Math 418, Spring 2024 – Homework 3

Due: Wednesday, February 7th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition*. All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 2 bonus points per assignment.

1. **Dummit and Foote #9.3.2:** *Prove that if $f(x)$ and $g(x)$ are polynomials with rational coefficients whose product $f(x)g(x)$ has integer coefficients, then the product of any coefficient of $g(x)$ with any coefficient of $f(x)$ is an integer.*

2. **Dummit and Foote #9.4.2d:** *Let p be an odd prime. Prove that the polynomial $f(x) = \frac{(x+2)^p - 2^p}{x}$ is irreducible in $\mathbb{Z}[x]$.*

3. **Dummit and Foote #9.4.10:** *Prove that the polynomial $p(x) = x^4 - 4x^2 + 8x + 2$ is irreducible over the quadratic field $F = \mathbb{Q}(\sqrt{-2}) = \{a + b\sqrt{-2} | a, b \in \mathbb{Q}\}$.*

4. **Dummit and Foote #9.4.12:** *Prove that $f(x) = x^{n-1} + x^{n-2} + \cdots + x + 1$ is irreducible over $\mathbb{Z}$ if and only if n is a prime.*

5. **Dummit and Foote #13.1.1:** *Show that $p(x) = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$. Let θ be a root of $p(x)$. Find the inverse of $1 + \theta$ in $\mathbb{Q}(\theta)$.*

6. **Dummit and Foote #13.1.3:** *Show that $p(x) = x^3 + x + 1$ is irreducible over $\mathbb{F}_2$ and let θ be a root. Compute the powers of θ in $\mathbb{F}_2(\theta)$.*

7. **Dummit and Foote #13.1.4:** *Prove directly that the map $a + b\sqrt{2} \mapsto a - \sqrt{2}$ is an isomorphism of $\mathbb{Q}(\sqrt{2})$ with itself.*