**Problem §9.5 - 3(c,d,e):** Which of these relations on the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

(c) $\{(f,g)\ :\ f(x)-g(x)=1 \text{ for all } x\in\mathbb{Z}\}$.

(d) $\{(f,g)\ :\ \text{for some } C\in\mathbb{Z},\ \text{for all } x\in\mathbb{Z}, f(x)-g(x)=C\}$.

(e) $\{(f,g)\mid f(0)=g(1) \text{ and } f(1)=g(0)\}$.

*Solution.* (c) This relation has *none* of the properties of an equivalence relation. It is not reflexive, since $f(x)-f(x)=0\neq 1$. To see that it is not symmetric, observe that if $f(x)-g(x)=1$ then $g(x)-f(x)=-(f(x)-g(x))=-1\neq 1$. Finally, to see that it is not transitive observe that if $f(x)-g(x)=1$ and $g(x)-h(x)=1$ then

$$f(x)-h(x)=(f(x)-(g(x))+(g(x)-h(x))=1+1=2\neq 1.$$

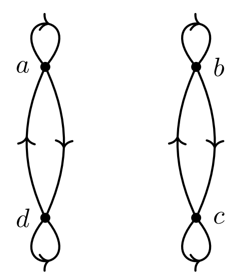
(d) This is an equivalence relation! It is reflexive, since $f(x)-f(x)=0\in\mathbb{Z}$. It is symmetric, since if $f(x)-g(x)=C\in\mathbb{Z}$ then $g(x)-f(x)=-C\in\mathbb{Z}$. Finally, to see that it is transitive observe that if $f(x)-g(x)=C_1\in\mathbb{Z}$ and $g(x)-h(x)=C_2\in\mathbb{Z}$ then

$$f(x)-h(x)=(f(x)-g(x))+(g(x)-h(x))=C_1+C_2\in\mathbb{Z}.$$

(e) This relation is symmetric, but is not reflexive or transitive. We can observe that it is symmetric by inspection of the definition of the relation, since the ordered pairs $(f,g)$ and $(g,f)$ simply swap the order of the conditions. To see that it is not reflexive, observe that if $f(x)=x$ then $f(0)\neq f(1)$ so the relation does not contain $(f,f)$. To see that it is not transitive, suppose $(f,g)$ and $(g,h)$ are ordered pairs in the relation with $f(0)=g(1)=h(0)=7$ and $f(1)=g(0)=h(1)=3$. The pair $(f,h)$ is not in the relation, because $f(0)=7\neq 3=h(1)$ and $f(1)=3\neq 7=h(0)$.

□

**Problem §9.5 - 22:** Determine whether the relation with the directed graph shown is an equivalence relation.




*Solution.* This directed graph does represent an equivalence relation! To see this, we need to check that it is reflexive (i.e., has a loop at each vertex), symmetric (if there is an edge $a\to b$ then there is also an edge $b\to a$), and transitive (if there is a path $a\to b\to c$ then there is also a "shortcut" edge $a\to c$). By inspection, we see that this directed graph does satisfy all three properties and therefore represents an equivalence relation. □

**Problem §9.5 - 24(a,b):** Determine whether the relations represented by these binary matrices are equivalence relations.

(a) $\begin{bmatrix}1&1&1\\0&1&1\\1&1&1\end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

*Solution.* (a) This matrix does not represent an equivalence relation because it's not symmetric (to see this, observe that there is a 1 in position $(1, 2)$ but a 0 in position $(2, 1)$). Hence, the relation it represents is not symmetric.

(b) This matrix does represent an equivalence relation! One can check that it has 1s along the main diagonal and therefore is reflexive and is symmetric. To see that it's transitive, the easiest thing to do is to write down the relation that it represents. If we let the underlying set be $\{1, 2, 3, 4\}$, then this matrix represents the relation

$$R = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\},$$

which one can verify is transitive.

□

**Problem §9.5 - 30(a,b):** What are the equivalence classes of these bit strings for the equivalence relation

$R = \{(x, y) : x$ and $y$ are binary strings of length three or more that agree in the first three bits$\}$

(a) 010

(b) 1011

*Solution.* (a) The equivalence class of 010 under $R$ is the set of all binary strings whose first three digits are 010.

(b) The equivalence class of 1011 under $R$ is the set of all binary strings whose first three digits are 101.

□

**Problem §9.5 - 44(a,b,e):** Which of these collections of subsets are partitions of the set of integers?

(a) the set of even integers and the set of odd integers.

(b) the set of positive integers and the set of negative integers.

(e) the set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6.

*Solution.* Recall that a collection of subsets $A_1, \ldots, A_n$ is a partition of $\mathbb{Z}$ if each $A_i$ is nonempty, $A_i \cap A_j = \emptyset$ if $i \neq j$, and $\cup_{i=1}^n A_i = \mathbb{Z}$.

(a) This is a partition. Both the set of even integers and set of odd integers are non-empty, the intersection of the two sets is empty, and their union is indeed the entire set of integers.

(b) This is not a partition, because the union of the set of positive integers and the set of negative integers is *not* the entire set of integers. It's missing 0!

