No announcements today

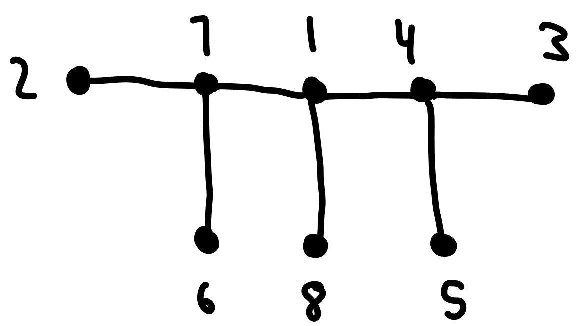
Recall:

Def: The <u>Prüfer Code</u> $f(T) = (a_1, \ldots, a_{n-2})$ of T is given by the following algorithm:

At step i:

- delete the leaf w/ the smallest label
- a_i is the label for the (unique) neighbor of the leaf

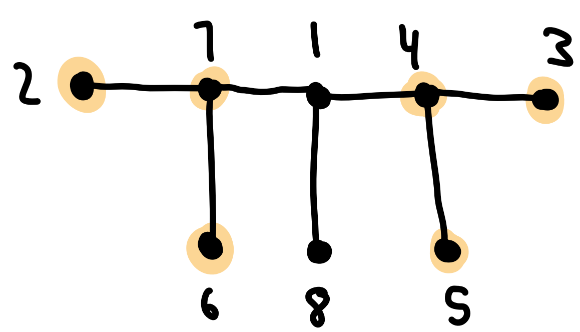
Ex:



$\text{Pru}(T) = 744171$

Can go backwards:

$\text{Pru}(T) = 744171$



Cayley's Formula (Thm 2.2.3): There are n^{n-2} labelled trees with n vertices

Pf: $n=1$ good

We prove that for $n \geq 2$

$$T \longleftrightarrow \mathrm{Pru}(T)$$

is a bijection.

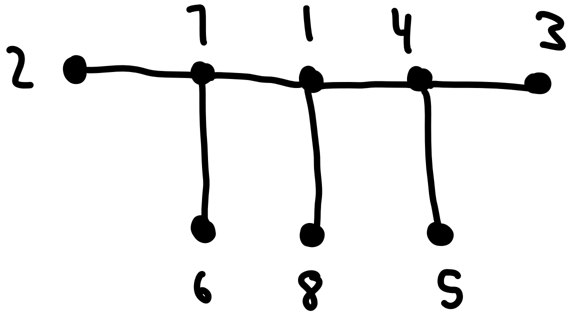
Base case: $n=2$

length zero

 $\longleftrightarrow \emptyset$

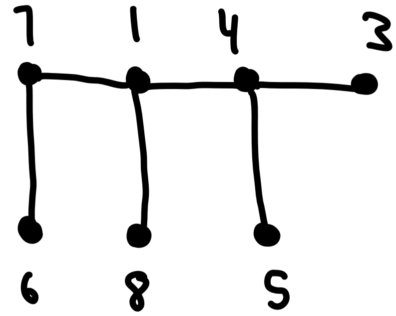
Inductive step: $n>2$.

T



$\text{Pru}(T) = 744171$

T'



$\text{Pru}(T') = 44171$

Cor 2.2.4: Let $d_1, \ldots, d_n \in \mathbb{Z}_{\geq 1}$ s.t. $d_1 + \cdots + d_n = 2n-2$. Then the number of trees w/ label set $\{1, \ldots, n\}$ s.t. vertex i has degree d_i is $\dfrac{(n-2)!}{\prod (d_i - 1)!}$

Pf sketch: Look at how many times i appears in $\text{Pru}(T)$

Further question: How many spanning trees does a graph G have?

