Solutions to Math 412 Final Exam — Dec. 14, 2023

1. (20 points) Let P be a polyhedron such that every vertex lies on the boundary of exactly one triangular face, two quadrilateral faces, and one pentagonal face. Determine the numbers n, e, and f of vertices, edges, and faces, respectively, of P.

Since P is a polyhedron, it can be represented as a plane graph G by pulling open the rear face, as we discussed in class. P and G have the same number of vertices, edges, and faces.

Let f_i denote the number of length-i faces of G. We have $f_3 + f_4 + f_5 = f$ and all other $f_i = 0$. Since each vertex lies on the boundary of exactly one triangular face, and each triangular face has three vertices, we have $1n = 3f_3$, so $f_3 = n/3$. Similarly, each vertex lies on the boundary of exactly one pentagonal face, and each pentagonal face has five vertices, so we have $1n = 5f_5$, so $f_5 = n/5$. Similarly, each vertex lies on the boundary of exactly two quadrilateral faces, and each quadrilateral face has four vertices, so we have $2n = 4f_4$, so $f_4 = n/2$. Adding these up, we have $f = 31n/30$.

G is 4-regular, so by the degree sum formula for G, $e = 2n$. Applying Euler's formula, we obtain

$$2 = n - e + f = n - 2n + \frac{31n}{30} = \frac{n}{30},$$

so $n = 60, e = 120$, and $f = 62$.

2. (30 points) Let G be a simple planar graph such that each vertex is contained in at most one triangle.

(a) (15 points) Prove that G must have a vertex of degree ≤ 4.

If $n(G) \leq 5$, since G is simple, every vertex must have degree ≤ 4. So assume $n(G) \geq 6$.

Let k be the number of triangles in G. Since each vertex is contained in at most one triangle and each triangle contains 3 vertices, $3k \leq n(G)$. We can remove one edge from each triangle to obtain a simple planar triangle-free graph G' such that $n(G') = n(G)$ and $e(G') \geq e(G) - k$. It is a consequence (Theorem 6.1.23) of Euler's formula that since G' is planar and triangle-free, $e(G') \leq 2n(G') - 4$, so

$$e(G) \leq e(G') + k \leq 2n(G') - 4 + k = 2n(G) - 4 + k < \frac{5n(G)}{2}$$

since $k < \frac{n}{2}$ By the degree-sum formula for G, we have

$$\sum_{v \in V(G)} d(v) = 2e(G) < 5n(G),$$

so G must have a vertex of degree less than 5.

(b) (15 points) Use part (a) to prove that G is four-colorable WITHOUT using the four-color theorem. *(We did a similar argument in class as part of a proof; you may NOT simply cite that argument without proof, but you may recreate it, if you wish).*

Suppose for a contradiction that G is a simple planar graph satisfying the triangle condition that is not 4-colorable, but every such graph with fewer vertices is 4-colorable. We must have $n(G) \geq 5$ since every graph with fewer than 5 vertices is 4-colorable. Let v be a vertex with degree ≤ 4. Since $G \setminus v$ is planar and satisfies the triangle condition, $G \setminus v$ is four-colorable.

Fix a proper 4-coloring of $G \setminus v$. Since G is not 4-colorable, $N(v)$ must consist of four vertices, each with a different color. Without loss of generality, assume that the neighbors of v are colored 1,2,3,4, clockwise from top; let v_i be the vertex of color i. Let G_{ij} be the subgraph of G induced by vertices of colors i and j, and let P_{ij} be any path in G_{ij} from v_i to v_j, if one exists.

If P_{13} doesn't exist, then we can swap colors 1 and 3 in the connected component of G_{13} containing v_1, followed by coloring v color 1, to obtain a proper 4-coloring of G. So P_{13} must exist, and by a similar argument, so must P_{24}. Let C be the cycle consisting of P_{13} along with the edges vv_1 and vv_3. Then v_2 is inside C and v_4 is outside, or vice-versa, so by the Jordan curve theorem, P_{24} must cross C. Since G is planar, this crossing must happen at a vertex, but that is impossible since P_{24} consists only of vertices of colors 2 and 4, whereas C has no such vertices. This is a contradiction, so G is 4-colorable.

3. (25 points) Let D be a digraph and let $x, y \in V(D)$ be vertices such that there is no edge from x to y in $E(D)$. Recall the following definitions:

   - $\kappa(x, y)$ is the minimum set of an x, y-vertex-cut in D (a set of vertices $U \subseteq V(D)$ such that $D \setminus U$ has no x, y-path).
   - $\lambda(x, y)$ is the maximum number of internally disjoint x, y-paths in D.

   Use network flows to prove that $\kappa(x, y) = \lambda(x, y)$ *(Note: this is an analogue of Menger's Theorem).*

   If there exist k internally-disjoint x, y-paths, then every x, y-vertex-cut must contain at least one vertex from each path, so contains at least k vertices. Thus, $\kappa(x, y) \geq \lambda(x, y)$.

