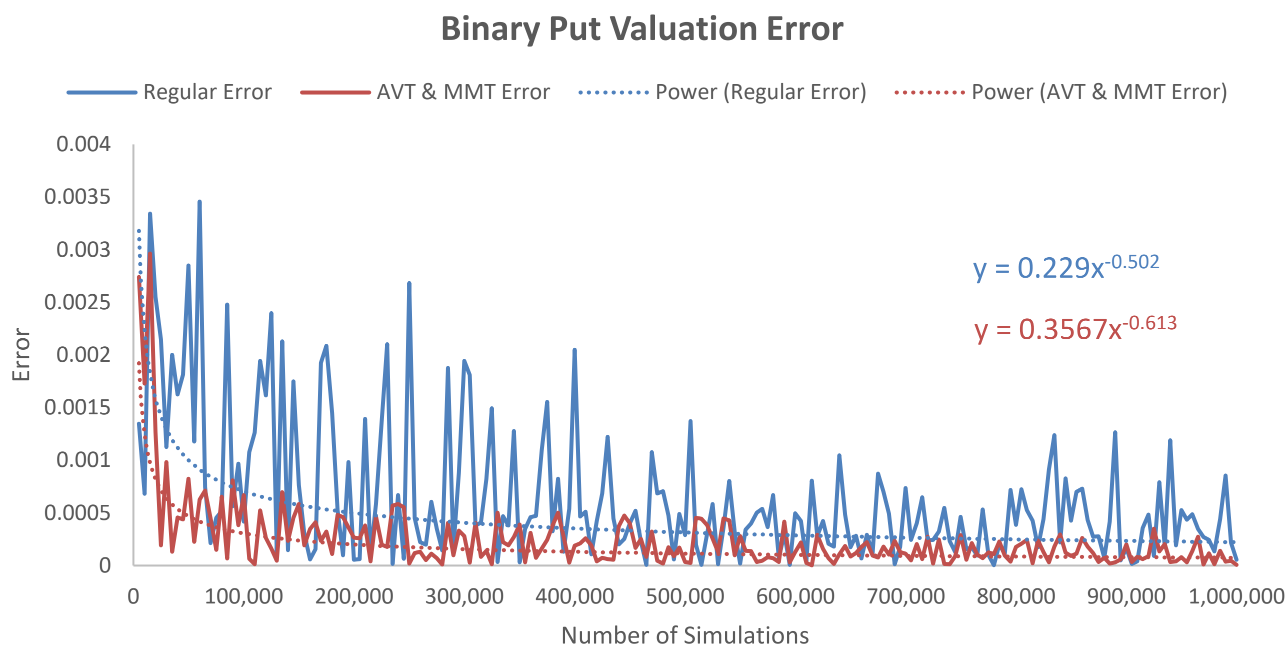


Plot 1.6: Binary Put Value with 252 Time-steps (Regular and AVT with MMT)

With the intention of showing further evidence of the fact that the implementation of AVT and MMT does improve the Monte Carlo valuation, the error behavior is charted in plots 1.7 and 1.8. In both cases (call and put) the convergence of the error towards zero is happening at a faster rate with the AVT & MMT valuation than with the regular Monte Carlo valuation. This convergence rate is quantified by the exponential of the equation describing the fitted power line. In both cases (call and put) the exponential is more negative for the AVT & MMT valuation than for the Monte Carlo valuation.



Plot 1.7: Error Size Binary Call Valuation with 252 Time-steps (Regular and AVT with MMT)



Plot 1.8: Error Size Binary Put Valuation with 252 Time-steps (Regular and AVT with MMT)

Table 1.2 shows the standard deviation of the regular Monte Carlo valuation and the AVT and MMT valuation for the call and the put. It also shows the standard deviation of the error of these valuations.

Standard Deviation

	Regular	AVT&MMT	Regular Error	AVT&MMT Error
Binary Call	0.000855074	0.000351513	0.000572736	0.00026874
Binary Put	0.000932648	0.00042353	0.000667191	0.000344717

Table 1.2: Standard Deviation of Valuation and Error for Binary Call and Put (Regular and AVT&MMT)

The fact that the standard deviation of the actual valuation is lower in the AVT & MMT valuation than in the regular Monte Carlo valuation shows that AVT & MMT is more consistent when approximating the option value when considering various levels of simulations. In addition to that, the fact that the standard deviation of the error is significantly lower in AVT&MMT than in the regular Monte Carlo simulation shows that the AVT&MMT method is substantially more accurate than the regular Monte Carlo valuation.

Part II: Lookback Options Valuation

Lookback options are exotic, path-dependent options whose payoff depends on the maximum or minimum realized value of the underlying asset over the life of the option. There are two types of Lookback options, which are fixed strike Lookback options and floating strike Lookback options. The payoff functions for Lookback options are the following:

$$Lookback\ Call\ Payoff_{Fixed\ Strike} = \max(S_{max} - K) \tag{7}$$

$$Lookback\ Put\ Payoff_{Fixed\ Strike} = \max(K - S_{min}) \tag{8}$$

$$Lookback\ Call\ Payoff_{Floating\ Strike} = \max(S_T - S_{min}) \tag{9}$$

$$Lookback\ Put\ Payoff_{Floating\ Strike} = \max(S_{max} - S_T) \tag{10}$$

where S_{max}, K, S_{min} and S_T are the maximum value realized by the underlying asset over the life of the option, the fixed strike price, the minimum value realized by the underlying asset over the life of the option and the value of the underlying asset at time T (option contract expiry).

