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Jaume Llibre, and Clàudia Valls

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Global analytic first integrals for the real planar Lotka-Volterra system

Jaume Llibre[a)]
Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, Barcelona, 08193 Catalonia, Spain

Clàudia Valls[b)]
Departamento de Matemática, Instituto Superior Técnico, Avenida Rovisco Pais, 1049-001 Lisboa, Portugal

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We provide the complete classification of all Lotka-Volterra systems of the form $\dot{x}=x(ax+by+c)$ and $\dot{y}=y(Ax+By+C)$ in $\mathbb{R}^2$ having a global analytic first integral. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2713076]

I. INTRODUCTION

The nonlinear ordinary differential equations appear in a natural way in many branches of applied mathematics, physics, chemistry, economy, etc. For a two-dimensional system, the existence of a first integral determines completely its phase portrait. For such systems the notion of integrability is based on the existence of a first integral. Then a natural question is: *Given a system of ordinary differential equations in $\mathbb{R}^2$ depending on parameters, how do we recognize the values of the parameters for which the system has a first integral*?

The easiest planar integrable systems are the Hamiltonian ones; i.e., the systems in $\mathbb{R}^2$ that can be written as

$$\dot{x} = -\frac{\partial H}{\partial y}, \quad \dot{y} = \frac{\partial H}{\partial x},$$

for some function $H:\mathbb{R}^2\to\mathbb{R}$ of class C^2. The planar integrable systems which are not Hamiltonian are in general very difficult to detect. The goal of this paper is to present the complete classification of the global analytic first integrals for the quadratic Lotka-Volterra systems,

$$\dot{x} = x(ax+by+c),$$

$$\dot{y} = y(Ax+By+C), \tag{1}$$

in $\mathbb{R}^2$. Here *a global analytic first integral* or simply *an analytic first integral* is a nonconstant analytic function $H:\mathbb{R}^2\to\mathbb{R}$, whose domain of definition is the whole $\mathbb{R}^2$, and it is constant on the solutions of system (1). This last assertion means that for any solution $(x(t),y(t))$ of Eq. (1), we have

$$\frac{dH}{dt}(x(t),y(t)) = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = 0.$$

[a)]Electronic mail: jllibre@mat.uab.cat
[b)]Electronic mail: cvalls@math.ist.utl.pt

We note that a complete characterization of the global analytic first integrals of polynomial differential systems has been made for very few families of differential systems, and as far as we know, this is the first time that such a classification is ended for a family of differential systems depending of six parameters.

The two-dimensional Lotka-Volterra system [Eq. (1)] has been a paradigmatic system for the study of the integrability. Thus recently for such systems, the classification of the ones having a Liouvillian first integral has been completed (see Refs. 2–4, 6, and 8–11 and for a definition of Liouvillian first integral, see Singer[13]).

The two-dimensional Lotka-Volterra system [Eq. (1)], which was introduced by Lotka[8] and Volterra,[14] appeared in ecology. It models two species in competition, and it has been widely used in applied mathematics, in chemistry, and in a large variety of problems in physics: laser physics, plasma physics, convective instabilities, neural networks, etc. (see, for instance, the references given in Almeida *et al.*[2]).

We reduce the study of the six parameter family of quadratic Lotka-Volterra system [Eq. (1)] to study 12 subfamilies having one, two, three, or four parameters. More precisely, we have the following result, whose proof is given in Sec. II.

Proposition 1: *All the quadratic Lotka-Volterra systems [Eq. (1)] can be transformed via an affine change of variables and a rescaling of the time to one of the following 12 systems:*

$$\text{(lv1)} \quad \dot{x} = x(ax+by+1), \quad \dot{y} = y((1+a)x-y+C) \quad \text{with } a(b+1)C \neq 0.$$

$$\text{(lv2)} \quad \dot{x} = cx, \quad \dot{y} = y(Ax+By+C) \quad \text{with } A^2+B^2 \neq 0.$$

$$\text{(lv3)} \quad \dot{x} = x(y+c), \quad \dot{y} = y(By+1).$$

$$\text{(lv4)} \quad \dot{x} = xy, \quad \dot{y} = y(x+By+1).$$

$$\text{(lv5)} \quad \dot{x} = xy, \quad \dot{y} = y(Ax+By).$$

$$\text{(lv6)} \quad \dot{x} = x(ax+by+c), \quad \dot{y} = By^2 \quad \text{with } a^2+b^2+B^2 \neq 0.$$

$$\text{(lv7)} \quad \dot{x} = x(y+1), \quad \dot{y} = y(x+By).$$

$$\text{(lv8)} \quad \dot{x} = x(ax+by+c), \quad \dot{y} = xy.$$

$$\text{(lv9)} \quad \dot{x} = x(ax+by), \quad \dot{y} = y(x+y).$$

$$\text{(lv10)} \quad \dot{x} = x(ax+by+1), \quad \dot{y} = y(x+y).$$

$$\text{(lv11)} \quad \dot{x} = x(x+c), \quad \dot{y} = y(Ax+1) \quad \text{with } cA = 0.$$

$$\text{(lv12)} \quad \dot{x} = x(x+y+1), \quad \dot{y} = y(Ax+y+1).$$

Our main result is the following.

Theorem 1: *The unique Lotka-Volterra systems [Eq. (1)] having a global analytic first integral $H=H(x,y)$ are the following ones.*

(a) *Systems (lv1) with $C=-p/q$, $b=p_1/q_1$, and $a=-qq_1/(p(p_1+q_1))$ with $p, q, p_1, q_1 \in \mathbb{N}$ and $pp_1-qq_1 \geqslant 0$, then $H=x^{pq_1}y^{qq_1}(pp_1+pq_1-qq_1x+p_1qy+qq_1y)^{pp_1-qq_1}$.*

(b) *Systems (lv2) with*

(b.1) $c=0$, *then* $H=x$;
(b.2) $c>0$ *and* $B=C=0$, *then* $H=e^{-Ax}y^{c}$;
(b.3) $c<0$ *and* $B=C=0$, *then* $H=e^{Ax}y^{-c}$; *and*
(b.4) $c/C=-p/q$ *and* $B=0$ *with* $p,q\in\mathbb{N}$, *then* $H=e^{Aqx/C}x^{q}y^{p}$.