   Conversely, let N be the following network, with source x and sink y. Start with D and replace every vertex $w \in V(D) \setminus \{x, y\}$ with a pair of vertices w^- and w^+, with an edge of capacity 1 from w^- to w^+. For each edge $e = uv \in V(D)$, let N have the edge u^+v^-, with capacity $n(D)$ (if one of the endpoints is x or y, that should also be the corresponding endpoint in N). By the Max-Flow, Min-Cut Theorem, the value k of a maximum feasible flow in N equals the capacity of a minimum source-sink edge cut $[S, T]$.

   Since N has integer capacities, the Integrality Theorem guarantees that there exists a maximum flow f with integer edge-flows. As we did in class, we can interpret f as a union of k x, y-paths in N, which after contracting along every edge of the form w^-w^+ become x, y-paths in D. These paths are internally disjoint in N (and therefore in D) since edges of the form w^-w^+ have capacity 1, so must have flow either 0 or 1, and flow conservation along with integrality means that at most a single edge going into or out of a vertex can have nonzero flow. Thus, D has k internally-disjoint x, y-paths, so $\lambda(x, y) \geq k$.

   Since x and y are not adjacent in D, they are not adjacent in N, so the set of edges

   $$F := \{w^-w^+ | w \in V(D) \setminus \{x, y\}\}.$$

   is a source-sink cut of capacity $n(G) - 2$, which is less than the capacities of every other edge (which is $n(G)$). Therefore, any minimum capacity source-sink edge cut $[S, T]$ must be a subset of F, meaning it's of the form

   $$F' := \{w^-w^+ | w \in U\}$$

   for some subset $U \subset V(G) \setminus \{x, y\}$. The set U is therefore a vertex cut in D since every x, y-path in N must go through w^- and w^+ for all $w \in U$, and therefore each x, y-path in D must go through w. The size of U is the capacity k of the cut, so $\kappa(x, y) \leq k$. Combining this with the statement $\lambda(x, y) \geq k$ gives $\kappa(x, y) \leq \lambda(x, y)$, proving the converse.

4. (20 points) Let G be a loopless graph with chromatic number k. Prove that there exists an ordering $v_1, \ldots, v_n$ of $V(G)$ such that greedy coloring with respect to that ordering results in a proper k-coloring of G.

Fix a proper k-coloring f of G, and order the vertices such that the colors are weakly increasing (put all the vertices of color 1 first, then the vertices of color 2, etc.). We claim that applying the greedy coloring algorithm with respect to this ordering gives a proper k-coloring of G. [Note that this coloring doesn't necessarily equal f.]

In particular, we claim that the resulting coloring g has the property $g(v_i) \leq f(v_i)$ for all i. Suppose not: then there exists a vertex v_j such that $g(v_j) > f(v_j)$, but for all $i < j$, $g(v_i) \leq f(v_i)$. Since $g(v_j) > f(v_j)$ and the greedy coloring algorithm doesn't color v_j color $f(v_j)$, this means that v_j has a neighbor $v_i, i < j$ such that $g(v_i) = f(v_j)$. However, this cannot happen: since $i < j$, $f(v_i) \leq f(v_j)$ by our choice of ordering, and since $g(v_i) \leq f(v_i)$ by assumption, $f(v_i) = f(v_j)$. But this contradicts the assumption that v_i and v_j are neighbors and f is a proper coloring. Therefore, g is a proper coloring (guaranteed by the greedy coloring algorithm) that has $g(v_n) \leq f(v_n) = k$, so it is a proper k-coloring of G.

5. (20 points) Recall the following definitions:

- $\alpha(G)$: maximum size of independent set
- $\alpha'(G)$: maximum size of matching
- $\beta(G)$: minimum size of vertex cover
- $\beta'(G)$: minimum size of edge cover

(a) (5 points) Prove that $\alpha'(G) \leq \beta(G)$ for all graphs G.

Let M be a matching of size $\alpha'(G)$. Then any vertex cover must contain at least one vertex from each edge of M, and no vertex can cover two edges of M since the edges are not adjacent to each other. Thus $\beta(G) \geq \alpha'(G)$.

(b) (5 points) Prove that $\alpha(G) \leq \beta'(G)$ for all graphs G without isolated vertices.

Let I be an independent set of size $\alpha(G)$. Then in any edge cover, there is an edge covering each vertex in I, but an edge can cover at most one such vertex since I is an independent set, so $\beta'(G) \geq \alpha(G)$.

(c) (5 points) Provide (with proof) a counterexample for the statement: $\alpha(G) \leq \alpha'(G)$ for all graphs G.

A graph G consisting of n isolated vertices is a counterexample since $\alpha(G) = n$ and $\alpha'(G) = 0$

(d) (5 points) Provide (with proof) a counterexample for the statement: $\beta(G) \leq \beta'(G)$ for all graphs G without isolated vertices.

K_{2n}, $n \geq 2$ is a counterexample since $\beta(K_{2n}) = 2n - 1$, while $\beta'(K_{2n}) = n$.