The purpose of this section of the report is to price all four types of Lookback options using Monte Carlo methods, evaluate how the accuracy of this derivative valuation varies with changes in the number of simulations (I) and changes in the number of time intervals in which the underlying asset is observed (M) throughout the life of the derivative. To simulate the paths, one must apply (3), again considering that δt is defined by T/M. The same valuation parameters ($S_0 = 100, K = 100, \sigma = 0.20, r = 0.05\ and\ T = 1$) used in Part I are used in this section. After generating each path, function (7), (8), (9) or (10) is applied (depending on the type of Lookback option) in order to find the payoff of the contract at maturity. The final step is to simply apply (4) but considering the payoff of the Lookback option.

Lookback Option Valuations with 100,000 Simulations

With the purpose of assessing the accuracy of the numerical valuations performed in this section, the analytical solutions are calculated using the closed-form formulae in Appendix V. It is worth mentioning the fact that in these formulas, $S_0 = S_{max} = S_{min} = 100$, because since the valuation is done at time t=0, the historical maximum (and minimum) underlying asset price observed is equal to the current price of the underlying asset (S_0). That being considered, the analytical closed form valuations for the Lookback options are 19.168, 12.340, 17.217 and 14.291 for the fixed strike call, fixed strike put, float strike call and float strike put, respectively.

The initial valuations are done with 252 time-steps and 100,000 simulations. Table 2.1 shows the results of the valuations using regular Monte Carlo approximations:

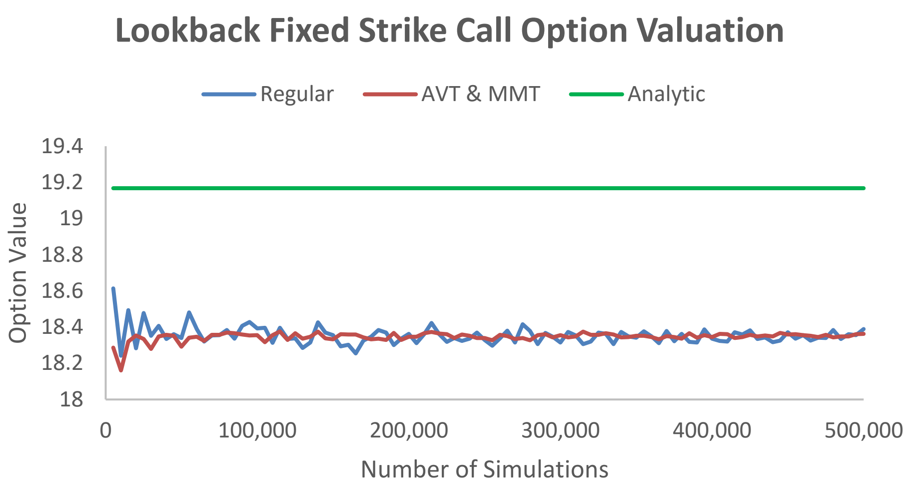
Fixed Strike	**Numerical**	**Analytical**	**Error**	**Relative Error**
Call	18.39187435	19.16762526	0.775750904	4.0472%
Put	11.71472127	12.33974469	0.625023414	5.0651%
Floating Strike	**Numerical**	**Analytical**	**Error**	**Relative Error**
Call	16.60118874	17.21680224	0.615613501	3.5757%
Put	13.47986261	14.29056771	0.810705102	5.6730%

Table 2.1: Lookback Options: Analytical Solution vs. Numerical Solution with 100,000 Simulations and 252 time-steps

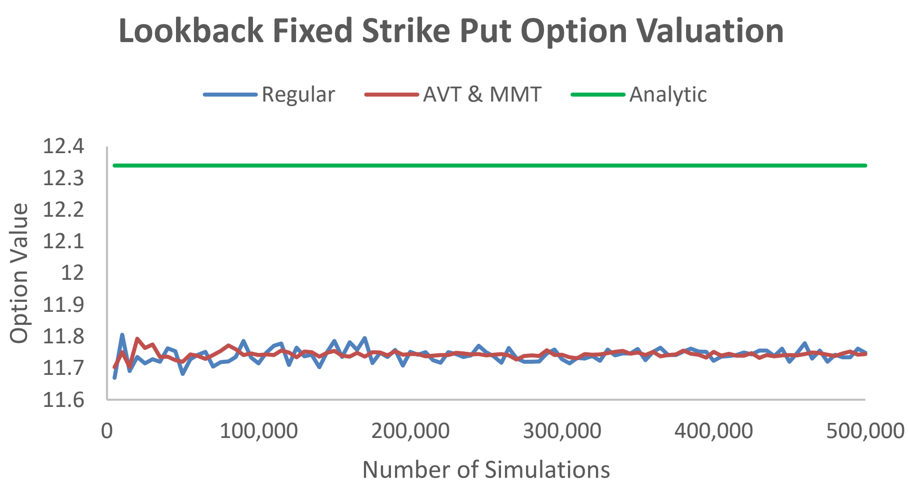
By observing the values illustrated in Table 2.1, it is evident the relative error of the numerical valuations is much higher for the Lookback options than for the Binary options observed in Table 1.1. This has to do with the fact that the accuracy of the valuation is not only a function of the number of simulations used for approximation, but also of the number of time steps (M) used, which is equivalent to the number of observations made to the price of the underlying asset throughout the life of the exotic option contract.

