Note: the distribution of these problems may not match the distribution of exam topics.

Problem §9.1 - 42: List the 16 different relations on the set $A = \{0, 1\}$.

Solution. $A \times A = \{(0,0),(0,1),(1,0),(1,1)\}$, so the 16 relations are the 16 possible subsets of this set: $\emptyset, \{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\}, \{(0,0),(0,1)\}, \{(0,0),(1,0)\}, \{(0,0),(1,1)\}, \{(0,1),(1,0)\}, \{(0,1),(1,1)\}, \{(1,0),(1,1)\}, \{(0,0),(0,1),(1,0)\}, \{(0,0),(0,1),(1,1)\}, \{(0,0),(1,0),(1,1)\}, \{(0,1),(1,0),(1,1)\}, \{(0,0),(0,1),(1,0),(1,1)\}$, □

Problem §9.1 - 44(a,c,d,f): Which of the 16 relations on $A = \{0, 1\}$, which you listed in Exercise 42, are reflexive? Symmetric? Antisymmetric? Transitive?

Solution. $\emptyset$: symmetric, antisymmetric, transitive.
$\{(0,0)\}$: symmetric, antisymmetric, transitive.
$\{(0,1)\}$: antisymmetric, transitive.
$\{(1,0)\}$: antisymmetric, transitive.
$\{(1,1)\}$: symmetric, antisymmetric, transitive.
$\{(0,0),(0,1)\}$: antisymmetric, transitive.
$\{(0,0),(1,0)\}$: antisymmetric, transitive.
$\{(0,0),(1,1)\}$: reflexive, symmetric, antisymmetric, transitive.
$\{(0,1),(1,0)\}$: symmetric.
$\{(0,1),(1,1)\}$: antisymmetric, transitive.
$\{(1,0),(1,1)\}$: antisymmetric, transitive.
$\{(0,0),(0,1),(1,0)\}$: symmetric.
$\{(0,0),(0,1),(1,1)\}$: reflexive, antisymmetric, transitive.
$\{(0,0),(1,0),(1,1)\}$: reflexive, antisymmetric, transitive.
$\{(0,1),(1,0),(1,1)\}$: symmetric.
$\{(0,0),(0,1),(1,0),(1,1)\}$: reflexive, symmetric, transitive.

□

Problem §9.3 - 6: How can the matrix representing a relation R on a set A be used to determine whether the relation is asymmetric?

Solution. R is asymmetric if and only if $m_{ij} + m_{ji} \leq 1$ for all i and j i.e. M_R has no zeroes on the diagonal, and for every off-diagonal entry m_{ij}, m_{ij} and m_{ji} *aren't* both 1. □

Problem §9.3 - 13: Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the matrices representing $R^{-1}, \overline{R}, R \circ R$.

Solution.

$$M_{R^{-1}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = M_R.$$

$$M_{\overline{R}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$M_{R \circ R} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

□

Problem §9.5 - 36(a,b): What is the congruence class $[4]_m$ when m is

(a) 2?

(b) 3?

Solution. In each case, the congruence class $[4]_m$ is the set of integers congruent to 4, modulo m.

(a) $[4]_2 = \{4 + 2n : n \in \mathbb{Z}\} = \{\cdots, -4, -2, 0, 2, 4, \cdots\}$

(b) $[4]_3 = \{4 + 3n : n \in \mathbb{Z}\} = \{\cdots, -2, 1, 4, 7, 10, \cdots\}$

$\square$

Problem §10.2 - 5: Can a simple graph exist with 15 vertices each of degree five?

Solution. No. Since the sum of all degrees is twice the number of edges, it must be even. Here, that sum would be 75. $\square$

Problem §10.3 - 35,37-39,41-44: Determine whether the given pair of graphs is isomorphic (see Rosen for the graphs). Exhibit an isomorphism or provide a rigorous argument that none exists.

Solution. These are mostly *sketches* that would NOT receive full credit. See Homework 10 solutions for full arguments for similar problems.

