Math 412, Fall 2023 – Homework 5

Due: Wednesday, October 4th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should complete **all three** of the following problems. Graduate students taking the course for four credits should contact the instructor. The first problem will count double in the grading. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. In this problem, we will prove the weighted Matrix Tree Theorem by using the unweighted analogue. Recall the definitions from class of the tree sum $\tau(G)$ and the Laplacian $L^i(G)$ for weighted graphs.

   (a) Let $f(x_1, \ldots, x_k) : \mathbb{R}^k \to \mathbb{R}$ and $g(x_1, \ldots, x_k) : \mathbb{R}^k \to \mathbb{R}$ be polynomials in k-variables $x_1, \ldots, x_k$. If for all k-tuples $(a_1, \ldots, a_k)$ of positive integers, we have $f(a_1, \ldots, a_k) = g(a_1, \ldots, a_k)$, prove that f and g are equal for *any* inputs. *[Hint: think of f and g as polynomials only in x_k, and consider $f - g$. Use induction on k.]*

   Proof. If $k = 1$, we have single variable functions f and g. Therefore, $f - g$ is a polynomial which has a root at every positive integer. Since the number of roots of a nonzero polynomial is less than or equal to its degree. Since $f - g$ has infinitely many roots, we must have $f - g = 0$, so $f = g$.
   If $k > 1$, let a be a (fixed) positive integer, and consider the $(k-1)$-variable functions $f_a(x_1, \ldots, x_{k-1}) := f(x_1, \ldots, x_{k-1}), a$ and $g_a(x_1, \ldots, x_{k-1}) := g(x_1, \ldots, x_{k-1}, a)$. Since a is a positive integer, by assumption, $f(a_1, \ldots, a_{k-1}) = g(a_1, \ldots, a_{k-1}$ whenever $a_1, \ldots, a_{k-1}$ are positive integers, so by the inductive hypothesis, f_a and g_a are the same function. This is true for any positive integer a, so we now know that $f(a_1, \ldots, a_k) = g(a_1, \ldots, a_k)$ for any $a_1, \ldots, a_{k-1} \in \mathbb{R}$ and for any positive integer a.
   Next, fix $a_1, \ldots, a_{k-1} \in \mathbb{R}$, and let $f(x) := f(a_1, \ldots, a_{k-1}, x)$, $g(x) := f(a_1, \ldots, a_{k-1}, x)$. These are single variable functions that are equal when x is a positive integer, so by the base case, $f(x) = g(x)$ as functions. Therefore, $f(a_1, \ldots, a_k) = g(a_1, \ldots, a_k)$ for all $a_1, \ldots, a_k \in \mathbb{R}$, so they are equal as functions. $\square$

   (b) Let G be a weighted loopless graph with positive integer weights. Using the (unweighted) Matrix Tree Theorem, prove that for any i, $\tau(G) = \det L^i(G)$. *[Hint: consider the unweighted graph G' formed from G by taking each edge e of G with weight w and, and letting G' have w edges with the same endpoints as e.]*

Proof. As in the hint, let G' be the following (unweighted) graph: $V(G') = V(G)$, and for each edge e of weight w and endpoints u and v, G' has w edges with endpoints u and v.

By construction, the degree of a vertex v in G' is the sum of the weights of all edges $e \in E(G)$ with v as an endpoint, and the number of edges with endpoints u and v in G' is the sum of the weights of all edges $e \in E(G)$ with endpoints u and v. Thus, G and G' have the same Laplacian matrix.

On the other hand, consider a spanning tree T of G. The weight of T is the product of the weights of its edges. A corresponding spanning tree T' in G' is chosen by for every edge e in T with weight w choosing one of the w edges in G' corresponding to e. Therefore, the number of such spanning trees is the product of the number of choices for each edge, with equals the weight of T. Therefore, $\tau(G) = \tau(G')$, so by the (unweighted) Matrix Tree Theorem, $L^i(G) = \tau(G)$ for any i. □

(c) Combine (a) and (b) to prove the weighted Matrix Tree Theorem: if G is a weighted loopless graph with any weights, then for any i, $\tau(G) = \det L^i(G)$. *[Hint: think of the edge weights as variables, and explain why part (a) holds.]*

Proof. Fix i, and if $|E(G)| = k$, let $x_1, \ldots, x_k$ be the edge weights of the edges of G. Consider $\tau(G)$ and $\det L^i(G)$ as functions of $x_1, \ldots, x_n$. Every spanning tree contributes a monomial to $\tau(G)$, so $\tau(G)$ is a polynomial in the x_i. Similarly, the entries of $L^i(G)$ are polynomials in the x_i, and determinants are polynomials in their entries, so $\det L^i(G)$ is also a polynomial in the x_i.

