Math 412, Fall 2023 – Homework 8

Due: Wednesday, November 8th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should choose four of the following five problems to solve and turn in–if you do all five, only the first four will be graded. Graduate students taking the course for four credits should solve all five. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. Let G be a 2-connected simple graph.

   (a) Prove that in every ear decomposition of G, the number of ears (including the initial cycle) is $|E(G)| - |V(G)| + 1$.

   Proof. Let $(P_0, P_1, \ldots, P_s)$ be an ear decomposition of G. For $i = 0, 1, \ldots, s$, let $G_i = \bigcup_{j=0}^{i} P_j$. In particular, $G = G_s$. We prove by induction on i that for $i = 0, 1, \ldots, s$,

   $$|E(G_i)| - |V(G_i)| = i \quad \text{(and } G_i \text{ has } i+1 \text{ ears).} \tag{1}$$

   Indeed, by definition, G_0 is the cycle P_0 which is the initial ear. Hence $|E(G_0)| = |V(G_0)|$. Suppose now that $|E(G_{i-1})| - |V(G_{i-1})| = i - 1$. Graph G_i is obtained from G_{i-1} by adding one year, which is a path such that all internal vertices of this path are new. Thus the number of new edges is by 1 more than the number of added vertices. This proves the induction step. Thus (1) holds for all i. And the validity of (1) for $i = s$ implies the result. □

   (b) Let $s, t \in V(G)$. Prove that the vertices of G can be linearly ordered so that each vertex other than s and t has a neighbor that is earlier in the order and a neighbor that is later in the order.

   Proof. By Theorem 4.2.4, G has a cycle C containing s and t. By the ear decomposition theorem, G has an ear decomposition $(P_0, P_1, \ldots, P_k)$ with $P_0 = C$. As in the proof of (a), for $i = 0, 1, \ldots, k$, let $G_i = \bigcup_{j=0}^{i} P_j$.
   We construct a linear ordering L_i of $V(G_i)$ satisfying (b) with the smallest element s and the largest element t by induction on i. For $i = 0$, we view the cycle P_0 as two internally disjoint s,t-paths, P' and P''. We place s as the smallest element in our order, then all vertices of $P' \setminus t$, and then all vertices of P'' with the last

vertex t. This is our order L_0. By construction, it satisfies the ordering condition from the problem statement.

Suppose now that we have constructed a linear order L_{i-1} satisfying the ordering condition with s as the smallest and t as the largest elements. By definition, P_i is a path $v_{i,0}, v_{i,1}, \ldots, v_{i,m_i}$, where $v_{i,0}$ and v_{i,m_i} are in $V(G_{i-1})$. By symmetry, we may assume that $v_{i,0}$ precedes v_{i,m_i} in the order L_{i-1}. We obtain L_i by inserting the new vertices $v_{i,1}, \ldots, v_{i,m_{i-1}}$ into L_{i-1} immediately after $v_{i,0}$ and in this order. Then L_i also satisfies the ordering condition.

□

2. Let G be a connected graph with at least 3 vertices. Prove that the following are equivalent:

   (A) G is 2-edge connected

   (B) Every edge of G appears in a cycle

   (C) Every pair of edges of G lie on a closed trail

   (D) Every pair of vertices of G lie on a closed trail

   Proof. $A \iff B$: A connected graph is 2-edge-connected if and only if it has no cut-edges. Cut-edges are precisely the edges belonging to no cycles.

   $A \implies D$: By Menger's Theorem, a 2-edge-connected graph G has two edge-disjoint x, y-paths, where $x, y \in V(G)$. Following one path and returning on the other yields a closed trail containing x and y.

   $D \implies B$: Let x, y be the endpoints of the edge. D yields a closed trail containing x and y. This breaks into two trails with endpoints x and y. At least one of them, T, does not contain the edge xy. Since T is an x, y-walk, it contains an x, y-path. Since T does not contain xy, this path completes a cycle with xy.

   $B \implies C$: Choose $e, f \in E(G)$; we want a closed trail through e and f. Subdivide e and f to obtain a new graph G', with x, y being the new vertices. Subdividing an edge does not destroy paths or cycles, although it may lengthen them. Thus G' is connected and has every edge on a cycle, because G has these properties. Because we have already proved the equivalence of B and D, we know that G' has a closed trail containing x and y. Replacing the edges incident to x and y on this trail with e and f yields a closed trail in G containing e and f.

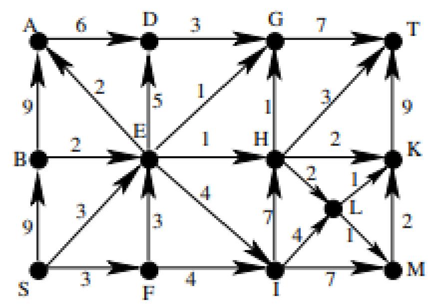
   $C \implies D$: Given a pair of vertices, choose edges incident to them. A closed trail containing these edges is a closed trail containing the original vertices.

