Math 412, Fall 2023 – Homework 7

Due: Wednesday, October 25th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should choose four of the following five problems to solve and turn in–if you do all five, only the first four will be graded. Graduate students taking the course for four credits should solve all five. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. Prove that every graph G with no isolated vertices has a matching of size at least $\frac{n(G)}{1+\Delta(G)}$.

   Proof. We use induction on the number of edges. To use the inductive hypothesis, we want to delete an edge and still have the resulting graph have no isolated vertices. So the base case covers situations where that is not possible.

   Base case: suppose that at least one endpoint of every edge has degree one. Then every component of G is a "star" a graph with one central vertex and one or more adjacent vertices. Let M be any matching of G with one edge from each component. In any component H of G, the number of vertices $n(H)$ in H is 1 plus the degree of the central vertex, so $n(H) = 1 + \Delta(H) \leq 1 + \Delta(G)$. If G has k components $H_1, \ldots, H_k$, then $n = \sum_i n(H_i) \leq \sum_i 1 + \Delta(G) = k(1 + \Delta(G))$, and so $k \geq \frac{n}{1+\Delta(G)}$. Since M has k edges, the result holds in this case.

   Inductive step: Now suppose G has at least one edge e where both endpoints have degree ≥ 2. Then $G' := G \setminus e$ has no isolated vertex, so by the inductive hypothesis, G' has a matching with at least $\frac{n(G')}{1+\Delta(G')}$ edges, and this is also a matching of G. Now, $n(G') = n(G)$ and $\Delta(G') \leq \Delta(G)$, so $\frac{n(G)}{1+\Delta(G)} \leq \frac{n(G')}{1+\Delta(G')}$, and so the result holds for this matching. □

2. Let G be a simple graph with $\Delta(G) \leq 3$. Prove that $\kappa(G) = \kappa'(G)$.

   Solution: We know that $\kappa(G) \leq \kappa'(G) \leq \delta(G) \leq \Delta(G)$ for all graphs by Whitney's Theorem (4.1.9), so we need only show the other direction, and specifically we only need to focus on cases where $\kappa'(G) \in \{0, 1, 2, 3\}$. Note that if G is complete, then $G \cong K_i$ for $i \in \{1, 2, 3, 4\}$. In these four cases it is easily checked that $\kappa(K_i) = \kappa'(K_i) = \Delta(K_i) = i - 1$, so let us assume G is not complete.

   Now, note that if $\kappa'(G) = 0$, then $0 \leq \kappa(G) \leq \kappa'(G)$ implies $\kappa(G) = 0$. Also, if $\kappa'(G) = 1$, this implies that G is connected, so $\kappa(G) \geq 1$, and thus $1 = \kappa'(G) \geq \kappa(G) \geq 1$ implies $\kappa'(G) = \kappa(G)$. Thus, the only cases left are when $\kappa'(G) \in \{2, 3\}$.

Let us consider the case when $\kappa'(G) = 2$, and note that this implies that G is connected. If $\kappa(G) \neq \kappa'(G)$, then $\kappa(G) = 1$, so there is a cut vertex v. Let H_1 and H_2 be two components in $G \setminus v$, and note that v must have at least one edge to a vertex in H_1 and at least one edge to a vertex in H_2. Then since $d(v) \leq 3$, this implies that v has exactly one edge to one of H_1 or H_2 (possibly both if there is a third component in $G \setminus v$), say without loss of generality that there is exactly one edge from v to H_1. Then removing this edge from G separates $V(H_1)$ from $V(G) \setminus H_1$, which contradicts the claim that $\kappa'(G) = 2$. Thus, when $\kappa'(G) = 2$, $\kappa(G) = 2 = \kappa'(G)$.

The final case to consider is when $\kappa'(G) = 3$, but in this case, since $\delta(G) \geq \kappa'(G) = 3$ and $\Delta(G) = 3$, this implies that G is 3-regular, which is Theorem 4.1.11

3. Let G be k-connected and let $U_1, U_2 \subseteq V(G)$ be disjoint sets with $|U_1| = |U_2| = k$. Prove that there are k pairwise disjoint paths, each of which have one endpoint in U_1 and the other endpoint in U_2.

   Solution: If G is k-connected and we add a vertex $v \notin V(G)$ and attach v to at least k other vertices, we claim that $H := G \cup v$ is still k-connected. To see this, note that since $d(v) \geq k$, v will not be isolated without deleting at least all of its $\geq k$ neighbors from H. Since G is k-connected, we can delete up to $k-1$ vertices from G and there is a path between every pair of vertices, so if we delete up to $k-1$ vertices from H, there will still be a path between any pair of vertices in G, and since v is not isolated, a path from v to any other vertex as well.

