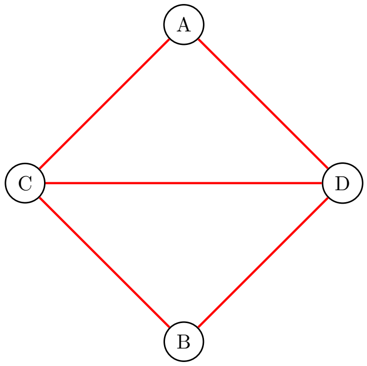
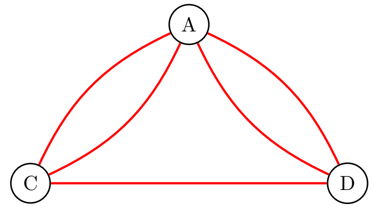
Solutions to Math 412 Midterm Exam 2 — Oct. 18, 2023

1. (25 points) Consider the following graphs.



G



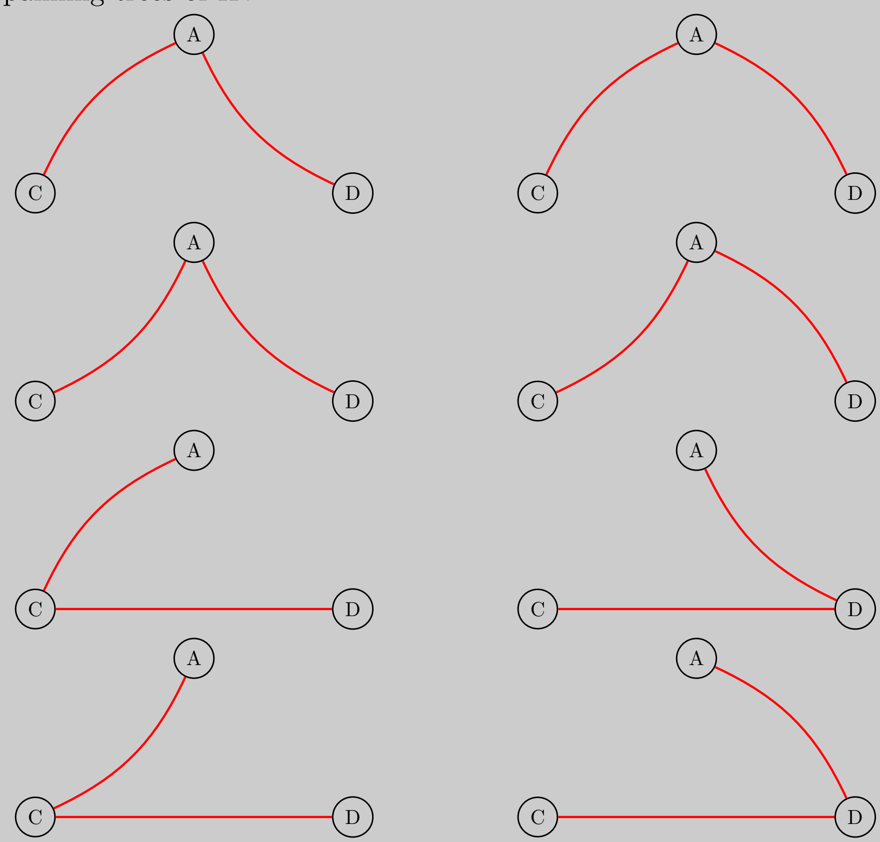
H

(a) (10 points) Determine the number $\tau(G)$ of spanning trees of G via the matrix tree theorem (i.e. by taking the determinant of the reduced Laplacian). *[Note: You will NOT receive credit for computing $\tau(G)$ directly]*

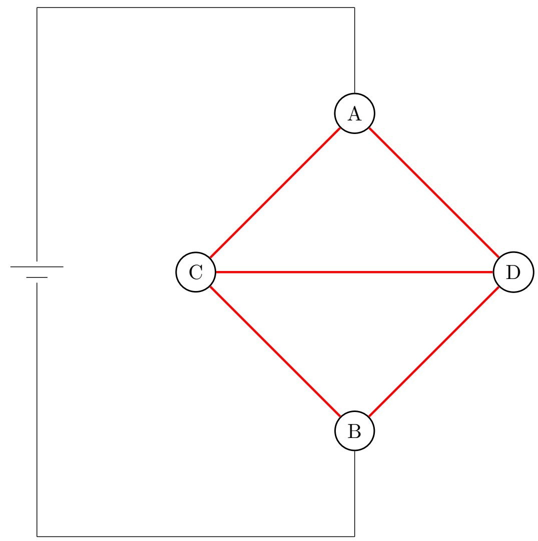
$$L(G) = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}, \quad L^4(G) = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \quad \det L^4(G) = 8.$$

(b) (10 points) Determine the number $\tau(H)$ of spanning trees of H *directly* (i.e. by drawing all of them–make clear which edges you are using!). *[Note: You will NOT receive credit for computing $\tau(H)$ via the matrix tree theorem]*

There are 8 spanning trees of H:



(c) (5 points) Finally, consider the following electrical circuit, involving the graph G.



