Math 121, Winter 2023, Galois Correspondence Project

This project entails computing the Galois correspondence and associated data for different polynomials. It will be due via Gradescope on Friday, March 3rd, 2023.

1. Let $f(x) = x^{13} - 1$. Let K be the splitting field of f over $\mathbb{Q}$, and let $G = \mathrm{Gal}(K/\mathbb{Q})$.
   (a) Compute K and its degree over $\mathbb{Q}$.
   (b) Compute G. Namely, give specific automorphisms that generate G, compute the relations between them, and prove that they generate G.
   (c) Compute the subgroup lattice of G.
   (d) Compute the fixed field of each subgroup, and thereby compute the lattice of intermediate fields between $\mathbb{Q} \subseteq E \subseteq K$.
2. Now, let $f(x) = x^8 - 3$ and again let K be the splitting field of f over $\mathbb{Q}$ and $G = \mathrm{Gal}(K/\mathbb{Q})$.
   (a) Prove that $K = \mathbb{Q}(\zeta_8, \sqrt[8]{3}) = \mathbb{Q}(i, \sqrt{2}, \sqrt[8]{3})$, and determine the degree of K over $\mathbb{Q}$. [You may take for granted that $\sqrt{2} \notin \mathbb{Q}(\sqrt[8]{3})$.]
   (b) Compute G. Namely, give specific automorphisms that generate G, compute the relations between them, and prove that they generate G.
   (c) Consider the group
   $$GL_2(\mathbb{Z}/8\mathbb{Z}) = \left\{ \sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{Z}/8\mathbb{Z}, \sigma \text{ invertible} \right\},$$
   and let G' be the set
   $$G' = \left\{ \sigma = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \middle| \, a \in (\mathbb{Z}/8\mathbb{Z})^*, b \in \mathbb{Z}/8\mathbb{Z} \right\}.$$
   Prove that $G' \subseteq GL_2(\mathbb{Z}/8\mathbb{Z})$ and that G' is a subgroup.
   (d) Prove that $G \cong G'$.
   (e) Compute that subgroups of G' fixing $\mathbb{Q}(\zeta_8)$, $\mathbb{Q}(\sqrt[8]{3})$, $\mathbb{Q}(i)$, and $\mathbb{Q}(\sqrt{2})$.
   (f) Compute at least 5 more subgroups of G' (trivial ones ok!) and compute the fixed field of each of these subgroups.
   (g) Using your computations, draw a partial lattice of subgroups of G and the corresponding partial lattice of intermediate fields between $\mathbb{Q}$ and K.