   Using this, we will form a graph G' with two new vertices, u_1 and u_2, where u_1 is adjacent to every vertex in U_1 and u_2 is adjacent to every vertex in u_2. Since $|U_1| = |U_2| = k$, G' is still k-connected. Then by Menger's Theorem, we have k pairwise internally disjoint u_1, u_2-paths. The second vertex in each of these paths must be from U_1 and the second last vertex in each path must be from U_2, so this gives us k pairwise disjoint paths, each of which has one endpoint in U_1 and the other endpoint in U_2.

4. Prove that the hypercube Q_k is k-connected.

   Solution: We will prove this via induction on k. Note for $k = 1$, $Q_1 \cong K_2$, and K_2 is 1-connected, so we are done. Similarly, for $k = 2$, $Q_2 \cong C_4$, and C_4 is 2-connected. Now assume $k \geq 3$ and assume Q_{k-1} is $(k-1)$-connected. Note that since $k \geq 3$, Q_{k-1} is not complete. Assume to the contrary that Q_k is not k-connected, so there is a set S with $|S| \leq k-1$ such that $Q_k - S$ is disconnected. Let x and y be vertices in $V(Q_k) \setminus S$ that are in different components in $Q_k - S$.

   Recall that Q_k contains two disjoint copies of Q_{k-1} where corresponding vertices in the two copies are adjacent. Let $Q_{k-1}^{(1)}$ and $Q_{k-1}^{(2)}$ be two copies of Q_{k-1} such that $V(Q_k) = V(Q_{k-1}^{(1)}) \cup V(Q_{k-1}^{(2)})$. By symmetry we can assume that $x \in V(Q_{k-1}^{(1)})$. Let us consider cases based on which copy of Q_{k-1} the vertex y is in.

   Case 1: $y \in V(Q_{k-1}^{(1)})$. If $|S \cap V(Q_{k-1}^{(1)})| < k-1$, then the induced graph $Q_k[V(Q_{k-1}^{(1)}) \setminus S]$ is connected since Q_{k-1} is $k-1$ connected, but this implies that x and y are in the same component of $Q_k - S$ and we are done. Thus, we may assume that $S \subseteq V(Q_{k-1}^{(1)})$. In this case then we can find a path from x to y through $Q_{k-1}^{(2)}$. Indeed, if x' and y' are the vertices corresponding to x and y in $Q_{k-1}^{(2)}$, then we can go from x to x', then take any path in $Q_{k-1}^{(2)}$ from x' to y', then go from y' to y. Thus, in all cases x and y are in the same component of $Q_k - S$.

Case 2: $y \in V(Q_{k-1}^{(2)})$. Let $y' \in V(Q_{k-1}^{(1)})$ be the vertex corresponding to y (and thus adjacent to y). By the previous case, if $y' \notin S$, then there is an x–y' path in $Q_k - S$, and thus since $y'y \in E(G)$, x and y are adjacent. Thus, $y' \in S$. This implies that $|S \cap Q_{k-1}^{(2)}| < k-1$, so $Q_k[V(Q_{k-1}^{(2)}) \setminus S]$ is connected since Q_{k-1} is $(k-1)$-connected. Thus, if there is a path from S to any vertex in $Q_{k-1}^{(2)}$ we are done. This implies that $x' \in S$, where x' is the vertex in $Q_{k-1}^{(2)}$ corresponding to x. Then since $x' \in S$, $|S \cap V(Q_{k-1}^{(1)})| < k-1$, so $Q_k[V(Q_{k-1}^{(1)}) \setminus S]$ is connected. Thus, the only way x and y are not in the same component of $Q_k - S$ is if at least one of u or u' are in S for each $u \in V(Q_{k-1}^{(1)})$, where $u' \in V(Q_{k-1}^{(2)})$ is the vertex corresponding to u. This implies that $k-1 \geq |S| \geq |V(Q_{k-1})| = 2^{k-1}$, but this is false for all $k \geq 2$. Thus we reach a contradiction and we are done.

5. Draw a graph G with n vertices and $\kappa(G) = k$ for the following values of n and k.

   (a) $n = 9, k = 3$

   (b) $n = 9, k = 4$

   (c) $n = 10, k = 3$

   (d) $n = 10, k = 4$

   [Hint: see Example 4.1.4]

   Solution: These are the Harary graphs $H_{3,9}, H_{4,9}, H_{3,10}$, and $H_{4,10}$, respectively.