Math 418, Spring 2025 – Practice Problems 3

14.2.3 *Determine the Galois group of* $(x^2-2)(x^2-3)(x^2-5)$*. Determine all the subfields of the splitting field of this polynomial.*

14.2.10 *Determine the Galois group of the splitting field over* $\mathbb{Q}$ *of* x^8-3*.*

14.2.13 *Prove that if the Galois group of the splitting field of a cubic over* $\mathbb{Q}$ *is the cyclic group of order 3 then all the roots of the cubic are real.*

14.3.1 *Factor* x^8-x *into irreducibles in* $\mathbb{Z}[x]$ *and in* $\mathbb{F}_2[x]$*.*

14.4.4 *Let* $f(x) \in F[x]$ *be an irreducible polynomial of degree* n *over the field* F*, let* L *be the splitting field of* $f(x)$ *over* F *and let* α *be a root of* $f(x)$ *in* L*. If* K *is any Galois extension of* F*, show that the polynomial* $f(x)$ *splits into a product of* m *irreducible polynomials each of degree* d *over* K*, where* $d = [K(\alpha) : K] = [(L \cap K)(\alpha) : L \cap K]$ *and* $m = n/d = [F(\alpha) \cap K : F]$*.*

14.5.2 *Determine the subfields of* $Q(\zeta_8)$ *generated by the periods of* ζ_8 *and in particular show that not every subfield has such a period as primitive element.*

14.6.2a *Determine the Galois group of* x^3-x^2-4

14.6.3 *Prove for any* $a, b \in \mathbb{F}_{p^n}$ *that if* $f(x) = x^3+ax+b$ *is irreducible then* $-4a^3-27b^2$ *is a square in* $\mathbb{F}_{p^n}$

14.7.3 *Let* F *be a field of characteristic* $\neq 2$*. State and prove a necessary and sufficient condition on* $\alpha, \beta \in F$ *so that* $F(\sqrt{\alpha}) = F(\sqrt{\beta})$*. Use this to determine whether* $\mathbb{Q}(\sqrt{1-\sqrt{2}}) = \mathbb{Q}(i, \sqrt{2})$