§6.5: Generalized Permutation and Combinations

$P(n,k)$ and $\binom{n}{k}$ refer to permutations/combinations <u>without</u> repetition and with <u>distinguishable</u> objects

e.g. ABCDEF

BCAE	$\{C, E\}$
4-permutation	2-combination

If we allow repetition, we allow examples like

BBBB	$\{C, C\}$
4-perm w/ rep.	2-comb. w/ rep.

For a set of size n, the number of

r-perms w/ repetition is	r-combs w/ repetition is
n^r	$\binom{n+r-1}{r}$ (*)
(by prod. rule)	

Idea behind (*): "sticks and stones".

Stones are the elements

Sticks are the "separators"

Ex 4: 4 different kinds of cookie. How many different ways are there to choose 6 cookies <u>with</u> (potential) repetition?

e.g.

$$* * | * | | * * *$$

2 of type 1   1 of type 2   0 of type 3   3 of type 4

6 stars (cookies)

4 types, so $4-1=3$ separators

Choose the spots for the 6 cookies (or 3 separators)

Num ways:

$$\binom{6+3}{6} = \binom{9}{6} = \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1} = 84$$

Ex 5: How many nonnegative integer sol'ns does the eqn. $x_1+x_2+x_3=11$ have

11 "stones" $3-1=2$ "sticks"

$$* * | * * * | * * * * * *$$

$$2 + 3 + 6 = 11$$

Num ways: $\binom{11+(3-1)}{11} = \binom{13}{11} = 78$

Permutations of partially indistinguishable objects:

e.g. AABBBC BABCBA

For n total objects, k types, n_i of the ith type, there are $\frac{n!}{n_1!\,n_2!\cdots n_k!} =: \binom{n}{n_1,\dots,n_k}$ permutations (**)

(Note: $\binom{n}{k,n-k} = \binom{n}{k}$)

Idea: take a permutation:

$n!$ ways BABCBA

Swapping around the first type of object doesn't change the permutation

$n_1!$ ways to do this

Divide, by division rule

Same for $n_2, n_3, \dots$

all the same

$$\left\{\begin{array}{l} \text{BABCBA} \\ \text{BABCBA} \\ \text{BABCBA} \\ \text{BABCBA} \\ \text{BABCBA} \\ \text{BABCBA} \end{array}\right.$$

Next, we want to put n objects into k boxes

Distinguishable objects into distinguishable boxes:

k^n ways

If we want $n_1, n_2, \dots, n_k$ elts. in box $1, 2, \dots, k$:

$\binom{n}{n_1, \dots, n_k}$ ways

Indistinguishable objects into distinguishable boxes:

Sticks and stones: $\binom{n+k-1}{n}$

Other two cases are harder: use ad hoc methods

Distinguishable objects and indistinguishable boxes:

Ex 10: How many ways are there to put four (distinguishable!) students into at most three groups?

Sol'n:

All four in one group: 1 way ABCD

3 in one group, 1 in another: 4 ways ABC, D ACD, B ABD, C BCD, A

2 in one group, 2 in another: 3 ways

AB, CD
AC, BD
AD, BC

not 6 since BC, AD is the same as AD, BC

2 in one group, 1 in another, 1 in a third: 6 ways

AB, C, D   BC, A, D
AC, B, D   BD, A, C
AD, B, C   CD, A, B

Total: $1+4+3+6=14$

Indistinguishable objects into indistinguishable boxes:

Ex 11: How many ways are there to pack 6 identical copies of a book into (at most) 4 identical boxes?

Soln: List the possibilities:

6   3,1,1,1
5,1   2,2,2
4,2   2,2,1,1
4,1,1
3,3
3,2,1

boxes listed in decreasing order

9 total ways