Lookback Option Valuation with Different Number of Simulations

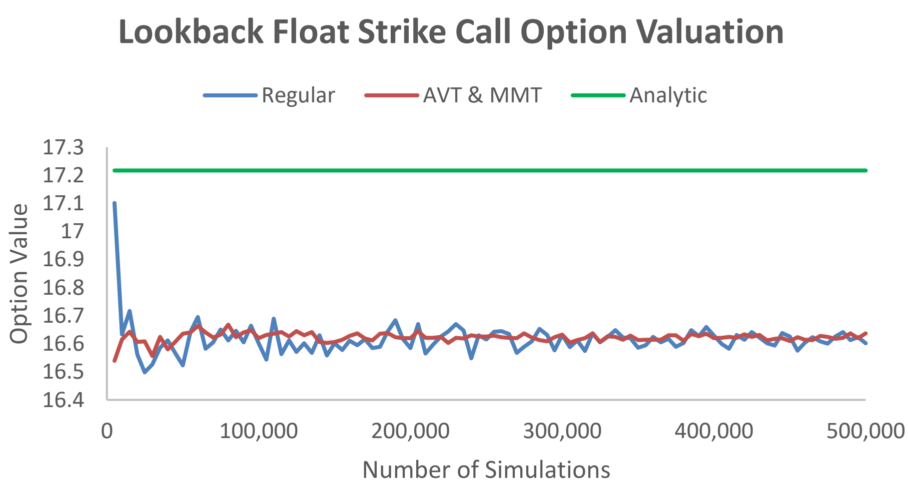
Since the variation reduction techniques AVT and MMT were already introduced in Part I, there is no need to explain these once again. Plots 2.1, 2.2, 2.3 and 2.4 illustrate the analytical and numerical valuation of the four different types of Lookback options using regular Monte Carlo and also the Monte Carlo with the AVT & MMT applied. All these plots show valuations that use 252 time-steps:



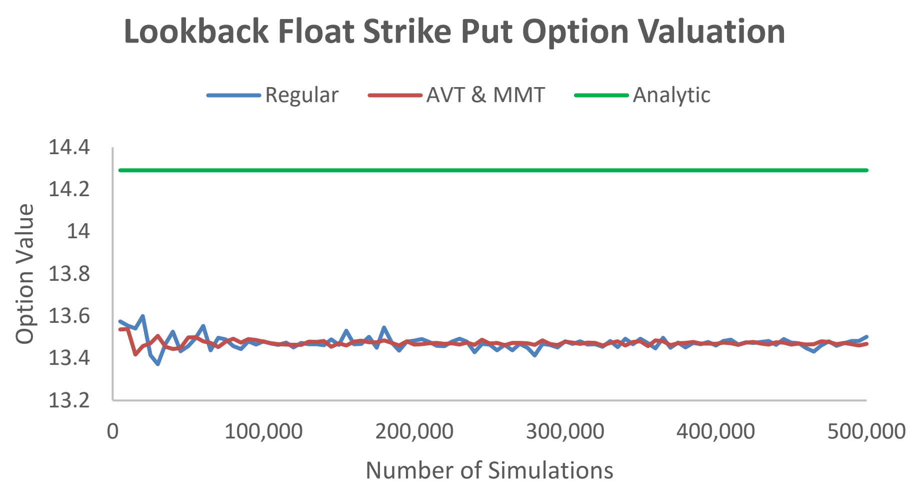
Plot 2.1: Lookback Fixed Strike Call Valuation with 252 Time-steps (Regular and AVT with MMT)



2.2: Lookback Fixed Strike Put Valuation with 252 Time-steps (Regular and AVT with MMT)

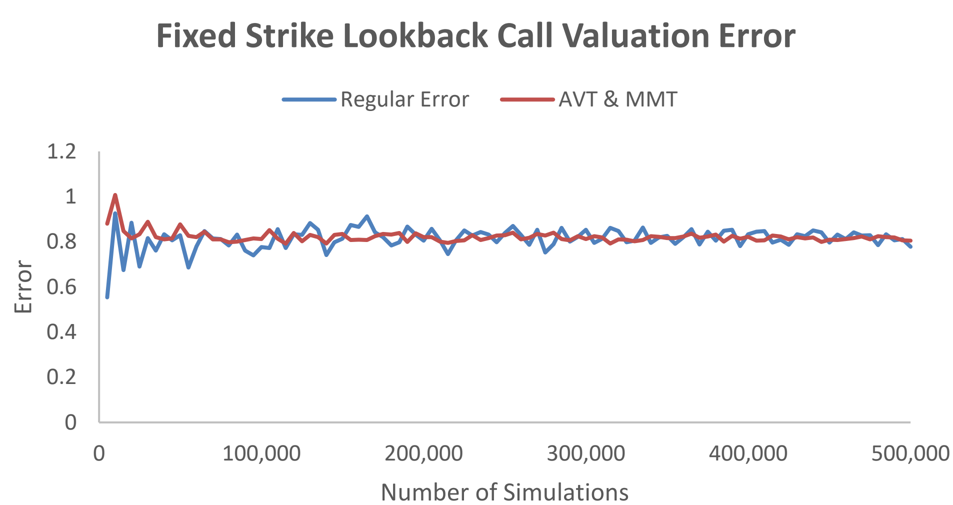


Plot 2.3: Lookback Float Strike Call Valuation with 252 Time-steps (Regular and AVT with MMT)



Plot 2.4: Lookback Float Strike Put Valuation with 252 Time-steps (Regular and AVT with MMT)

As it is clearly evidenced by the plots above, the numerical approximation doesn't really improve much as the number of simulations increases. As a matter of fact, the approximation error doesn't really show any convergence behavior towards zero as it is evidenced by Plot 2.5:

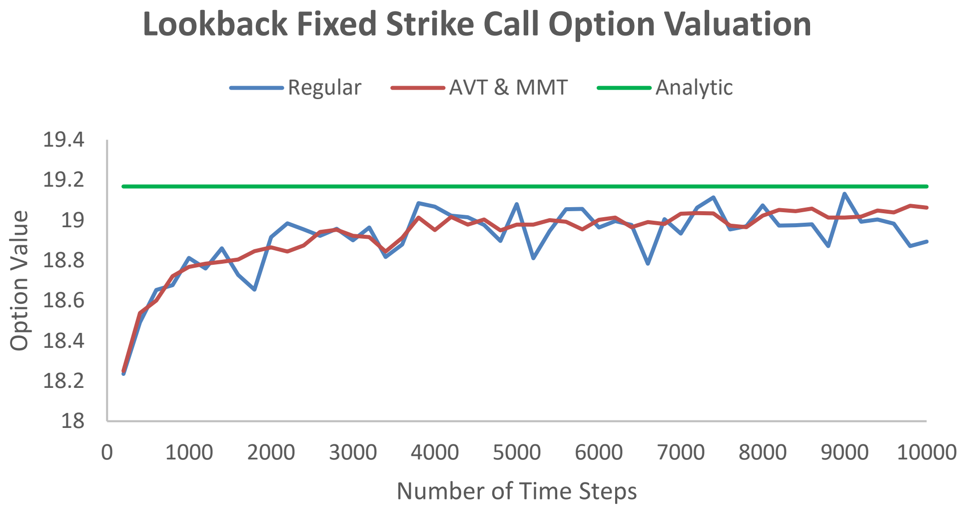


Plot 2.5: Error Size (Regular and AVT with MMT) for Fixed Strike Lookback Call Considering 252 Time-steps

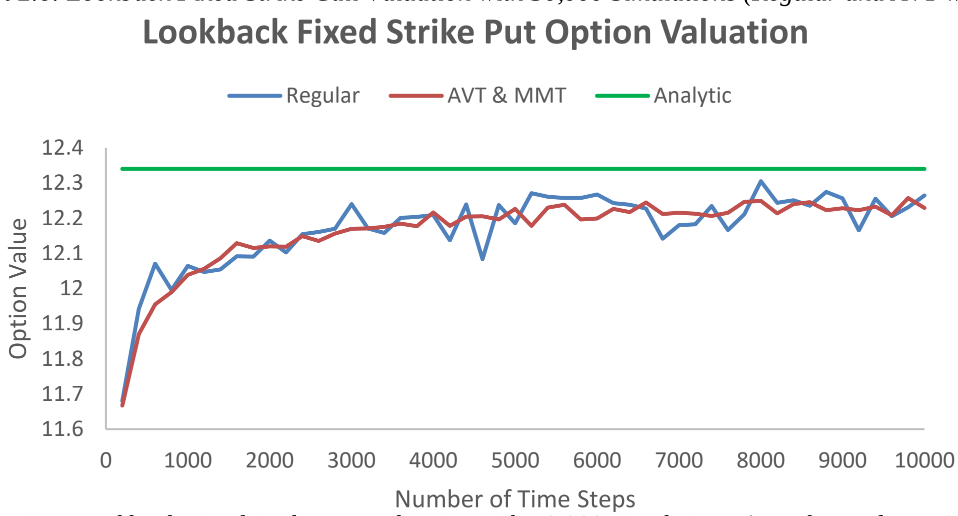
Plot 2.5 illustrates the fact that the accuracy of the numerical approximation is not very sensitive to changes in the number of simulations generated for the valuation. The next section explores the sensitivity of the valuation accuracy with respect to changes in the number of time steps.

Lookback Option Valuation with Different Number of Time-Steps

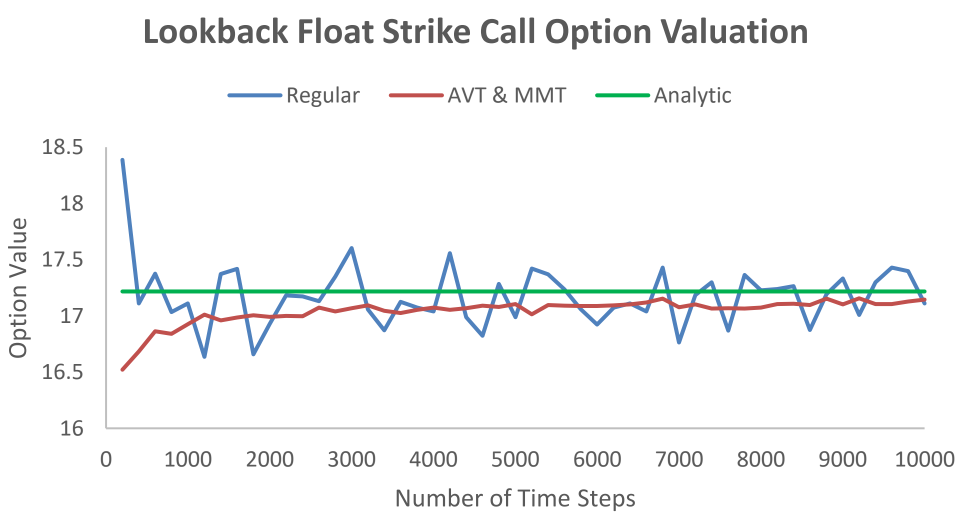
With the purpose of investigating the sensitivity of the numerical valuation behavior with respect to changes in the number of time steps (or number of observations throughout the life of the contract) the numerical valuation is executed using various levels of time steps (M). However, considering the fact that the computation time increases as the number of simulations (and time steps) increase, a fixed number of 50,000 simulations is chosen in order to test how the approximation and its error behave as the number of time steps increases from 200 to 10,000. This behavior is evaluated using the regular Monte Carlo method and the AVT & MMT Monte Carlo method. Plots 2.6, 2.7, 2.8 and 2.9 illustrate the behavior of the numerical approximations as the number of time steps increase:



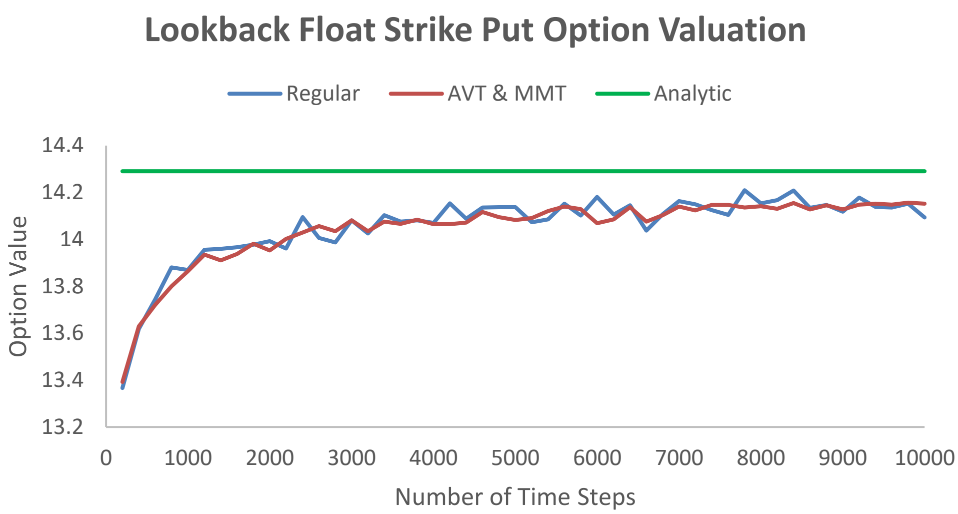
Plot 2.6: Lookback Fixed Strike Call Valuation with 50,000 Simulations (Regular and AVT with MMT)



Plot 2.7: Lookback Fixed Strike Put Valuation with 50,000 Simulations (Regular and AVT with MMT)



Plot 2.8: Lookback Float Strike Call Valuation with 50,000 Simulations (Regular and AVT with MMT)

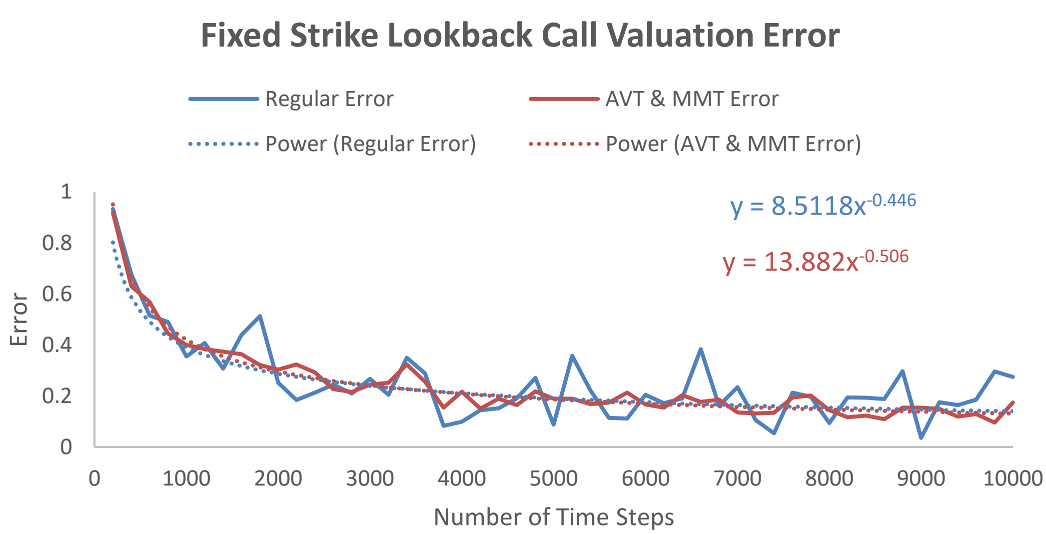


Plot 2.9: Lookback Float Strike Put Valuation with 50,000 Simulations (Regular and AVT with MMT)

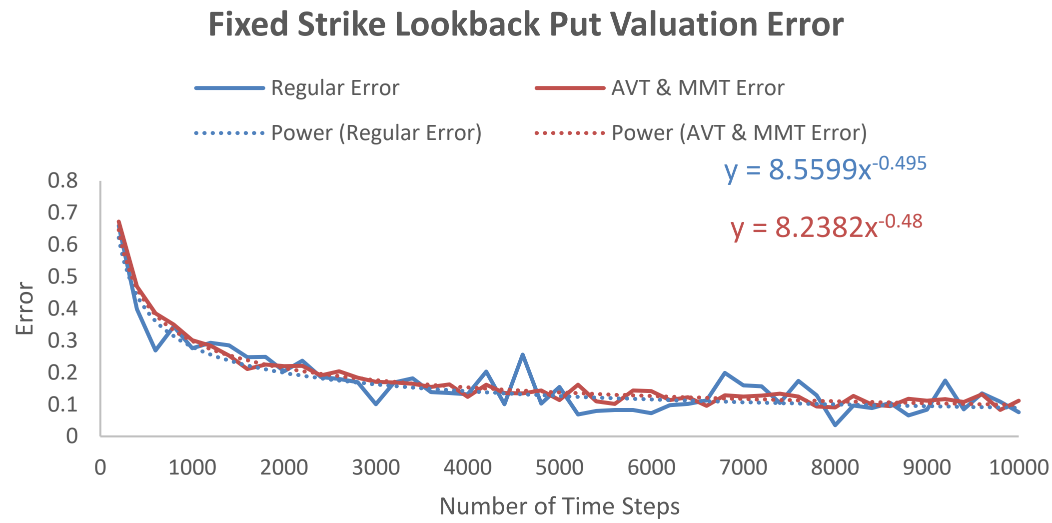
If we compare the convergence behavior of Plots 2.1, 2.2, 2.3 and 2.4 with the convergence behavior of 2.6, 2.7, 2.8 and 2.9 it is quite evident that the convergence improves significantly as one increases the number of time steps of the numerical valuation. According to these plots the accuracy of the valuation appears to be more sensitive to increases in the number of time steps than to increases in the number of simulations. In addition to that, the convergence of the AVT & MMT Monte Carlo approximation towards the analytical solution is much smoother, meaning that the deviations from the previous calculation (with less time steps) is less pronounced when compared to those of the regular Monte Carlo valuation. This proves once again, the superiority of the former method in terms of accuracy.