By part (b), $\tau(G) = \det L^i(G)$ for all positive integer values of $x_1, \ldots, x_k$, and by part (a), this means they are equal for all $x_1, \ldots, x_k \in \mathbb{R}$. Therefore, $\tau(G) = \det L^i(G)$ for any weighted loopless graph G. □

2. Let D be a weakly-connected weighted loopless digraph with underlying graph G.

For any walk W in G, define the signed weight of W to be

$$\operatorname{sgwt}(W) = \sum_{e \in E(W)} \pm \operatorname{wt}(e),$$

where the sign on $\operatorname{wt}(e)$ is positive if and only if we traverse e from tail to head.

Fix a vertex $v \in V(G)$. Prove that the following conditions are equivalent.

(K2) For any closed walk C in G, $\operatorname{sgwt}(C) = 0$.

(K2') There exists a unique function $U : V(G) \to \mathbb{R}$ such that $U(v) = 0$ and for all $e \in E(D)$,

$$\operatorname{wt}(e) = U(\text{head of } e) - U(\text{tail of } e).$$

[Hint: if $U(v) = 0$ and W is a walk from v to u, what must be true about $U(u)$?]

Proof. (K2) implies (K2'): Since D is weakly-connected, for all $u \in V(G)$, there exists a walk from v to u in G. Choose such a walk P_u, and define $U(v) := \text{sgwt}(P_u)$.

Next, we show that $U(u)$ is independent of the trail P_u. Suppose we have another walk P'_u, and consider the closed walk C starting at u formed by traversing P_u backwards, followed by P'_u. Since the edges of P_u are done backwards, $\text{sgwt}(C) = \text{sgwt}(P'_u) - \text{sgwt}(P_u)$ By (K2), we have $\text{sgwt}(C) = 0$, so $\text{sgwt}(P_u) = \text{sgwt}(P'_u)$, and the definition of U was independent of the walk we chose.

If $e \in E(G)$, there is some walk W starting at v containing e. If u and w are the endpoints of e, without loss of generality suppose that W first traverses e from u to w. Let P_u be W cut off right before e is used for the first time, and P_w be W cut off right afterwards. Then

$$U(w) - U(u) = \text{sgwt}(P_w) - \text{sgwt}(P_u) = \pm\text{wt}(e),$$

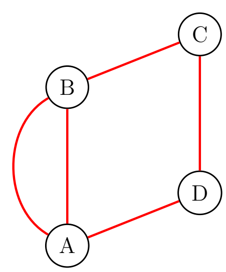
where the sign is positive precisely if w is the head of e, proving that such a function U exists. To see it is unique, note that any such function must satisfy $U(u) = \text{sgwt}(P_u)$ for all u and any choice of paths P_u from v to u, so U is unique.

(K2') implies (K2): If there exists such a function U, then if W is a walk with vertices $v_0, v_1, \ldots, v_k$, we have

$$\text{sgwt}(W) = (U(v_1) - U(v_0)) + (U(v_2) - U(v_1)) + \cdots + (U(v_k) - U(v_{k-1})) = U(v_k) - U(v_0).$$

If W is closed, its signed weight is therefore 0. □

3. For the following (unweighted) graph G, compute $\tau(G)$, $L(G)$, $\det L^1(G)$ and $\det L^2(G)$, and thus confirm the Matrix Tree Theorem for this example.



Solution: Let AB and $\widetilde{AB}$ represent the two edges between A and B. Then, G has spanning trees $\{AB, BC, CD\}$, $\{\widetilde{AB}, BC, CD\}$, $\{AB, BC, AD\}$, $\{\widetilde{AB}, BC, AD\}$, $\{AB, CD, AD\}$, $\{\widetilde{AB}, CD, AD\}$, and $\{BC, CD, AD\}$, so $\tau(G) = 7$.

Next,

$$L(G) = D(G) - A(G) = \begin{bmatrix} 3 & & & \\ & 3 & & \\ & & 2 & \\ & & & 2 \end{bmatrix} - \begin{bmatrix} & 2 & & 1 \\ 2 & & 1 & \\ & 1 & & 1 \\ 1 & & 1 & \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 & -1 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

We have

$$L^1(G) = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad L^2(G) = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

and computing the determinants, $\det L^1(G) = \det L^2(G) = 7 = \tau(G)$.