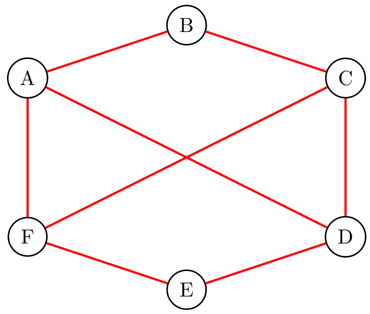
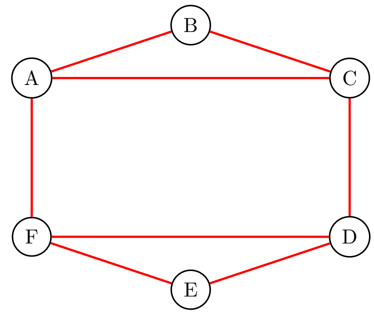
Solutions to Math 412 Midterm Exam 1 — Sept. 20, 2023

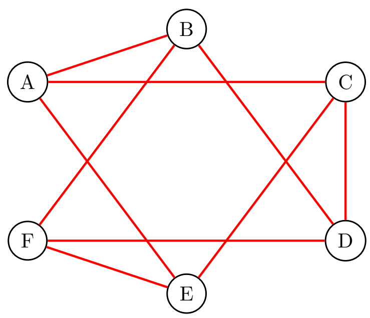
1. (20 points) Determine, with proof, which of the graphs below are isomorphic and which are not.



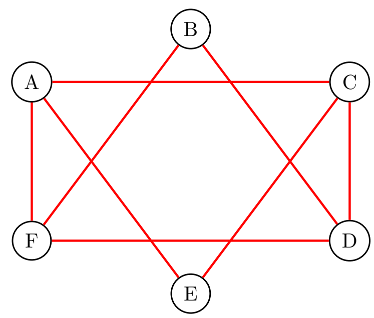
G



H



K



L

K is 3-regular, while the rest are not, so K isn't isomorphic to any of them. H and L have 3-cycles, while G does not, so G isn't isomorphic to the others either.

We are left with H and L, and we will show that they are isomorphic. Let $f : V(H) \to V(K)$ be the following bijection:

$$f(A) = A,\ f(B) = E,\ f(C) = C,\ f(D) = D,\ f(E) = B,\ f(F) = F.$$

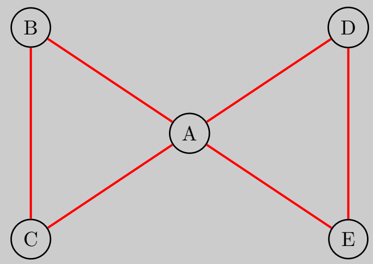
We will show that f is an isomorphism by showing that $uv \in E(H)$ if and only if $f(u)f(v) \in E(L)$.

$$AB \in E(H) \leftrightarrow f(A)f(B) = AE \in E(L)$$
$$BC \in E(H) \leftrightarrow f(B)f(C) = EC \in E(L)$$
$$CD \in E(H) \leftrightarrow f(C)f(D) = CD \in E(L)$$
$$DE \in E(H) \leftrightarrow f(D)f(E) = DB \in E(L)$$
$$EF \in E(H) \leftrightarrow f(E)f(F) = BF \in E(L)$$
$$FA \in E(H) \leftrightarrow f(F)f(A) = FA \in E(L)$$
$$AC \in E(H) \leftrightarrow f(A)f(C) = AC \in E(L)$$
$$DF \in E(H) \leftrightarrow f(D)f(F) = DF \in E(L)$$

This is all the edges in both H and L, so f is an isomorphism.

2. (15 points) Prove or disprove: Let G be a graph with an Eulerian circuit. If e and f are edges of G which share an endpoint, then G has an Eulerian circuit in which edges e and f appear consecutively.

This is false. For a counterexample, consider the following graph:



This graph contains an Eulerian circuit: A, B, C, A, D, E, A. However, it does not have an Eulerian circuit using AD and AE consecutively, even though they share an endpoint. Up to equivalence, such a circuit would start $E, A, D, \ldots$. However, the circuit cannot contain (for example) AB since there is no path from D to A that doesn't use the edge AD or the edge AE.

3. (15 points) Recall that G is H-free if G has no *induced* subgraph isomorphic to H. Prove that a simple graph G is bipartite if and only if it is C_k-free for all odd k.

If G is bipartite, then by Konig's Theorem it has no odd cycles, so it certainly had no induced subgraphs which are odd cycles.

