

Cubic formula

Thm (Cardano, 1545): The cubic eqn. is solvable by radicals.

$$f(x) = x^3 + ax^2 + bx + c$$

$$g(y) = f\left(x + \frac{a}{3}\right) = y^3 + py + q \quad \text{"depressed cubic"}$$

$$p = \frac{1}{3}(3b - a^2) \quad q = \frac{1}{27}(2a^3 - 9ab + 27c)$$

$$\text{Let } \zeta = \zeta_3 \quad \zeta^2 + \zeta + 1 = 0$$

Let $g(y)$ have roots α, β, γ

$$\text{Have } e_1 = \alpha + \beta + \gamma = -\gamma^2\text{-coeff} = 0$$

$$e_2 = p \quad e_3 = -q$$

$$0 = \alpha + \beta + \gamma$$

$$\left. \begin{aligned} \Theta_1 &:= \alpha + \zeta\beta + \zeta^2\gamma \\ \Theta_2 &:= \alpha + \zeta^2\beta + \zeta\gamma \end{aligned} \right\} \text{"Lagrange resolvents"}$$

$$\theta_1 + \theta_2 = 3\alpha$$

$$\gamma^2 \theta_1 + \gamma \theta_2 = 3\beta$$

$$\gamma \theta_1 + \gamma^2 \theta_2 = 3\gamma$$

$$\sqrt{D} = (\alpha - \beta)(\alpha - \gamma)(\beta - \gamma) = \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha - \alpha\beta^2 - \beta\gamma^2 - \gamma\alpha^2$$

So if $S = \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha + \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2$, then

$$\begin{aligned} \theta_1^3 &= \alpha^3 + \beta^3 + \gamma^3 + 3\gamma(\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha) \\ &\quad + 3\gamma^2(\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2) + 6\alpha\beta\gamma \end{aligned}$$

$$= \alpha^3 + \beta^3 + \gamma^3 + \frac{3}{2}S(S + \sqrt{D}) + \frac{3}{2}\gamma^2(S - \sqrt{D}) + 6\alpha\beta\gamma$$

Can show (given that $\alpha + \beta + \gamma = 0$)

$$\alpha^3 + \beta^3 + \gamma^3 = -3q, \quad S = 3q$$

So

$$\theta_1^3 = -3q + \frac{3}{2}\gamma(3q + \sqrt{D}) + \frac{3}{2}\gamma^2(3q - \sqrt{D}) - 6q$$

$$= -\frac{27}{2}q + \frac{3}{2}\sqrt{-3D} \quad \left(\text{since } \gamma + \gamma^2 = -1 \right. \\ \left. \gamma - \gamma^2 = \sqrt{-3} \right)$$

Similarly,

$$\Theta_2^3 = -\frac{27}{2}q - \frac{3}{2}\sqrt{-3D}$$

(need $\Theta, \Theta_2 = -3p$)

Choose

$$A = \sqrt[3]{-\frac{27}{2}q + \frac{3}{2}\sqrt{-3D}}$$

$$B = \sqrt[3]{-\frac{27}{2}q - \frac{3}{2}\sqrt{-3D}}$$

s.t. $AB = -3p$. Then,

$$\alpha = \frac{A+B}{3} \quad \beta = \frac{\wp^2 A + \wp B}{3}$$

$$\gamma = \frac{\wp A + \wp^2 B}{3}$$

(Quartic formula follows from this and the resolvent cubic)

Solvability by radicals

Def: $f(x) \in F[x]$ is solvable by radicals if \exists

$$F = K_0 \subseteq K_1 \subseteq \dots \subseteq K_s = \text{Sp}_F f$$

where $K_{i+1} = K_i(\alpha_i)$ w/ α_i a root of $x^{n_i} - a_i$

Assume $\text{char } F = 0$ (or just let it not divide anything)
(we don't want it to)

Thm (ancients, Cardano, Ferrari): All $\deg \leq 4$ polys are solvable

Thm (Abel-Ruffini): There is no general formula by radicals for $f_{\text{gen}}^{(n)}$, $n \geq 5$.

Thm (Galois):

a) $f(x)$ is solvable by radicals $\Leftrightarrow \text{Gal } f$ is a 'solvable gp'

b) \exists a degree 5 poly. which is not solvable by radicals.

Def: A finite gp. G is solvable (UK: "soluble") if

$$\{1\} = G_s \triangleleft G_{s-1} \triangleleft \dots \triangleleft G_0 = G$$

where G_i/G_{i+1} is cyclic.

Examples:

- abelian gps.
- dihedral gps.

$$1 \triangleleft C_n \triangleleft D_{2n}$$

\nwarrow
 C_2

- p-gps. ($|G| = p^k$)

$$S_4: 1 \triangleleft V_4 \triangleleft A_4 \triangleleft S_4$$

$\nwarrow \quad \nwarrow \quad \nwarrow$
 $V_4 \quad C_2 \quad C_2$

Non-examples:

- S_n or A_n for $n \geq 5$ (DEF Thm 4.24)
no normal subgps! i.e. "simple"

- Other finite simple gps. (e.g. the monster)

Cor! If $n = 5$, $K = S_{\mathbb{P}_F} f$,

$\text{Gal}(K/F) = S_n \text{ or } A_n \Rightarrow f$ is not solvable by radicals

So Galois \Rightarrow Abel-Ruffini

Prop:

a) If $H \leq G$, then G solvable $\Rightarrow H$ solvable

b) If $H \trianglelefteq G$, then H solvable, G/H solvable $\Rightarrow G$ solvable

Pf:

a) Let $\{1\} = G_s \triangleleft G_{s-1} \triangleleft \dots \triangleleft G_0 = G$

where G_i/G_{i+1} is cyclic, and let $H_i = H \cap G_i$

Then $H_{i+1} \triangleleft H_i$ and H_{i+1}/H_i is isom to a subgp. of G_{i+1}/G_i , so is cyclic.

b) $1 = H_s \triangleleft H_{s-1} \triangleleft \dots \triangleleft H_0 = H$

$1 = J_r \triangleleft J_{r-1} \triangleleft \dots \triangleleft J_1 = G/H$

If $\pi: G \rightarrow G/H$, then

$1 = H_s \triangleleft \dots \triangleleft H_0 = \pi^{-1}(J_r) \triangleleft \pi^{-1}(J_{r-1}) \triangleleft \dots \triangleleft \pi^{-1}(J_1) = G$

\nwarrow cyclic \nearrow

□