Announcements:

Exam 1 graded

Median: 75/95

Mean: 71.4/95

Std. dev.: 17.3

Regrade request deadline: next Wed. (10/4)

Recall: Matrix tree thm.: For any loopless graph G, and for any i,

$$\tau(G) = \det \overbrace{L^i(G)}^{\text{reduced Laplacian}},$$

where $L(G) = D(G) - A(G)$ is the Laplacian matrix of G.

Pf (Godsil-Royle, Algebraic Graph Theory):

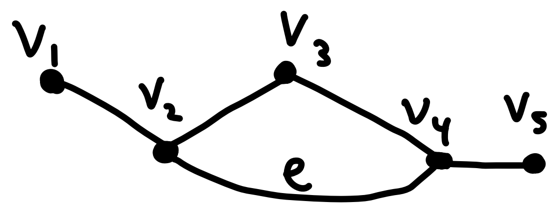
Induction on $|E(G)|$, using Prop. 2.2.8:

$$\tau(G) = \tau(G \setminus e) + \tau(G \cdot e)$$

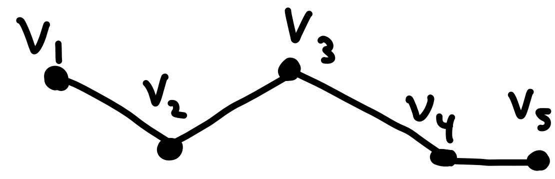
Base case: no edges:

$$\tau(G) = \begin{cases} 0, & n=1 \\ 1, & n>1 \end{cases} = \det L^i(G). \checkmark$$

Inductive step:



G



$G \setminus e$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

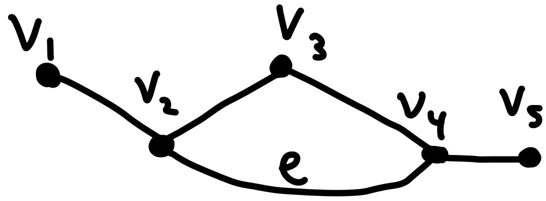
$L(G) \qquad L(G \setminus e) \qquad E$

$$\begin{array}{c|c} & \begin{matrix} v_1 & v_3 & v_4 & v_5 \end{matrix} \\ \begin{matrix} v_1 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \quad \begin{array}{c|c} & \begin{matrix} v_1 & v_3 & v_4 & v_5 \end{matrix} \\ \begin{matrix} v_1 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \quad \begin{bmatrix} & \\ & 1 \end{bmatrix}$$

$L^2(G) \qquad L^2(G \setminus e) \qquad E'$

$$\det L^2(G) = \det L^2(G \setminus e) + 1 \cdot L^{2,4}(G \setminus e)$$

Return to example:



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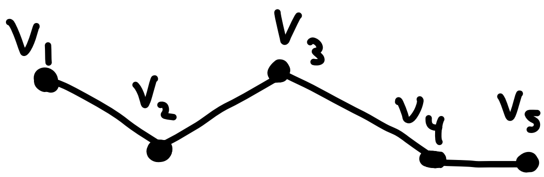
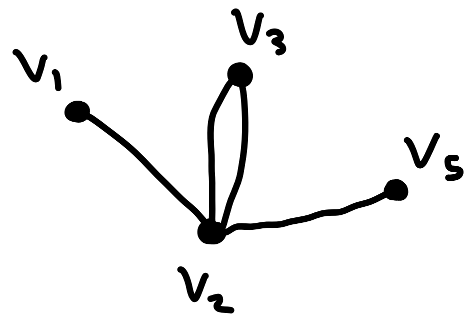


$G \setminus e$



$G \cdot e$

$$\begin{array}{c|cccc} & v_1 & v_3 & v_4 & v_5 \\ \hline v_1 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 2 & -1 & 0 \\ v_4 & 0 & -1 & 3 & -1 \\ v_5 & 0 & 0 & -1 & 1 \end{array}$$

$L^2(G)$

$$\begin{array}{c|cccc} & v_1 & v_3 & v_4 & v_5 \\ \hline v_1 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 2 & -1 & 0 \\ v_4 & 0 & -1 & 2 & -1 \\ v_5 & 0 & 0 & -1 & 1 \end{array}$$

