Math 412, Fall 2023 – Homework 2

Due: Wednesday, September 6th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should choose four of the following five problems to solve and turn in–if you do all five, only the first four will be graded. Graduate students taking the course for four credits should solve all five. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. Recall that K_n denotes the complete graph on n vertices. Prove or disprove.

   a. For every $e, f \in E(K_n)$, $(K_n \setminus e) \cong (K_n \setminus f)$.
   Solution. This is true

   Proof. Since K_n is a simple graph, any subgraph of K_n is also simple (in fact, being a subgraph of K_n is both a necessary and sufficient condition for a graph to be simple), so by Problem 2a of Homework 1, $(K_n \setminus e) \cong (K_n \setminus f)$ if and only if their complements are isomorphic to each other.

   Let G be the complement of $K_n \setminus e$ and let H be the complement of $K_n \setminus f$. G has a single edge, say with endpoints u and v; while H has a single edge, say with endpoints u' and v'. Since both G and H have n vertices, let $f : V(G) \to V(H)$ be a bijection such that $f(u) = u', f(v) = v'$. The induced map on edges maps uv to $f(u)f(v) = u'v' \in E(H)$, so f satisfies the condition that $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$, so f is a graph isomorphism between G and H, and therefore $(K_n \setminus e) \cong (K_n \setminus f)$. $\square$

   b. For every $e_1, e_2, f_1, f_2 \in E(K_n)$ such that $e_1 \neq e_2$ and $f_1 \neq f_2$,

   $$((K_n \setminus e_1) \setminus e_2) \cong ((K_n \setminus f_1) \setminus f_2).$$

   Solution. This is false for any $n \geq 4$. As a counterexample, suppose that e_1 and e_2 share an endpoint v, while f_1 and f_2 do not share any endpoints (this is possible for $n \geq 4$ since K_n has at least 4 vertices, with an edge connecting any pair). The degree of v in $((K_n \setminus e_1) \setminus e_2)$ is $n - 2$, while in $((K_n \setminus f_1) \setminus f_2)$ every vertex has degree n or $n - 1$. Therefore, the two graphs cannot be isomorphic.

   [For $n < 3$, the statement is vacuously true, since there is at most one edge. For $n = 3$, it is true via a quick check.]

2. Let G be a graph, let e be an edge of G and let W be a closed walk in G such that e is in W an odd number of times. Prove that W contains a cycle that contains the edge e.

   Solution: Let us proceed by induction on the length ℓ of the walk w. If $\ell = 1$, then e is a loop, which is a cycle of length 1, so we are done. Now assume $\ell \geq 2$, and assume that this property holds for all walks of length $\ell' < \ell$. If W does not contain any repeated vertices (except the first/last vertex), then either W is a cycle, so we are done, or W has length 2, with the same edge twice, which violates the hypothesis that e appears an odd number of times in W.

   Thus, W must contain a vertex v that appears at least twice. Let x be the first vertex of W. Then there exists values k, k' with $0 \leq k < k' \leq \ell$, with not both $k = 0$ and $k' = \ell$ such that

   $$W = (x = v_0, v_1, \ldots, v = v_k, v_{k+1}, \ldots, v = v_{k'}, v_{k'+1}, \ldots, x = v_\ell).$$

   Then we have two closed walks:

   $$W_1 = (x = v_0, v_1, \ldots, v = v_k = v_{k'}, v_{k'+1}, \ldots, x = v_\ell),$$

   and

   $$W_2 = (v = v_k, v_{k+1}, \ldots, v = v_{k'}).$$

   Both walks have length less than ℓ, and they partition W, so one of the walks must contain e an odd number of times. Thus, by induction, that walk contains a cycle through e and we are done.

3. Determine for which values m and n the complete bipartite graph $K_{m,n}$ has an Eulerian circuit.

   Solution $K_{m,n}$ has an Eulerian circuit if and only if either (a) m and n are both even or (b) one of m and n is zero.

