Math 121, Winter 2023, Homework 2 Solutions

Section 13.2

Problem 1. *Let $\mathbb{F}$ be a finite field of characteristic p. Prove that $|\mathbb{F}| = p^n$ for some positive integer n.*

Solution. Since $\mathbb{F}$ is characteristic p, the prime field of $\mathbb{F}$ is isomorphic to $\mathbb{F}_p$. Therefore, $F/\mathbb{F}_p$ is a field extension, so $\mathbb{F}$ is a vector space over $\mathbb{F}_p$, and so $\mathbb{F} = \{a_1v_1+\ldots+a_nv_n | a_n \in \mathbb{F}_p\}$ has order p^n.

Problem 4. *Determine the degree over $\mathbb{Q}$ of $2+\sqrt{3}$ and of $1+\sqrt[3]{2}+\sqrt[3]{4}$.*

Solution. For the first problem, since $2+\sqrt{3} \in \mathbb{Q}(\sqrt{3})$ and $\sqrt{3} \in \mathbb{Q}(2+\sqrt{3})$, we have $\mathbb{Q}(2+\sqrt{3}) = \mathbb{Q}(\sqrt{3})$. By Proposition 11, $\sqrt{3}$, the extension $\mathbb{Q}(\sqrt{3})/\mathbb{Q}$, and $2+\sqrt{3}$ all have the same degree, and since x^2-3 is the minimal polynomial for $\sqrt{3}$, this degree is 2.

We approach the second problem similarly. Let $\theta = 1+\sqrt[3]{2}+\sqrt[3]{4}$. $\theta \in \mathbb{Q}(\sqrt[3]{2})$ since $\sqrt[3]{4} = (\sqrt[3]{2})^2$. On the other hand, $\theta^2 = 5+4\sqrt[3]{2}+3\sqrt[3]{4}$, so $\sqrt[3]{2} = \theta^2 - 3\theta - 2 \in \mathbb{Q}(\theta)$ Therefore, θ has the same degree as $\sqrt[3]{2}$ i.e. 3.

Problem 5. *Let $F = \mathbb{Q}(i)$. Prove that x^3-2 and x^3-3 are irreducible over F.*

Solution. We'll consider the polynomial $p(x) = x^3-2$, and the other one is similar. By Proposition 11, we can prove the result by showing that $[\mathbb{Q}(i,\sqrt[3]{2}) : \mathbb{Q}(i)] = 3$ (see also Lemma 16). $p(x)$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion, so it's the minimal polynomial for $\sqrt[3]{2}$, and by Proposition 11, $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$. Also, $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ since i has minimal polynomial x^2+1. The Tower Law then says

$$[\mathbb{Q}(i,\sqrt[3]{2}) : \mathbb{Q}] = [\mathbb{Q}(i,\sqrt[3]{2}) : \mathbb{Q}(i)][\mathbb{Q}(i) : \mathbb{Q}] = [\mathbb{Q}(i,\sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})][\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}],$$

so

$$[\mathbb{Q}(i,\sqrt[3]{2}) : \mathbb{Q}(i)] = \frac{3[\mathbb{Q}(i,\sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})]}{2}$$

is a multiple of 3.

Problem 7. *Prove that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$. Conclude that $[\mathbb{Q}(\sqrt{2}+\sqrt{3}) : \mathbb{Q}] = 4$. Find an irreducible polynomial satisfied by $\sqrt{2}+\sqrt{3}$.*

Solution. Since $\theta := \sqrt{2} + \sqrt{3} \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$, we have containment one way. For the other direction, note that $\theta^3 = 11\sqrt{2} + 9\sqrt{3}$, so both $\sqrt{2} = \frac{1}{2}(\theta^3 - 9\theta)$ and $\sqrt{3} = -\frac{1}{2}(\theta^3 - 11\theta)$ are in $\mathbb{Q}(\theta)$.