Conversely, if G is not bipartite, Konig's Theorem tells us that G has an odd cycle C with k edges. If the induced subgraph $G[V(C)]$ is C itself (i.e. G has no edges between vertices of C except for the edges of C), we are done, so suppose there exists an edge $e \in E(G) \setminus E(C)$ with endpoints in C. More concretely, if as a walk C is $v_1, v_2, \ldots, v_k, v_1$ we can say without loss of generality that e has endpoints v_1 and v_j. Then $C_1 = v_1, v_2, \ldots, v_j, v_1$ and $C_2 = v_1, v_j, v_{j+1}, \ldots, v_k, v_1$ are cycles in G of shorter length, and since they have a total of $k + 2$ edges, one of them must be odd. Therefore, we take C to be the shortest odd cycle in G; this must be an induced subgraph, so G is not C_k-free for all odd k.

4. (20 points) Which of the following are graphic sequences (i.e. the degree sequence of a simple graph)? If a degree sequence is graphic, draw a graph with that degree sequence (no proof needed in this case), and if not, prove that no such graph exists.

(a) (10 points) $(7, 6, 5, 4, 2, 2, 2, 1, 1)$

In all of these cases, we use the criterion in the Havel-Hakimi Theorem repeatedly, until we obtain either a contradiction or the one-vertex simple graph. Let d_0 be the given degree sequence, and $d_1, d_2, \ldots$ be the sequences such that $d_i = d'_{i-1}$ in the Havel-Hakimi theorem. We have

$$d_0 = (7, 6, 5, 4, 2, 2, 2, 1, 1)$$

$$d_1 = (5, 4, 3, 1, 1, 1, 0, 1) = (5, 4, 3, 1, 1, 1, 1, 0)$$

$$d_2 = (3, 2, 0, 0, 0, 1, 0) = (3, 2, 1, 0, 0, 0, 0)$$

$$d_3 = (1, 0, -1, 0, 0, 0),$$

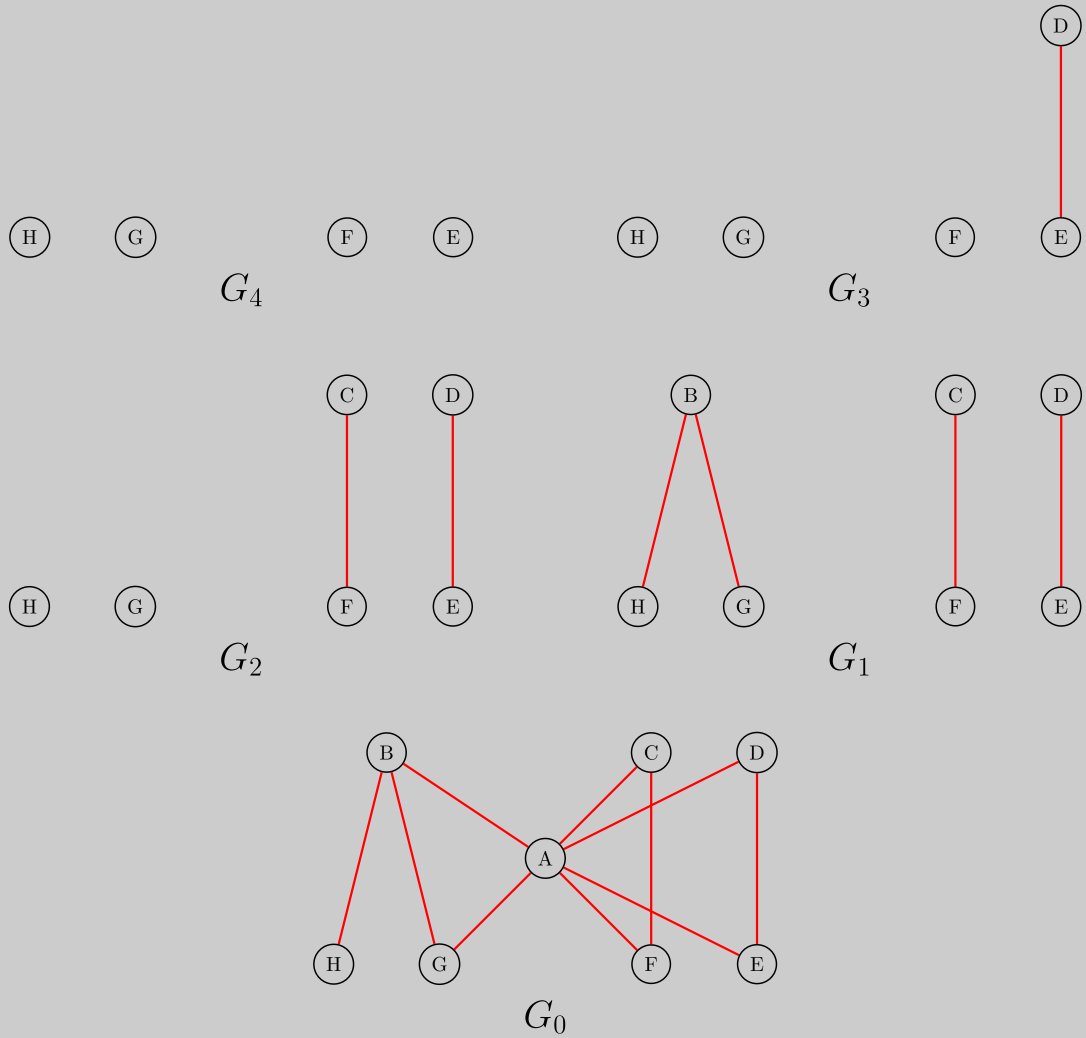
a contradiction.

