Announcements

Course evaluations at go.illinois.edu/ices-online

Final exam: Tues. 5/13 8:00am–11:00am,
1047 Sidney Lu Mech. E. Bldg. (lecture room, not the midterm room)
(email ASAP w/ any issues)

Wednesday's class: review

Policy email coming soon w/ office hours & review session

What is better for the review session: Sunday or Monday?

Introduction to Schemes

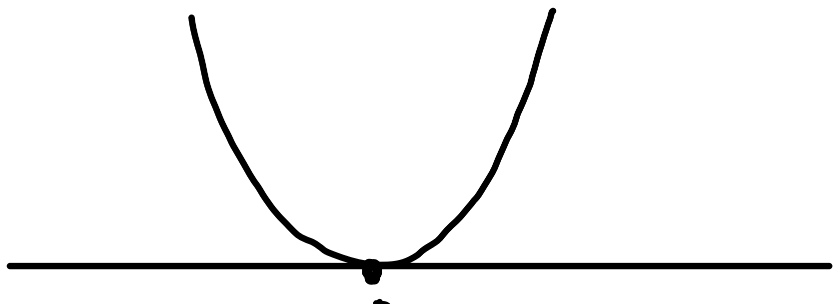
Motivating examples:

a) On $\mathbb{C}^1$, the varieties $V(x)$ and $V(x^2)$ are equal

$\left(\begin{array}{l}\text{prove over } \mathbb{C} \\ \text{draw over } \mathbb{R}\end{array}\right)$

but the ideals (x) and (x^2) are different. Is there any way we can tell them apart?

b) Consider the intersection $V(y-x^2) \cap V(y) \subseteq \mathbb{C}^2$



This is just a single point, the origin. But in some sense, this point should have "multiplicity 2".

Let's think about varieties for a bit longer.

1) We have already seen:

$$\left\{ \begin{matrix} \text{points} \\ \text{in } \mathbb{C}^n \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{maximal ideals} \\ \text{in } \mathbb{C}[x_1, \ldots, x_n] \end{matrix} \right\}$$

$$a \longleftrightarrow I(a) = (x_1 - a_1, \ldots, x_n - a_n)$$

If $f \in \mathbb{C}[x_1, \ldots, x_n]$, we can evaluate $f(a)$ by reducing it modulo $I(a)$:

$$\mathbb{C}[x_1, \ldots, x_n] \longrightarrow \mathbb{C}[x_1, \ldots, x_n] / I(a) \cong \mathbb{C}$$

$$f \longmapsto f \bmod I(a) = f(a)$$

e.g. $f = xy$, $a = (1,2)$ $I(a) = (x-1, y-2)$

$$f = (x-1)(y-2) + 2(x-1) + (y-2) + 2 \mapsto 2 = f(1,2)$$

2) The set of functions on a variety $V \subseteq \mathbb{C}^n$ is

$$\mathbb{C}[x_1, \dots, x_n] / I(V)$$

(coord. ring, see lecture 38)

All the poly. functions on $\mathbb{C}^n$, but two functions which differ by an elt. of $I(V)$ are equal on V

3) If $f \in \mathbb{C}[x_1, \dots, x_n]$, let

$$D(f) = \{ a \in \mathbb{C}^n \mid f(a) \neq 0 \} = \mathbb{C}^n \setminus V(f)$$

"doesn't vanish set"

Since we know $f(a) \neq 0$ for $a \in D(f)$, we can now <u>divide</u> by f ("localization")

The functions on $D(f)$ are therefore all rat'l funs. of the form:

$$\frac{g(x_1, \dots, x_n)}{h(x_1, \dots, x_n)}, \quad g, h \in \mathbb{C}[x_1, \dots, x_n], \quad h \text{ is a power of } f$$

Now we're ready to talk about schemes. By necessity, we'll have to be

a) somewhat imprecise, and
b) not fully general.

Def: Let A be any (commutative, unital) ring. The <u>scheme</u> Spec A consists of

- The set of prime ideals of A (also called Spec A)
  "points of Spec A"

- A description of "functions" on Spec A, as follows

If $f \in A$, $p \in \operatorname{Spec} A$, let

$$f(p) := f \bmod p$$

Note that $f(p) = 0 \Leftrightarrow f \in p$

Let

$$D(f) = \{ p \in \operatorname{Spec} A \mid f \notin p \}$$

$$V(f) = \{ p \in \operatorname{Spec} A \mid f \in p \} = \operatorname{Spec} A \setminus D(f)$$

The <u>structure sheaf</u> for Spec A is a map

$$\mathcal{O}_{\operatorname{Spec} A} : \left\{ \begin{matrix} \text{certain subsets} \\ \text{of Spec } A \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{functions on} \\ \text{the given subset} \end{matrix} \right\}$$

where

$$\mathcal{O}_{\mathrm{Spec}\,A}(A) = A$$

$$\mathcal{O}_{\mathrm{Spec}\,A}(D(f)) = \left\{ \frac{g}{h} \,\middle|\, g,h \in A,\ h \text{ is a power of } f \right\}$$