$\tau(G) :=$ number of spanning trees of G

Cases we know so far:

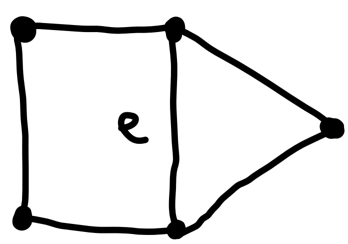
- $\tau(\text{tree}) = \underline{1}$ Def'n
- $\tau(\text{disconn. graph}) = 0$ Cor 2.1.5
- $\tau(C_n) = n$
- $\tau(K_n) = n^{n-2}$ Cayley's Formula
- $\tau(G) = \tau(G \setminus \text{loops})$

Matrix Tree Theorem (2.2.12) $\tau(G)$ can be given as the determinant of a certain matrix.

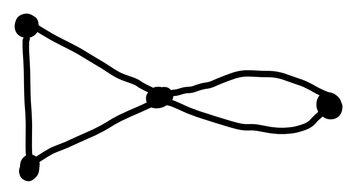
Need a recursive tool first:

Def 2.2.7: Let $e \in E(G)$ have endpoints u and v. The contraction $G \cdot e$ is the graph obtained from G by replacing u and v with a single vertex whose incident edges are the edges other than e that were incident to u or v.