   [Note that if we "graph" the implications in this proof, we'll get an ear decomposition with a cycle plus two ears!] □

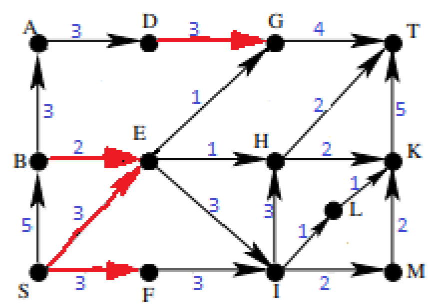
3. Use (an analogue of) Menger's Theorem [see end of Lecture 28 notes] to prove that $\kappa(G) = \kappa'(G)$ when G is 3-regular (Theorem 4.1.11).

Proof. By the analogue of Menger's Theorem, denoted b),i) in the notes, for each $x, y \in V(G)$ there are $\kappa'(G)$ pairwise edge-disjoint x, y-paths. These paths cannot share internal edges, and since G is 3-regular, they also cannot share internal vertices because that would force four distinct edges at a vertex. Hence for each $x, y \in V(G)$ there are $\kappa'(G)$ pairwise internally disjoint x, y-paths, so $\kappa(G) \geq \kappa'(G)$, and by Whitney's Theorem the reverse inequality also holds. □

4. Find a minimum capacity source-to-sink edge cut in the following network (make sure to also prove it is indeed of minimum capacity):



Solution: Below the red edges constitute a source-to-sink edge cut with capacity 11. Provided as well is a feasible flow with value 11 (edges with flow 0 have been deleted to make the whole graph a little easier to look at). By the Max-flow Min-cut theorem (or even Corollary 4.3.8), the fact that this flow and this source-to-sink edge cut have the same value tell us that the source-to-sink edge cut is minimum capacity.



5. Let B be an X, Y-bigraph that satisfies Hall's condition for X (i.e. $|N(S)| \geq S$ for all $S \subseteq X$). WITHOUT USING HALL'S THEOREM, and instead using network flow, prove that there is a matching that saturates X.

   [Hint: Create a digraph with consists of an orientation of B, along with a source and sink. Choose capacities so that a minimum cut won't contain any edges of B, and use the max-flow, min-cut theorem.]

   Proof. Let D be the network formed as follows:

- $V(D) = X \cup Y \cup \{s, t\}$, where $s, t \notin V(B)$.
- $E(D) = \{sx \mid x \in X\} \cup \{yt \mid y \in Y\} \cup \{xy \mid x \in X, y \in Y, xy \in E(B)\}$.
- $c(sx) = 1$ for each vertex $x \in X$, $c(yt) = 1$ for all $y \in Y$, while $c(e) = |X| + 1$ for every other edge $e \in E(D)$.

Starting from a flow that is zero everywhere, let us repeatedly run the Ford-Fulkerson labeling algorithm on D until we get a maximum flow f and a minimum capacity source-to-sink edge cut $[S, T]$. Note that since 1 is the least common denominator over all capacities of D, the flow on every edge is an integer. Furthermore, since $c(sx) = 1$ for all $x \in X$, and sx is the only edge going into x, this implies that each edge leaving x has flow at most one, and there is at most one edge leaving x with flow 1 for all $x \in X$. Similarly, since $c(yt) = 1$ for all $y \in Y$, and yt is the only edge leaving y, each edge entering y has flow at most 1, and there is at most one edge of flow 1 entering y for all $y \in Y$. This implies that the edges $xy \in E(B)$ with $f(xy) = 1$ form a matching in B. We want to show this matching is size $|X|$, which is equivalent to showing that $\text{val}(f) = |X|$.

Recall that $[S, T]$ is the source-to-sink edge cut of minimum capacity that we got from the Ford-Fulkerson labeling algorithm, and note that

$$\text{cap}(S, T) \leq \text{cap}(\{s\}, V(D) \setminus \{s\}) = |X|.$$

Let $A := S \cap X$. Note that $N(A) \subseteq S$ since every edge in $[A, N(A)]$ has capacity $|X| + 1 > \text{cap}(S, T)$. Thus,

$$[\{s\}, X \setminus A] \cup [N(A), \{y\}] \subseteq [S, T].$$

Note that $[\{s\}, X \setminus A]$ and $[N(A), \{y\}]$ are disjoint, so

$$\text{cap}(S, T) \geq \text{cap}(\{s\}, X \setminus A) + \text{cap}(N(A), \{y\}) = (|X| - |A|) + |N(A)| \geq |X|,$$

where the final inequality follows from Hall's Condition. Thus $\text{cap}(S, T) = |X|$, which completes the proof since $\text{val}(f) = \text{cap}(S, T)$. $\square$