(e) This is not a partition because the first two sets - the set of integers not divisible by 3 and the set of even integers - are not disjoint. For example, the integer 4 is both even and not divisible by 3.

□

**Problem §9.5 - 48(a):** List the ordered pairs in the equivalence relation produced by the following partition of $\{a, b, c, d, e, f, g\}$:

$$\{a, b\}, \{c, d\}, \{e, f, g\}$$

*Solution.* This partition corresponds to the equivalence relation

$$\left\{ \begin{array}{c} (a, a),\ (a, b),\ (b, a),\ (b, b),\ (c, c),\ (c, d),\ (d, c),\ (c, c), \\ (e, e),\ (e, f),\ (e, g),\ (f, f),\ (f, e),\ (f, g),\ (g, g),\ (g, e),\ (g, f) \end{array} \right\}$$

□

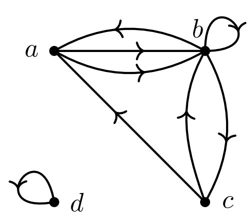
**Problem §10.1 - 28:** Describe a graph model that represents a subway system in a large city. Should edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

*Solution.* We can represent the subway stations as vertices, with an edge from station $u$ to $v$ if there is a subway train that runs from $u$ to $v$ with no intermediate stops (i.e., the edges represent direct trips). These edges should be directed, since it's possible that some subway lines will only run one way. There should be no loops, because no reasonable passenger would want to board a train at a station and ride ... back to that station, without stopping anywhere.

We may or may not want to allow multiple edges, depending on the particular subway system that we're modeling. If there are multiple train lines that share some common stops, we may want to allow multiple edges between stations and color each edge according to the line that it represents.

For example, in Urbana-Champaign there are multiple bus lines that share some common stops. If we were modeling the bus system on Wright Street, we might want to have multiple edges between Green & Wright and Wright & Healey: a brown edge for the 9B Brown, a yellow edge for the 1N Yellow, etc. Depending on their destination and origin, someone using the bus system may be able to either use a single bus line or may need to transfer. Including multiple edges allows passengers to distinguish between these situations. □

**Problem §10.2 - 8:** Determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the following graph:




*Solution.* By inspection, we can see that this graph contains four vertices and eight edges. The in-degrees and out-degrees of each vertex are as follows:

$$\deg^-(a) = 2 \qquad \deg^+(a) = 2$$
$$\deg^-(b) = 3 \qquad \deg^+(b) = 4$$
$$\deg^-(c) = 2 \qquad \deg^+(c) = 1$$
$$\deg^-(d) = 1 \qquad \deg^+(d) = 1$$

□

**Problem §10.2 - 10:** For the graph in Problem §10.2 − 8, determine the sum of the in-degrees of the vertices and the sum of the out-degree of the vertices directly. Show that they are both equal to the number of edges in the graph.

*Solution.* We can compute the sum of the in-degrees and out-degrees of the vertices in that graph as

$$\deg^-(a) + \deg^-(b) + \deg^-(c) + \deg^-(d) = 2 + 3 + 2 + 1 = 8$$
$$\deg^+(a) + \deg^+(b) + \deg^+(c) + \deg^+(d) = 2 + 4 + 1 + 1 = 8$$

which we observe matches the number of edges, as expected. (This is, after all, the Directed Handshake Theorem!) □

**Problem §10.2 - 35(a,b,c):** How many vertices and how many edges do these graphs have?

(a) $K_n$.

(b) $C_n$.

(c) $W_n$.

*Solution.* (a) By definition, $K_n$ has $n$ vertices. Because we know that it has an edge for each pair of distinct vertices, it has $\binom{n}{2} = \frac{n(n-1)}{2}$. edges.

(b) By definition, $C_n$ has $n$ vertices. If those vertices are labeled $v_1, \ldots, v_n$ in a cyclic order, then $C_1$ has $n - 1$ edges of the form $\{v_i, v_{i+1}\}$, for $1 \leq i < n$, and the edge $\{v_1, v_n\}$. Hence, it has a total of $n$ edges.

(c) Recall that $W_n$ is constructed from $C_n$ by adding a central vertex connected to each of the vertices of $C_n$. As such, it has $n + 1$ vertices and $n + n = 2n$ edges.

□

**Problem §10.2 - 37(a,b,c):** Find the degree sequence of each of the following graphs.

(a) $K_4$.

(b) $C_4$.

(c) $W_4$.

*Solution.* (a) By definition, each vertex of $K_4$ is adjacent to each of the other three vertices of $K_4$. As such, every vertex has degree 3 and $K_4$ has degree sequence $(3, 3, 3, 3)$.

(b) Each vertex of $C_4$ is connected to two neighboring vertices. Hence, $C_4$ has degree sequence $(2, 2, 2, 2)$.

(c) The central vertex of $W_4$ is connected to all four vertices of the cycle portion of the wheel and therefore has degree 4. Each of the vertices in the cycle portion of $W_4$ is adjacent to its two neighbors in the cycle and the central vertex. Hence, each of those vertices has degree 3 and the entire graph has degree sequence $\deg(W_4) = (4, 3, 3, 3, 3)$

□