A factorial analogue of the boson-fermion correspondence

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arXiv:2410.06582 (double factorial); in preparation (factorial, algebraic proofs)

Joint with Daniel Bump and Travis Scrimshaw

Some (brief) background

Isomorphism between two representations of the infinite Heisenberg algebra, the *fermionic* and *bosonic* Fock spaces

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- Kyoto school: Sato, Jimbo, Miwa, Date, Kashiwara etc.
- Frenkel, Kac, Peterson, etc.

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- Skyrme
- Kyoto school: Sato, Jimbo, Miwa, Date, Kashiwara etc.
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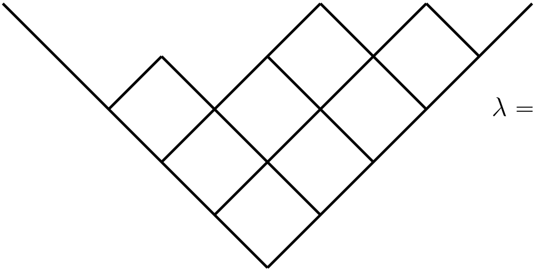
Various generalizations:

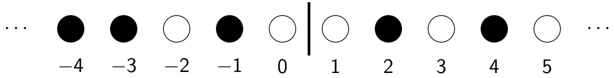
- Jing (Hall-Littlewood)
- Kashiwara-Miwa-Stern (quantum Fock space)

The classical boson-fermion correspondence

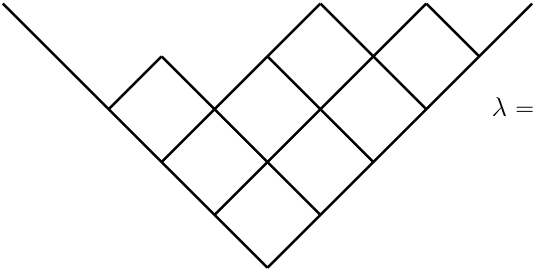


$\lambda = (4, 3, 1)$

$|\lambda\rangle =$



Fermionic Fock space



$\lambda = (4, 3, 1)$

$|\lambda\rangle = \quad \cdots$ ● ● ○ ● ○ | ○ ● ○ ● ○ $\cdots$

−4	−3	−2	−1	0	1	2	3	4	5
●	●	○	●	○	○	●	○	●	○

$$\mathfrak{F}^{(0)} := \mathbb{C}\text{-span of } \{|\lambda\rangle\},$$

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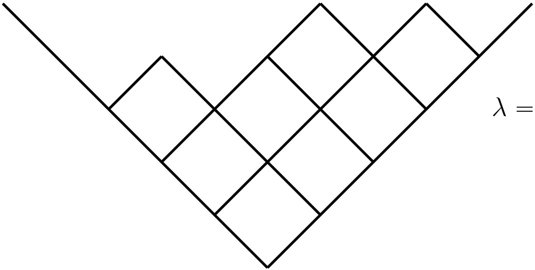

$$|\lambda\rangle = \quad \cdots \quad \bullet_{-4} \; \bullet_{-3} \; \circ_{-2} \; \bullet_{-1} \; \circ_{0} \; \Big| \; \circ_{1} \; \bullet_{2} \; \circ_{3} \; \bullet_{4} \; \circ_{5} \quad \cdots$$

$$\mathfrak{F}^{(0)} := \mathbb{C}\text{-span of } \{|\lambda\rangle\}, \qquad \mathfrak{F} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{F}^{(m)}$$

$\mathfrak{F}^{(m)} := \mathbb{C}$-span of $\{\Sigma^m|\lambda\rangle\}$, where Σ shifts all particles one unit right

Fermionic Fock space

ψ_i : *creation* at position i,

$$
\begin{array}{c}
- \quad \cdots \; \bullet \; \bullet \; \circ \; \bullet \; \bullet \; | \; \circ \; \circ \; \circ \; \bullet \; \circ \; \cdots \in \mathfrak{F}^{(0)} \\
\Big\downarrow \psi_{-2} \\
\cdots \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; | \; \circ \; \circ \; \circ \; \bullet \; \circ \; \cdots \in \mathfrak{F}^{(1)}
\end{array}
$$

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\psi_3^* \downarrow \\
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 & & & & & & \psi_3^* & \downarrow & & & & & & & \\
 & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & | & \circ & \circ & \circ & \circ & \circ & \cdots & \in \mathfrak{F}^{(0)}
\end{array}
$$

These operators form a Clifford algebra:

$$\psi_i\psi_j + \psi_j\psi_i = \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0, \qquad \psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}$$

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Fermion fields:

$$\psi(z) = \sum_{i\in\mathbb{Z}} \psi_i z^i, \qquad \psi^*(w) = \sum_{j\in\mathbb{Z}} \psi_j^* w^{-j},$$

$\mathfrak{gl}_\infty$-representation

$$\mathfrak{gl}_\infty = \left\{ \sum_{i,j} a_{ij} E_{ij} \middle| a_{ij} = 0 \text{ for } |i-j| \gg 0 \right\}$$