(c) *Systems (lv3) with*
(c.1) $c=B=0$, *then* $H=xe^{-y}$;
(c.2) *with* $c=0$ *and* $B=-p/q$ *with* $p,q\in\mathbb{N}$, *then* $H=x^{p}(q-py)^{q}$;
(c.3) $c=-p/q$ *with* $p,q\in\mathbb{N}$ *and* $B=0$, *then* $H=e^{-qy}x^{q}y^{p}$; *and*
(c.4) $c=-p/q$ *and* $B=-p_1/q_1$ *with* $p, q, p_1, q_1\in\mathbb{N}$ *and* $qq_1-pp_1\geqslant 0$, *then* $H=x^{p_1q}y^{pp_1}(q_1-p_1y)^{qq_1-pp_1}$.

(d) *Systems (lv4) with*
(d.1) $B=0$, *then* $H=xe^{x-y}$;
(d.2) $B=-p/q$ *with* $p,q\in\mathbb{N}$, *then* $H=x^{p}\,(y-(q/p)-[q/(q+p)]x)^{q}$.

(e) *Systems (lv5) with*
(e.1) $B=0$, *then* $H=Ax-y$ *and*
(e.2) $B=-p/q$ *with* $p,q\in\mathbb{N}$, *then* $H=x^{p}(Aqx-(p+q)y)^{q}$.

(f) *Systems (lv6) with*
(f.1) $B=0$, *then* $H=y$;
(f.2) $a=b=0$, *then* $H=x^{|B|}$;
(f.3) $a=0$, $b>0$, *and* $B<0$, *then* $H=y^{b}/x^{B}$; *and*
(f.4) $a=0$, $b<0$, *and* $B>0$, *then* $H=x^{B}/y^{b}$.

(g) *Systems (lv7) with* $B=0$, *then* $H=ye^{y-x}$.

(h) *Systems (lv8) with*
(h.1) $a=0$ *and* $c<0$, *then* $H=y^{-c}e^{x-by}$;
(h.2) $a=0$ *and* $c\geqslant 0$, *then* $H=y^{c}e^{-x+by}$; *and*
(h.3) $a=-p/q$ *with* $p,q\in\mathbb{N}$, *then* $H=y^{p}(-c-p/qc+p/qx+p^2/q^2x-pb/qy)^{q}$.

(i) *Systems (lv9) with* $(a,b)\neq(1,1)$ *and* $a-1$, $(1-b)a$, *and* $b-a$ *have all the same signs, then* $H=x^{|a-1|}y^{|a(1-b)|}((a-1)x+(b-1)y)^{|b-a|}$.

The proof of Theorem 1 is given in Secs. IV and V. For clarity in the exposition, Sec. V deals with system (lv1) and Sec. IV deals with systems (lv2)–(lv12).

Since all the global analytic first integrals described in Theorem 1 are given by elementary functions, it follows that all the analytic first integrals are Liouvillian (see for more details Ref. 12).

II. NORMAL FORMS

In this section we prove Proposition 1.

Proof of Proposition 1: If $c(a-A)B\neq 0$, then system (1) becomes the system

$$\dot{x}=x(-\bar{B}x+(\bar{C}-1)y+1),$$

$$\dot{y}=y((1-\bar{B})x-y+\bar{A}), \tag{2}$$

with the following rescaling of the variables:

$$(x,y,t)\rightarrow\left(\frac{c}{A-a}x,-\frac{c}{B}y,\frac{1}{c}t\right), \tag{3}$$

where

$$\bar{A}=\frac{C}{c},\quad \bar{B}=\frac{a}{a-A},\quad \bar{C}=\frac{B-b}{B}. \tag{4}$$

We divide systems (1) into three subclasses:

(i) $a(B-b)C\neq 0$ and $c(a-A)B\neq 0$;
(ii) $a(B-b)C=0$ and $c(a-A)B\neq 0$; and
(ii) $c(a-A)B=0$.

The two first classes can be transformed in systems (2). Moreover, the first class becomes the class (lv1) if we redefine with $a=-\bar{B}$, $b=\bar{C}-1$, and $C=\bar{A}$.

We subdivide case (iii) into the following two subcases:

(iii.1) $a(B-b)C\neq 0$ and $c(a-A)B=0$ and

(iii.2) $a(B-b)C=0$ and $c(a-A)B=0$.

Doing the change of variables $(x,y)\to(y,x)$, it is immediate to check that the expressions $a(B-b)C$ and $c(a-A)B$ are interchanged. Therefore, this change of variables interchanges cases (ii) and (iii.1). In other words, case (ii) is contained in case (iii.1). Again due to the change of variables $(x,y)\to(y,x)$ to study case (iii) is equivalent to study the case $a(B-b)C=0$.

Now we shall reduce the study of systems (1) with $a(B-b)C=0$ to analyze the subclasses (lv$k$) of systems (1) for $k=2,\ldots,12$.

We divide the class of systems (1) satisfying $a(B-b)C=0$ into the following four subclasses.

Case 1: $a=0$, $C\neq 0$, and $B-b$ arbitrary. Then, doing the rescaling $(x,y,t)\to(\alpha x,\beta y,t/C)$ in system (1), we obtain the system

$$\dot{x}=x\left(\frac{b\beta}{C}y+\frac{c}{C}\right),\quad \dot{y}=y\left(\frac{A\alpha}{C}x+\frac{B\beta}{C}y+1\right). \tag{5}$$

Now we consider the following three subcases:

(1.1) $b=0$. Then, system (1) is contained into system (lv2) after a redefinition of the parameters.

(1.2) $b\neq 0$ and $A=0$. Taking $\beta=C/b$, system (5) becomes $\dot{x}=x(y+c/C)$ and $\dot{y}=y(By/b+1)$; i.e., system (lv3) after a redefinition of the parameters.