Every edge from G represents a resistor with resistance 1. Determine the effective resistance from A to B. *[Hint: use the previous parts. It is allowed, but not recommended, to use other methods for this problem–if you do so, it must be mathematically valid and justified with no assumptions beyond what we've covered in this class.]*

Luckily for us, the resistances and the conductances are all 1 here. By Kirchoff's Theorem, the effective resistance is

$$\frac{\tau((G \cup AB) \cdot AB)}{\tau(G)} = \frac{\tau(H)}{\tau(G)} = \frac{8}{8} = 1.$$

2. (15 points) Let G be a 3-regular graph with at most two cut-edges. Prove that G has a perfect matching.

The proof is similar to the proof of the first Petersen's Theorem proved in class (Cor. 3.3.8). Suppose that G has no perfect matching. Let $n = |V(G)|$. Since G is 3-regular, n is even, so if G doesn't have a perfect matching, every matching in G has at least 2 vertices that it does not saturate. Then by Berge-Tutte Formula, there exists an $S \subseteq V(G)$ such that $o(G \setminus S) \geq |S| + 2$. Suppose that $S = \{v_1, \ldots v_s\}$ and that $M_1, \ldots, M_{s_2}$ are the vertex sets of $s + 2$ of the odd components of $G \setminus S$. For $j = 1, \ldots, s + 2$, let a_j denote the number of edges with one endpoint in M_j and the other in S. Since the sum of degrees of vertices in M_j (equal to $3|M_j|$) is odd, a_j is odd for each j. Since G has at most two cut-edges, at most two of the a_j are equal to 1, and every other a_j must be at least 3. Thus,

$$\sum_{j=1}^{s+2} a_j \geq 1 + 1 + 3s.$$

On the other hand, $\sum_{j=1}^{s+2} a_j$ does not exceed the total number A of edges connecting S with $V(G) \setminus S$ in G. Since G is 3-regular, $|A| \leq 3|S| = 3s$, a contradiction to $\sum_{j=1}^{s+2} a_j \leq 2 + 3s$.

3. (15 points) For $n \geq 3$, determine the minimum number of edges in a maximal triangle-free n-vertex simple graph.

If a graph is disconnected, then adding an edge connecting vertices in distinct components will not create a triangle, so any maximal triangle-free graph must be connected, and thus has at least $n - 1$ edges. On the other hand, the "star" tree with a single vertex incident to every other vertex is a maximal triangle-free n-vertex simple graph and has $n - 1$ edges. Thus, the answer is $n - 1$.

4. (15 points) **Prim's Algorithm** grows a spanning tree of a graph G from a fixed vertex v of G by iteratively adding the cheapest edge from a vertex already reached to a vertex not yet reached, finishing when all vertices of G have been reached:

   Start: $S = \{v\}$, T is the graph with $V(T) = V(G),\ E(T) = \emptyset$
   While $S \neq V(G)$:
      Let e be the cheapest edge from a vertex u in S to a vertex w not in S
      Add e to T and w to S

   Prove that Prim's Algorithm produces a minimum-weight spanning tree of G *[Note: there are two things to prove here. You may assume that all edge weights of G are distinct.]*

   Let T be the tree produced by Prim's Algorithm, and let T^* be a minimum-weight tree with the property that no other minimum-weight tree agrees with T for more steps than T^* does. If $T \neq T^*$, let e be the first edge chosen for T that does not appear in T^*, and let U be the set of vertices reached before e is added. Adding e to T^* creates a cycle C; since e links U to $V(G) \setminus U$, C must contain another edge e' from U to $V(G) \setminus U$. Then, $(T^* \cup e) \setminus e'$ is another spanning tree which agrees with T for more steps than T' does, so by assumption it is not minimal, and so $\mathrm{wt}(e') < \mathrm{wt}(e)$. Since e' is incident to U, e' is available for consideration when e is chosen by the algorithm; since the algorithm chose e, we have $\mathrm{wt}(e) \leq \mathrm{wt}(e')$, a contradiction.

5. (20 points) Consider the following preference lists:

X $= \{u, v, x, y, z\}$	Y $= \{a, b, c, d, e\}$
$u:\ c > b > e > d > a$	$a:\ y > x > u > v > z$
$v:\ c > d > e > a > b$	$b:\ u > x > v > z > y$
$x:\ a > b > c > d > e$	$c:\ z > x > y > u > v$
$y:\ a > e > d > b > c$	$d:\ v > x > u > z > y$
$z:\ c > e > b > a > d$	$e:\ v > u > y > x > z.$

   determine by going step-by-step the stable matchings resulting from the Gale-Shapley Algorithm when:

   (a) (10 points) The vertices of X are the puppies and the vertices of Y are the children (i.e. the vertices of X propose to / bound up to the vertices of Y).

   $$M = \{ub, vd, xe, ya, zc\}.$$

   (Steps needed for full credit)

   (b) (10 points) The vertices of Y are the puppies and the vertices of X are the children.

   $$M = \{ay, bu, cz, dv, ex\}$$

   (Steps needed for full credit)