Class activity: Find $G \cdot e$



G

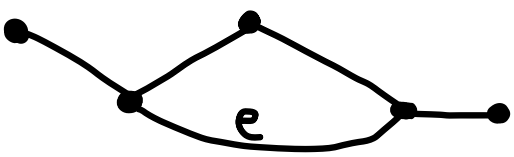


$G \cdot e$

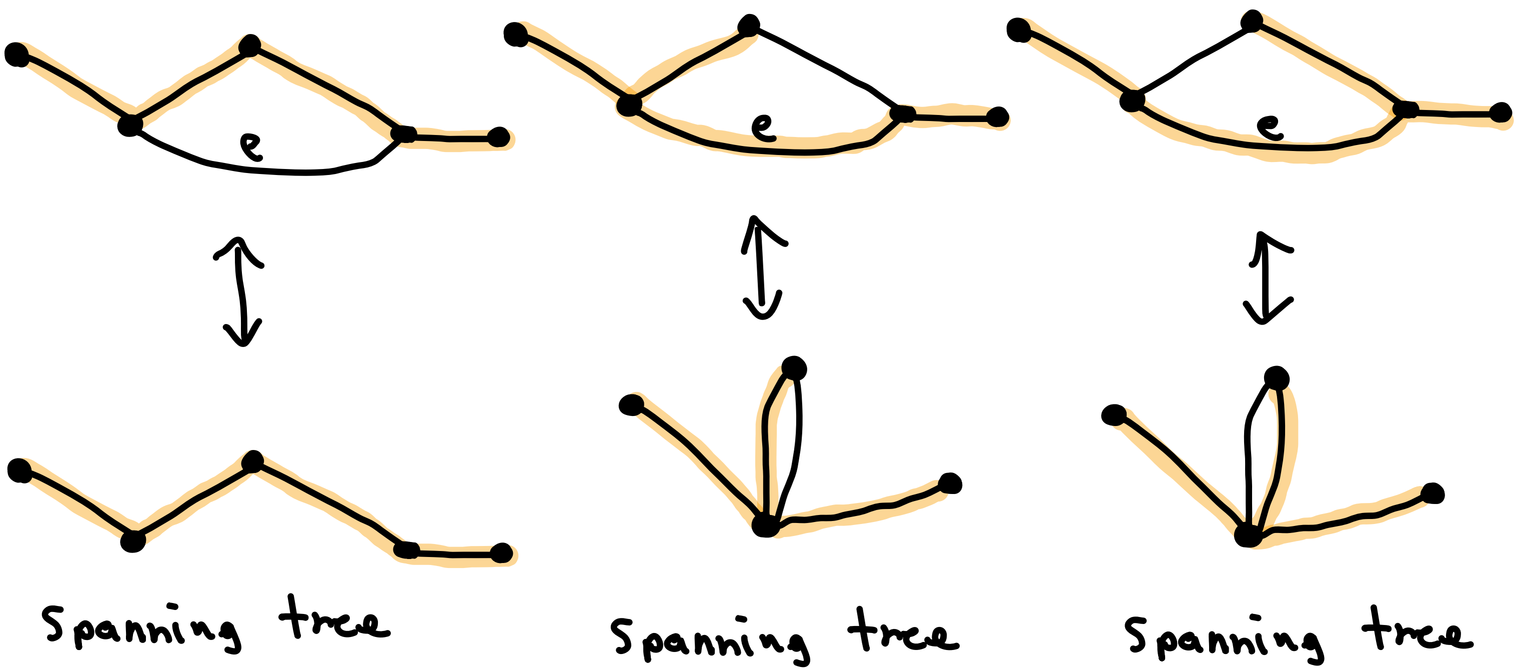
Prop 2.2.8: If e is not a loop, then

$$\tau(G) = \tau(G \setminus e) + \tau(G \cdot e)$$

Pf:



G



Spanning tree of $G \setminus e$

Spanning tree of $G \cdot e$

Spanning tree of $G \cdot e$

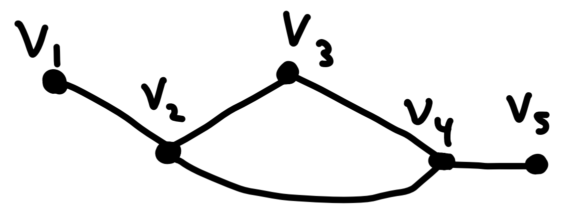
Def:

a) The <u>degree matrix</u> $D(G)$ is the diagonal matrix with (i,i)-entry equal to $d(v_i)$

b) The <u>Laplacian matrix</u> of G is the matrix

$$L(G) = D(G) - \underbrace{A(G)}_{\text{adjacency matrix}}$$

c) The <u>reduced Laplacian</u> $L^i(G)$ is $L(G)$ with the ith row and column deleted



G

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{D(G)} - \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{A(G)} = \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}}_{L(G)}$$

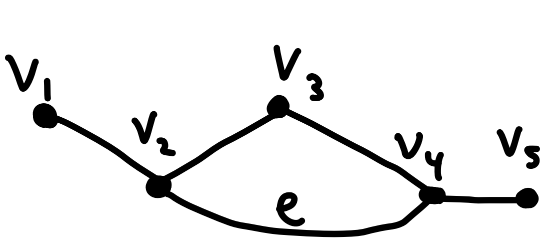
$$\begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$L^1(G)$ $\qquad$ $L^2(G)$ $\qquad$ $L^{1,2}(G)$

Matrix Tree Theorem: For any loopless graph G, and for any i,

$$\tau(G) = \det L^i(G)$$

Pf (Godsil-Royle, Algebraic Graph Theory):



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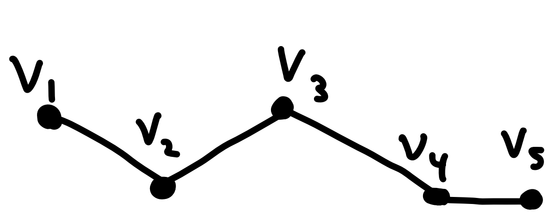

$G \setminus e$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} & & & & \\ & 1 & & -1 & \\ & & & & \\ & -1 & & 1 & \\ & & & & \end{bmatrix}$$

$$L(G) \qquad\qquad L(G \setminus e) \qquad\qquad E$$

$$\begin{array}{c} \\ v_1 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ v_1 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \qquad \begin{bmatrix} & & & \\ & & & \\ & & 1 & \\ & & & \end{bmatrix}$$

$$L^2(G) \qquad\qquad L^2(G \setminus e) \qquad\qquad E'$$

$$\det L^2(G) = \det L^2(G \setminus e) + 1 \cdot L^{2,4}(G \setminus e)$$