(1.3) $bA\neq 0$. Taking $\alpha=C/A$ and $\beta=C/b$, system (5) becomes $\dot{x}=x(y+c/C)$ and $\dot{y}=y(x+By/b+1)$, or equivalently $\dot{x}=x(y+c)$ and $\dot{y}=y(x+By+1)$. Now, if $c\neq 0$, then doing the changes of variables $x=c/Y$, $y=cX/Y$, the rescaling of the independent variable $t\to Yt/c$, and finally $X=-x$, $Y=-y$, we get system (lv10). If $c=0$, then we obtain system (lv4).

Case 2: $a=0$, $C=0$, and $B-b$ arbitrary. Then, doing the rescaling $(x,y,t)\to(\alpha x,\beta y,\gamma t)$ in system (1), we obtain the system

$$\dot{x}=x(b\beta\gamma y+c\gamma),\quad \dot{y}=y(A\alpha\gamma x+B\beta\gamma y). \tag{6}$$

We deal with the following three subcases.

(2.1) $b=0$. We obtain a particular case of system (lv2).

(2.2) $b\neq 0$ and $c=0$. Taking $\beta=1/b$ and $\gamma=1$ in Eq. (6), we get system (lv5).

(2.3) $bc\neq 0$. Taking $\beta=c/b$ and $\gamma=1/c$ in Eq. (6), we get a particular case of system (lv6) if $A=0$. Additionally, if $A\neq 0$ taking $\alpha=c/A$, we obtain system (lv7).

Case 3: $a\neq 0$, $C=0$, and $B-b$ arbitrary. Then, doing the rescaling $(x,y,t)\to(\alpha x,\beta y,\gamma t)$ in system (1), we obtain the system

$$\dot{x}=x(a\alpha\gamma x+b\beta\gamma y+c\gamma),\quad \dot{y}=y(A\alpha\gamma x+B\beta\gamma y).$$

We consider the following four subcases.

(3.1) $A=0$. Then we obtain system (lv6).

(3.2) $A\neq 0$ and $B=0$. Taking $\alpha\gamma=1/A$, we get system (lv8).

(3.3) $AB\neq 0$ and $c=0$. Taking $\alpha\gamma=1/A$ and $\beta\gamma=1/B$, we have system (lv9).

(3.4) $ABc\neq 0$. Taking $\gamma=1/c$, $\alpha=c/A$, and $\beta=c/B$, we obtain system (lv10).

Case 4: $a\neq 0$, $C\neq 0$, and $B-b=0$. Then, doing the rescaling $(x,y,t)\to(Cx/a,\beta y,t/C)$ in system (1), we obtain the system

$$\dot{x} = x\left(x + \frac{b\beta}{C}y + \frac{c}{C}\right), \quad \dot{y} = y\left(\frac{A}{a}x + \frac{b\beta}{C}y + 1\right). \tag{7}$$

We distinguish the following two subcases.

(4.1) $b=0$. Then, we obtain system $\dot{x}=x(x+c)$ and $\dot{y}=y(Ax+1)$. If $cA\neq 0$, then doing the change of variables $x=Y$, $y=X$, and a rescaling of the independent variable, this system becomes system (lv3). If $cA=0$, then we get system (lv11).

(4.2) $b\neq 0$. Taking $\beta=C/b$, system (7) goes over to system $\dot{x}=x(x+y+c)$ and $\dot{y}=y(Ax+y+1)$. If $c\neq 1$ and $A\neq 1$, then doing the change of variables $x=(1-c)Y/((A-1)X)$, $y=(1-c)/X$, and the rescaling of the independent variable $t\to Xt/(c-1)$, we obtain system (lv10). If $c\neq 1$ and $A=1$, then doing the change of variables $x=1/X$, $y=Y/X$, and the rescaling of the independent variable $t\to Xt/(1-c)$, we obtain system (lv8). If $c=1$, we get system (lv12). ∎

III. PRELIMINARY RESULTS

In this section we introduce two auxiliary results that will be used throughout the paper.

We write Eq. (1) as the autonomous system,

$$\dot{x} = f_1(x,y), \quad \dot{y} = f_2(x,y). \tag{8}$$

Let $f(x,y)=(f_1(x,y),f_2(x,y))$. We will denote by $Df(0)$ the Jacobian matrix of system (8) at $(x,y)=(0,0)$ and by Df the Jacobian matrix of system (8) at an arbitrary point (x,y) that will be explicitly especified.

The following result is due to Poincaré (see Ref. 1) and its proof can be found in Ref. 5.

Throughout this paper $\mathbb{Z}^+$ will denote the set of non-negative integers.

Theorem 2: *Assume that the eigenvalues λ_1 and λ_2 of Df at some singular point $(\bar{x},\bar{y})$ do not satisfy any resonance condition of the form*

$$\lambda_1 k_1 + \lambda_2 k_2 = 0 \quad \text{for } k_1,k_2 \in \mathbb{Z}^+ \quad \text{with } k_1 + k_2 > 0.$$

Then system (8) has no global analytic first integrals.

The following result is due to Li *et al.* (see Ref. 7).

Theorem 3: *Assume that the eigenvalues λ_1 and λ_2 of Df at some singular point $(x,y)=(\bar{x},\bar{y})$ satisfy that $\lambda_1=0$ and $\lambda_2\neq 0$. Then, system (8) has no global analytic first integrals if the singular point $(x,y)=(\bar{x},\bar{y})$ is isolated.*

IV. ANALYTIC FIRST INTEGRALS FOR SYSTEM (LV1)

This section is dedicated to classify the analytic first integrals of system (lv1).