(b) (10 points) $(6, 3, 2, 2, 2, 2, 2, 1)$

Again using the Havel-Hakimi Theorem,

$$d_0 = (6, 3, 2, 2, 2, 2, 2, 1)$$
$$d_1 = (2, 1, 1, 1, 1, 1, 1)$$
$$d_2 = (0, 0, 1, 1, 1, 1) = (1, 1, 1, 1, 0, 0)$$
$$d_3 = (0, 1, 1, 0, 0) = (1, 1, 0, 0, 0)$$
$$d_4 = (0, 0, 0, 0)$$

and d_4 is the degree sequence for the 4-vertex graph with no edges. Therefore, we can reconstruct a graph $G := G_0$ by letting G_i be a graph with degree sequence d_i. Then G_{i-1} is formed from G_i by adding a new vertex, and making it adjacent to the corresponding degree vertices of G_i.



5. (25 points) Recall the hypercube graph Q_k, which has 2^k vertices with labels corresponding to the length k binary strings, $a_1 \cdots a_k, a_i \in \{0, 1\}$, and an edge between two vertices precisely when their labels differ in only one entry.

   Let D_k be the unique orientation of Q_k such that the binary string corresponding to the head of every edge is smaller as a base-2 integer (or base-10 integer) than its tail. For example, D_3 has an edge with tail 110 and head 010.

   Fix a vertex v with label $a_1 \cdots a_k$, and let p be the number of 1's in $a_1 \cdots a_k$ and $q := k - p$ be the number of 0's.

(a) (10 points) Find and prove expressions for the in-degree and out-degree of v.

There is an edge from $a_1 \cdots a_k$ to $b_1 \cdots b_k$ precisely if for some i, $a_i = 1$ and $b_i = 0$, while for all other j, $a_j = b_j$. Then there are p strings $b_1 \cdots b_k$ with this property, so v has out-degree p. Since Q_k is k-regular, v has in-degree $q = k - p$.

(b) (10 points) Find and prove an expression for the number of paths in D_k from v to the vertex with label all 0's.

If P is such a path, each edge of P corresponds to changing a 1 to a 0, and as seen above, from any vertex w with a 1 in position i, there is an edge with tail w and head the same string except a 0 in position i. Therefore, the number of paths is equal to the number of ways to sequentially change every 1 in $a_1 \cdots a_k$ to 0. If $a_1 \cdots a_k$ has p 1's, this is $p!$.

Alternatively, we use induction, combined with the result in part a. The base case is that $00 \cdots 0$ has precisely one path to itself, which is the case for any vertex in any graph using the trivial (one-vertex path). For the inductive step, assume that there are $(p-1)!$ paths from any vertex w with $p-1$ 1's to the origin. Every out-neighbor of v is such a vertex, and v has p out-neighbors by part a, so there are $p \cdot (p-1)! = p!$ paths from v to $00 \cdots 0$.

(c) (5 points) Find (no proof necessary!) an expression for the number of maximal paths in D_k passing through v.

This is just the number of paths from $11 \cdots 1$ to $a_1 \cdots a_k$ times the number of paths from $a_1 \cdots a_k$ to $00 \cdots 0$. The former is $p!$ from the previous problem, and by symmetry the latter is $q! = (k-p)!$, so the total is $p!(k-p)!$. (Interestingly, this means that the proportion of maximal paths passing through v is $\binom{k}{p}^{-1}$, which can be seen by observing that after q steps, we arrive at one of $\binom{k}{p} = \binom{k}{q}$ possible vertices, and each one is equally likely).