Recall: M^λ : span of tabloids

S^λ : span of polytabloids (Specht module)

Today: finish irreducibility of S^λ

and decomposition of M^λ

Thm 30 (Submodule Theorem):

a) Let U be a submodule of M^μ.

Then $U \supseteq S^\mu$ or $U \subseteq (S^\mu)^\perp$.

b) S^μ is irreducible

Lemma 31: Let $u \in M^\mu$, and let T be a tableau w/ shape λ.

a) If $\kappa_T u \neq 0$, then $\lambda \trianglerighteq \mu$

b) If $\lambda = \mu$, then $\kappa_T u$ is a multiple of e_T.

Pf: u is a linear combination of μ-tabloids, so we can reduce to the case where $u = \{S\}$ for some λ-tableau S, and extend by linearity.

a) Last time

b) If there exist two entries i, j in the same row of S that appear in the same col. of T, the argument for part a) shows that $k_T\{S\} = 0$. Otherwise, we can permute each col of T and obtain a tableau which is row equiv. to S
(Pf: Look at the first col of T, and proceed by induction)
i.e. $\exists\, \sigma \in C_T$ s.t. $\omega\{T\} = \{S\}$.
Then,

$$k_T\{S\} = k_T\,\sigma\{T\} = \sum_{\omega \in C_T} (-1)^{\omega}\, \omega\sigma\{T\}$$

$$= \pm \sum_{\omega \in C_T} (-1)^{\omega\sigma}\, \omega\sigma\{T\}$$

$$= \pm \sum_{\omega' \in C_T} (-1)^{\omega'}\, \omega'\{T\}$$

$$= \pm k_T\{T\} = \pm e_T. \qquad \square$$

Pf of Submodule Thm:

Let $u \in U$, and let T be a μ-tableau.

By Lemma 31, $k_T u = f e_T$ for some $f \in \mathbb{C}$.

Since U is S_n-invariant, this means $f e_T \in U$.

If for any choice of u and T, $f \neq 0$, then $e_T \in U$, so since e_T generates S^λ, $S^\mu \subseteq U$.

Otherwise, $k_T u = 0$ $\forall u, T$. We have

$$\langle u, e_T \rangle = \langle u, k_T \{T\} \rangle$$

$$= \sum_{w \in C_T} (-1)^w \langle u, w\{T\} \rangle$$

$$= \sum_{w^{-1} \in C_T} (-1)^w \langle u, w^{-1}\{T\} \rangle \qquad \text{(inverting } w\text{)}$$

$$= \sum_{w \in C_T} (-1)^w \langle wu, \{T\} \rangle \qquad \text{(by } S_n \text{ invariance)}$$

$$= \langle k_T u, \{T\} \rangle$$

$$= 0,$$

so $u \in (S^\mu)^\perp$ $\forall u \in U$.

$\square$

Thm 32 (Decomposition Theorem):

The S^λ are mutually inequivalent, and therefore form a complete set of S_n-irreps. M^μ decomposes as:

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} m_{\lambda,\mu} S^\lambda$$

where $m_{\mu,\mu} = 1$.

Pf: Let $\phi \in \mathrm{Hom}_{S_n}(S^\lambda, M^\mu)$.

This extends to an S_n-homom $M^\lambda \to M^\mu$ by setting $\phi((S^\lambda)^\perp) = 0$. We have

$$\phi(e_T) = \phi(K_T\{T\}) = K_T\,\phi(\{T\}),$$

and since $\phi(\{T\})$ is a linear combination of μ-tabloids, by Lemma 31a, this is 0 unless $\lambda \trianglerighteq \mu$.

In particular, since $S^\lambda \leq M^\lambda$, if $S^\lambda \cong S^\mu$, then $\mu \trianglerighteq \lambda$ and $\lambda \trianglerighteq \mu$, so $\lambda = \mu$.

If $\lambda = \mu$, by Lemma 31b, $\phi(e_T) = c_T e_T$ for some $c_T \in \mathbb{C}$. However, c_T is independent of T since

$$\phi(e_{wT}) = \phi(w e_T) = w\phi(e_T) = w \cdot c_T e_T = c_T e_{wT},$$

so ϕ is mult. by a scalar, and therefore $\dim \mathrm{Hom}_{S_n}(S^\lambda, M^\mu) = 1$.

By Schur's Lemma, in the decomposition

$$M^\mu = \bigoplus_\lambda m_{\lambda,\mu} S^\lambda,$$

we have $m_{\lambda,\mu} = \dim \mathrm{Hom}_{S_n}(S^\lambda, M^\mu)$, so the above shows that $m_{\mu,\mu} = 1$ and $m_{\lambda,\mu} = 0$ unless $\lambda \trianglerighteq \mu$. $\square$

Next, want to find a basis for S^λ.

Recall that a std. tableau has entries $1, \dots, n$, which increase along rows and down columns

Heading towards:

Thm 33: The set

$$\{ e_T \mid T \text{ is a std. tableau of shape } \lambda \}$$

is a basis for S^λ.

Composition sequence of nonneg. integers

$$\lambda = (\lambda_1, \dots, \lambda_k) \vDash n,$$

s.t. $\lambda_1 + \dots + \lambda_k = n$.

For any tabloid T of shape λ, let

same for poly tab. and (row inc.) tab

$\{T^i\}$ be the tabloid formed by all elts. $\le i$ in $\{T\}$

Forms a composition $\lambda^i := \lambda^i(T) := \text{shape}(\{T^i\})$

We say that $\{S\}$ dominates $\{T\}$ $(\{S\} \trianglerighteq \{T\})$

if $\lambda^i(S) \trianglerighteq \lambda^i(T)$ $\forall i$.

If time:

Class activity: draw the poset of

tabloids of shape