Announcements:

- HW2 posted (due Wed. 9am)
- No class Monday!

Recall: Konig's Theorem [1936]: G: graph

G is bipartite $\Longleftrightarrow$ G has no odd cycle

Proved $\Rightarrow$

When G is connected, reduced $\Leftarrow$ to the following claim:

Claim: Every closed odd walk contains an odd cycle.

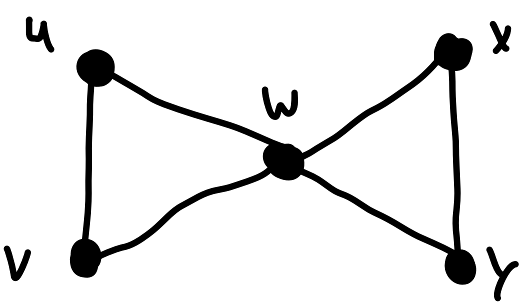
Pf of Claim: Induction on the length ℓ of a closed odd walk W:

Eulerian circuits

Def 1.2.24:

a) A circuit is a closed trail. Two circuits are equivalent if they're the same up to cyclic order and reversal (book slightly different)

Class activity: same or different?



a) u, v, w, x, y, w, u

b) w, y, x, w, v, u, w

c) v, w, x, y, w, u, v

d) u, v, w, y, x, w, u

b) An Eulerian $\left\{\begin{array}{l}\text{trail}\\ \text{circuit}\end{array}\right.$ is a $\left\{\begin{array}{l}\text{trail}\\ \text{circuit}\end{array}\right.$ containing all the edges

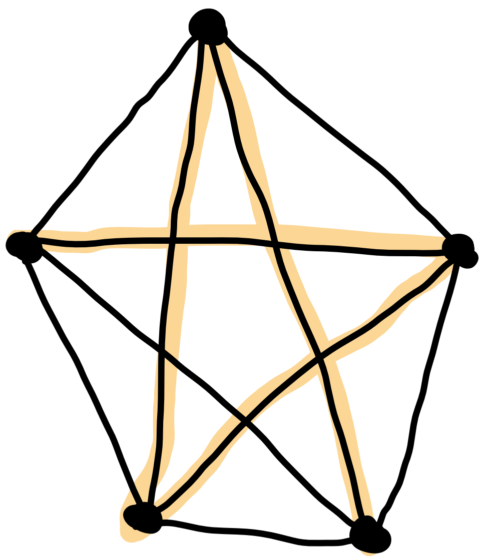
c) A graph is $\left\{\begin{array}{l}\text{even}\\ \text{odd}\end{array}\right.$ if all vertex degrees are $\left\{\begin{array}{l}\text{even}\\ \text{odd}\end{array}\right.$

(Note: loops count double for degree)

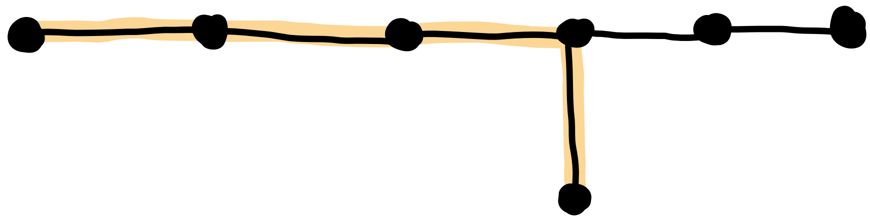
d) A <u>maximal path</u> is a path not contained in a longer path

Class activity: Which of these are maximal paths?

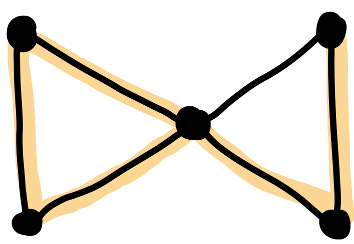
a)



b)



c)



Lemma 1.2.25: If $\deg v \geq 2$ for all $v \in V(G)$, then G contains a cycle.

