

Network flow:

Def 4.3.1/4.3.2:

a) A network is a digraph N w/

- A source vertex s
- A sink vertex t
- A nonneg. capacity $c(e)$ for each edge e

b) A flow f is an assignment of a value $f(e)$ to every edge e .

c) Let

$$f^+(v) = \sum_{\substack{\bullet \rightarrow \bullet \\ \downarrow v}} f(e)$$

"outflow"

$$f^-(v) = \sum_{\substack{\bullet \rightarrow \bullet \\ \uparrow v}} f(e)$$

"inflow"

d) f is feasible if

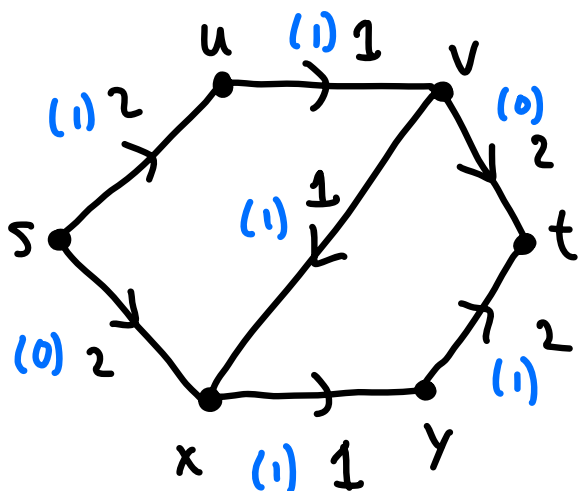
- $0 \leq f(e) \leq c(e)$ for all $e \in E(N)$ (capacity constraints)
- $f^+(v) = f^-(v)$ for all $v \in V(N) \setminus \{s, t\}$ (conservation constraints)

e) The value of f is $\text{val}(f) := f^-(t) - f^+(t)$

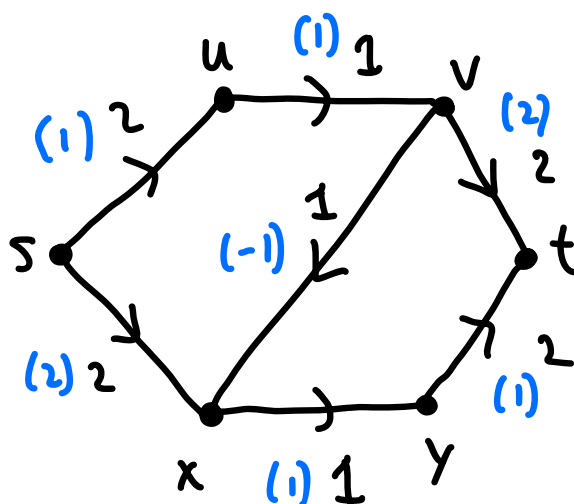
f) A maximum flow is a feasible flow of maximum value

Class activity: Is this flow feasible? Maximum?
What is its value?

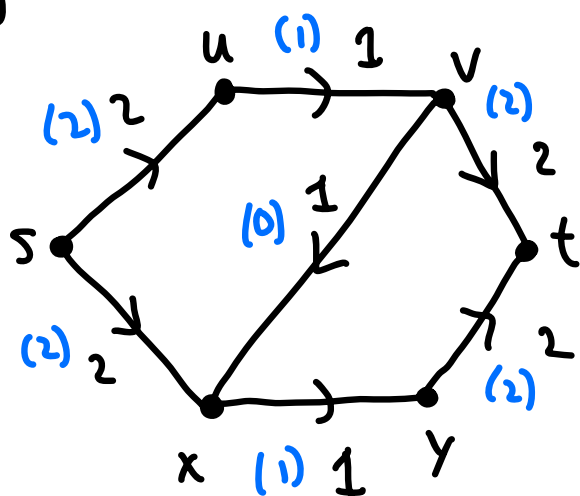
a)



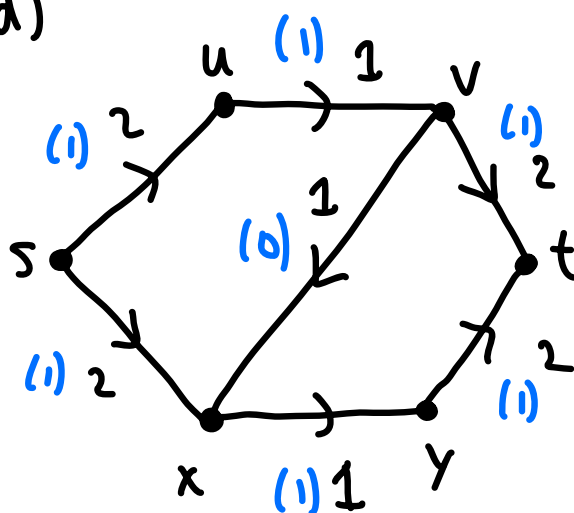
b)



