Announcements:

- H/W 1 posted (due 9am Wed. 8/30 via Gradescope)
- Midterm etc. times posted to course website

Last time: Def'n of graph, chromatic #, path/cycle, etc.

Today: Isomorphism classes, special graphs

Adjacency Matrix

Let G be a loopless graph

Write $V(G) = \{v_1, \ldots, v_n\}$

$E(G) = \{e_1, \ldots, e_m\}$

Def 1.1.17

a) $v \in V(G)$ and $e \in E(G)$ are <u>incident</u> if v is an endpoint of e

b) The <u>degree</u> of $v \in V(G)$ is the number of edges incident to v

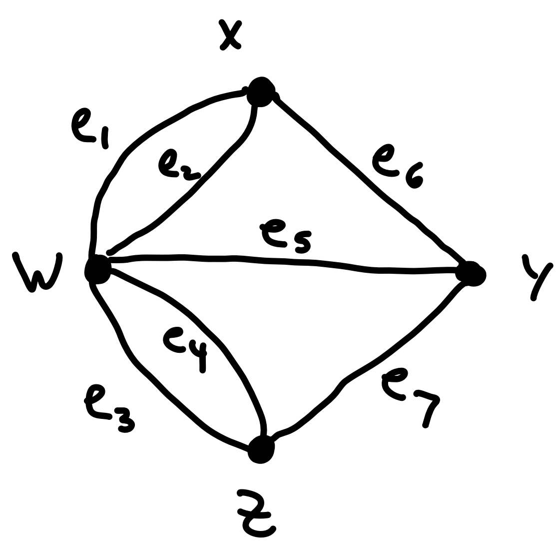
c) The <u>adjacency matrix</u> $A(G)$ is the $n \times n$ matrix where

$$a_{ij} = \text{number of edges w/ endpoints } v_i \text{ and } v_j$$

d) The incidence matrix $M(G)$ is the $n \times m$ matrix where

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is an endpoint of } e_j \\ 0 & \text{otherwise} \end{cases}$$

G:



$$A(G) = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

symmetric

$$M(G) = \begin{array}{c|ccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \hline w & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ x & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ y & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ z & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}$$

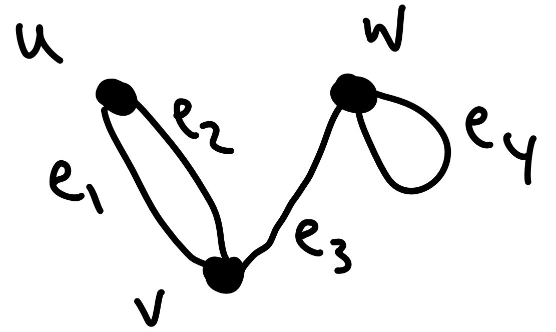
Def 1.1.20: An isomorphism from a graph G to a graph H consists of bijections

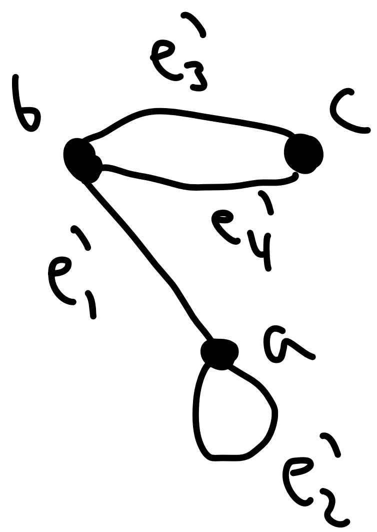
$$f : V(G) \to V(H)$$

$$g : E(G) \to E(H)$$

such that if $e \in E(G)$ has endpoints u and v, $g(e) \in E(H)$ has endpoints $f(u)$ and $f(v)$. We write $G \cong H$.

