

Solutions to Math 213-X1 Midterm Exam 2 — Oct. 25, 2024

1. (20 points) Find all solutions to the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 3^n$.

This is a linear inhomogeneous recurrence relations, so all solutions are of the form $a_n = a_n^{(h)} + a_n^{(p)}$, where $a_n^{(p)}$ is any particular solution and $a_n^{(h)}$ is a solution to the homogeneous recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2}.$$

The characteristic equation is $r^2 - 4r + 4 = (r - 2)^2$, so the only root is $r = 2$, with multiplicity 2. Therefore, (by Theorem 8.2.4) the general solution to the homogeneous equation is $a_n^{(h)} = (\alpha + \beta n)2^n$, where α and β are arbitrary.

Next, for the particular solution. Since the inhomogeneous part is $F(n) = 3^n$ and 3 is not a root of the characteristic equation, we know (Theorem 8.2.6) that there is a particular solution of the form $a_n^{(p)} = p3^n$, for some (but not all!) values of p .

Finally, we plug this particular solution into the recurrence relation to obtain

$$p3^n = 4p3^{n-1} - 4p3^{n-2} + 3^n,$$

so factoring out 3^{n-2} , $9p = 12p - 4p + 9$, and solving for p we get $p = 9$.

Therefore, the general solution is

$$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha + \beta n)2^n + 9 \cdot 3^n,$$

for arbitrary α and β .

2. (10 points) Suppose that the word “bitcoin” appears in 200 out of 1,000 spam email messages and in 1 out of 1,000 legitimate email messages. If a randomly chosen message is just as likely to be spam as to be legitimate, what is the probability that a given message containing the word “bitcoin” is spam?

We use Bayes’ Theorem. Given an email, let E be the event that it is spam, and let F be the event that it contains the word “bitcoin”. We want to find $p(E|F)$

From the problem statement, $p(E) = p(\overline{E}) = 0.5$, $p(F|E) = 200/1000 = 0.2$, and $p(F|\overline{E}) = 1/1000 = 0.001$.

Applying Bayes Theorem, we have

$$p(E|F) = \frac{p(F|E)p(E)}{p(F)} = \frac{p(F|E)p(E)}{p(F|E)p(E) + p(F|\overline{E})p(\overline{E})} = \frac{0.2 \cdot 0.5}{0.2 \cdot 0.5 + 0.001 \cdot 0.5} = 0.995.$$

(Since no calculators are allowed, $\frac{0.2 \cdot 0.5}{0.2 \cdot 0.5 + 0.001 \cdot 0.5}$ is an acceptable answer)

3. (10 points) Let A , B , and C be sets with $|A| = 9$, $|B| = 8$, $|C| = 10$, $|A \cap B| = 6$, $|A \cap C| = 3$, $|B \cap C| = 4$, and $|A \cap B \cap C| = 2$. Find $|A \cup B \cup C|$.

By inclusion-exclusion,

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 9 + 8 + 10 - 6 - 3 - 4 + 2 \\ &= 16 \end{aligned}$$

4. (10 points) Recall that a standard deck contains a total of 52 cards, 4 each of the kinds: 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace. The kinds Jack, Queen, and King are called *face cards*.

What is the probability that a random five-card hand contains exactly 2 face cards?

(For this problem, you may leave your answer in terms of binomial coefficients.)

There are a total of $\binom{52}{5}$ poker hands. Each deck contains $3 \cdot 4 = 12$ face cards, and 40 non-face cards, so to choose a hand with exactly 2 face cards, we choose 2 of the 12 face cards and 3 of the 40 non-face cards; thus there are $\binom{12}{2}\binom{40}{3}$ possible such hands.

Therefore, the probability that a given hand has exactly two face cards is

$$\frac{\binom{12}{2}\binom{40}{3}}{\binom{52}{5}}.$$

5. (20 points) Answer the following questions.

(No work necessary for this problem! Only your answer will be graded.)

(For this problem, you may leave your answer in terms of binomial coefficients.)

- (a) (5 points) How many ways are there to buy a total of 10 cookies when there are 4 different flavors?

Sticks-and-stones: $\binom{10+4-1}{10} = \binom{13}{10}$.

- (b) (5 points) How many ways are there to pack 4 identical copies of a book into any number of indistinguishable boxes?

This is just the number of ways to write 4 as a sum of positive integers in decreasing order: 4; 3 + 1; 2 + 2; 2 + 1 + 1; 1 + 1 + 1 + 1. Total: 4

- (c) (5 points) How many ways are there to pack 16 distinguishable objects into 20 distinguishable boxes?

Repeated product rule gives 20^{16}

- (d) (5 points) How many permutations of the letters *ABCDEF* contain either the string *AB* or the string *BA*? (Without any other letters in between)

Since we can't have both AB and BA, no need to do inclusion-exclusion. For strings containing AB, just treat that as a chunk. Then we have 5 chunks, so there are 5! such permutations. Same for BA, so we have a total of $2 \cdot 5! = 240$ valid permutations.

6. (15 points) Let n and k be positive integers. **Using a combinatorial argument**, prove the following identity:

$$\binom{n+1}{2k+1} = \sum_{j=k}^{n-k} \binom{j}{k} \binom{n-j}{k}$$

We count the number of binary strings of length $n+1$ with $2k+1$ 1's and $n-2k$ 0's. We do this in two ways. On one hand, choosing the positions of the 0's clearly gives $\binom{n+1}{2k+1}$.

On the other hand, let $j+1$ be the position of the *middle* 1, i.e. the $(k+1)$ st 1 reading from left to right. There are k 1's both before and after position $j+1$, so we must have $k < j+1 \leq n+1-k$,

so $j \leq j \leq n - k$. Then we can (independently) choose the k 1's before position $j + 1$, and the k 1's after position $j + 1$. The former choice has $\binom{j}{k}$ possibilities, while the latter has $\binom{n-j}{k}$ possibilities. Since we are counting the same set in multiple ways, we have

$$\binom{n+1}{2k+1} = \sum_{j=k}^{n-k} \binom{j}{k} \binom{n-j}{k},$$

as desired.