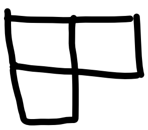
Recall: M^λ : span of tabloids

S^λ : span of polytabloids (Specht module)

Today: irreducibility of S^λ and decomposition of M^λ

Ex:

a) $\lambda =$ 

$$e_{\substack{12\\3}} = \begin{array}{c}\overline{1\ 2}\\ \underline{3\ \ }\end{array} - \begin{array}{c}\overline{2\ 3}\\ \underline{1\ \ }\end{array} = -e_{\substack{32\\1}}$$

$$e_{\substack{13\\2}} = \begin{array}{c}\overline{1\ 3}\\ \underline{2\ \ }\end{array} - \begin{array}{c}\overline{2\ 3}\\ \underline{1\ \ }\end{array} = -e_{\substack{23\\1}} \qquad e_{\substack{21\\3}} = e_{\substack{12\\3}} + e_{\substack{13\\2}}$$

$$e_{\substack{21\\3}} = \begin{array}{c}\overline{1\ 2}\\ \underline{3\ \ }\end{array} - \begin{array}{c}\overline{1\ 3}\\ \underline{2\ \ }\end{array} = -e_{\substack{31\\2}}$$

So $S^\lambda = \mathbb{C}\left[e_{\substack{12\\3}}, e_{\substack{13\\2}}\right]$

b) $S^{(n)} = M^{(n)}$ is the trivial repn.

c) $S^{(1^n)}$ is the sign repn. since if $T = \begin{array}{|c|}\hline a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_n \\ \hline \end{array}$,

then $$e_T = \sum_{w \in S_n} (-1)^w \left(\begin{array}{c} \hline a_{w(1)} \\ \hline a_{w(2)} \\ \hline \vdots \\ \hline a_{w(n)} \\ \hline \end{array} \right) = \pm e_{\begin{smallmatrix} 1 \\ 2 \\ \vdots \\ n \end{smallmatrix}}$$

d) $S^{(n-1,n)}$ is the submodule of $M^{(n-1,1)}$ spanned by $\{e_{ik}, i<k\}$ where

$$e_{ik} := e_{\begin{smallmatrix} i & \cdots \\ k & \end{smallmatrix}} = \begin{array}{c} \hline 1 \cdots (k-1)\ (k+1) \cdots n \\ \hline k \\ \hline \end{array} - \begin{array}{c} \hline 1 \cdots (i-1)\ (i+1) \cdots n \\ \hline i \\ \hline \end{array}$$

This is the <u>reflection repn</u> of S_n.

We have $S^{(n-1,1)} \oplus$ triv. repn $= M^{(n-1,1)}$.

We'll use the S_n-invariant inner product on M^λ:

$$\langle \{T\}, \{T'\} \rangle := \delta_{\{T\},\{T'\}}$$

Thm 30 (Submodule Theorem):

a) Let U be a submodule of M^μ. Then $U \supseteq S^\mu$ or $U \subseteq (S^\mu)^\perp$.

b) S^μ is irreducible

Remark: b) follows immediately from a), since $S^\mu \cap (S^\mu)^\perp = \{0\}$, so if U is irreducible, either $U = S^\mu$ or $U \cap S^\mu = \{0\}$.

However, a) actually holds over any field! When working in char p, $S^\mu \cap (S^\mu)^\perp$ may be nonzero, and $S^\mu / S^\mu \cap (S^\mu)^\perp$ form a complete set of irreps. (See [James])

Lemma 31: Let $u \in M^\mu$, and let T be a tableau w/ shape λ.

a) If $\kappa_T u \neq 0$, then $\lambda \trianglerighteq \mu$

b) If $\lambda = \mu$, then $\kappa_T u$ is a multiple of e_T.