Example:





$f(u) = c$ $\qquad$ $g(e_1) = e_3'$

$g(e_2) = e_4'$

$f(v) = b$

$g(e_3) = e_1'$

$f(w) = a$

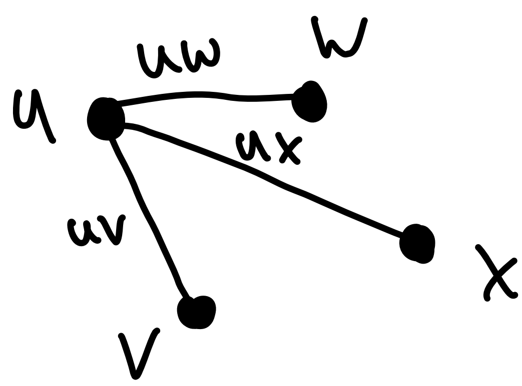
$g(e_4) = e_2'$

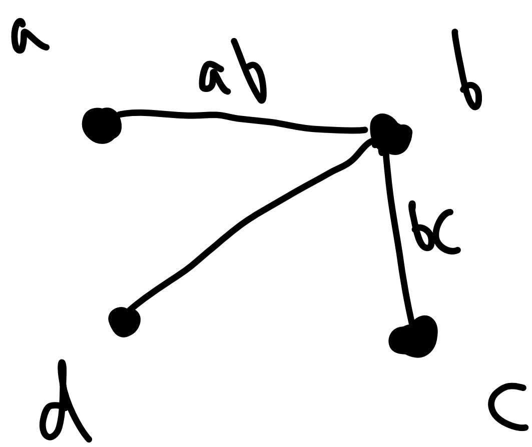
endpoints of e_3: v, w

endpoints of e_1': $b = f(v)$, $a = f(w)$ $\checkmark$

When we have a simple graph, the map g is implied

Ex:





$f(u) = b$

$f(v) = a$

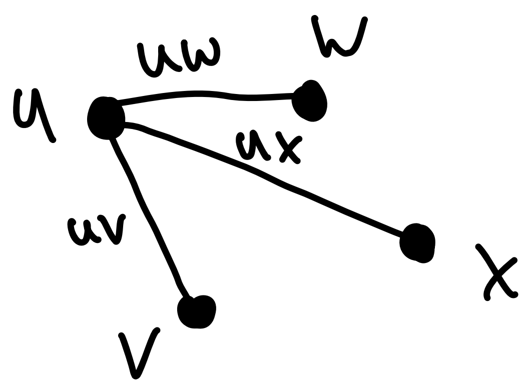
$f(w) = c$

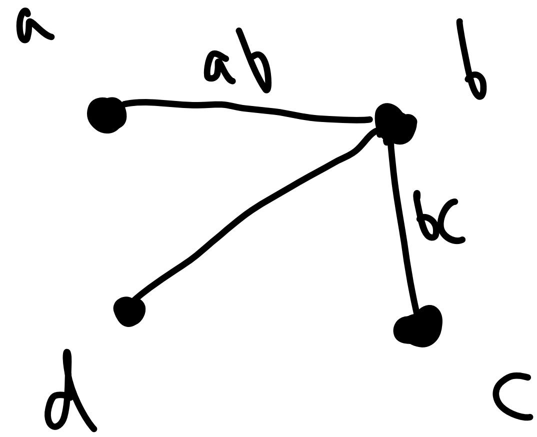
$f(x) = d$

so $g(uv) = f(u)f(v) = ba$

etc.

Ex:





$f(u) = a$

$f(v) = b$

$f(w) = c$

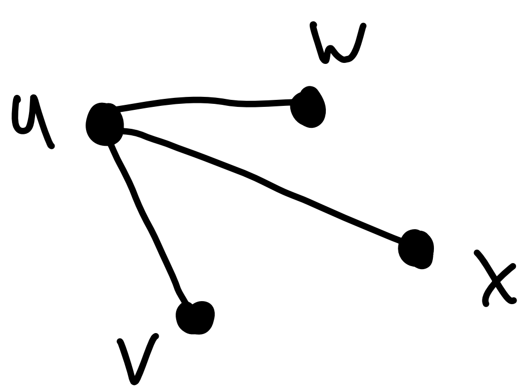
$f(x) = d$

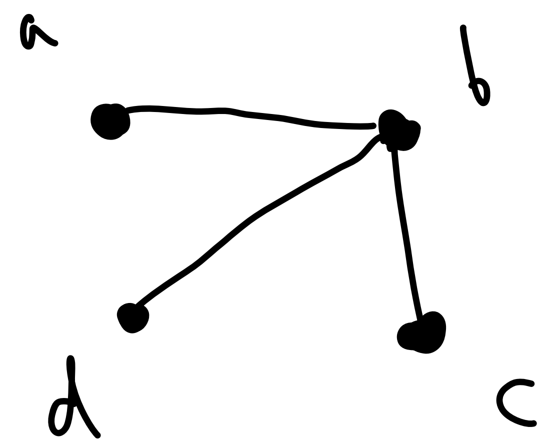
so $g(uw) = f(u)f(w) = ac$

not an isom.

Remark: (loopless) $G \cong H$ if and only if there exists a permutation σ such that applying σ to both the rows and columns of $A(G)$ gives $A(H)$

Ex (cont.)





$$\begin{array}{c} \\ u \\ v \\ w \\ x \end{array} \begin{array}{c} \begin{array}{cccc} u & v & w & x \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Swap u & v rows and columns:

$$\begin{array}{c|cccc} & v & u & w & x \\ \hline v & 0 & 1 & 0 & 0 \\ u & 1 & 0 & 1 & 1 \\ w & 0 & 1 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \end{array}$$

Same!

Pf sketch: If we have a permutation σ on $V(G) = \{v_1, \ldots, v_n\}$ s.t. applying σ to the rows and colums of $A(G)$ gives $A(H)$, then let

$f(v_i) = v'_{\sigma(i)}$ where $V(H) = \{v'_1, \ldots, v'_n\}$

$$\left[\begin{array}{ll} u \mapsto b & \\ v \mapsto a & \text{in example} \\ w \mapsto c & \\ x \mapsto d & \end{array}\right]$$

Then check that if $v_i v_j \in E(G)$,

$$g(v_i v_j) = f(v_i) f(v_j) \in E(H)$$

This holds since $A(G) = A(H)$

Prop 1.1.24: Isomorphism is an equivalence rel'n on (simple) graphs.

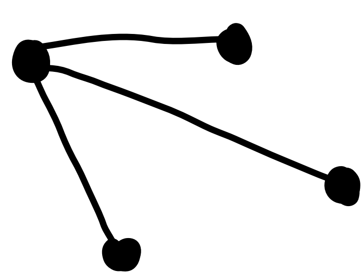
Reflexivity: $G \cong G$ (identity isom.)

Symmetry: If $G \cong H$, then $H \cong G$ (inverse of bijection f^{-1})

Transitivity: If $G \cong H$, $H \cong K$, then $G \cong K$ (compose bijections)

Pf (in simple case): see textbook

Def: An <u>unlabelled graph</u> is an isomorphism class of graphs



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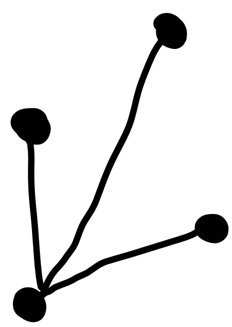
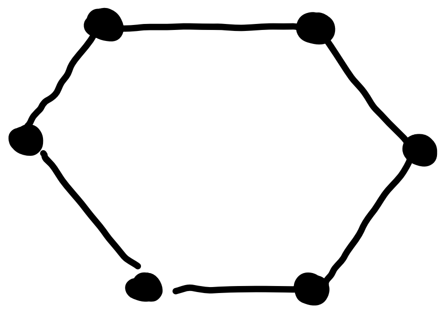


Special (unlabelled, simple) graphs:

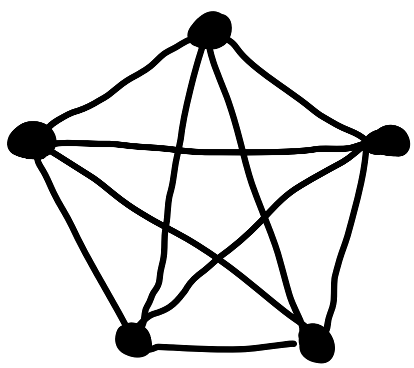
P_n: path on n vertices

P_4: 