6. (20 points) For an n-vertex graph G, prove that the following are equivalent.

(a) There exists an edge $e \in E(G)$ such that $G \setminus e$ is a tree.

(b) G is connected and has exactly one cycle

(c) G is connected and has exactly n edges

(d) G has exactly one cycle and exactly n edges

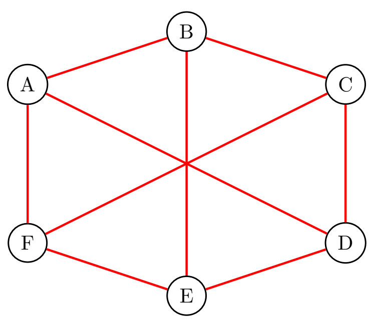
(Hint: prove that everything is equivalent to (a). Our several equivalent definitions of trees may come in handy here)

We use Theorem 2.1.4, which says that a graph T is a tree if and only if T satisfies any of two of the conditions (i) T has $n-1$ edges, (ii) T is acyclic, (iii) T is connected.

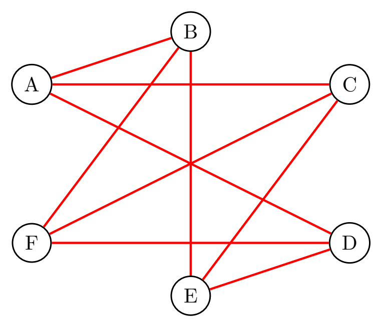
First we prove that (a) implies the other three conditions. Let $e \in E(G)$ be an edge such that $G \setminus e$ is a tree. Then G is connected since $G \setminus e$ is connected, it has n edges since $G \setminus e$ has $n-1$ edges, and it has exactly one cycle since $G \setminus e$ is acyclic, so adding e to the unique path between the endpoints of e in $G \setminus e$ gives the unique cycle in G.

Conversely, if G satisfies one of the other conditions, we determine an edge e such that $G \setminus e$ is a tree. If G is connected and has exactly one cycle, let e be any edge in that cycle; removing it leaves a connected, acyclic graph; hence, a tree. If G is connected and has exactly n edges, then G is not a tree, so is not acyclic, so choose any edge e that is part of a cycle. Then $G \setminus e$ is connected and has $n-1$ edges, so is a tree. Finally, if G has exactly one cycle and exactly n edges, let e be any edge in that cycle; removing it leaves an acyclic graph with $n-1$ edges; hence, a tree.

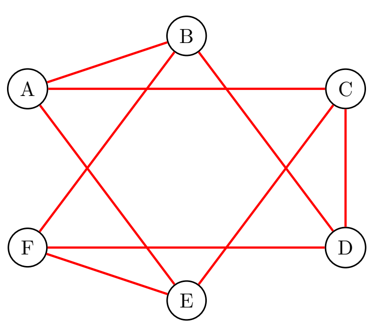
7. (20 points) Determine, with proof, which of the graphs below are isomorphic and which are not.



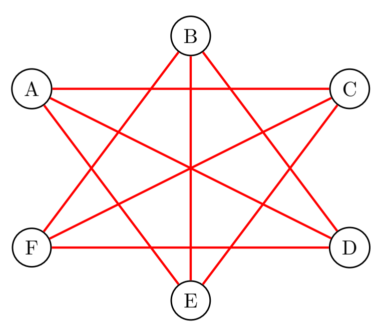
G



H



K



L

Both G and H are isomorphic to $K_{3,3}$, so they are isomorphic. Neither K or L is bipartite, since both have a 3-cycle, so neither are isomorphic to G or H.

We construct as isomorphism $f : V(K) \to V(L)$ explicitly (note that both graphs are simple). Let $f : V(K) \to V(L)$ be the following bijection:

$$f(A) = A,\ f(B) = D,\ f(C) = C,\ f(D) = F,\ f(E) = E,\ f(F) = B.$$

We will show that f is an isomorphism by showing that $uv \in E(K)$ if and only if $f(u)f(v) \in E(L)$.

$$AB \in E(K) \leftrightarrow f(A)f(B) = AD \in E(L)$$
$$AC \in E(K) \leftrightarrow f(A)f(C) = AC \in E(L)$$
$$AE \in E(K) \leftrightarrow f(A)f(E) = AE \in E(L)$$
$$BD \in E(K) \leftrightarrow f(B)f(D) = DF \in E(L)$$
$$BF \in E(K) \leftrightarrow f(B)f(F) = DB \in E(L)$$
$$CD \in E(K) \leftrightarrow f(C)f(D) = CF \in E(L)$$
$$CE \in E(K) \leftrightarrow f(C)f(E) = CE \in E(L)$$
$$DF \in E(K) \leftrightarrow f(D)f(F) = FB \in E(L)$$
$$EF \in E(K) \leftrightarrow f(E)f(F) = EB \in E(L)$$

This is all the edges in both K and L, so f is an isomorphism.