$\mathcal{O}_{\mathrm{Spec}\,A}$ of other "open sets" is det'd by the above

Example: Let $A = \mathbb{C}[x,y]$. Then Spec A consists of

- $I(a)$ for $a \in \mathbb{C}^2$
- (f) for irred. $f \in A$
- (0)

We have $f(I(a)) = f(a)$

$$f((g)) = \begin{cases} 1, & \text{if } f \text{ is a mult. of } g \\ 0, & \text{otherwise} \end{cases}$$

$$f((0)) = 0 \qquad (\text{so } (0) \in V(f)\ \forall f \in A)$$

Exercise (for home): use this information to determine

$$\mathcal{O}_{\mathrm{Spec}\,A}(D(f)) \qquad \text{for all } f \in A$$

Recall: Let I be an ideal in A. $\exists$ bijection

$$\left\{ \begin{matrix} \text{(prime) ideals} \\ \text{in } A \\ \text{containing } I \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{(prime) ideals} \\ \text{in } A/I \end{matrix} \right\}$$

$$J \longmapsto J/I$$

Therefore,

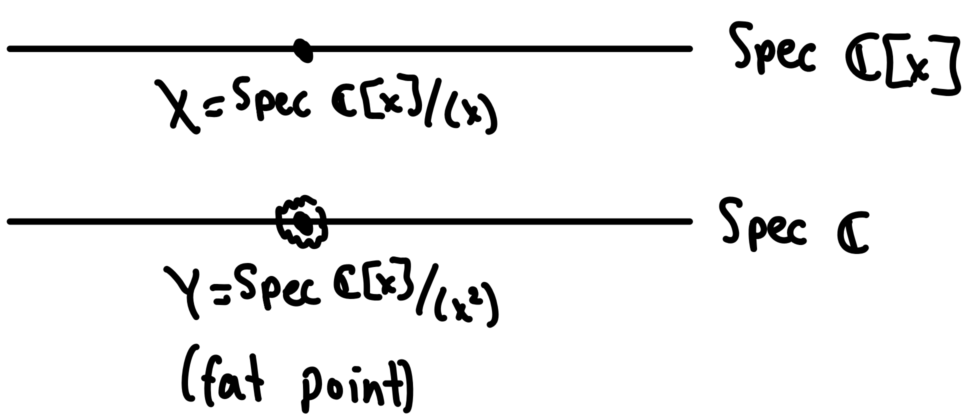
$$\operatorname{Spec} A/I \subseteq \operatorname{Spec} A$$

(as sets, and also as a "closed subscheme")

$$\mathcal{O}_{\operatorname{Spec} A/I}(\operatorname{Spec} A/I) = A/I \quad \text{(similar to varieties)}$$

Ex:

a) Inside $\operatorname{Spec} \mathbb{C}[x]$



Both equal the origin as sets (along with (0))

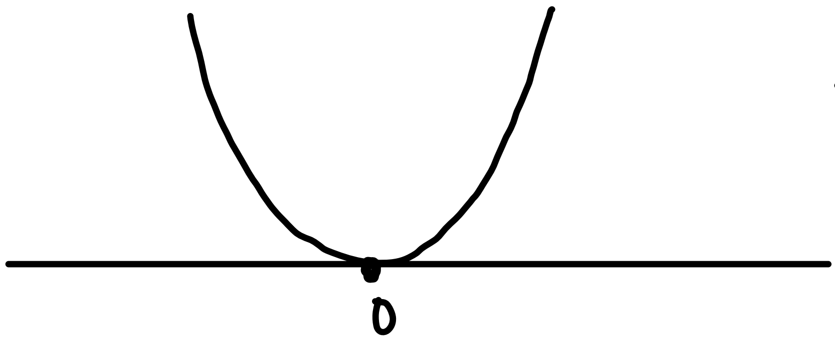
But the set of functions on X is $\mathbb{C}[x]/(x) \cong \mathbb{C}$

and the set of functions on Y is $\mathbb{C}[x]/(x^2) = \{a+bx \mid a,b \in \mathbb{C}\}$

think "tangent vectors at the origin"

b) Consider the intersection $V(y-x^2) \cap V(y) \subseteq \operatorname{Spec} \mathbb{C}[x,y]$

subschemes, come with functions



The "scheme-theoretic" intersection is defined to be

$$V(I) \cap V(J) := V(I+J)$$

$$V((y-x^2)) \cap V((y)) = V((y-x^2)+(y)) = V(x^2, y) =: X$$

Again, this is just the origin (and (0))

But

$$\mathcal{O}_X = \{a+bx \mid a,b \in \mathbb{C}\}$$

So we see linear information in x, but not in y