C_n: cycle on n vertices

C_6: 

K_n: complete graph on n vertices
(every vertex is adjacent to every other vertex)

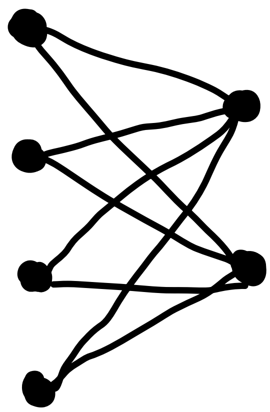
K_5: 

$$A(K_5) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$K_{r,s}$: complete bipartite graph with parts of size r and s $(= K_{s,r})$

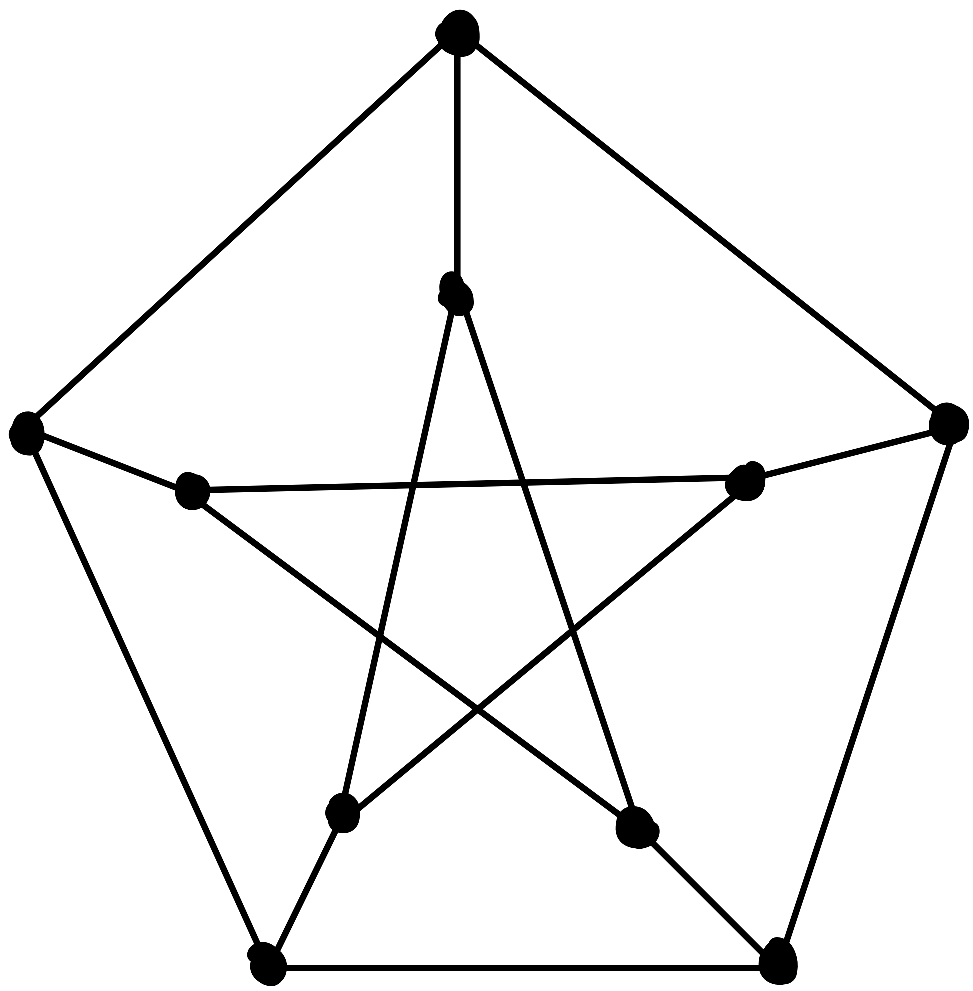
(all vertices in opposite parts are adjacent)

$K_{4,2}$:



Note: $K_{r,s}$ is <u>not</u> a complete graph

Petersen graph: $S = \{a, b, c, d, e\}$



Idea for thought:

How can we describe this graph using subsets of a 5-element set?

(Book has the answer)

Next week: Königsberg bridge problem