Pf:

Thm 1.2.26 [Euler]:

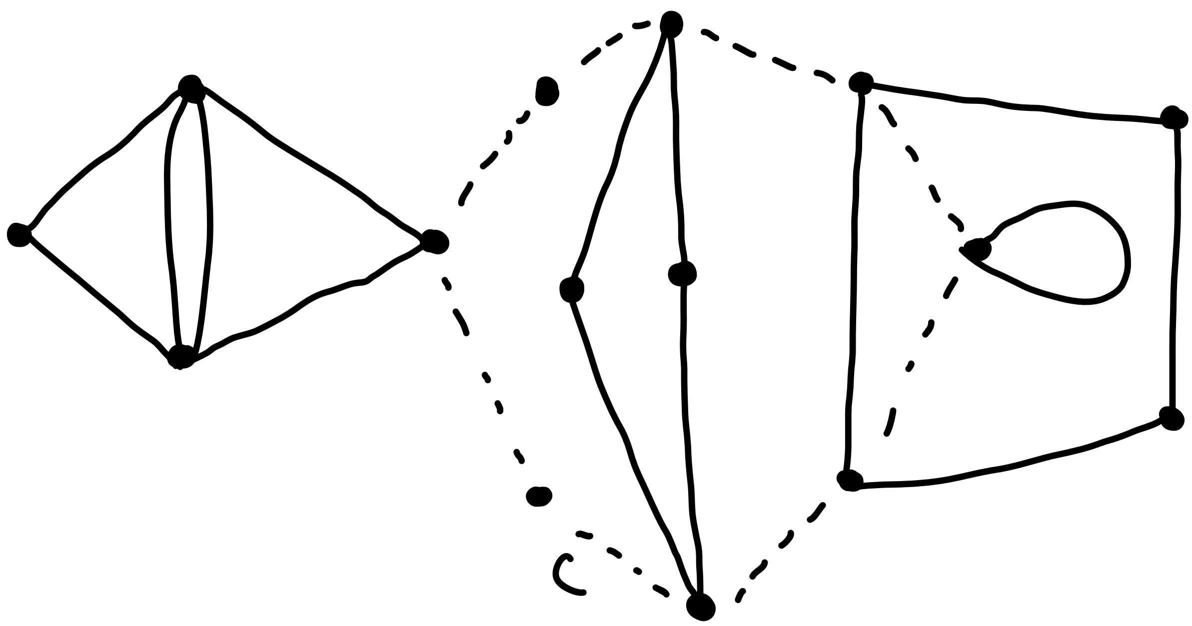
G has an Eulerian circuit $\Longleftrightarrow$ a) G has ≤ 1 "nontrivial" connected component

containing edges

AND

b) G is even

Pf:

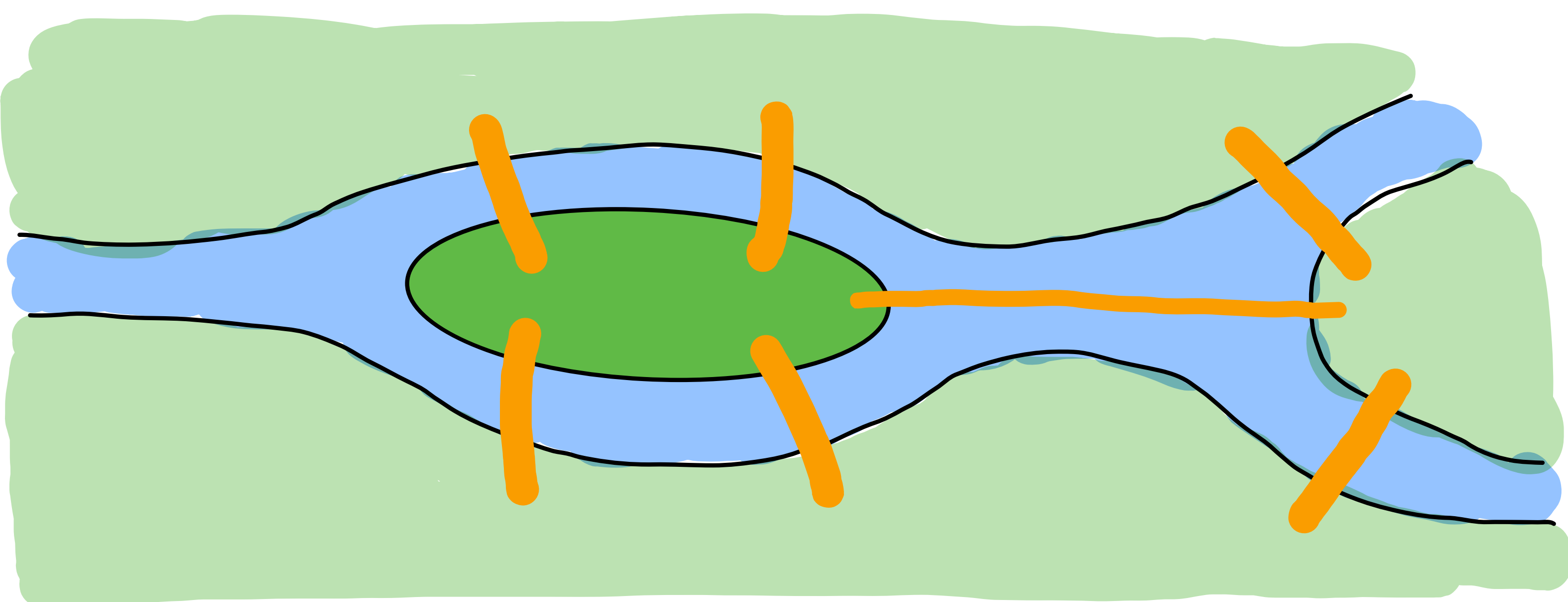


Def 1.1.32: A decomposition of G is a list of subgraphs s.t. each edge appears in exactly one subgraph from the list

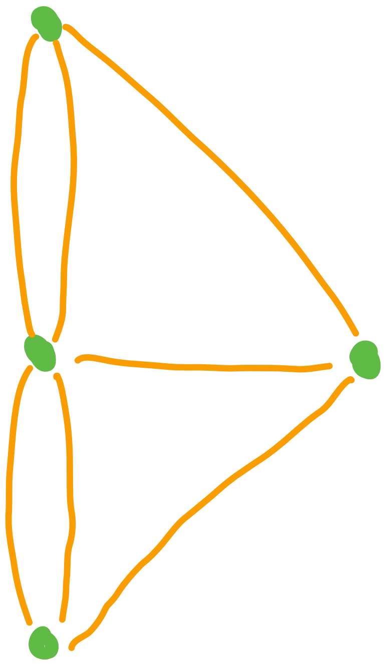
Corollary (Prop 1.2.27): Every even graph decomposes into cycles.

Pf: In the previous proof, G decomposes into G' and C; use induction on $|E(G)|$. $\square$

<u>Bridges of Königsberg (redux)</u>



Question: Can we cross each bridge exactly once?



Answer: No, since the corresponding graph is not even (in fact, it's odd).

Cor:

G has an Eulerian ~~circuit~~ trail $\iff$ a) G has ≤ 1 "nontrivial" connected component AND b) ~~G is even~~ G has at most two $\underbrace{\text{odd vertices}}_{\text{vertices of odd degree}}$

Pf: $\Rightarrow$) If the trail is closed, it's a circuit. Otherwise, the starting and ending vertices have odd degree; add an edge between them and apply Thm. 1.2.26.

$\Leftarrow$) If G has no odd vertices, by Thm. 1.2.26 it has an Euler circuit. Otherwise, add an edge between the two odd vertices, and the resulting graph has an Euler circuit (again, by Thm. 1.2.26). Remove the edge you just added, and it becomes an Euler trail. $\square$

Cor: The Königsberg bridge graph doesn't have an Euler trail. $\square$