Math 506, Spring 2026 – Homework 2

Due: Wednesday, February 25th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. All assertions require proof unless otherwise stated. Typesetting your homework using LaTeX is recommended.

For this homework, unless otherwise stated all groups are finite and all representations are finite dimensional and complex.

More problems to come!

1. Let V and W be G-representations, with W irreducible. Prove that the following map $V \to V$ is a G-equivariant projection onto the W-isotypic component of V:
$$\pi_W := \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \cdot g.$$
(*In the case where W is the trivial representation, this is Proposition 9.*)

2. Prove the last statement in Corollary 16. That is, given that the irreducible characters form an orthonormal basis of $\mathbb{C}_{cl}(G)$ with respect to the Hermitian inner product $(\cdot, \cdot)$, show that
$$\sum_{\chi:\text{irred}} \overline{\chi(g)}\chi(h) = \begin{cases} |C(g)|, & \text{if } g = h, \\ 0, & \text{otherwise,} \end{cases}$$
where $C(g)$ is the centralizer of g in G.

   (*Hint: use properties of unitary matrices.*)

3. Let G and H be finite groups. If V is an irreducible representation of G and W is an irreducible representation of H, show that $V \otimes W$ is an irreducible representation of $G \times H$ (i.e. define an action of $G \times H$ on $V \otimes W$ and show that the resulting representation is irreducible). (This is called the *external tensor product*; don't confuse it with the *internal tensor product* we've already looked it.) Show that every irreducible representation of $G \times H$ is of this form.

   (*Hint: use character theory.*)

4. Prove *directly* that the induced representation given in Definition 17b is unique up to isomorphism. That is, fix two sets of coset representatives $\sigma_1, \ldots, \sigma_k$ and $\sigma'_1, \ldots, \sigma'_k$ for G/H with $\sigma_i^{-1}\sigma'_i \in H$. Let

$$V = \bigoplus_{i=1}^{k} W_i, \qquad W_i = \{w_i | w \in W\}, \qquad \text{with action } g{\cdot}w_i := (h{\cdot}w)_j, \text{ where } g\sigma_i = \sigma_j h,$$

$$V' = \bigoplus_{i=1}^{k} W_i, \qquad W'_i = \{w'_i | w \in W\}, \qquad \text{with action } g{\cdot}w'_i := (h'{\cdot}w)_j, \text{ where } g\sigma'_i = \sigma'_j h'.$$

Give an explicit G-isomorphism $V \to V'$.

5. There is another definition of induced representation that is more natural in some settings. Fix an H-representation (ρ, W). Let $\mathrm{Ind}_H^G \rho$ refer to the definition from class of the induced representation, corresponding to a fixed set $\sigma_1, \ldots, \sigma_k$ of (left) coset representatives for G/H. Let

$$\mathrm{ind}_H^G \rho = \{f : G \to W | f(hg) = \rho(h)f(g) \text{ for all } h \in H, g \in G\},$$

with G-action given by

$$g \cdot f(g') := f(g'g).$$

   (a) Prove that $\mathrm{ind}_H^G \rho$ is a G-representation. Specifically, prove that $g \cdot f(g') := f(g'g)$ is a G-action on the space of functions $G \to W$, and that if $f \in \mathrm{ind}_H^G \rho$, then so is $g \cdot f$.

   (b) Prove that $\mathrm{Ind}_H^G \rho \cong \mathrm{ind}_H^G \rho$ by showing that

$$w_i \mapsto f_{w,i}, \qquad \text{where} \qquad \underset{(h \in H)}{f_{w,i}(h\sigma_j^{-1})} = \begin{cases} h \cdot w, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

   is a G-isomorphism.

More problems to come!