Math 412, Fall 2023 – Homework 9

Due: Thursday, November 30th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should choose four of the following five problems to solve and turn in–if you do all five, only the first four will be graded. Graduate students taking the course for four credits should solve all five. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. For a chess piece Q, *the Q-graph* is the graph whose vertices are the squares of the chess board and the two squares are adjacent if Q can move from one of them to the other in one move. Find the chromatic number of the Q-graph when Q is (a) the king, (b) a rook, (c) a bishop, (d) a knight.

   Solution: Let G be the Q-graph. Then we may assume $V(G) = \{(i, j) : 1 \leq i, j \leq 8\}$.

   (a) Q is the king. The subgraph of G induced by the vertices $(1, 1), (1, 2), (2, 1), (2, 2)$ is the complete graph K_4. Thus $\chi(G) \geq \omega(G) \geq 4$. On the other hand, let $f(i, j) = i+2j$ mod 4. By definition, this coloring uses four colors. If two vertices (i, j) and (i', j') are adjacent in G, then $|i - i'| \leq 1$ and $|j - j'| \leq 1$. Then $(i + 2j) - (i' + 2j')$ is not divisible by 4, and so the colors of (i, j) and (i', j') are distinct. Thus, f is a proper 4-coloring, which yields $\chi(G) = 4$.

   (b) Q is a rook. The subgraph of G induced by the vertices $(1, 1), (1, 2), \dots, (1, 8)$ is the complete graph K_8. Thus $\chi(G) \geq \omega(G) \geq 8$. On the other hand, let $f(i, j) = i+j$ mod 8. By definition, this coloring uses eight colors. If (i, j) and (i', j') are adjacent in G, then $i = i'$ or $j = j'$. So $(i + j) - (i' + j')$ is either $j - j'$ or $i - i'$; in both cases not divisible by 8. Thus, f is a proper 8-coloring, which yields $\chi(G) = 8$.

   (c) Q is a bishop. The subgraph of G induced by the vertices $(1, 1), (2, 2), \dots, (8, 8)$ is the complete graph K_8. Thus $\chi(G) \geq \omega(G) \geq 8$. On the other hand, let $f(i, j) = i$. By definition, this coloring uses eight colors. If (i, j) and (i', j') are adjacent in G, then $|i - i'| = |j - j'|$. So every color class is an independent set. Thus, f is a proper 8-coloring, which yields $\chi(G) = 8$.

   (d) Q is a knight. The graph has edges, so $\chi(G) \geq 2$. But the original coloring of the board is a proper 2-coloring of G, since the knight always jumps from a black square to a white one and vice versa. Thus, $\chi(G) = 2$.

2. Prove or disprove: for every n-vertex graph G, $\chi(G) \leq \omega(G) + \frac{n}{\alpha(G)}$.

Solution. The statement is false. Let G be the triangle-free graph with chromatic number k obtained via Mycielski's construction from any triangle-free $(k-1)$-chromatic simple graph with at least 2 vertices, and let $n = |V(G)|$. By construction U is an independent set of size $\frac{n-1}{2}$, so $\alpha(G) \geq \frac{n-1}{2}$. Since G is triangle-free, $\omega(G) = 2$. Thus $\omega(G) + \frac{n}{\alpha(G)} \leq 2 + \frac{2n}{n-1} < 5$, but k can be arbitrarily large.

3. Let G be a simple graph. Prove that the chromatic polynomial $p(k) := \chi(G; k)$ of G satisfies $p'(0) \neq 0$ if and only if G is connected. (*Hint: Use Theorem 5.3.8 and its proof. Note that that result doesn't guarantee that coefficients are nonzero*).

   Proof. $p'(0)$ is just the coefficient for the x^1 term in $p(k)$, so we must show this is nonzero precisely when G is connected. If G is disconnected, proper colorings of its connected components are independent, so $\chi(G; k)$ is the product of the chromatic polynomials of its connected components. Since every chromatic polynomial has 0 constant term, the lowest degree (nonzero) term in $\chi(G; k)$ is at least the number of connected components of G.