35. Isomorphic. $f(u_1) = v_1, f(u_2) = v_3, f(u_3) = v_5, f(u_4) = v_2, f(u_5) = v_4$

37. Isomorphic. Similar to previous problem.

38. Isomorphic. $f(u_1) = v_1, f(u_2) = v_3, f(u_3) = v_2, f(u_4) = v_5, f(u_5) = v_4$

39. Isomorphic. Just push v_5 up and v_6 down; the isomorphism comes from matching up the vertices.

41. Not isomorphic. The first graph has a degree-1 vertex which is adjacent to a degree-2 vertex, while the second graph does not.

42. Not isomorphic. In the first graph, the two degree-4 vertices are adjacent to each other, while in the second graph they are not.

43. Isomorphic. We draw the adjacency matrices:

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}
\end{array}$$

$$\begin{array}{cc} & \begin{array}{cccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Next, we rearrange the rows and columns of the first matrix to try to get the second. Remember that you must rearrange rows and columns in exactly the same way. First, note that $u_1, u_2, u_3, u_4, u_7, u_{10}, u_1$ is a circuit of length 6, so let's put those first to try to match the "hexagon" in the second graph.

$$\begin{array}{cc} & \begin{array}{cccccccccc} u_1 & u_2 & u_3 & u_4 & u_7 & u_{10} & u_5 & u_6 & u_8 & u_9 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_7 \\ u_{10} \\ u_5 \\ u_6 \\ u_8 \\ u_9 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

We're pretty close. Let's move around the last four rows/columns to make the entries match. (For instance, we have a 1 in row 1 column 7, and we want it in row 1, column 9.

$$\begin{array}{cc} & \begin{array}{cccccccccc} u_1 & u_2 & u_3 & u_4 & u_7 & u_{10} & u_8 & u_9 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_7 \\ u_{10} \\ u_8 \\ u_9 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

This matches the adjacency matrix for the v's, so the graphs are isomorphic.

44. Not isomorphic. One way to see this is to take the complement of both graphs. The complement of the left graph is two copies of C_4, while the complement of the right graph is C_8. Since the complements are not isomorphic, neither are the original graphs.

    We can also look at it directly. Notice that every vertex (in either graph) is adjacent to all but two of the other vertices. Call these vertices the *non-neighbors*. In the left graph, choose a vertex (e.g. u_1), and take its non-neighbors (in the example, u_3 and u_7). The *other* non-neighbors of both of these vertices are the same vertex (in the example, u_5). In the right graph, choose a vertex (e.g. v_1), and take its non-neighbors (in the example, v_4 and v_6). The *other* non-neighbors of these vertices are different vertices (in the example, v_7 versus v_3). Thus, the graphs are not isomorphic.

□

Problem §10.4 - 11a: Determine whether this graphs is strongly connected (see Rosen for the graphs) and if not, whether it is weakly connected.

Solution. Weakly connected, but not strongly connected since e.g. there is no path from a to c. □

Problem §10.4 - 37: Show that a simple graph with at least two vertices has at least two vertices that are not cut vertices.

Solution. Let the *distance* between two vertices be the number of edges in the shortest path between them (the is like the shortest path problems of Section 10.6, with weight-1 edges). Choose two vertices u and v that are the maximal distance apart (other pairs of vertices may be this far apart, but none farther). We claim that neither u nor v is a cut vertex.

Suppose that u is a cut vertex. Then after we remove u, the resulting graph is no longer connected. Thus there are two vertices w and x such that there is no longer a path from w to x, so in the original graph every path from w to x passes through u. Note that there must be a path (in the original graph) from w to v that doesn't pass through u, since if there weren't, then the distance from w to v would be longer than the distance from u to v. Similarly, there must be a path from v to x that doesn't pass through u. Combining these paths, there is a path from w to x not passing through u, and this path will still exist once we remove u, so u cannot be a cut vertex. The same argument shows that v also cannot be a cut vertex. □

Problem §10.5 - 10: Can someone cross all the bridges shown in this map (see Rosen) exactly once and return to the starting point?

Solution. Yes, in the corresponding graph, every vertex has even degree, so there is an Eulerian circuit. □

Problem §10.5 - 36: Determine whether the given graph (see Rosen) has a Hamilton circuit.

Solution. $a, d, g, h, e, i, f, c, b, a$ is such a circuit. □