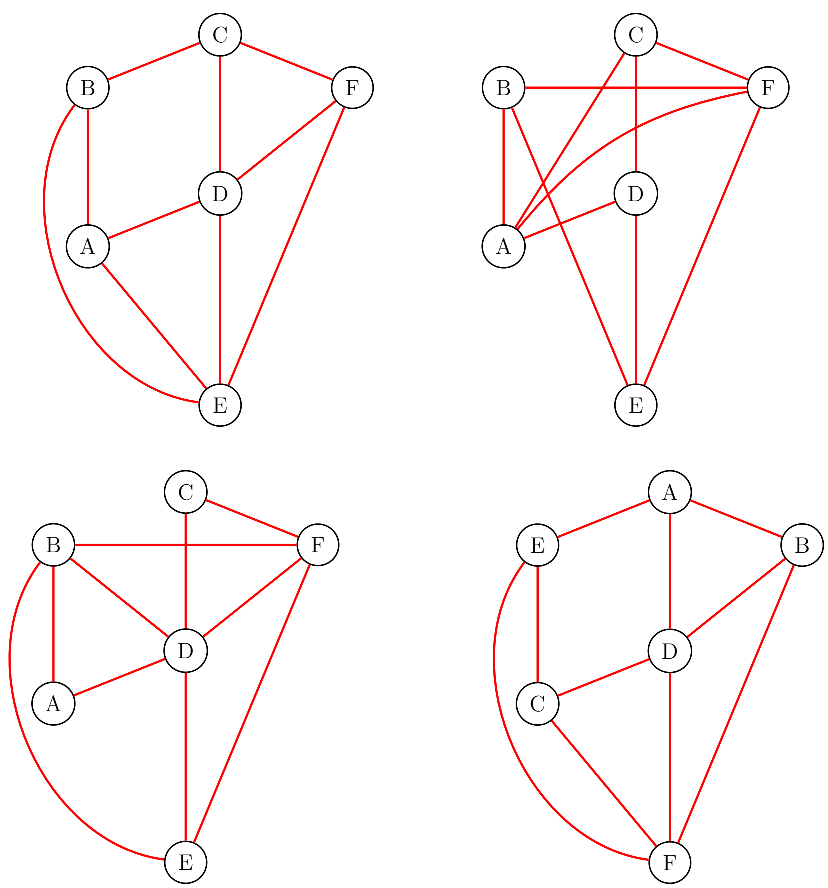
Math 412, Fall 2023 – Homework 1

Due: Wednesday, August 30th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should choose four of the following five problems to solve and turn in–if you do all five, only the first four will be graded. Graduate students taking the course for four credits should solve all five. Problems that use the word “describe”, “determine”, ”show”, or ”prove” require proof for all claims.

1. Of the following graphs, determine which pairs of graphs are isomorphic (and which are nonisomorphic).



Solution: The top left, top right, and bottom right graphs are isomorphic to each other, and the bottom left graph is not isomorphic to any of them.

Proof. In the bottom left graph, D has degree 5, whereas no vertex in any of the other graphs has degree 5, so the bottom left graph can't be isomorphic to any of the others.

We next show that the top left and bottom right graphs are isomorphic via explicit isomorphism. The map f from the vertex set of the top left graph to the vertex set of the bottom right graph is as follows

$$f(A) = C,\ f(B) = E,\ f(C) = A,\ f(D) = D,\ f(E) = F,\ f(F) = B.$$

Due to the layout of the graphs, it is clear from this map that the induced map g on edges is bijective, and that the endpoints of $g(e)$ are f applied to the endpoints of e.

To show that the other top left and top right graphs are isomorphic, we examine their adjacency matrices, ordering the vertices in both graphs as: A,B,C,D,E,F. These are

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Swapping the first and fifth rows and columns of A_1 gives

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

and further swapping the fourth and sixth rows and columns gives

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Swapping the third and fifth rows and columns gives

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix},$$

and finally, swapping the second and fourth rows and columns gives

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

which equals A_2. By what we discussed in class, this means that the two graphs are isomorphic.

Since isomorphism is an equivalence relation, we also know that the top right and bottom right graphs are isomorphic. $\square$

2. Let G, H, K be simple graphs.

   (a) If $G \cong H$, prove that $\overline{G} \cong \overline{H}$.

   Proof. Since G and H are isomorphic, there is a bijection $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ if and only if $uv \in E(G)$.
   We use the same map f to exhibit an isomorphism $\overline{G} \cong \overline{H}$. Since $V(\overline{G}) = V(G)$ and $V(\overline{H}) = V(H)$. f is a bijection from $V(\overline{G})$ to $V(\overline{H})$. If $uv \in E(\overline{G})$ then $uv \notin E(G)$, so as noted in the first paragraph, $f(u)f(v) \notin E(H)$, so $f(u)f(v) \in E(\overline{H})$. The converse is similar: if $uv \notin E(\overline{G})$, then $uv \in E(G)$, so $f(u)f(v) \in E(H)$, so $f(u)f(v) \notin E(\overline{H})$. Therefore, f is an isomorphism from $\overline{G}$ to $\overline{H}$, and so $\overline{G} \cong \overline{H}$. $\square$