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$$\begin{array}{cc} & j \\ E_{ij} := i & \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} \end{array} \mapsto \psi_i \psi_j^*$$

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The *current operators* $J_k := \sum_{i \in \mathbb{Z}} :\psi_{i-k}\psi_i^*:$ form a Heisenberg algebra:

$$[J_k, J_\ell] = k\delta_{k,-\ell} \cdot 1$$

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Half-vertex operator:

$$e^{H_\pm(\mathbf{p})} := \exp\left(\sum_{k \geqslant 1} \frac{1}{k} p_k J_{\pm k}\right) \qquad \left(\text{Think: } p_k = \sum_{i \geqslant 1} x_i^k\right)$$

Bosonic Fock Space

Bosonic Fock space:

$$B^{(m)} = \mathbb{C}[p_1, p_2, \ldots] \cdot s^m, \qquad B = \bigoplus_{m \in \mathbb{Z}} B^{(m)}$$

(Think: symmetric functions in $x_1, x_2, \ldots$)

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Heisenberg algebra acts via

$$J_k \mapsto \begin{cases} k\frac{\partial}{\partial p_k}, & \text{if } k > 0 \\ p_{|k|}, & \text{if } k < 0 \end{cases}$$

Boson-fermion correspondence

Theorem (Boson-Fermion Correspondence, first part)

There exist vertex operators

$$\psi(z)|_{\mathfrak{F}^{(m)}} = z^m e^{H_-(z)} \Sigma e^{-H_+(z^{-1})}$$

and

$$\psi^*(z)|_{\mathfrak{F}^{(m)}} = z^{-m} e^{-H_-(z)} \Sigma^{-1} e^{H_+(z^{-1})}$$

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Theorem (Boson-Fermion Correspondence, second part)

The map $\mathfrak{F}^{(0)} \to B^{(0)}$,

$$|\lambda\rangle \mapsto \langle\varnothing| e^{H_+(\mathbf{p})} |\lambda\rangle$$

is an isomorphism of Heisenberg algebra modules. Under this map,

$$|\lambda\rangle \mapsto s_\lambda.$$

The factorial boson-fermion correspondence

(Supersymmetric) factorial Schur functions

Factorial Schur function (Biedenharn-Louck, Macdonald 6th variation):

$$s_\lambda(x|\alpha) = \frac{\det\left((x_i - \alpha_1)\cdots(x_i - \alpha_{\lambda_j + j})\right)_{1\leqslant i,j\leqslant n}}{\prod_{i<j}(x_i - x_j)}.$$

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$$s_\lambda(x|\alpha) \xrightarrow{\alpha_i \mapsto 0} s_\lambda(x)$$

factorial Schur	Schur

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$$s_\lambda(x/y|\alpha) \xrightarrow{y_i\mapsto\alpha_i} s_\lambda(x|\alpha) \xrightarrow{\alpha_i\mapsto 0} s_\lambda(x)$$

supersymmetric factorial Schur (Molev)	factorial Schur	Schur

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$$s_\lambda(\mathbf{p}|\alpha) \xrightarrow{p_k \mapsto p_k(x/y)} s_\lambda(x/y|\alpha) \xrightarrow{y_i \mapsto \alpha_i} s_\lambda(x|\alpha) \xrightarrow{\alpha_i \mapsto 0} s_\lambda(x)$$

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Deformed fermion fields

Deform the fermion fields:

$$\psi(z|\boldsymbol{\alpha}) = \sum_{i\in\mathbb{Z}} \frac{z^i}{(1-z\alpha_1)\cdots(1-z\alpha_i)}\psi_i$$

$$\psi^*(w|\boldsymbol{\alpha}) = \sum_{j\in\mathbb{Z}} \frac{(1-w\alpha_1)\cdots(1-w\alpha_{j-1})}{w^j}\psi^*_j.$$

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This induces an automorphism of the Clifford algebra

Deformed current operators

$$J_k^{(\boldsymbol{\alpha})} := \sum_{i,j} A_{ij}^k E_{ij}, \text{ where } A_{ij}^k = \begin{cases} e_{j-i-k}(-\alpha_{i+1}, \ldots, -\alpha_{j-1}) & \text{if } j \geqslant i+k \text{ and } k > 0, \\ h_{j-i-k}(\alpha_j, \ldots, \alpha_i) & \text{if } j \leqslant i \leqslant j-k \text{ and } k \leqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

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These live in the space

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The operators $J_k^{(\boldsymbol{\alpha})}$ still satisfy the (undeformed) Heisenberg relations

$$[J_k^{(\boldsymbol{\alpha})}, J_\ell^{(\boldsymbol{\alpha})}] = k\delta_{k,-\ell} \cdot 1$$

Deformed boson-fermion correspondence

Theorem (Deformed Boson-Fermion Correspondence, first part)

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is an isomorphism of Heisenberg algebra modules. Under this map,

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Consequences

The structure of the boson-fermion correspondence allows us to turn the crank and prove various identities for the (skew) factorial Schur functions:

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In addition,

- Our half vertex operator matches the tranfer matrix of a solvable, free fermionic lattice model of Bump-McNamara-Nakasuji
- The factorial Schur functions are tau-functions of the KP-hierarchy

Murnaghan-Nakayama rule

Theorem (Factorial Murnaghan–Nakayama rule)

$$p_k s_\lambda(\mathbf{p}|\boldsymbol{\alpha}) = \ast \cdot s_\lambda(\mathbf{p}|\boldsymbol{\alpha}) + \sum_\nu (-1)^{ht(\nu/\lambda)-1} A_{j,i}^{-k} s_\nu(\mathbf{p}|\boldsymbol{\alpha}),$$

where the sum is over all ν such that ν/λ is a nonzero ribbon of size at most k and the contents of the boxes in ν/λ are $i, i+1, \ldots, j-1$

(Similar rule for differentiation)

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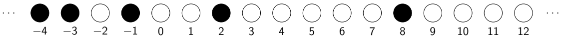
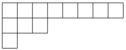
Example: $\lambda = (8,3,1)$. Compute $p_3 s_\lambda(\mathbf{p}|\boldsymbol{\alpha})$:





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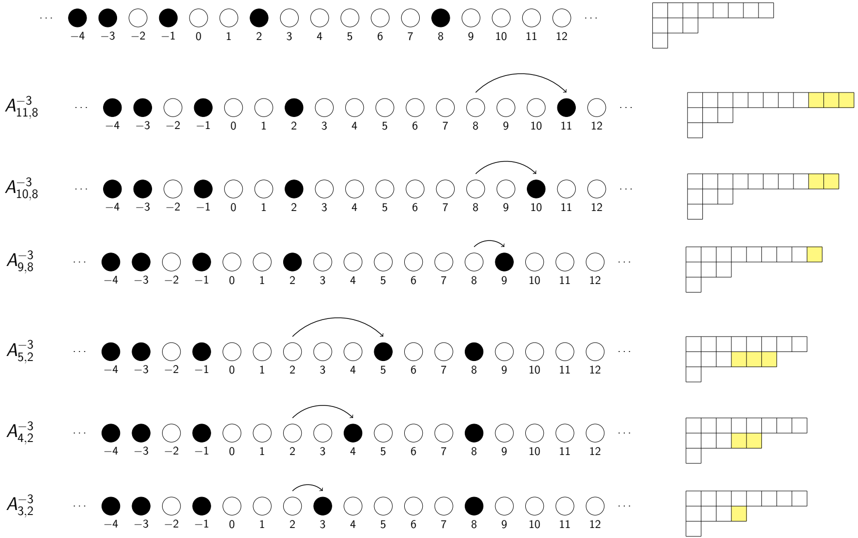
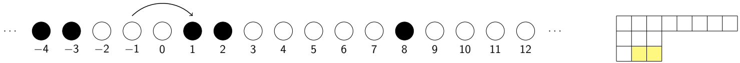
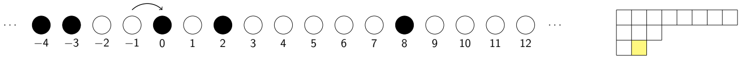


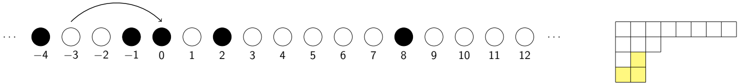
$A^{-3}_{1,-1}$



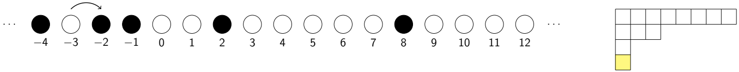
$A^{-3}_{0,-1}$



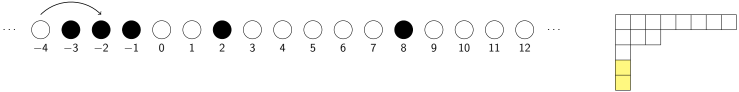
$-A^{-3}_{0,-3}$



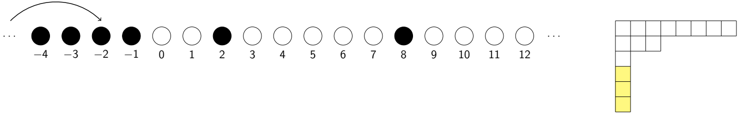
$A^{-3}_{-2,-3}$



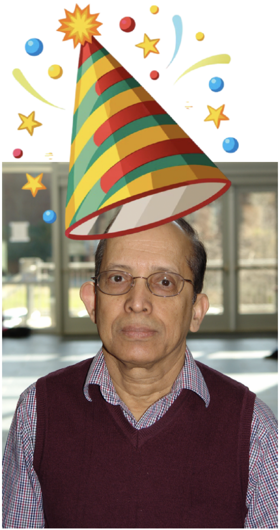
$A^{-3}_{-2,-4}$



$A^{-3}_{-2,-5}$



Thank You



Happy Birthday, Kailash Misra!