

Math 418, Spring 2025 – Homework 1

Due: Wednesday, January 29th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition*. All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 1 bonus point per assignment.

1. **Dummit and Foote #7.1.3:** Let R be a ring with identity and let S be a subring of R containing the identity. Prove that if u is a unit in S then u is a unit in R . Show by example that the converse is false.
2. **Dummit and Foote #7.1.11:** Prove that if R is an integral domain and $x^2 = 1$ for some $x \in R$ then $x = \pm 1$.
3. **Dummit and Foote #7.2.1:** Let $p(x) = 2x^3 - 3x^2 + 4x - 5$ and let $q(x) = 7x^3 + 33x - 4$. In each of parts (a), (b) and (c) compute $p(x) + q(x)$ and $p(x)q(x)$ under the assumption that the coefficients of the two given polynomials are taken from the specified ring (where the integer coefficients are taken mod n in parts (b) and (c)).
 - (a) $R = \mathbb{Z}$.
 - (b) $R = \mathbb{Z}/2\mathbb{Z}$.
 - (c) $R = \mathbb{Z}/3\mathbb{Z}$.
4. **Dummit and Foote #7.3.2:** Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.
5. **Dummit and Foote #7.4.15:** Let $x^2 + x + 1$ be an element of the polynomial ring $E = \mathbb{F}_2[x]$ and use the bar notation to denote passage to the quotient ring $\mathbb{F}_2[x]/(x^2 + x + 1)$.
 - (a) Prove that E has 4 elements: $\bar{0}, \bar{1}, \bar{x}$, and $\overline{x+1}$.
 - (b) Write out the 4×4 addition table for E and deduce that the additive group E is isomorphic to the Klein 4-group.
 - (c) Write out the 4×4 multiplication table for E and prove that E^\times is isomorphic to the cyclic group of order 3. Deduce that E is a field.
6. Consider $R = \mathbb{Z}[\sqrt{-5}]$ with the (non-Euclidean) norm $N : R \rightarrow \mathbb{Z}_{\geq 0}$ given by $N(a) = |a|^2$. Note that $N(a \cdot b) = N(a)N(b)$.

- (a) Prove that $a \in R$ is a unit if and only if $N(a) = 1$. Find all the units in R .
 - (b) Recall that $r \in R$ is irreducible if whenever $r = ab$ then one of a or b is a unit. Use the norm to show that 2, 3, $1 + \sqrt{-5}$, and $1 - \sqrt{-5}$ are all irreducible elements of R .
 - (c) Show that 2, 3, $1 + \sqrt{-5}$, and $1 - \sqrt{-5}$ are not unit multiples of one another, proving that R lacks unique factorization since $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.
7. Let R be an integral domain. Recall that g is a greatest common divisor of two elements $a, b \in R$ if g divides a and b , and if d divides a and b then d divides g .
- (a) Show that if g and g' are two gcds of $a, b \in R$, $g' = ug$ for some unit u .
 - (b) Let $R = \mathbb{Z}[\sqrt{-5}]$. Prove that 6 and $2 + 2\sqrt{-5}$ have no gcd. (*Hint: Use the fact that 2 and $1 + \sqrt{-5}$ are both common divisors of these elements*)