Pf: u is a linear combination of μ-tabloids, so we can reduce to the case where $u = \{S\}$ for some λ-tableau S, and extend by linearity.

a) Suppose $\lambda \not\trianglerighteq \mu$. By the Dominance Lemma (Lem. 21), there exist two entries i, j in the same row of S that appear in the same col. of T. We have

$$k_T = \sum_{w \in C_T} (-1)^w w = \left[\sum_{\substack{w \in C_T \\ w(i) < w(j)}} (-1)^w w \right] \big((1) - (i, j) \big)$$

and since $(i, j)\{S\} = \{S\} = (1)\{S\}$, $k_T\{S\} = 0$.

b) If there exist two entries i, j in the same row of S that appear in the same col. of T, the argument for part a) shows that $k_T\{S\} = 0$. Otherwise, we can permute each col of T and obtain a tableau which is row equiv. to S
(Pf: Look at the first col of T, and proceed by induction)
i.e. $\exists\, \sigma \in C_T$ s.t. $w\{T\} = \{S\}$.
Then,

$$k_T\{S\} = k_T \sigma\{T\} = \sum_{w \in C_T} (-1)^w w\sigma\{T\}$$

$$= \pm \sum_{w \in C_T} (-1)^{w\sigma} w\sigma \{T\}$$

$$= \pm \sum_{w' \in C_T} (-1)^{w'} w' \{T\}$$

$$= \pm \kappa_T \{T\} = \pm e_T. \qquad \square$$

Pf of Submodule Thm:

Let $u \in U$, and let T be a μ-tableau.

By Lemma 31, $\kappa_T u = f e_T$ for some $f \in \mathbb{C}$.

Since U is S_n-invariant, this means $f e_T \in U$.

If for any choice of u and T, $f \neq 0$,

then $e_T \in U$, so since e_T generates S^λ,

$S^\mu \subseteq U$.

Otherwise, $\kappa_T u = 0$ $\forall u, T$. We have

$$\langle u, e_T \rangle = \langle u, \kappa_T \{T\} \rangle$$

$$= \sum_{w \in C_T} (-1)^w \langle u, w\{T\} \rangle$$

$$= \sum_{w^{-1} \in C_T} (-1)^w \langle u, w^{-1}\{T\} \rangle \qquad \text{(inverting } w\text{)}$$

$$= \sum_{w \in C_T} (-1)^w \langle wu, \{T\} \rangle \qquad \text{(by } S_n \text{ invariance)}$$

$$= \langle K_T u, \{T\} \rangle$$

$$= 0,$$

so $u \in (S^\mu)^\perp$ $\forall u \in U$.

$\square$

Thm 32 (Decomposition Theorem):

The S^λ are mutually inequivalent, and therefore form a complete set of S_n-irreps. M^μ decomposes as:

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} m_{\lambda,\mu} S^\lambda$$

where $m_{\mu,\mu} = 1$.

Pf: Let $\phi \in \mathrm{Hom}_{S_n}(S^\lambda, M^\mu)$.

This extends to an S_n-homom $M^\lambda \to M^\mu$ by setting $\phi((S^\lambda)^\perp) = 0$. We have

$$\phi(e_T) = \phi(K_T \{T\}) = K_T \phi(\{T\}),$$

and since $\phi(\{T\})$ is a linear combination of μ-tabloids, by Lemma 31a, this is 0 unless $\lambda \trianglerighteq \mu$.

In particular, since $S^\lambda \subseteq M^\lambda$, if $S^\lambda \cong S^\mu$, then $\mu \trianglerighteq \lambda$ and $\lambda \trianglerighteq \mu$, so $\lambda = \mu$.

If $\lambda = \mu$, by Lemma 31b, $\phi(e_T) = c_T e_T$ for some $c_T \in \mathbb{C}$. However, c_T is independent of T since

$\phi(e_{wT}) = \phi(w e_T) = w\phi(e_T) = w \cdot c_T e_T = c_T e_{wT}$,

so ϕ is mult. by a scalar, and therefore

$\dim \operatorname{Hom}_{S_n}(S^\lambda, M^\mu) = 1$.

By Schur's Lemma, in the decomposition

$$M^\mu = \bigoplus_\lambda m_{\lambda,\mu} S^\lambda,$$

we have $m_{\lambda,\mu} = \dim \operatorname{Hom}_{S_n}(S^\lambda, M^\mu)$, so the above shows that $m_{\mu,\mu} = 1$ and $m_{\lambda,\mu} = 0$ unless $\lambda \trianglerighteq \mu$. $\square$