

Announcements

HW10 updated with remaining problems (due Wed. 4/7)

Final exam: Tues. 5/13 8:00am-11:00am,

1047 Sidney Lu Mech. E. Bldg. (lecture room, not the
(email ASAP w/ any issues) midterm room)

Exam will be cumulative

Schedule:

Today, Friday: projective space, projective varieties

Monday: topic TBD; see poll in email

Wednesday: review

We'll also have a review session closer to the exam

Midterm 3 graded

Q1: 70%

Median: 53/70

Q2: 74%

Mean: 52.6/70

Q3: 86%

Std. dev: 11.3

Q4: 80%

Gradelines: A-/A: 54 to 70

B+/B/B-: 33 to 54 -E

C+/C/C-: 14 to 33 -E

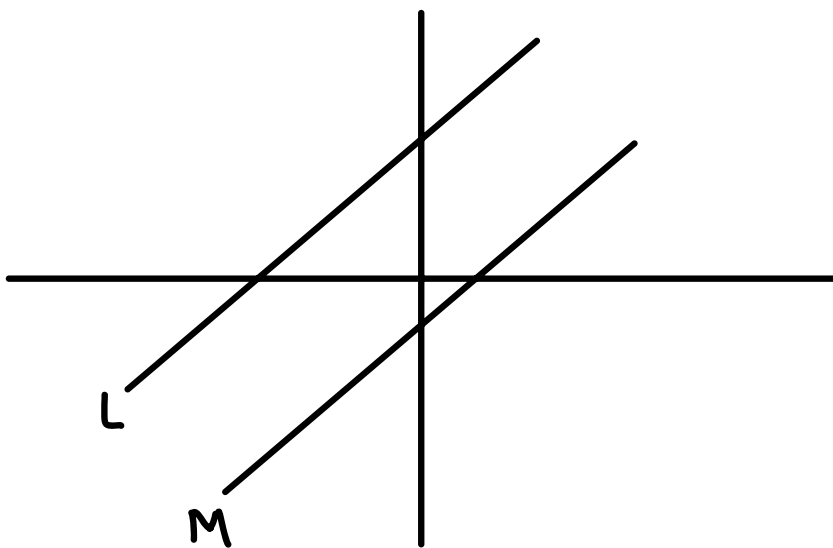
Sol's posted to website + gradeline spreadsheet updated

Projective space (see Cox, Little, O'Shea: Ideals, Varieties, and Algorithms, Ch 8.)

Motivation: recall

Bézout's Thm: The "usual" situation is that two poly. in $\mathbb{C}[x,y]$ of degrees m and n have $m \cdot n$ intersection points in \mathbb{C}

But what about parallel lines?



$(\deg L)(\deg M) = 1 \cdot 1 = 1$, but L and M don't intersect

Fix: add pts. "at ∞ " where parallel lines meet

Consider equiv. classes of parallel lines

Def (version 1): The (complex) projective plane is the set

$\nearrow \widetilde{\mathbb{P}^2(\mathbb{C})} = \mathbb{C}^2 \cup \{\text{one pt "at } \infty" \text{ for each equiv. class of parallel lines}\}$

for now, to distinguish from def. 2 H_∞

Works, but kind of a weird def'n

For a nicer one, let's define homogenous coords. in \mathbb{C}^3

We say that $\underbrace{(a_0, a_1, a_2)}_{\in \mathbb{C}^3} \sim (b_0, b_1, b_2)$

if $(b_0, b_1, b_2) = (\lambda a_0, \lambda a_1, \lambda a_2)$ for some $\lambda \in \mathbb{C} \setminus \{0\}$

i.e. if all the ratios are the same: $\frac{a_0}{a_1} = \frac{b_0}{b_1}, \frac{a_0}{a_2} = \frac{b_0}{b_2}, \frac{a_1}{a_2} = \frac{b_1}{b_2}$

i.e. if $a, b \neq 0, a \sim b \iff a$ and b are on the same line thru. origin in \mathbb{C}^3

Denote equiv. classes $[a_0 : a_1 : a_2]$

Def (version 2): The complex proj. plane is the set of equivalence classes

$$\mathbb{P}^2(\mathbb{C}) = (\mathbb{C}^3 \setminus \{0\}) / \sim$$

i.e. the set of 1D subspaces of \mathbb{C}^3

Prop: There is a (nice) bijection

$$\begin{array}{ccc} \mathbb{P}^2(\mathbb{C}) & \longrightarrow & \widetilde{\mathbb{P}^2(\mathbb{C})} \\ \text{def 2} & & \text{def 1} \end{array}$$

$$\text{Pf: } \mathbb{P}^2(\mathbb{C}) = \underbrace{\{[1:x:y] \mid x, y \in \mathbb{C}\}}_{S_1} \cup \underbrace{\{[0:1:y] \mid y \in \mathbb{C}\}}_{S_2} \cup \underbrace{\{[0:0:1]\}}_{S_3}$$

