Math 418, Spring 2024 – Homework 8

Due: Wednesday, April 10th, at 9:00am via Gradescope.

MORE PROBLEMS TO COME!!!

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition*. All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 2 bonus points per assignment.

1. **Dummit and Foote #14.2.6:** *Let $K = \mathbb{Q}(\sqrt[8]{2}, i)$ and let $F_1 = \mathbb{Q}(i), F_2 = \mathbb{Q}(\sqrt{2}), F_3 = \mathbb{Q}(\sqrt{-2})$. Prove that $Gal(K/F_1) = \mathbb{Z}/8\mathbb{Z}, Gal(K/F_2) = D_8, Gal(K/F_3) = Q_8$. (Hint: use the example in Section 14.2, and the diagrams on pages 580-1)*

2. **Dummit and Foote #14.2.7:** *Determine all the subfields of the splitting field of $x^8 - 2$ which are Galois over $\mathbb{Q}$. (Hint: use the example in Section 14.2, and the diagrams on pages 580-1)*

3. **Dummit and Foote #14.2.14:** *Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ is a cyclic quartic field, i.e., is a Galois extension of degree 4 with cyclic Galois group.*

4. *Let K/F be a Galois extension of degree n with $G = Gal(K/F)$. For $\alpha \in K$, define the norm and trace of α by*

$$N_{K/F}(\alpha) := \prod_{\sigma \in G} \sigma(\alpha), \qquad \textit{and} \qquad Tr_{K/F}(\alpha) = \sum_{\sigma \in G} \sigma(\alpha).$$

   Let $m_{\alpha,F}(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$.

   (a) Show that $N_{K/F}(\alpha) = (-1)^n a_0^{n/d}$ and $Tr_{K/F}(\alpha) = -\frac{n}{d} a_{d-1}$.

   (b) Show that

$$N_{K/F}(\alpha\beta) = N_{K/F}(\alpha) N_{K/F}(\beta) \qquad \text{and} \qquad Tr_{K/F}(\alpha+\beta) = Tr_{K/F}(\alpha) + Tr_{K/F}(\beta).$$

   (c) Show that $N_{K/F}(a\alpha) = a^n N_{K/F}(\alpha)$ and $Tr_{K/F}(a\alpha) = a Tr_{K/F}(\alpha)$ for all $a \in F$, In particular show that $N_{K/F}(a) = a^n$ and $Tr_{K/F}(a) = na$ for all $a \in F$.

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