$L^2(G \setminus e)$

$$\begin{array}{c|ccc} & v_1 & v_3 & v_5 \\ \hline v_1 & 1 & 0 & 0 \\ v_3 & 0 & 2 & 0 \\ v_5 & 0 & 0 & 1 \end{array}$$

$L^2(G \cdot e)$

$$\det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} + \det\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 2 & | & 0 \\ - & - & 1 & - \\ 0 & 0 & | & 1 \end{bmatrix}$$

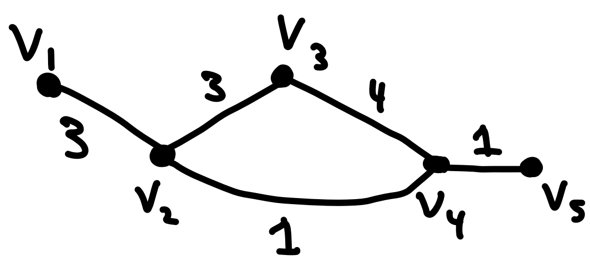
$$= \quad 1 \quad + \quad 2 \quad = 3$$

There are many generalizations of the Matrix Tree Theorem. Here's one:

Def:

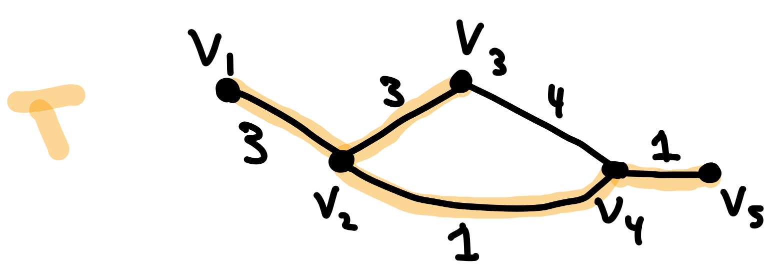
a) A weighted graph G is a graph together with a function $wt: E(G) \longrightarrow \mathbb{R}$

$$wt(v_1 v_2) = 3$$
$$\vdots$$



b) If T is a spanning tree of G, the weight of T is

$$wt(T) := \prod_{e \in E(T)} wt(e)$$



$$wt(T) = 3 \cdot 3 \cdot 1 \cdot 1 = 9$$

c) The tree sum $\tau(G)$ of G is

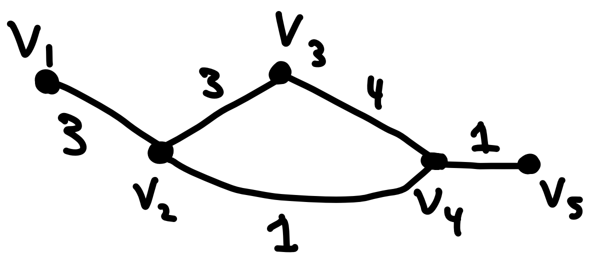
$$\tau(G) = \sum_{T \text{ sp. tree of } G} wt(T)$$

d) The (weighted) Laplacian matrix $L(G)$ of G is given by:

$$L_{ij} = \begin{cases} \displaystyle\sum_{\substack{e \text{ incident} \\ \text{to } i}} wt(e), & \text{if } i = j \\ \displaystyle -\sum_{\substack{e \text{ has} \\ \text{endpoints } i,j}} wt(e), & \text{if } i \neq j \end{cases}$$

Class activity:

Find $\tau(G)$ and $L(G)$ for:



Weighted Matrix Tree Theorem: For any loopless weighted graph G and any i,

$$\tau(G) = \det L^i(G)$$

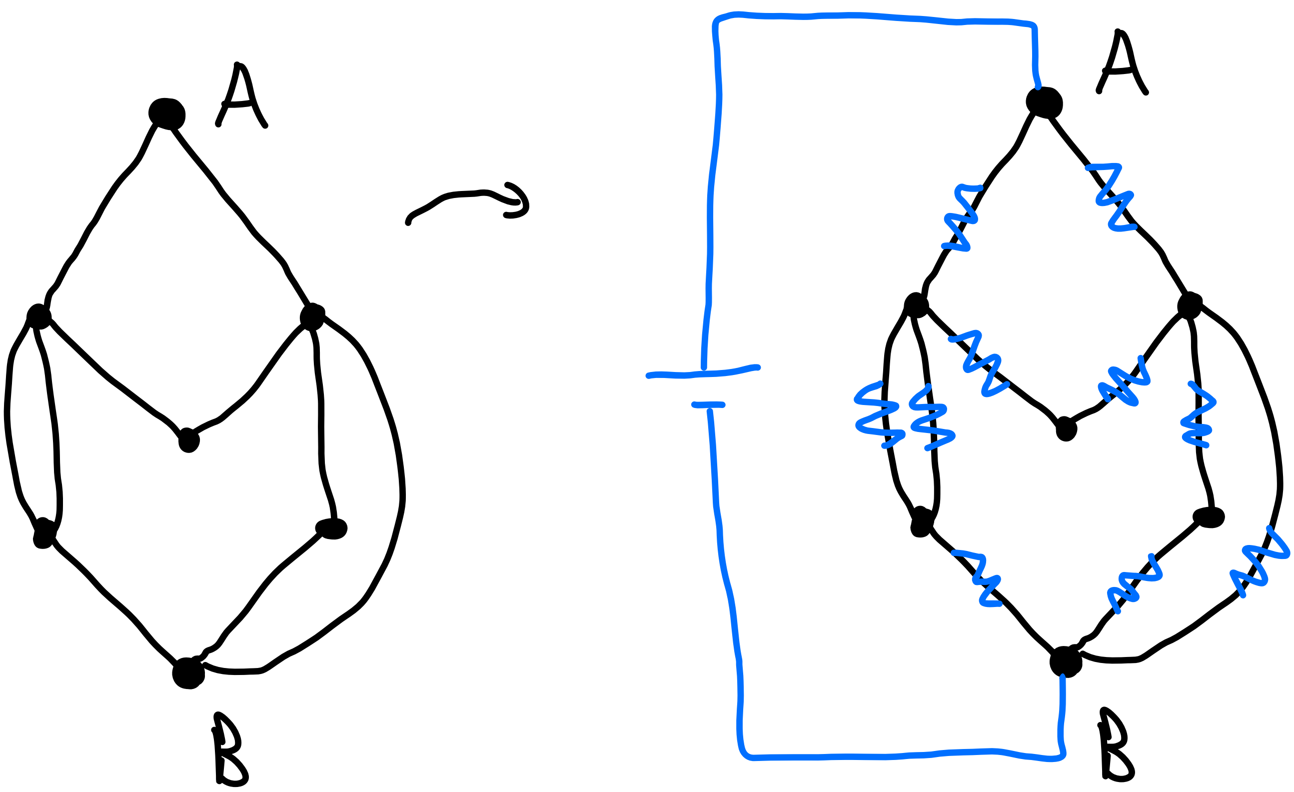
Pf: Homework!

Application/motivation:

Kirchoff's laws for electrical circuits

Source: Postnikov lecture notes (link on 412 course website)

Let G be a (loopless) graph, and consider edges of G to represent resistors.
Choose vertices A and B to be connected to a source of electricity



Choose any orientation D of G

(doesn't matter which)

Quantities associated to each edge e:

- Current I_e through e
- Voltage (or potential difference) V_e across e
- Resistance R_e of e $(R_e > 0)$
- Conductance $C_e := \frac{1}{R_e}$

Three laws:

K1: At any vertex v, the sum of the in-currents equals the sum of the out-currents:

$$\sum_{\substack{e \text{ has} \\ \text{head } v}} I_e = \sum_{\substack{e \text{ has} \\ \text{tail } v}} I_e$$

K2: For any cycle C in G, the (signed) sum of voltages is 0:

$$\sum_{e \in E(C)} \pm V_e = 0,$$

where we traverse C in either direction, and the term involving V_e is positive iff we traverse e in the way it's oriented in D.

Ohm's Law: $\forall e \in E(D)$,

$$V_e = I_e R_e \qquad (I_e = V_e C_e)$$

Prop: K_2 is equivalent to the following condition:

K_2': There exists a (unique) function

$$U : V(G) \to \mathbb{R},$$

called the potential function, s.t.

a) $\forall \; u \xrightarrow{e} v, \qquad V_e = U(v) - U(u)$

b) $U(B) = 0$

Pf: Homework!