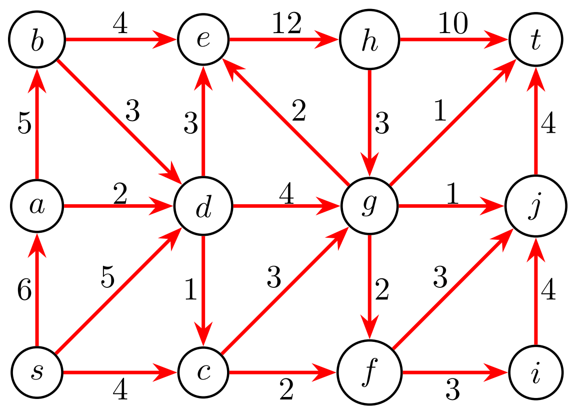
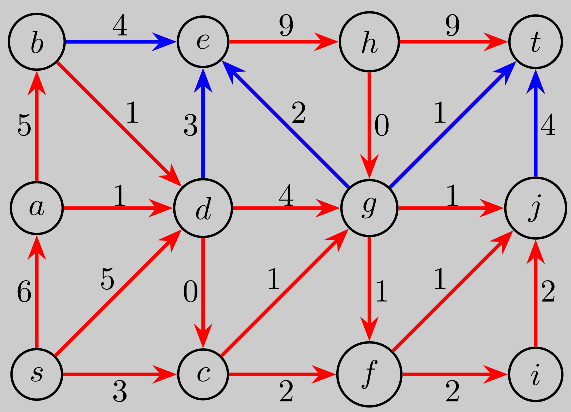
Solutions to Math 412 Midterm Exam 3 — Nov. 15, 2023

1. (25 points) Find a minimum capacity source-to-sink edge cut in the following network, and prove that it is indeed minimum capacity:





The above graph gives a feasible flow with value 14. The blue edges are an edge cut with capacity 14. Therefore, by the max-flow, min-cut theorem, the flow is maximum and the cut is minimum.

2. (20 points) Let G be a simple color-critical graph (i.e. k-critical for some value of k). Prove that the graph $G' := \mathrm{Myc}(G)$ obtained by applying Mycielski's construction to G is also color-critical.

We have proven (Theorem 5.2.2) that if $\chi(G) = k$, then $\chi(G') = k + 1$. Since removing edges and vertices never makes the chromatic number go up, and since G' has no isolated vertices, we need only show that removing any edge e from G' creates a graph $G'' := G \setminus e$ that is k-colorable

Let $V(G) = \{v_1, \ldots, v_n\}$ and $U = \{u_1, \ldots, u_n\}, w$ be as in Mycielski's construction. We do a case-by-case analysis based on the edge e:

Case 1: $e = v_i v_j$ i.e. $e \in E(G)$. Since G is k-critical, $\chi(G \setminus e) = k - 1$, so let f be a $(k-1)$-coloring of G. By coloring every vertex in U color k and coloring w color 1, we create a k-coloring of G''.

Case 2: $e = v_i u_j$. Let f be a proper $(k-1)$-coloring of $G \setminus v_i v_j$. Since $\chi(G) = k$, we must have $f(v_i) = f(v_j)$ since otherwise adding $v_i v_j$ back in would create a $(k-1)$-coloring of G. Let G be the k-coloring of G'' given by $g(v_a) = g(u_a) = f(v_a), g(w) = k$. This is not a proper coloring since v_i and v_j share the same color, but it is a proper coloring of $G'' \setminus v_i v_j$. (Proof: w has a different color than all other vertices, so g has no problem with edges incident to w; f is a proper coloring of $G \setminus v_i v_j$, so g has no problem with edges in $G \setminus v_i v_j$; for any edge with one endpoint in U and the other in $V(G)$, there is a corresponding edge in $G \setminus v_iv_j$ with the same colors on endpoints, so g has no problem with these edges). Furthermore, w is the only vertex of color k by construction. Therefore, changing v_j to color k gives a proper k-coloring of G''.

Case 3: $e = wu_j$. Let f be a k-coloring of G formed by taking a $(k-1)$-coloring of $G \setminus v_j$, and setting $f(v_j) = k$, so that v_j is the only vertex in G of color k. By coloring u_i the same color as v_i for all i, and coloring w color k, we create a proper k-coloring of G''.

3. (20 points) Prove that a simple graph G with at least three vertices is 2-connected if and only if for every triple (x, y, z) of distinct vertices, G has an x, y-path containing z.

First assume G is 2-connected and fix a triple (x, y, z). Construct a graph G' by adding a new vertex w with neighbors x and y. We have added an ear to G, so G' is 2-connected (in more detail, we have added an edge, which doesn't reduce connectivity, and subdivided that edge, which maintains 2-connectivity by Corollary 4.2.6). By Theorem 4.2.4 (the equivalent definitions for 2-connectivity), there are 2 internally disjoint z, w-paths in G', and since x and y are the only neighbors of w, removing w gives us an x, y path in G through z.

Now assume to the contrary that G is not 2-connected but satisfies the path condition in the hypothesis. If G is not connected, then there are pairs that are not connected by a path, so we can assume G is connected, thus $\kappa(G) = 1$, so G has a cut vertex, say x. Then let y and z be vertices in separate components of $G \setminus x$. This implies that there is no x, y-path through z since this would give us a y, z-path in $G \setminus x$, a contradiction. Thus we are done.

4. (30 points) (a) (15 points) Let G be a simple graph. Prove that $\chi(G) \cdot \chi(\overline{G}) \geq n(G)$.

Let $\alpha(G)$ be the largest size of an independent set in G, and let $\omega(G)$ be the largest size of a clique. By Proposition 5.1.7, $\chi(G) \geq n(G)/\alpha(G)$ (since every color class has size $\leq \alpha(G)$), and $\chi(\overline{G}) \geq \omega(\overline{G})$ (since every vertex in a clique much be colored a different color). Therefore,

$$\chi(G) \cdot \chi(\overline{G}) \geq \frac{n}{\alpha(G)}\omega(\overline{G}) = \frac{n}{\alpha(G)}\alpha(G) = n,$$

since $\alpha(G) = \omega(\overline{G})$.

(b) (15 points) Prove that the chromatic polynomial of the n-cycle C_n is $(k-1)^n + (-1)^n(k-1)$.

Let $f_n(k) = (k-1)^n + (-1)^n(k-1)$. We use induction on n. We only need $n = 1$ as a base case, but we can easily do the first three cases:

$$\chi(C_1; k) = 0 = f_1(k)$$

$$\chi(C_2; k) = \chi(P_2; k) = k(k-1) = (k-1)^2 + (k-1) = f_2(k)$$

$$\chi(C_3; k) = \chi(K_3; k) = k(k-1)(k-2) = (k-1)((k-1)^2 - 1) = k-1)^n + (-1)^n(k-1) = f_3(k).$$

For the inductive step, we use the deletion-contraction formula (Proposition 5.3.6):

$$\chi(C_n; k) = \chi(C_n \setminus e; k) - \chi(C_n \cdot e; k) = \chi(P_n; k) - \chi(C_{n-1}; k),$$

where e is any edge in C_n. Using the inductive hypothesis and the chromatic polynomial for trees: $\chi(P_n; k) = k(k-1)^{n-1}$, we obtain

$$\chi(C_n) = k(k-1)^{n-1} - (k-1)^{n-1} - (-1)^{n-1}(k-1) = (k-1)^n + (-1)^n(k-1) = f_n(k).$$