Proposition 2: *Assume that $C=-p/q$, $b=p_1/q_1$,* and *$a=-p_2/q_2$, where $p,q,p_1,q_1,p_2,q_2\in\mathbb{N}$ and either $pp_1-qq_1<0$ or $qq_2-p_2(p+q)<0$, then system* (lv1) *has no analytic first integrals.*

Proof: System (lv1) becomes

$$\dot{x} = x\left(-\frac{p_2}{q_2}x + \frac{p_1}{q_1}y + 1\right), \quad \dot{y} = y\left(\frac{q_2-p_2}{q_2}x - y - \frac{p}{q}\right). \tag{9}$$

We will proceed by contradiction. Assume that $F(x,y)$ is an analytic first integral of system (9). Then $F(x,y)$ must satisfy

$$\frac{\partial F}{\partial x}x\left(-\frac{p_2}{q_2}x + \frac{p_1}{q_1}y + 1\right) + \frac{\partial F}{\partial y}y\left(\frac{q_2-p_2}{q_2}x - y - \frac{p}{q}\right) = 0.$$

We denote by $\bar{F}=\bar{F}(y)$ the restriction of F to $x=0$ and by $\hat{F}=\hat{F}(x)$ the restriction of F to $y=0$. Then, clearly, $\bar{F}$ satisfies

$$y\left(-y-\frac{p}{q}\right)\frac{d\bar{F}}{dy}=0 \quad \text{that is } \bar{F}=c_0\in\mathbb{R}$$

and $\hat{F}$ satisfies

$$x\left(-\frac{p_2}{q_2}x+1\right)\frac{d\hat{F}}{dx}=0 \quad \text{that is } \hat{F}=c_1\in\mathbb{R}.$$

Then, since $\bar{F}_{|y=0}=\hat{F}_{|x=0}$, we get that $c_1=c_0$ and $F=c_0+xyG(x,y)$ for some function $G=G(x,y)$, which is a formal power series in the variables x and y. Then G satisfies

$$\frac{\partial G}{\partial x}x\left(-\frac{p_2}{q_2}x+\frac{p_1}{q_1}y+1\right)+\frac{\partial G}{\partial y}y\left(\frac{q_2-p_2}{q_2}x-y-\frac{p}{q}\right)=-\left(\frac{q_2-2p_2}{q_2}x+\frac{p_1-q_1}{q_1}y+1-\frac{p}{q}\right)G. \tag{10}$$

Now we consider two different cases.

Case 1: $pp_1-qq_1<0$. In this case, we write

$$G(x,y)=\sum_{k=0}^{\infty}G_k(y)x^k. \tag{11}$$

We will show by induction that

$$G_k(y)=0 \quad \text{for } k\geqslant 0. \tag{12}$$

We restrict G to $x=0$. We note that $G_{|x=0}=G_0(y)$. Then we have

$$y\left(-y-\frac{p}{q}\right)\frac{dG_0}{dy}=-\left(\frac{p_1-q_1}{q_1}y+1-\frac{p}{q}\right)G_0.$$

Therefore,

$$\frac{dG_0}{G_0}=-\left(\frac{p_1-q_1}{q_1}y+1-\frac{p}{q}\right)\frac{dy}{y(-y-(p/q))},$$

which yields

$$G_0(y)=K_0y^{(q-p)/p}(p+qy)^{(pp_1-qq_1)/pq_1}, \quad K_0\in\mathbb{R}.$$

Since $pp_1-qq_1<0$ and $G_0(y)$ must be global analytic, it follows that $K_0=0$ and Eq. (12) is proved for $k=0$. Now we assume that Eq. (12) is true for $k=N$ and we will prove it for $k=N+1$. In view of Eq. (11), we have

$$G(x,y)=x^{N+1}H(x,y),$$

where $H(x,y)$ is a formal power series in the variables x and y and satisfies

$$\frac{\partial H}{\partial x}x\left(-\frac{p_1}{q_1}x+\frac{p_1}{p_1+q_1}y+1\right)+\frac{\partial H}{\partial y}y\left(\frac{q_1-p_1}{q_1}x-y-\frac{p}{q}\right)$$
$$=-\left(\frac{q_2-(N+3)p_2}{q_2}x+\frac{(N+2)p_1-q_1}{q_1}y+N+2-\frac{p}{q}\right)H.$$

We restrict H to $x=0$ and we get $G_{N+1}=G_{N+1}(y)$. Then,

$$y\left(-y-\frac{p}{q}\right)\frac{dG_{N+1}}{dy}=-\left(\frac{(N+2)p_1-q_1}{q_1}y+N+2-\frac{p}{q}\right)G_{N+1},$$

that is,

$$\frac{dG_{N+1}}{G_{N+1}}=-\left(\frac{(N+2)p_1-q_1}{q_1}y+N+2-\frac{p}{q}\right)\frac{dy}{y(-y-(p/q))},$$

which yields

$$G_{N+1}(y)=K_{N+1}y^{((N+2)q-p)/p}(p+qy)^{(N+2)[(pp_1-qq_1)/pq_1]},\quad K_{N+1}\in\mathbb{R}.$$

Since $pp_1-qq_1<0$ and $G_{N+1}(y)$ must be global analytic, it follows that $K_{N+1}=0$ and consequently $G_{N+1}=0$, which concludes the proof of Eq. (12). Then, the proprosition is proved in this case.

Case 2: $qq_2-p_2(p+q)<0$. In this case we write

$$G(x,y)=\sum_{k=0}^{\infty}G_k(x)y^k. \tag{13}$$

We will show by induction that

$$G_k(x)=0\quad\text{for } k\geqslant 0. \tag{14}$$

We restrict G to $y=0$. We note that $G_{|y=0}=G_0(x)$. Then, from Eq. (10) we have

$$x\left(-\frac{p_2}{q_2}x+1\right)\frac{dG_0}{dx}=-\left(\frac{q_2-2p_2}{q_2}x+1-\frac{p}{q}\right)G_0.$$

Therefore,

$$\frac{dG_0}{G_0}=-\left(\frac{q_2-2p_2}{q_2}x+1-\frac{p}{q}\right)\frac{dx}{x(-(p_2/q_2)x+1)},$$

which yields

$$G_0(x)=C_0x^{(p-q)/q}(q_2-p_2x)^{(qq_2-p_2(p+q))/qp_2},\quad C_0\in\mathbb{R}.$$

Since $qq_2-p_2(p+q)<0$ and $G_0(x)$ must be global analytic, it follows that $C_0=0$ and Eq. (14) is proved for $k=0$. Now we assume Eq. (14) is true for $k=N$ and we will prove it for $k=N+1$. In view of Eq. (13), we have

$$G(x,y)=y^{N+1}H(x,y),$$

where $H(x,y)$ is a formal power series in the variables x and y that satisfies

$$\frac{\partial H}{\partial x}x\left(-\frac{p_2}{q_2}x+\frac{p_1}{q_1}y+1\right)+\frac{\partial H}{\partial y}y\left(\frac{q_2-p_2}{q_2}x-y-\frac{p}{q}\right)$$
$$=-\left(\frac{(N+2)q_2-(N+3)p_2}{q_2}x+\frac{p_1-(N+2)q_1}{q_1}y+1-(N+2)\frac{p}{q}\right)H.$$

