Math and Proofs Class 8

November 14th, 2017

Summary of the class so far

- First two weeks: looked at a couple different axiom systems
- Since then: set theory
- Last time: talked about the Axiom of Choice
- Goodstein's Theorem
  - Can be *stated* using the Peano axioms (normal arithmetic)
  - Can only be *proven* using ordinals (which are *not* part of normal arithmetic)

Hilbert's Program

- Can we build a complete axiom system?
- "Complete" means that everything we can state using our axioms we can prove true or false using our axioms
    - We don't have to prove *everything* true or false, just everything we can state using the axioms
    - For instance, "I like ice cream" is not a likely candidate for proof in any axiom system we come up with, but that's OK because we can't state it in our usual axiom systems
- So in a sense, it's really easy to make a complete axiom system (no axioms or clock math)
- But we want axioms that are powerful, like our normal math

Hilbert's Program (cont.)

- In 1900, David Hilbert published a list of 23 open problems throughout mathematics that he considered to be the most important problems of the day
- Even now, many have not been solved
- Hilbert's Second Problem: Prove that the axioms of arithmetic are consistent (no contradictions)

Hilbert's Program (cont.)

- Further, Hilbert envisioned a program to secure the foundations of mathematics, namely (copied from Wikipedia):
  - A formulation of all mathematics; in other words all mathematical statements should be written in a precise formal language, and manipulated according to well defined rules.
  - Completeness: a proof that all true mathematical statements can be proved in the formalism.
  - Consistency: a proof that no contradiction can be obtained in the formalism of mathematics. This consistency proof should preferably use only "finitistic" reasoning about finite mathematical objects.
  - Conservation: a proof that any result about "real objects" obtained using reasoning about "ideal objects" (such as uncountable sets) can be proved without using ideal objects.
  - Decidability: there should be an algorithm for deciding the truth or falsity of any mathematical statement.

Godel's Incompleteness Theorem

- Kurt Godel (1931): Hilbert's program is impossible
- There is no consistent system of axioms that is capable of proving true or false all statements about the natural numbers
- What this means: there are tons and tons more "theorems" like Goodstein's Theorem, that need extra axioms to prove
- No axiom system can prove its own consistency

Godel's Incompleteness Theorem (cont.)

- Example: The axiom of choice cannot be proved or disproved
- Example: The continuum hypothesis cannot be proved or disproved (whether we assume the axiom of choice or not)
- Proof idea: encode statement as symbols, and use that to encode statements as numbers
- Define a statement that roughly means "I am not provable" (not easy to do, but Godel did it)
- This is a paradox, which means that not every statement can be proved or disproved

Implications from Godel's Incompleteness Theorem

- We can never make an axiom system that can prove everything we want it to prove
- We can also never prove that an axiom system is consistent
- So we need to be really careful in choosing our axioms
- However, we can use a more sophisticated viewpoint to make this much less of a problem than it seems. If we assume certain things about ordinals, then we can use this to prove the consistency of our axioms
- So as long as these ordinal assumptions are correct, we don't have to worry

Conclusion

- What this *doesn't* mean is that math is useless or wrong
- Math is still the most successful and accurate way we have of understanding the universe
- It is highly likely that our axioms are consistent; we just can never prove it
- We may just need to think of it more as a science, instead of some greater form of understanding, and with this viewpoint, it's by far the most accurate science of them all