   Conversely, if G is connected, we use induction on the number of vertices n. If $n = 1$, since G is simple, it is an isolated vertex with chromatic polynomial k. If $n > 1$, choose any edge e and use deletion contraction: $\chi(G; k) = \chi(G \setminus e; k) - \chi(G \cdot e; k)$. By Theorem 5.3.8, we can write

$$\chi(G \setminus e; k) = k^n - a_{n-1}k^{n-1} + a_{n-2}k^{n-2} + \cdots + (-1)^{n-1}a_1 k$$

   and

$$\chi(G \cdot e; k) = k^{n-1} - b_{n-2}k^{n-2} + b_{n-3}k^{n-3} + \cdots + (-1)^{n-2}b_1 k,$$

   where all of a_i, b_i are nonnegative integers. Therefore, by deletion-contraction, $\chi(G; k)$ has x^1-coefficient $(-1)^{n-2}a_1 - (-1)^{n-2}b_1 = (-1)^{n-1}(a_1 + b_1)$. Both a_1 and b_1 are nonnegative, and by the inductive hypothesis a_1 is nonzero, so the x^1-coefficient of $\chi(G; k)$ is nonzero. □

4. Let G be a plane graph, and let G^* be its dual graph. Prove the following:

   (a) G^* is connected.

   Proof. One vertex of G^* is placed in each face of G. If $u, v \in V(G^*)$, then any curve in the plane between u and v (avoiding vertices) crosses face boundaries of G in its passage from the face of G containing u to the face of G containing v. This yields a u, v-walk in G^*, which contains a u, v-path in G^*. □

   (b) If G is connected, then each face of G^* contains exactly one vertex of G. (*Hint: Use Euler's Formula*)

Proof. The edges incident to a vertex $v \in V(G)$ appear in some order around v. Their duals form a cycle in G^* in this order. This cycle is a face of G^*. If w is another vertex of G, then there is a v,w-path because G is connected, and this path crosses the boundary of this face exactly once. Hence every face of G^* contains at most one vertex of G. Equality holds because the number of faces of G^* equals the number of vertices of G: since both G and G^* are connected, Euler's formula yields $n - e + f = 2$ and at $n^* - e^* + f^* = 2$. We have $e = e^*$ and $f = n^*$ by construction, which yields $n = f^*$. □

(c) $(G^*)^* \cong G$ if and only if G is connected.

Proof. Since G^{**} is the dual of the plane graph G^*, part (a) implies that G^{**} is connected. Hence if G^{**} is isomorphic to G, then G is connected.

Conversely, suppose that G is connected. By part (b), the usual drawing of G^* over the picture of G has exactly one vertex of G inside each face of G^*. Associate each vertex $x \in V(G)$ with the vertex $x' \in V(G^{**})$ contained in the face of G that contains x; by part (b), this is a bijection.

Consider $xy \in E(G)$. Because the only edge of G^* crossing xy is the edge of G^* dual to it, we conclude that the faces of G^* that contain x and y have this edge as a common boundary edge. When we take the dual of G^*, we thus obtain $x'y'$ as an edge. Hence the vertex bijection from G to G^{**} that takes x to x' preserves edges. Since the number of edges doesn't change when we take the dual, G^{**} has no other edges and thus is isomorphic to G. □

5. Let P be a polyhedron such that

- The faces of P are all either pentagons or hexagons.
- Each vertex of P has degree 3, and lies on the boundary of one pentagon and two hexagons.

Determine how many vertices, edges, and faces P has, and how many of these faces are pentagonal vs. hexagonal.

Solution: Let v be the number of vertices, e be the number of edges, f be the number of faces, p be the number of pentagons, and h be the number of hexagons. By Euler's formula, $v - e + f = 2$, and since every face is a pentagon or hexagon, $p + h = f$. Since every vertex is on the boundary of 1 pentagon, and each pentagon has 5 vertices, we have $p = 5v$. Since every vertex is on the boundary of 2 hexagons, and each hexagon has 6 vertices, we have $2h = 6v$. Additionally, since P is 3-regular, $2e = 3v$. Solving these equations gives $h = 20, p = 12, f = 32, e = 90, v = 60$.