By Corollary 15, $[\mathbb{Q}(\theta) : \mathbb{Q}(\sqrt{2})] \leq 2$, and since $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$ this degree must equal 2. Therefore, by the tower law,

$$\mathbb{Q}(\theta) : \mathbb{Q}] = [\mathbb{Q}(\theta) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 \cdot 2 = 4.$$

Finally, we compute $\theta^2 = 5 + 2\sqrt{6}$ and $\theta^4 = 49 + 20\sqrt{6}$, and conclude that $\theta^4 - 10\theta^2 + 1 = 0$.

Problem 12. *Suppose the degree of the extension K/F is a prime p. Show that any subfield E of K containing F is either K or F.*

Solution. This is a straightforward consequence of the tower law. First note that a degree one field extension is trivial, since the extension field is a dimension-one vector space over the base field, and thus the same field. Then we have $p = [K : F] = [K : E][E : F]$, and since these are all integers one of $[K : E]$ and $[E : F]$ must be p, and the other must be 1.

Problem 15. *A field F is said to be* formally real *if -1 is not expressible as a sum of squares in F. Let F be a formally real field, let $f(x) \in F[x]$ be an irreducible polynomial of odd degree and let α be a root of $f(x)$. Prove that $F(\alpha)$ is also formally real.*

Solution. Suppose otherwise, and a counterexample α such that the degree of α over F is the minimum possible. Then $-1 = \beta_1^2 + \cdots + \beta_m^2$ for some choice of $\beta_i \in F(\alpha)$, Let the coset $p_i(x) + (f(x))$ be the image of β_i under the (inverse of the) isomorphism given in Theorem 6, and we choose the representatives p_i to have $\deg p_i < \deg f$ (otherwise, divide with remainder). Then we have

$$-1 + (f) = p_1^2 + \cdots + p_m^2,$$

where we have collected copies of the ideal (f). Pulling this back to the polynomial ring $F[x]$, we see that

$$-1 + f(x)g(x) = p_1^2 + \cdots + p_m^2 \tag{1}$$

for some polynomial $g \in F[x]$. Since the degree of the right side of (1) is even and less that $2 \deg f$, so must be the degree of the left side, so $\deg g$ is odd and less than $\deg f$.

Now, g may not be irreducible, but at least one of its irreducible factors must have odd degree. Let β be a root of such a factor $h(x)$; then β has odd degree over F. Under the maps $F[x] \to F[x]/(h) \to F(\beta)$, (1) becomes

$$-1 = \gamma_1^2 + \cdots \gamma_m^2$$

for elements $\gamma_i \in F(\beta)$. This means that $F(\beta)$ is also not formally real, and since $\deg \beta < \deg \alpha$, this contradicts the minimality of α.

Section 13.3

Problem 2. *Prove that Archimedes' construction actually trisects the angle θ. (See the book for the construction).*

Solution. Let ϕ be the third angle of the triangle lying within the circle, ϵ be the angle supplementary to β, and η be the remaining angle of the other triangle. We have $\beta = \gamma$ and $\alpha = \eta$ since these pairs of angles are each part of the same isosceles triangle. Adding up the angles in the two triangles gives $\epsilon + 2\alpha = 180°$ and $\phi + 2\beta = 180°$. Decomposing straight line angles gives $\epsilon + \beta = 180°$ and $\alpha + \phi + \theta = 180°$; in particular, $\beta = 2\alpha$. Solving this last equation for θ and substituting, we get

$$\theta = 180° - \phi - \alpha = 2\beta - \alpha = 3\alpha.$$

Problem 4. *The construction of the regular 7-gon amounts to the constructibility of* $\cos(2\pi/7)$. *We shall see later (Section 14.5 and Exercise 2 of Section 14. 7) that $\alpha = 2\cos(2\pi/7)$ satisfies the equation $p(x) = x^3 + x^2 - 2x - 1 = 0$. Use this to prove that the regular 7-gon is not constructible by straightedge and compass.*

Solution. This problem amounts to showing that the degree of $\cos(2\pi/7)$ over $\mathbb{Q}$ is not a power of 2, for which it suffices to show that $p(x)$ is irreducible. Since $p(x)$ is cubic, by Propositions 9 and 10 of Chapter 9, $p(x)$ is reducible if and only if it has a root. By the rational root theorem, such a root must be ± 1, and plugging in shows neither is a root.