We restrict H to $y=0$ and we get $G_{N+1}=G_{N+1}(x)$. Then we have

$$x\left(-\frac{p_2}{q_2}x+1\right)\frac{dG_{N+1}}{dx}=-\left(\frac{(N+2)q_2-(N+3)p_2}{q_2}x+1-(N+2)\frac{p}{q}\right)G_{N+1}.$$

Therefore,

$$\frac{dG_{N+1}}{G_{N+1}} = -\left(\frac{(N+2)q_2-(N+3)p_2}{q_2}x + 1 - (N+2)\frac{p}{q}\right)\frac{dx}{x(-(p_2/q_2)x+1)},$$

which yields

$$G_{N+1}(x) = C_{N+1}x^{((N+2)p-q)/q}(q_2-p_2x)^{(N+2)(q_2q-p_2p-p_2q)/p_2q}, \quad C_{N+1} \in \mathbb{R}.$$

Since $qq_2-p_2(p+q)<0$ and $G_{N+1}(x)$ must be global analytic, it follows that $C_{N+1}=0$ and consequently $G_{N+1}=0$, which concludes the proof of Eq. (14). So the proof of the proposition is done.■

We note that the conditions of Proposition 5 can be written as $C\in\mathbb{Q}^-$, $a\in\mathbb{Q}^-$, $b\in\mathbb{Q}^+\setminus\{0\}$, and $1+bC>0$ or $-1+a(C-1)>0$.

Proposition 3: *For system (lv1) with a, b, and C not satisfying the hypotheses of Proposition 2, the following statements hold.*

(a) $x^{pq_1}y^{qq_1}(pp_1+pq_1-qq_1x+p_1qy+qq_1y)^{pp_1-qq_1}$ *is an analytic first integral if* $C=-p/q$, $b=p_1/q_1$, *and* $a=-qq_1/(p(p_1+q_1))$ *with* $p,q,p_1,q_1\in\mathbb{N}$ *and* $pp_1-qq_1\geqslant 0$.

(b) *For any other value of the parameters a, b, and C, system (lv1) has no analytic first integrals.*

Proof: It is easy to check that statement (a) holds. We note that statement (a) contains the case $C=-p/q$, $b=p_1/q_1$, and $a\in\mathbb{Q}^-$ with $1+bC=0$ and $-1+a(C-1)=0$.

The cases which are not covered by statement (a) are as follows.

(i) $C\notin\mathbb{Q}^-\cup\{0\}$;
(ii) $C\in\mathbb{Q}^-$ and $a\notin\mathbb{Q}^-\cup\{0\}$;
(iii) $C\in\mathbb{Q}^-$, $a\in\mathbb{Q}^-$, and $b\notin\mathbb{Q}^+$;
(iv) $C=-p/q$, $b=p_1/q_1$, and $a=-qq_1/(p(p_1+q_1))$ with $p,q,p_1,q_1\in\mathbb{N}$ and $pp_1-qq_1<0$;
(v) $C=-p/q$, $b=p_1/q_1$, and $a\in\mathbb{Q}^-$ with $a\neq -qq_1/(p(p_1+q_1))$ and $1+bC=0$ and $-1+a(C-1)<0$, where $p,q,p_1,q_1\in\mathbb{N}$;
(vi) $C=-p/q$, $b=p_1/q_1$, and $a\in\mathbb{Q}^-$ with $a\neq -qq_1/(p(p_1+q_1))$ and $1+bC<0$ and $-1+a(C-1)=0$, where $p,q,p_1,q_1\in\mathbb{N}$; and
(vii) $C=-p/q$, $b=p_1/q_1$, and $a\in\mathbb{Q}^-$ with $a\neq -qq_1/(p(p_1+q_1))$ and $1+bC<0$ and $-1+a(C-1)<0$, where $p,q,p_1,q_1\in\mathbb{N}$.

Case (i): $C\notin\mathbb{Q}^-\cup\{0\}$. Then the eigenvalues of $Df(0)$ are 1 and C. By the hypotheses, for any $k_1,k_2\in\mathbb{Z}^+$ with $k_1+k_2>0$, we have $k_1+k_2C\neq 0$. Therefore, Theorem 2 implies that system (lv1) has no analytic first integrals.

Case (ii): $C\in\mathbb{Q}^-$ and $a\notin\mathbb{Q}^-\cup\{0\}$. Then system (lv1) has the singular point $x=-1/a$ and $y=0$. The eigenvalues of Df at this singular point are -1 and $C-(1+a)/a\neq 0$. By the hypotheses, for any $k_1,k_2\in\mathbb{Z}^+$ with $k_1+k_2>0$, we have $-k_1+k_2C-k_2(1+a)/a\neq 0$. Therefore, Theorem 2 implies that system (lv1) has no analytic first integrals.

Case (iii): $C\in\mathbb{Q}^-$, $a\in\mathbb{Q}^-$, and $b\notin\mathbb{Q}^+$. Then system (lv1) has the singular point $x=0$ and $y=C$, and the eigenvalues of Df at this singular point are $-C$ and $1+bC\neq 0$. By the hypotheses, for any $k_1,k_2\in\mathbb{Z}^+$ with $k_1+k_2>0$, we have $-k_1C+k_2(1+bC)\neq 0$. Therefore, Theorem 2 implies that system (lv1) has no analytic first integrals.

Case (iv): $C=-p/q$, and $b=p_1/q_1$, $a=-qq_1/(p(p_1+q_1))$ with $p,q,p_1,q_1\in\mathbb{N}$ and $pp_1-qq_1<0$. Then it is easy to see that system (lv1) has the first integral

$$x^{pq_1}y^{qq_1}(pp_1+pq_1-qq_1x+p_1qy+qq_1y)^{pp_1-qq_1},$$

which is never global analytic.