   (b) If $K \subseteq G$ and $V(K) = V(G)$, prove that $\overline{G} \subseteq \overline{K}$ (NOTE: corrected from original)

   Proof. The vertex set of a graph complement equals the vertex set of the graph itself, so $V(\overline{G}) = V(\overline{K}) = V(G) = V(K)$. Consider an edge uv of $\overline{G}$. By the definition of complement, uv is not an edge of G. Since $K \subseteq G$, every edge of K is an edge of G, so uv is not an edge of K. Therefore, uv is an edge of $\overline{K}$, and we have proven $E(\overline{G}) \subseteq E(\overline{K})$. Thus, $\overline{G} \subseteq \overline{K}$. $\square$

3. Let G be a simple graph with incidence matrix M.

   (a) What meanings to the entries in position (i, j) of MM^T and M^TM have in terms of the edges and vertices of G? I.e. what does entry (i, j) tell you about G?
   Solution: Let $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$ Let m_{ij} denote the (i, j)-entry of M, so that m_{ji} is the (i, j)-entry of M^T. By the definition of the incidence matrix, $m_{i,j}$ is 1 if the edge e_j has the vertex v_i as an endpoint. Observe the matrix product:

   $$(MM^T)_{i,j} = \sum_{k=1}^{m} m_{i,k} m_{j,k}.$$

When $i \neq j$, the product $m_{i,k}m_{j,k}$ is 1 precisely if e_k has endpoints i and j, and $m_{i,k}m_{i,k}$ is 1 precisely if v_i is one of the endpoints of e_k. Taking the sum, when $i \neq j$, $(MM^T)_{i,j}$ is the number of edges with endpoints v_i and v_j, and $(MM^T)_{i,i}$ is the degree of v_i.

On the other hand,

$$(M^TM)_{i,j} = \sum_{k=1}^{m} m_{k,i}m_{k,j},$$

and the product $m_{k,i}m_{k,j}$ is 1 precisely if both e_i and e_j have v_k as an endpoint. Summing, $(M^TM)_{i,j}$ is the number of endpoints e_i and e_j share.

(b) Prove that all diagonal entries of M^TM are 2.

Proof. This follows from the previous part, but let's do it explicitly.

$$(M^TM)_{i,i} = \sum_{k=1}^{m} m_{k,i}m_{k,i} = \sum_{k=1}^{m} m_{k,i} = 2,$$

where the second equality is because $0^2 = 0, 1^2 = 1$, and the third equality is because every edge has precisely two endpoints. □

4. Prove that every group of six people contains either a set of three people who all know each other or a set of three people who all do not know each other.

Proof. This is a graph theory problem is disguise. Label the six people A, B, C, D, E, F, and draw a vertex for each of them. Draw an edge between two people if (and only if) they know each other. Then, we need only show that the resulting simple graph G must have either a clique of size 3 or an independent set of size 3.

Consider vertex A. First suppose that the degree of A is at least 3. If any of its neighbors are adjacent to each other, then along with A itself this will form a clique of size 3. So the neighbors of A must not be adjacent to each other, and since there are at least 3 of them, any three form an independent set of size 3.

On the other hand, suppose that the degree of A is at most 2, and consider the 3 vertices that are not adjacent to A. If any two of these vertices are not adjacent to each other, then along with A itself this will form an independent set of size 3. So these vertices must all be adjacent to each other, and since there are at least 3 of them, any three form a clique of size 3. [Note that this paragraph is just the same argument as the previous paragraph, applied to $\overline{G}$].

Therefore, in any case, G must have either a clique or indepdent set of size 3. □

5. Prove that the Petersen graph has no cycle of length 7 (i.e. a subgraph that is a cycle consisting of exactly 7 vertices).

Proof. We use the interpretation discussed in class. The vertices of the Petersen graph are labelled by two-element subsets of $S = \{1, 2, 3, 4, 5\}$, and there is an edge between two vertices if the corresponding subsets are disjoint. Each vertex in the Petersen graph has degree 3, corresponding to the 3 two-element subsets of the remaining three elements of S.

Suppose that the Petersen graph has a cycle C of length 7. Since any two vertices u and v of C are connected by a path of length at most 3 on C, any additional edge with endpoints on C would create a cycle of length at most 4, the ≤ 3 edges forming the path from u to v, plus the additional edge.

By Corollary 1.1.40, the Petersen graph has no cycles of length less than five, so there cannot be an additional edge with endpoints on C. Hence the third neighbor of each vertex on C is not on C. Thus there are seven edges from the vertices in C to the remaining three vertices of the Petersen graph. One of the remaining vertices receives at least three of these edges since this is the only way to get a total of 7. This vertex $x \notin V(C)$ has three neighbors on C. For any three vertices on C, either two are adjacent (call these u and v or two (again call these u and v) have a common neighbor on C (call this vertex w). Then, x, u, v, and potentially w form a cycle of length at most 4, which is a contradiction, to the Petersen graph cannot have a cycle of length 7. $\square$