

Math 418, Spring 2024 – Homework 8

Due: Wednesday, April 10th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition*. All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 2 bonus points per assignment.

1. **Dummit and Foote #14.2.6:** Let $K = \mathbb{Q}(\sqrt[8]{2}, i)$ and let $F_1 = \mathbb{Q}(i)$, $F_2 = \mathbb{Q}(\sqrt{2})$, $F_3 = \mathbb{Q}(\sqrt{-2})$. Prove that $\text{Gal}(K/F_1) = \mathbb{Z}/8\mathbb{Z}$, $\text{Gal}(K/F_2) = D_8$, $\text{Gal}(K/F_3) = Q_8$. (Hint: use the example in Section 14.2, and the diagrams on pages 580-1)

Solution. Use the example in Section 14.2, and the diagrams on pages 580-1. These diagrams (along with the Galois correspondence) tell us that $\text{Gal}(K/F_1) = \langle \sigma \rangle$, $\text{Gal}(K/F_2) = \langle \sigma^2, \tau \rangle$, $\text{Gal}(K/F_3) = \langle \sigma^2, \tau \sigma^3 \rangle$. σ has order 8, so $\text{Gal}(K/F_1) = \mathbb{Z}/8\mathbb{Z}$.

The dihedral group of order 8 has the presentation $\langle s, t \mid s^4 = t^2 = 1, tst = s^{-1} \rangle$. We have $(\sigma^2)^4 = 1$, $\tau^2 = 1$, and $\tau \sigma^2 \tau = \tau^2 (\sigma^3)^2 = (\sigma^2)^{-1}$, so since $\text{Gal}(K/F_2)$ has order 8, it must be D_8 .

Finally, the quaternion group has the presentation $\langle n, i, j, k \mid n^2 = 1, i^2 = j^2 = k^2 = ijk = n \rangle$. Let $n = \sigma^4$, $i = \sigma^2$, $j = \sigma\tau = \tau\sigma^3$, $k = \tau\sigma = \sigma^3\tau$. Then all the relations of Q_8 are satisfied (for example, $ijk = \sigma^2\sigma\tau\tau\sigma = \sigma^4 = n$), so since $\text{Gal}(K/F_3)$ has order 8, it must be Q_8 .

(There are simpler ways to do this if we assume knowledge about finite groups of order 8. In particular, D_8 and Q_8 are the only nonabelian groups of order 8, and we can distinguish between them by comparing the number of elements of order 2.)

2. **Dummit and Foote #14.2.7:** Determine all the subfields of the splitting field of $x^8 - 2$ which are Galois over \mathbb{Q} . (Hint: use the example in Section 14.2, and the diagrams on pages 580-1)

Solution. Use the example in Section 14.2, and the diagrams on pages 580-1. Let K be the splitting field of $x^8 - 2$. By the Fundamental Theorem of Galois theory, $E \subset K$ is Galois over the base field \mathbb{Q} if and only if $\text{Aut}(K/E)$ (the corresponding group to E in the diagrams) is normal in $G = \text{Gal}(K/\mathbb{Q})$. This is a group-by-group check. Use the relation $\sigma\tau\sigma^{-1} = \tau\sigma^2$, and the fact that any index-2 subgroup is normal; conjugate the generators by both σ and τ to see if their images generate the same group. To be conjugate, subgroups must have the same number of generators, which must have the same orders.

We conclude that the normal subgroups in G are $G, \langle \sigma^2, \tau \rangle, \langle \sigma \rangle, \langle \sigma^2, \tau \sigma^3 \rangle, \langle \sigma^2 \rangle, \langle \sigma^4 \rangle$, and 1, and therefore the intermediate fields which are Galois over \mathbb{Q} are $\mathbb{Q}, \mathbb{Q}(\sqrt{2}), \mathbb{Q}(i), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(i, \sqrt{2}), \mathbb{Q}(i, \sqrt[4]{2})$, and $K = \mathbb{Q}(i, \sqrt[8]{2})$.