Case (v): $C=-p/q$, $b=p_1/q_1$, and $a\in\mathbb{Q}^-$ with $a\neq -qq_1/(p(p_1+q_1))$ and $1+bC=0$ and $-1+a(C-1)<0$, where $p,q,p_1,q_1\in\mathbb{N}$. In this case, system (lv1) has the singular point $x=0$ and $y=C$, and the eigenvalues of Df at this singular point are $-C$ and 0. Since the singular point is isolated, Theorem 3 implies that system (lv1) has no analytic first integrals.

Case (vi): $C=-p/q$, $b=p_1/q_1$, and $a\in\mathbb{Q}^-$ with $a\neq -qq_1/(p(p_1+q_1))$ and $1+bC<0$ and -1

$+a(C-1)=0$, *where* $p,q,p_1,q_1\in\mathbb{N}$. In this case, system (lv1) has the singular point $x=-1/a$ and $y=0$. The eigenvalues of Df at this singular point are -1 and 0. Since the singular point is isolated, Theorem 3 implies that system (lv1) has no analytic first integrals.

Case (vii). $C=-p/q$, $b=p_1/q_1$, *and* $a\in\mathbb{Q}^-$ *with* $a\neq -qq_1/(p(p_1+q_1))$ *and* $1+bC<0$ $-1+a(C-1)<0$, *where* $p,q,p_1,q_1\in\mathbb{N}$. In this case we claim that $a+b+ab>0$. Indeed, since $a>1/(C-1)$ and $C<-1/b$, we have

$$a>\frac{1}{C-1}>\frac{1}{-1/b-1}=-\frac{b}{1+b}.$$

Thus, $a(1+b)>-b$ which implies that $a(1+b)+b>0$. So the claim is proved. From the assumptions, we have that $a(1+b)C\neq 1$. Then system (lv1) has the singular point

$$x=-\frac{1+bC}{a+b+ab},\quad y=-\frac{1+a-aC}{a+b+ab}.$$

The eigenvalues of Df at this singular point are

$$\lambda_1=\frac{\alpha+\sqrt{\beta}}{2(a+b+ab)},\quad \lambda_2=\frac{\alpha-\sqrt{\beta}}{2(a+b+ab)},$$

with $\alpha=1-a(1+b)C$ and

$$\beta=\alpha^2-4(a+b+ab)(-1+a(C-1))\ (1+bC)).$$

By the hypothesis, we have that $1+bC<0$ and $-1+a(C-1)<0$ and $a+b+ab>0$ and then $-4(a+b+ab)(-1+a(C-1)(1+bC)<0$ and then $\beta<\alpha^2$. This implies that $\lambda_1\lambda_2>0$ and thus for any $k_1,k_2\in\mathbb{Z}^+$ with $k_1+k_2>0$, we have that $k_1\lambda_1+k_2\lambda_2\neq 0$. By Theorem 2, system (lv1) has no analytic first integrals. This concludes the proof of the proposition. ■

V. ANALYTIC FIRST INTEGRALS FOR SYSTEMS (LV2)–(LV12)

We will study separately each of the systems (lv$k$) for $k=2,\ldots,12$. We will denote by $\mathbb{Q}^-$ the set of negative rationals.

Proposition 4: *For system (lv2) the following statements hold.*

(a) *x is an analytic first integral if $c=0$;*
(b) *$e^{-Ax}y^c$ is an analytic first integral if $c>0$ and $B=C=0$;*
(c) *$e^{Ax}y^{-c}$ is an analytic first integral if $c<0$ and $B=C=0$;*
(d) *$e^{Aqx/C}x^qy^p$ is an analytic first integral if $c/C=-p/q$ and $B=0$ with $p,q\in\mathbb{N}$; and*
(e) *for any other values of A, B, C, and c, system (lv2) has no analytic first integrals.*

Proof: If $c=0$ then x is an analytic first integral. So statement (a) follows. In the rest of this proof, we assume that $c\neq 0$.

Now we can assume that we are in the assumptions of statements (b) and (c). In this case, system (lv2), after a rescaling of the time, becomes $\dot{x}=c$ and $\dot{y}=Ay$, for which it is easy to check that statements (b) and (c) hold.

We suppose that we are under the hypotheses of statement (d). Then, system (lv2) becomes $\dot{x}=-pCx/q$ and $\dot{y}=y(Ax+C)$, for which it is easy to check that statement (d) follows.

The cases which are not covered by statements (a)–(d) are $C=0$ and $B\neq 0$, $C\neq 0$ and $c/C\notin\mathbb{Q}^-$, and $c/C\in\mathbb{Q}^-$ and $B\neq 0$.

We assume that $C=0$ and $B\neq 0$. In this case, the eigenvalues of $Df(0)$ are c and 0. Thus, since $(x,y)=(0,0)$ is isolated, in view of Theorem 4, the system has no analytic first integrals.

We assume that $C\neq 0$ and $c/C\notin\mathbb{Q}^-$. Then, we note that since the eigenvalues of $Df(0)$ are, respectively, c and C, then by the hypotheses for any $k_1,k_2\in\mathbb{Z}^+$ with $k_1+k_2>0$, we have $k_1c+k_2C\neq 0$. Then, Theorem 2 implies that system (lv2) has no analytic first integrals.

Finally, we consider the case $B\neq 0$ and $c/C\in\mathbb{Q}^-$. Then system (lv2) has the singular point $x=0$ and $y=-C/B$. The eigenvalues of Df at this singular point are c and C. Again by the hypotheses for any $k_1,k_2\in\mathbb{Z}^+$ with $k_1+k_2>0$, we have $k_1c+k_2C\neq 0$. Therefore, Theorem 2 implies that system (lv2) has no analytic first integrals. This concludes the proof of the proposition. ■

Proposition 5: *For system (lv3) the following statements hold.*

(a) xe^{-y} *is an analyic first integral if* $c=B=0$;
(b) $x^p(q-py)^q$ *is an analytic first integral if* $c=0$ *and* $B=-p/q$ *with* $p,q\in\mathbb{N}$;
(c) $e^{-qy}x^qy^p$ *is an analytic first integral if* $c=-p/q$ *with* $p,q\in\mathbb{N}$ *and* $B=0$;
(d) $x^{p_1q}y^{pp_1}(q_1-p_1y)^{qq_1-pp_1}$ *is an analytic first integral if* $c=-p/q$ *and* $B=-p_1/q_1$ *with* $p,q,p_1,q_1\in\mathbb{N}$ *and* $qq_1-pp_1\geqslant 0$; *and*
(e) *for any other values of the parameters c and B, system (lv3) has no analytic first integrals.*

Proof: If $B=c=0$, after a rescaling of the time, system (lv3) becomes $\dot{x}=x$ and $\dot{y}=1$, and it is easy to check that statement (a) follows.

