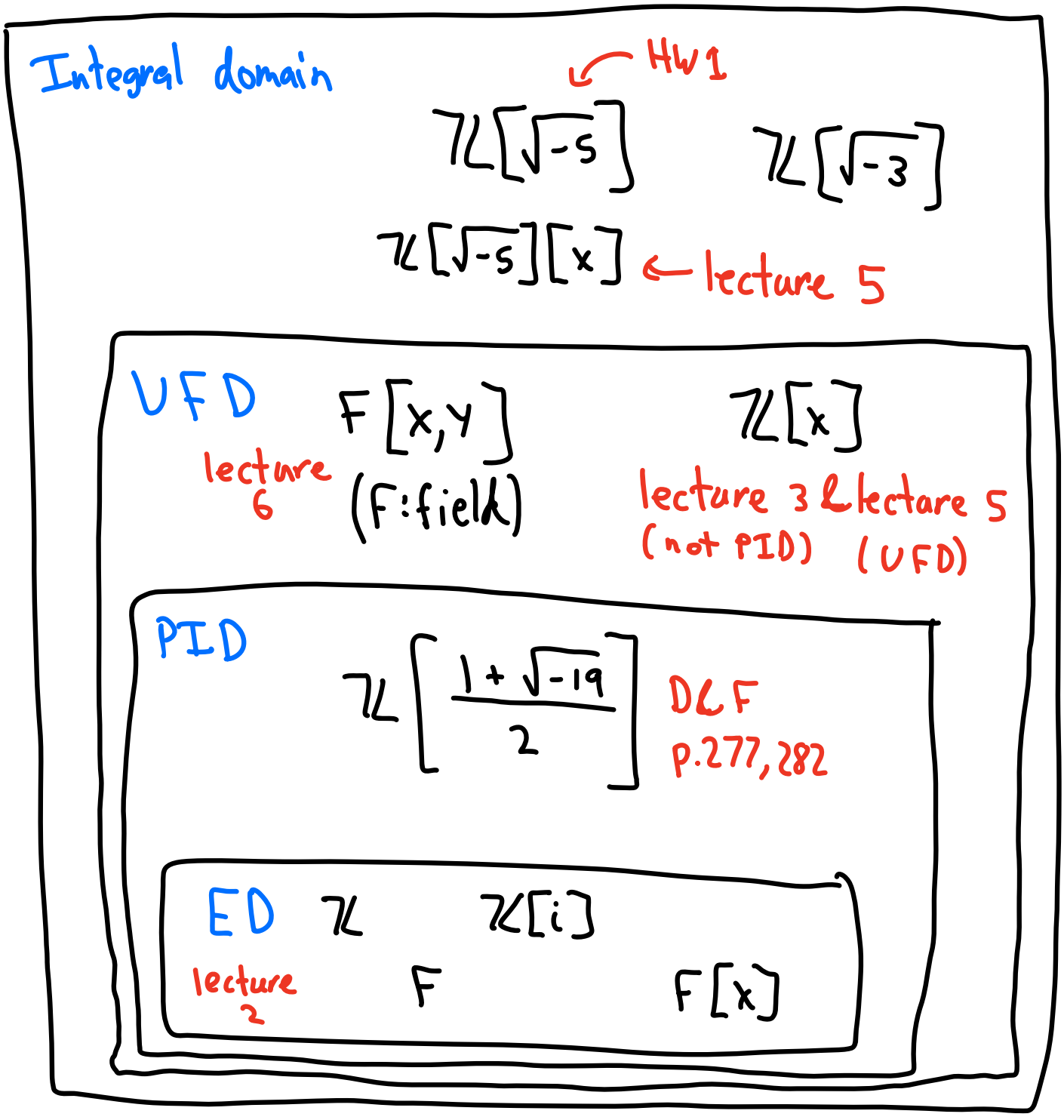
Announcements

HW1 due Wednesday @ 9am via Gradescope

Don't be late! (see syllabus) (entry code: 2BZDK7)

HW2 will be posted soon (due next Wed.)

Fill out midterm conflict survey



Principal Ideal Domains

Recall: A Euclidean domain is an int. domain R w/ a norm $N: R \to \mathbb{Z}_{\geq 0}$ s.t. $N(0) = 0$ and $\forall a, b \in R$, $b \neq 0$, $\exists q, r \in R$ with $a = qb + r$ and $r = 0$ or $N(r) < N(b)$

Def: A principal ideal domain (PID) is an integral domain in which every ideal is principal.