Error Behavior

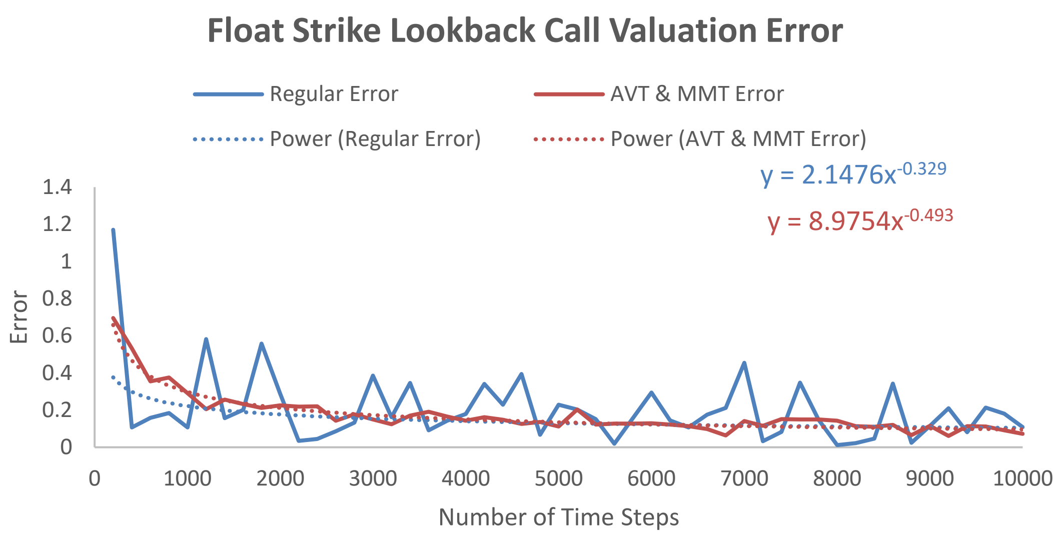
With the intention of further investigating the convergence behavior and quantifying it, error plots are created for each of the four options being priced. Plots 2.10, 2.11, 2.12 and 2.13 show the error behavior for each option:



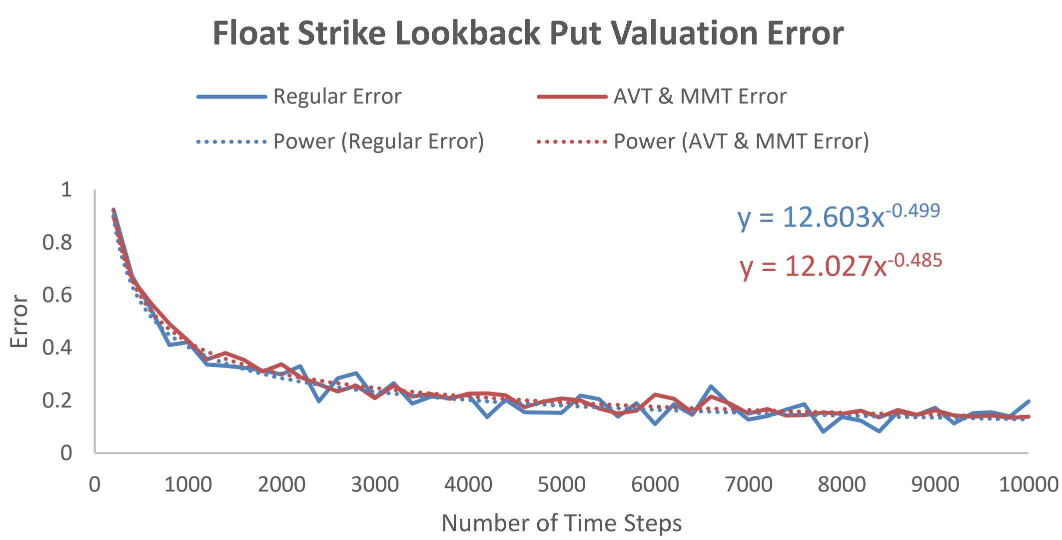
Plot 2.10: Error Size Fixed Strike Lookback Call Valuation with 50,000 Simulations



Plot 2.11: Error Size Fixed Strike Lookback Put Valuation with 50,000 Simulations



Plot 2.12: Error Size Floating Strike Lookback Call Valuation with 50,000 Simulations



Plot 2.13: Error Size Floating Strike Lookback Put Valuation with 50,000 Simulations

By observing Plots 2.10, 2.11, 2.12 and 2.13 it is clearly evident that the approximation error converges to zero as the number of time steps increases. However, it is worth noticing that the accuracy of the approximation increases at a decreasing rate if one looks at how much the error shrinks from 1,000 to 2,000 time-steps and compare this observation to how much the error shrinks from 9,000 to 10,000 time-steps. Once again, the AVT & MMT valuation produces an error that converges towards zero with much less variance than the regular Monte Carlo valuation. However, the convergence rate of the regular Monte Carlo valuation is faster than the convergence rate of the AVT & MMT valuation in both Lookback put options, as it is clearly displayed by the exponent of the equations representing the fitted power line describing the data.

One of the main reasons why the accuracy of the valuation increases as the number of time steps increases, has to do with the fact that the analytical closed-form solution considers continuous time trading. Consequently, as the number of time steps increases, the size of each time step converges towards zero, and as the size of each time step converges towards zero the valuation transitions slowly from discrete to continuous time. Therefore, one must make sure to consider a large number of time steps when trying to value a Lookback option, otherwise the numerical valuation can differ significantly from the closed-form solution, due to the continuous vs. discrete time phenomena already mentioned.