If $c=0$ and $B=-p/q$ with $p,q\in\mathbb{N}$, then after a rescaling of the time, sytem (lv3) becomes $\dot{x}=x$ and $\dot{y}=-py/q+1$. Then, it is straightforward to see that statement (b) holds.

If we are under assumptions (c), then system (lv3) is $\dot{x}=x(y-p/q)$ and $\dot{y}=y$, and it is easy to check that statement (c) is satisfied.

If we are under hypotheses (d), then system (lv3) is $\dot{x}=x(y-p/q)$ and $\dot{y}=y(-p_1y/q_1+1)$, and then we can check in an easy way that statement (d) follows.

Finally the cases which are not covered by statements (a)–(d) are $c=0$ and $B\notin\mathbb{Q}^-\cup\{0\}$, $c\in\mathbb{Q}^-\cup\{0\}$, $c\in\mathbb{Q}^-$ and $B\notin\mathbb{Q}^-\cup\{0\}$, and $c=-p/q$ and $B=-p_1/q_1$ with $p,q,p_1,q_1\in\mathbb{N}$ and $qq_1-pp_1<0$.

If $c=0$ and $B\notin\mathbb{Q}^-\cup\{0\}$, after a rescaling of the time, system (lv3) becomes $\dot{x}=x$ and $\dot{y}=By+1$. This system has the singular point $x=0$ and $y=-1/B$. The eigenvalues of Df at this singular point are 1 and B. Then by the hypotheses for any $k_1,k_2\in\mathbb{Z}^+$ with $k_1+k_2>0$, $k_1+k_2B\neq 0$, and by Theorem 2, system (lv3) has no analytic first integrals.

We assume that $c\notin\mathbb{Q}^-\cup\{0\}$. The eigenvalues of $Df(0)$ are 1 and c. By the hypotheses, for any $k_1,k_2\in\mathbb{Z}^+$ with $k_1+k_2>0$, we have $k_1+k_2c\neq 0$, and by Theorem 2 we conclude that system (lv3) has no analytic first integrals.

We suppose that $c\in\mathbb{Q}^-$ and $B\notin\mathbb{Q}^-\cup\{0\}$. Then, system (lv3) has the singular point $x=0$ and $y=-1/B$. The eigenvalues of Df at this singular point are -1 and $c-1/B$. By the hypotheses, for any $k_1,k_2\in\mathbb{Z}^+$ with $k_1+k_2>0$, we have $-k_1+k_2(c-1/B)\neq 0$. This last inequality can be written as $B\neq k_2/(k_2c-k_1)\in\mathbb{Q}^-$. Therefore, by Theorem 2 system (lv3) has no analytic first integrals.

If $c=-p/q$ and $B=-p_1/q_1$ with $p,q,p_1,q_1\in\mathbb{N}$ and $qq_1-pp_1<0$, then the system has the rational first integral $x^{p_1q}y^{pp_1}/(q_1-p_1y)^{pp_1-qq_1}$, which is not globally analytic. This concludes the proof of the proposition. ■

Proposition 6: *For system (lv4) the following statements hold.*

(a) xe^{x-y} *is an analytic first integral if* $B=0$;
(b) $x^p(y-(q/p)-[q/(q+p)]x)^q$ *is an analytic first integral if* $B=-p/q$ *with* $p,q\in\mathbb{N}$; *and*
(c) *for any other values of* B, *system (lv4) has no analytic first integrals.*

Proof: After a rescaling of the time, system (lv4) is equivalent to

$$\dot{x}=x,\quad \dot{y}=x+By+1. \tag{15}$$

If $B=0$ it is easy to check that statement (a) follows.

We now assume that $B=-p/q$ with $p,q\in\mathbb{N}$. In this case, it is also easy to check that statement (b) holds.

To conclude the proof of the proposition, we should study when $B\notin\mathbb{Q}^-\cup\{0\}$. Since $B\neq 0$, system (15) has the singular point $x=0$ and $y=-1/B$. The eigenvalues of Df at the singular point are 1 and B. Thus, by the hypotheses for any $k_1,k_2\in\mathbb{Z}^+$ with $k_1+k_2>0$, we have $k_2+k_2B\neq 0$. Thus by Theorem 2, system (15) has no analytic first integrals. This concludes the proof of the proposition. ■

Proposition 7: *For system (lv5) the following statements hold.*

(a) *$Ax-y$ is an analytic first integral if $B=0$;*
(b) *$x^p(Aqx-(p+q)y)^q$ is an analytic first integral if $B=-p/q$ with $p,q\in\mathbb{N}$; and*
(c) *for any other value of the parameters A and B, system (lv5) has no analytic first integrals.*

Proof: After a rescaling of the time system (lv5) becomes

$$\dot{x}=x,\quad \dot{y}=Ax+By. \tag{16}$$

If $B=0$, system (16) becomes, after a rescaling of the time, $\dot{x}=1$ and $\dot{y}=A$. The statement (a) follows.

It is easy to check for system (16) that statement (b) holds.

Now we assume that we are under the assumptions of statement (c); i.e., $B\notin\mathbb{Q}^-\cup\{0\}$. The eigenvalues of $Df(0)$ are 1 and B. Then by the hypotheses for any $k_1,k_2\in\mathbb{Z}^+$ with $k_1+k_2>0$ and $k_1+k_2B\neq 0$. By Theorem 2, system (16) has no analytic first integrals. This concludes the proof of the proposition. ■

Proposition 8: *For system (lv6) the following statements hold.*

(a) *y is an analytic first integral if $B=0$;*
(b) *$x^{|B|}$ is an analytic first integral if $a=b=0$;*
(c) *y^b/x^B is an analytic first integral if $a=0$, $b>0$, and $B<0$;*
(d) *x^B/y^b is an analytic first integral if $a=0$, $b<0$, and $B>0$; and*
(e) *for any other value of the parameters a, b, c, and B, system (lv6) has no analytic first integrals.*

Proof: Statement (a) is immediate. In the rest of the proof we assume that $B\neq 0$.