Last time: Euclidean domain $\Rightarrow$ PID

Next time: PID $\Rightarrow$ "unique factorization domain" (UFD)

Def: $R[a] = \{ r_0 + r_1 a + r_2 a^2 + \dots + r_n a^n \mid r_i \in R, n \in \mathbb{Z}_{\geq 0} \}/_{\text{equiv.}}$

Prop: R : PID. Let $a, b \in R$, $(a, b) = (d)$.

Then,

a) $d = sa + tb$ for some $s, t \in R$

b) d is a gcd of a and b

Pf: a) is a consequence of $d \in (d) \subseteq (a, b) = \{sa + tb\}$.

b) Since $a, b \in d$, d is a common divisor of a & b. If $d' | a$, $d' | b$, then $d' | sa + tb = d$, so d is a gcd of a & b. $\square$

Ex: $F[x,y]$ is not a PID since (x,y) is not principal.
We have $1 = \gcd(x,y)$, but can't have $1 = sx + ty$.

Def: Let $r \in R$: integral domain

a) r is a unit if $\exists s \in R$ w/ $rs = sr = 1$

If r not unit, $r \neq 0$

b) r is irreducible if $r = ab \Rightarrow a$ or b is a unit

c) r is prime if $r \mid ab \Rightarrow r \mid a$ or $r \mid b$

Prop: r is prime $\Rightarrow$ r is irreducible

Pf: Let $r = ab$, and assume WLOG that $r \mid a$, $a = rt$.

Then $r = ab = rtb \Rightarrow r(1-tb) = 0 \Rightarrow tb = 1 \Rightarrow b$ is a unit. □

Converse doesn't hold: 3 is irred. in $\mathbb{Z}[\sqrt{-5}]$, but

$3^2 = 9 = (2+\sqrt{-5})(2-\sqrt{-5})$ and $3 \nmid 2 \pm \sqrt{-5}$, so

3 is not prime.

Def: Let I be an ideal in R

a) I is maximal if either/both:
- $\nexists$ ideal J s.t. $I \subsetneq J \subsetneq R$
- R/I is a field

b) I is prime if either/both:
- $ab \in I \Rightarrow a \in I$ or $b \in I$
- R/I is an integral domain

So maximal $\Rightarrow$ prime

Lemma: If $r \neq 0$, (r) prime ideal $\Leftrightarrow$ r prime elt.

Pf: $a \in (r) \Leftrightarrow a$ is a multiple of r. So,

$$\underbrace{[ab \in (r) \Rightarrow a \in (r) \text{ or } b \in (r)]}_{\text{prime ideal}} \Leftrightarrow \underbrace{[r \mid ab \Rightarrow r \mid a \text{ or } r \mid b]}_{\text{prime elt.}}$$

□

Prop: Every nonzero prime ideal in a PID is maximal

Pf: Let $0 \subsetneq (p) \subseteq (m) \subsetneq R$, (p): prime.

By the previous results, (p) prime $\Rightarrow$ p prime $\Rightarrow$ p irred.

Since $(p) \subseteq (m)$, $p = am$ for some a, so either

- a is a unit $\Rightarrow (m) = (p)$
- m is a unit $\Rightarrow (m) = R$

Therefore, (p) is maximal. □

Cor: If $r \in R$: PID, r prime $\Leftrightarrow$ r irred.

Pf: $\Rightarrow$ holds in any int. dom. (see earlier)

$\Leftarrow$) By previous pf,

r irred. $\Rightarrow (r)$ maximal $\Rightarrow (r)$ prime $\Rightarrow r$ prime. □

Ex: $\mathbb{Z}[x]$ is not a PID since $(2, x)$ is not principal

Prop: $R[x]$: PID $\Leftrightarrow$ R: field

Pf: $\Leftarrow$) If R: field, $R[x]$ is Euclidean (last time), hence a PID.

$\Rightarrow$) $R[x]$ integral domain $\Rightarrow$ R integral domain

$\Rightarrow (x)$ prime (since $R[x]/(x) \cong R$)

$\Rightarrow (x)$ maximal (since it is a prime ideal in a PID)

$\Rightarrow R \cong R[x]/(x)$ field □