Lookback Option Valuation with Discrete Sampling

Considering the fact that in the real world the underlying asset is observed and sampled a finite and discrete number of times, tables that consider different number of discrete samplings throughout the life of the option contract are created. Since the time to maturity of the option is one year, the table considers monthly (M=12), weekly (M=52), twice-a-week (M=104) and daily (M=252) sampling and compares the performance of these valuations with the approximation that uses 10,000 time steps and the analytical closed-form solution. Table 2.2 reflects the valuations done with the regular Monte Carlo method, while table 2.3 reflects the valuations done with the AVT and MMT Monte Carlo method:

Time Steps (M)	**12**	**52**	**104**	**252**	**10,000**	**Analytic**
Fixed Strike						
Call Valuation	15.6958	17.3738	17.8820	18.3393	18.8932	19.1676
Relative Error	18.11%	9.36%	6.71%	4.32%	1.43%	
Put Valuation	9.8296	11.0298	11.4113	11.6813	12.2640	12.3397
Relative Error	20.34%	10.62%	7.52%	5.34%	0.61%	
Floating Strike						
Call Valuation	14.6815	16.0356	16.2751	16.5227	17.1092	17.2168
Relative Error	14.73%	6.86%	5.47%	4.03%	0.62%	
Put Valuation	10.8587	12.4896	13.0521	13.4583	14.0939	14.2906
Relative Error	24.01%	12.60%	8.67%	5.82%	1.38%	

Table 2.2: Regular Monte Carlo Lookback Option Valuation with 50,000 Simulations and Different Number of Time Steps

Time Steps (M)	12	52	104	252	10,000	Analytic
Fixed Strike						
Call Valuation	15.7558	17.3881	17.9247	18.2912	19.0616	19.1676
Relative Error	17.80%	9.28%	6.48%	4.57%	0.55%	
Put Valuation	9.8156	11.0506	11.3950	11.7200	12.2285	12.3397
Relative Error	20.46%	10.45%	7.66%	5.02%	0.90%	
Floating Strike						
Call Valuation	14.6735	15.9453	16.3129	16.6354	17.1451	17.2168
Relative Error	14.77%	7.39%	5.25%	3.38%	0.42%	
Put Valuation	10.8745	12.5170	13.0223	13.4982	14.1529	14.2906
Relative Error	23.90%	12.41%	8.87%	5.54%	0.96%	

Table 2.3: AVT & MMT Monte Carlo Lookback Option Valuation with 50,000 Simulations and Different Number of Time Steps

Both tables clearly corroborate the fact that the number of time steps is crucial to achieve a decent level of accuracy when pricing a Lookback option via Monte Carlo simulations. It is evident that the numerical Monte Carlo approximation performs quite poorly when considering discrete sampling (monthly, weekly or even daily). Furthermore, out of the 20 numerical valuations exposed on each table, the AVT & MMT Monte Carlo method has a lower relative error than the relative error of the regular Monte Carlo method in 13 occasions, which further confirms the benefits of using variance reduction techniques when pricing derivatives numerically via Monte Carlo.

Conclusion

During the development of this project the value of a Binary and Lookback options was obtained through the implementation of numerical Monte Carlo approximation methods. The accuracy of the various results obtained was evaluated by changing the number of simulations performed per valuation, analyzing the impact of these changes, measuring the precision achieved and applying other techniques such as the AVT and MMT with the purpose of minimizing variance in the results of the derivative valuation. One of the main conclusions that can be drawn from the results yielded is that the accuracy of the approximation increases at a decreasing rate as the number of simulations generated (I) increases. However, this conclusion is far more relevant in the case of Binary options than in the case of Lookback options. Additionally, it was proved by the results obtained in Parts I and II that by implementing different variance reduction techniques such as AVT and MMT, the accuracy of the numerical approximation can be improved quite significantly. Furthermore, after investigating the sensitivity of the convergence behavior of the Lookback options numerical approximation towards the value of the closed-form solution, it was proven that the accuracy improves significantly as the number of time-steps (M) in the valuation increases. This is due to the fact that the analytical formula considers a continuous time framework. On the other hand, the accuracy of the approximation for the Lookback options doesn't really improve much by increasing the number of simulations (I) above 100,000. Nevertheless, it improves up to 100,000. Finally, it is appropriate to conclude that Monte Carlo methods are indeed a very effective technique when it comes to the valuation of financial derivatives. However, the individual or machine generating the valuation will always have to face a trade off between accuracy and speed, given that these two are inversely related when implementing numerical Monte Carlo approximation methods.

References

Wilmott, Paul. *Paul Wilmott on Quantitative Finance-3 Volume Set*. John Wiley & Sons, 2006.

Hull, John C. *Options futures and other derivatives*. Pearson Education India, 2003.

Appendix

Appendix I

Binary Call Option Formula:

$$Call_{Binary} = e^{-rT}N(d_2)$$

Binary Put Option Formula:

$$Put_{Binary} = e^{-rT}N(-d_2)$$

Where,

$$d_2 = \frac{\ln\left(\frac{S}{X}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Appendix II

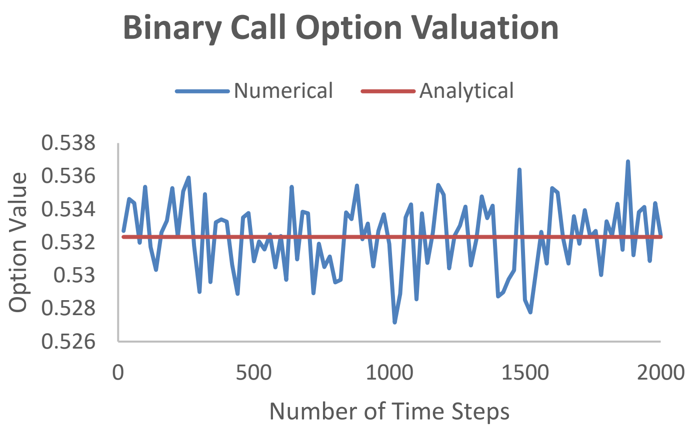
Central Limit Theorem:

Central Limit Theorem provides such a characterization, and more:

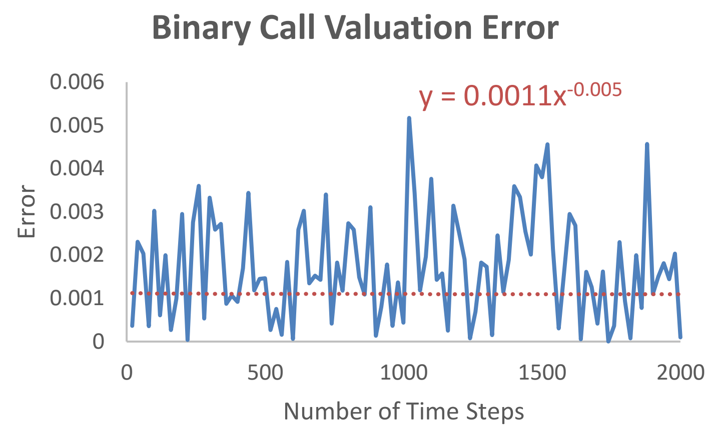
$$\sqrt{n}(\bar{X}_n - \mu) \rightarrow N(0, \sigma^2)$$

where σ^2 and μ are the population variance and mean respectively. This is saying that taking average of random samples is shrinking the standard deviation at a rate of $\frac{1}{\sqrt{n}}$, and the distribution of sample average is roughly normal when *n* is large.

Appendix III



Binary Call Valuation with 50,000 Simulations



Error Size Binary Call Valuation with 50,000 Simulations

Appendix IV

Antithetic Variable Technique (Hull, 2003):

In the antithetic variable technique, a simulation trial involves calculating two values of the derivative. The first value f_1 is calculated in the usual way; the second value f_2 is calculated by changing the sign of all the random samples from standard normal distributions. (If ϕ is a sample used to calculate f_1, then -ϕ is the corresponding sample used to calculate f_2.) The sample value of the derivative calculated from a simulation trial is the average of f_1 and f_2. This works well because when one value is above the true value, the other tends to be below, and vice versa.

Denote $\bar{f}$ as the average of f_1 andf_2:

$$\bar{f} = \frac{f_1 + f_2}{2}$$

The final estimate of the value of the derivative is the average of the $\bar{f}$'s.

Moment Matching (Hull, 2012):

Moment matching involves adjusting the samples taken from a standardized normal distribution so that the first, second, and possibly higher moments are matched. Suppose that we sample from a normal distribution with mean 0 and standard deviation 1 to calculate the change in the value of a particular variable over a particular time period. Suppose that the samples are $\phi_i (1 \le i \le n)$. To match the first two moments, we calculate the mean of the samples, m, and the standard deviation of the samples, σ. We then define adjusted samples ϕ_i^* $(1 \le i \le n)$ as

$$\phi_i^* = \frac{\phi_i - m}{\sigma}$$

These adjusted samples have the correct mean of 0 and the correct standard deviation of 1.0. We use the adjusted samples for all calculations. Moment matching saves computation time but can lead to memory problems because every number sampled must be stored until the end of the simulation. Moment matching is sometimes termed quadratic resampling. It is often used in conjunction with the antithetic variable technique. Because the latter automatically matches all odd moments, the goal of moment matching then becomes that of matching the second moment and, possibly, the fourth moment.

Appendix V

FIXED STRIKE LOOKBACK OPTIONS:

Fixed Strike Lookback Call Option Formula:

$$Call_{Lookback\ Fixed} = SN(d_1) - Ke^{-rT}N(d_2) + Se^{-rT}\frac{\sigma^2}{2r}\left[-\left(\frac{S}{K}\right)^{-\left(\frac{2r}{\sigma^2}\right)} N\left(d_1 - \frac{2r\sqrt{T}}{\sigma}\right) + e^{rT}N(d_1)\right]$$

Fixed Strike Lookback Put Option Formula:

$$Put_{Lookback\ Fixed} = Ke^{-rT}N(-d_2) - SN(-d_1) + Se^{-rT}\frac{\sigma^2}{2r}\left[\left(\frac{S}{K}\right)^{-\left(\frac{2r}{\sigma^2}\right)} N\left(-d_1 + \frac{2r\sqrt{T}}{\sigma}\right) - e^{rT}N(-d_1)\right]$$

where $d_1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

FLOATING STRIKE LOOKBACK OPTIONS:

Floating Strike Lookback Call Option Formula:

$$Call_{Lookback\ Float} = SN(d_1) - S_{min}e^{-rT}N(d_2) + Se^{-rT}\frac{\sigma^2}{2r}\left[\left(\frac{S}{S_{min}}\right)^{-\left(\frac{2r}{\sigma^2}\right)} N\left(-d_1 + \frac{2r\sqrt{T}}{\sigma}\right) - e^{rT}N(-d_1)\right]$$

where $d_1 = \frac{\ln\left(\frac{S}{S_{min}}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln\left(\frac{S}{S_{min}}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

Floating Strike Lookback Put Option Formula:

$$Put_{Lookback\ Float}$$

$$= S_{max}e^{-rT}N(-d_2) - SN(-d_1) + Se^{-rT}\frac{\sigma^2}{2r}\left[-\left(\frac{S}{S_{max}}\right)^{-\left(\frac{2r}{\sigma^2}\right)}N\left(d_1 - \frac{2r\sqrt{T}}{\sigma}\right) + e^{rT}N(d_1)\right]$$

where $d_1 = \frac{\ln\left(\frac{S}{S_{max}}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln\left(\frac{S}{S_{max}}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$