$[1:x:y] \mapsto (x, y)$ is a bij. $S_1 \rightarrow \mathbb{C}^2$

Let $a_m \in H_\infty$, $m \in \mathbb{C} \cup \{\infty\}$ be the equiv. class of lines in \mathbb{C}^2 of slope m

Then $[0:1:m] \mapsto a_m$

$[0:0:1] \mapsto a_\infty$

gives a bijection $S_2 \cup S_3 \rightarrow H_\infty$

□

Def: (complex) projective space is the set

$$\mathbb{P}^n(\mathbb{C}) = \{ \text{lines thru. origin in } \mathbb{C}^{n+1} \}$$

$$= \{ a = (a_0, \dots, a_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\} \} / (a \sim \lambda a, \lambda \in \mathbb{C})$$

$$= \{ [a_0 : \dots : a_n] \}$$

$$\text{Cor: } \mathbb{P}^n(\mathbb{C}) = \mathbb{C}^n \cup \mathbb{P}^{n-1}(\mathbb{C})$$

Pf: Use the maps from the previous prop:

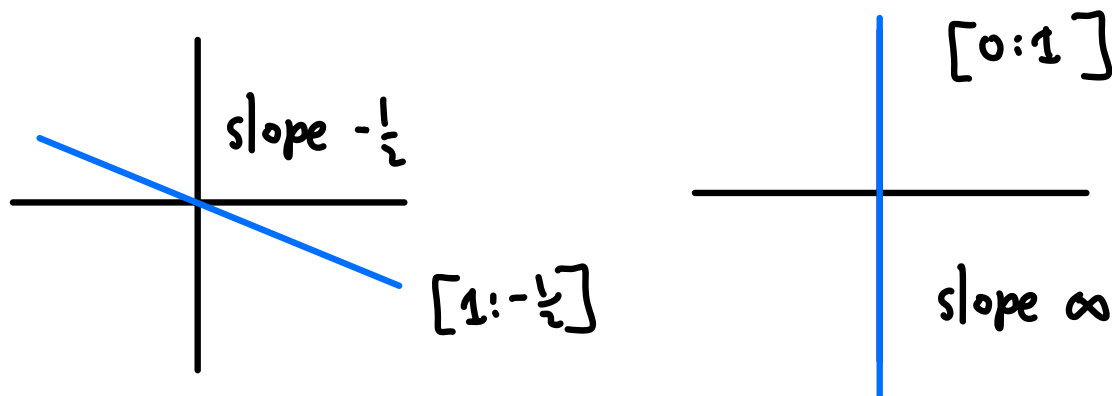
$$[1 : a_1 : \dots : a_n] \mapsto (a_1, \dots, a_n) \in \mathbb{C}^n$$

$$[0 : a_1 : \dots : a_n] \mapsto \underbrace{[a_1 : \dots : a_n]}_{\text{not all 0}} \in \mathbb{P}^{n-1}(\mathbb{C})$$

□

$$\text{Ex: } \mathbb{P}^1(\mathbb{C}) = \{ \text{lines in } \mathbb{C}^2 \} = \{ [x : y] \}$$

$$= \{ [1 : m] \mid m \in \mathbb{C} \} \cup \{ [0 : 1] \}$$



Also called the Riemann sphere

Want to define projective varieties in $\mathbb{P}^n(\mathbb{C})$

$$\text{Let } f(x, y, z) = xy - z$$

$$\text{Then } f(1, 1, 1) = 0$$

$$f(2, 2, 2) = 2$$

So what does $f([1:1:1])$ mean?

Problem: When we scaled the variables, we doubled z but quadrupled xy

Fix:

Def: $f(x_0, \dots, x_n) \in \mathbb{C}^{n+1}$ is homogeneous of degree d if every term has degree d

If f homog. of degree d

$$f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n)$$

$$\text{If } \lambda \neq 0, f(\lambda a_0, \dots, \lambda a_n) = 0 \iff f(a_0, \dots, a_n) = 0$$

Def: If $f \in \mathbb{C}[x_0, \dots, x_n]$ homog.,

$$V(f) := \{[a_0 : \dots : a_n] \in \mathbb{P}^n(\mathbb{C}) \mid f(a_0, \dots, a_n) = 0\}$$

is the projective variety assoc. to f .

Next time: $V(I)$ for "homog. ideal" I