3. **Dummit and Foote #14.2.14:** Show that $\mathbb{Q}(\sqrt{2 + \sqrt{2}})$ is a cyclic quartic field, i.e., is a Galois extension of degree 4 with cyclic Galois group.

Solution. Let $f(x) = x^4 - 4x^2 + 2$. This is irreducible over \mathbb{Q} by Eisenstein's criterion with the prime 2, and its roots (over a splitting field) are $\pm\theta_{\pm} := \pm\sqrt{2 \pm \sqrt{2}}$ (note: distinct). Since $\theta_+\theta_- = \sqrt{2}$, $\theta_- = \frac{\theta_+^2 - 2}{\theta_+} \in \mathbb{Q}(\theta_+)$, so $\mathbb{Q}(\theta_+)$ is the splitting field for f , and therefore is Galois over \mathbb{Q} .

Since the degree of the extension is 4, to show that the Galois group is cyclic, we just need to show that it has an element of degree 4. One such possibility is the unique automorphism sending θ_+ to θ_- since then $\sqrt{2} = \theta_+^2 - 2 \mapsto \theta_-^2 - 2 = -\sqrt{2}$, and so $\theta_- \mapsto \frac{-\sqrt{2}}{\theta_-} = -\theta_+$, so the order of this automorphism is 4.

4. Let K/F be a Galois extension of degree n with $G = \text{Gal}(K/F)$. For $\alpha \in K$, define the norm and trace of α by

$$N_{K/F}(\alpha) := \prod_{\sigma \in G} \sigma(\alpha), \quad \text{and} \quad \text{Tr}_{K/F}(\alpha) = \sum_{\sigma \in G} \sigma(\alpha).$$

Let $m_{\alpha, F}(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0$.

- (a) Show that $N_{K/F}(\alpha) = (-1)^n a_0^{n/d}$ and $\text{Tr}_{K/F}(\alpha) = -\frac{n}{d} a_{d-1}$.

Solution. Every Galois conjugate of α is $\sigma(\alpha)$ for precisely n/d values of σ . Therefore, $N_{K/F}(\alpha)$ is the (n/d) -th power of the product of the roots of $m_{\alpha, F}$, which is $(-1)^n a_0^{n/d}$. The result for the trace is similar.

- (b) Show that

$$N_{K/F}(\alpha\beta) = N_{K/F}(\alpha)N_{K/F}(\beta) \quad \text{and} \quad \text{Tr}_{K/F}(\alpha+\beta) = \text{Tr}_{K/F}(\alpha) + \text{Tr}_{K/F}(\beta).$$

Solution.

$$N_{K/F}(\alpha\beta) = \prod_{\sigma \in G} \sigma(\alpha\beta) = \prod_{\sigma \in G} \sigma(\alpha)\sigma(\beta) = \prod_{\sigma \in G} \sigma(\alpha) \prod_{\sigma \in G} \sigma(\beta) = N_{K/F}(\alpha)N_{K/F}(\beta).$$

The result for the trace is similar.

- (c) Show that $N_{K/F}(a\alpha) = a^n N_{K/F}(\alpha)$ and $\text{Tr}_{K/F}(a\alpha) = a \text{Tr}_{K/F}(\alpha)$ for all $a \in F$, In particular show that $N_{K/F}(a) = a^n$ and $\text{Tr}_{K/F}(a) = na$ for all $a \in F$.

Solution. $N_{K/F}(\alpha)$ is a product of n Galois conjugates of α , while $\text{Tr}_{K/F}(\alpha)$ is their sum. If $\sigma \in \text{Gal}(K/F)$, then $\sigma(a\alpha) = a\sigma(\alpha)$, so when we multiply α by a , each Galois conjugate is multiplied by a . The formulas in the first sentence of the problem follow.

The second sentence follows from the first, plus the fact that $N_{K/F}(1) = \text{Tr}_{K/F}(1) = 1$, since every automorphism fixes 1.