To prove statement (b) we first assume that $c\neq 0$. In this case the eigenvalues of $Df(0)$ are c and 0. Then, since $(x,y)=(0,0)$ is isolated, in view of Theorem 3, system (lv6) has no analytic first integrals.

We now assume that $c=0$. It is easy to check that

$$H(x,y)=\begin{cases}\dfrac{x}{y} & \text{if } a=0, B=b\\[2mm] \dfrac{y^b}{x^B} & \text{if } a=0, B\neq b\\[2mm] e^{-By/(ax)}/y & \text{if } a\neq 0, B=b\\ y^b((ax+(b-B)y)/xy)^B & \text{if } a\neq 0, B\neq b.\end{cases}$$

When $a=0$ and $bB\leqslant 0$, $H(x,y)$ is globally analytic. This proves statements (b)–(d). In any other case, $H(x,y)$ is not globally analytic. This concludes the proof of the proposition. ■

Proposition 9: *For system (lv7) the following statements hold.*

(a) *ye^{y-x} is an analytic first integral if $B=0$ and*
(b) *for $B\neq 0$, system (lv7) has no analytic first integrals.*

Proof: If $B=0$, system (lv7) becomes $\dot{x}=y+1$ and $\dot{y}=y$. It is easy to check that statement (a) holds.

Now we assume that $B\neq 0$. In this case the eigenvalues of $Df(0)$ are 0 and 1. Taking into account that $(x,y)=(0,0)$ is isolated, by Theorem 3 we obtain that system (lv7) does not have analytic first integrals. This concludes the proof of the proposition. ■

Proposition 10: *For system (lv8) the following statements hold.*

(a) *$y^{-c}e^{x-by}$ is an analytic first integral if $a=0$, $c<0$;*

(b) $y^c e^{-x+by}$ *is an analytic first integral if* $a=0$, $c \geqslant 0$;
(c) $y^p(-c-(p/q)c+(p/q)x+(p^2/q^2)x-(pb/q)y)^q$ *is an analytic first integral if* $a=-p/q$ *with* $p,q \in \mathbb{N}$; *and*
(d) *for any other values of the parameters* a, b, *and* c, *system (lv8) has no analytic first integrals.*

Proof: After a rescaling of the time, system (lv8) becomes

$$\dot{x} = ax + by + c, \quad \dot{y} = y. \tag{17}$$

If $a=0$, system (17) becomes $\dot{x}=by+c$ and $\dot{y}=y$ and statements (a) and (b) are easy to check.

It is also easy to verify that statement (c) follows.

The cases not covered by statements (a)–(c) are $a \notin \mathbb{Q}^- \cup \{0\}$. In this case, system (17) has the singular point $x=-c/a$ and $y=0$. The eigenvalues of Df at this singular point are 1 and a. Then, by the hypotheses, for any $k_1, k_2 \in \mathbb{Z}^+$ with $k_1+k_2>0$, we get that $k_1+k_2a \neq 0$. By Theorem 2, system (17) has no analytic first integrals. ■

Proposition 11: *For system (lv9) the following statements hold.*

(a) $x^{|a-1|}y^{|a(1-b)|}((a-1)x+(b-1)y)^{|b-a|}$ *is an analytic first integral if* $(a,b) \neq (1,1)$ *and* $a-1$, $(1-b)a$, *and* $b-a$ *have all the same sign*; *and*
(b) *for any other values of the parameters* a *and* b, *system (lv9) has no analytic first integrals.*

Proof: If $a=b=1$, then, after rescaling of the time, system (lv9) is $\dot{x}=x$ and $\dot{y}=y$. Thus, y/x is a first integral which is not globally analytic. For any other values of a and b, it is easy to check that

$$x^{a-1}y^{a(1-b)}((a-1)x+(b-1)y)^{b-a}$$

is a first integral for system (lv9). Then it is straightforward that the first integral of statement (a) is analytic if and only if $a-1$, $a(1-b)$, and $b-a$ have the same sign. This concludes the proof of the proposition. ■

Proposition 12: *System (lv10) has no analytic first integrals.*

Proof: The eigenvalues of $Df(0)$ for system (lv10) are 0 and 1. Thus, the statement of the lemma follows by Theorem 3 taking into account that $(x,y)=(0,0)$ is isolated. ■

Proposition 13. *System (lv11) has no analytic first integrals.*

Proof: If $c=-p/q$ with $p,q \in \mathbb{N}$, then system (lv11) becomes $\dot{x}=x(x-p/q)$ and $\dot{y}=y$ (note that $cA=0$ and $c \neq 0$). For this system it is easy to check that $x^q y^p/(p-qx)^q$ is a first integral which is not global analytic.

It remains to consider the cases $c=0$ and $A \neq 0$ and $c \in \mathbb{Q}^- \cup \{0\}$ and $A=0$.

We assume that $c=0$ and $A \neq 0$. In this case the eigenvalues of $Df(0)$ are 1 and 0. Therefore, since $(x,y)=(0,0)$ is isolated, by Theorem 3 we get that system (lv11) has no analytic first integrals.

Finally, we consider the case $c \in \mathbb{Q}^- \cup \{0\}$ and $A=0$. The eigenvalues of $Df(0)$ are 1 and c. By the hypotheses for any $k_1,k_2 \in \mathbb{Z}^+$ with $k_1+k_2>0$, we have $k_1+k_2c \neq 0$. Thus, by Theorem 2 system (lv11) has no analytic first integrals. This concludes the proof of the proposition. ■

Proposition 14. *System (lv12) has no analytic first integrals.*

Proof: The eigenvalues of $Df(0)$ are 1 and 1. Thus, the statement follows by Theorem 2 taking into account that for $k_1,k_2 \in \mathbb{Z}^+$ with $k_1+k_2>0$, we have $k_1+k_2 \neq 0$ ■

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