   Proof. We use Theorem 1.2.26 from West (proved by Euler, but *not* called Euler's Theorem): A graph G has an Eulerian circuit if and only if G has at most one nontrivial connected component and all its vertices have even degree.

   If m and n are both nonzero, $K_{m,n}$ is connected, so has one nontrivial component. By definition, $K_{m,n}$ is the union of two independent sets, S of size m and T of size n, and every vertex in S is adjacent to every vertex in T. Therefore, vertices in S have degree n and vertices in T have degree m; the graph $K_{m,n}$ is even precisely when both m and n are even, so $K_{m,n}$ has an Eulerian circuit in precisely this case.

   Finally, if $m = 0$ (resp. $n = 0$), then $K_{m,n}$ consists of n (resp. m) isolated vertices, so it has an Eulerian circuit since it has no edges. □

4. Prove that a loopless graph G is bipartite if and only if every subgraph H of G has an independent set of size at least $|V(H)|/2$.

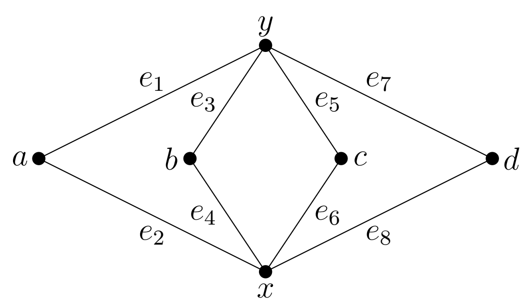
Proof. ($\Rightarrow$) We'll begin with the forward direction. Assume that G is a loopless bipartite graph and consider $H \subseteq G$. The subgraph H must also be bipartite. Let U_1 and U_2 be a bipartition of G, i.e. independent sets such that $V(H) = U_1 \sqcup U_2$. Because $|U_1| + |U_2| = |V(H)|$, we must have $\max(|U_1|, |U_2|) \geq V(H)/2$. Hence, all subgraphs H of G must contain an independent set of size at least $|V(H)|/2$.

($\Leftarrow$) To prove the other direction, we'll actually prove the contrapositive statement: "If G is not bipartite, then there exists some subgraph H of G whose independent sets are all strictly smaller than $|V(H)|/2$." Suppose G is not bipartite. Then by König's Theorem, G contains an odd cycle, C_{2n+1}, and the largest possible independent set in C_{2n+1} has size $n \leq |V(C_{2n+1})| = |2n+1|/2$. $\square$

5. Two Eulerian circuits in a graph G are *equivalent* if they have the same cyclic sequence of edges or if one cyclic sequence is the reverse of the other. For example,

$$(v_0, e_1, v_1, e_2, v_3, e_3, v_0) \quad \text{and} \quad (v_1, e_2, v_3, e_3, v_0, e_1, v_1) \quad \text{and} \quad (v_0, e_3, v_3, e_2, v_1, e_1, v_0)$$

are all equivalent Eulerian circuits. How many equivalence classes of Eulerian circuits are there for the following graph G? This problem does *not* require proof; however, you *do* need to write down an Eulerian circuit for each equivalence class.



Solution. In any Eulerian circuit of G, the pairs of edges $\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}$, and $\{e_7, e_8\}$ must appear consecutively. To count Eulerian circuits up to equivalence, we can count the number of distinct Eulerian circuits with first edge e_1 and second edge e_2. Such Eulerian circuits begin at y. After traversing edges e_1 and e_2, we are then at vertex x.

We then have three choices of pairs of edges to traverse to y: $\{e_4, e_3\}, \{e_6, e_5\}$, and $\{e_8, e_7\}$. After choosing a pair and traversing it, we are at y and have exactly two choices of paths for returning to x. Once back at x, there is then a single remaining path for returning to y. Hence, there are exactly $3 \cdot 2 = 6$ equivalence classes of Eulerian circuits.