Math 412, Fall 2023 – Homework 8

Due: Wednesday, November 8th, at 9:00AM via Gradescope

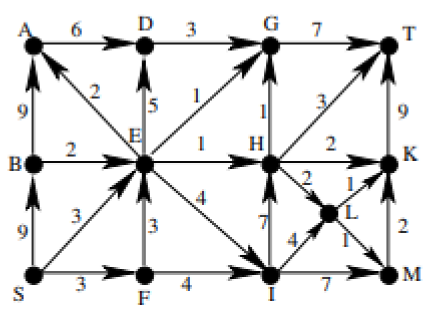
Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should choose four of the following five problems to solve and turn in–if you do all five, only the first four will be graded. Graduate students taking the course for four credits should solve all five. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. Let G be a 2-connected simple graph.
   (a) Prove that in every ear decomposition of G, the number of ears (including the initial cycle) is $|E(G)| - |V(G)| + 1$.
   (b) Let $s, t \in E(G)$. Prove that the vertices of G can be linearly ordered so that each vertex apart from s and t has a neighbor that is earlier in the order and a neighbor that is later in the order.

2. Let G be a connected graph with at least 3 vertices. Prove that the following are equivalent:
   (A) G is 2-edge connected
   (B) Every edge of G appears in a cycle
   (C) Every pair of edges of G lie on a closed trail
   (D) Every pair of vertices of G lie on a closed trail

3. Use Menger's Theorem to prove that $\kappa(G) = \kappa'(G)$ when G is 3-regular (Theorem 4.1.11).

4. Find a minimum capacity source-to-sink edge cut in the following network (make sure to also prove it is indeed of minimum capacity):



5. Let B be an X, Y-bigraph that satisfies Hall's condition for X (i.e. $|N(S)| \geq S$ for all $S \subseteq X$). WITHOUT USING HALL'S THEOREM, and instead using network flow, prove that there is a matching that saturates X.

   [Hint: Create a digraph with consists of an orientation of B, along with a source and sink. Choose capacities so that a minimum cut won't contain any edges of B, and use the max-flow, min-cut theorem.]