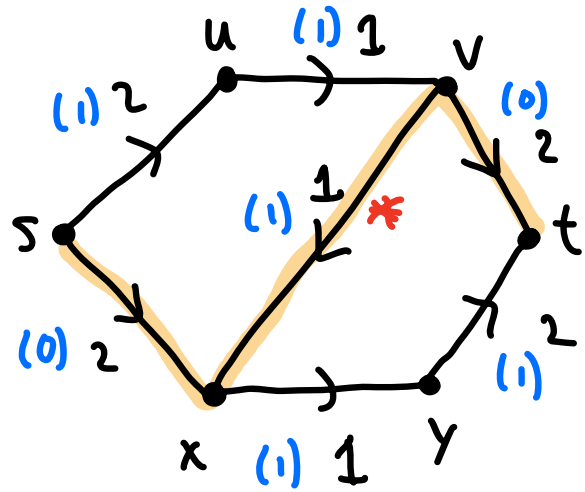
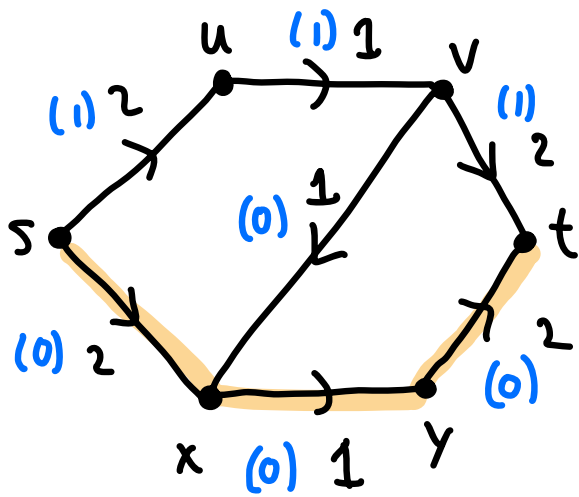
c)



d)

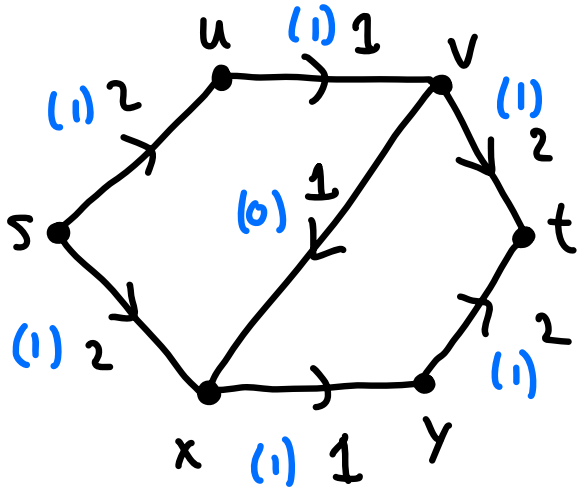
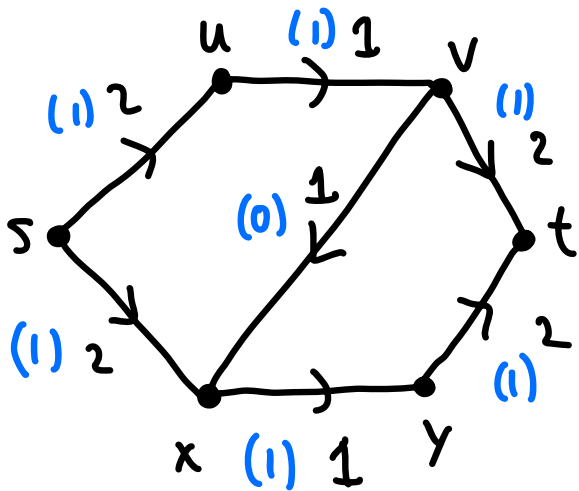


When we have an s, t -path with extra capacity...



We can augment the flow

* need to consider more than just paths



Def 4.3.4: Let f be a feasible flow in a network N .

a) An f -augmenting path is an s,t -path in the underlying graph G s.t.

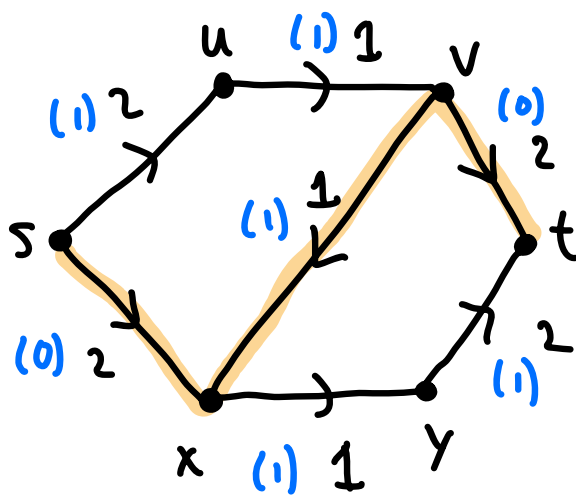
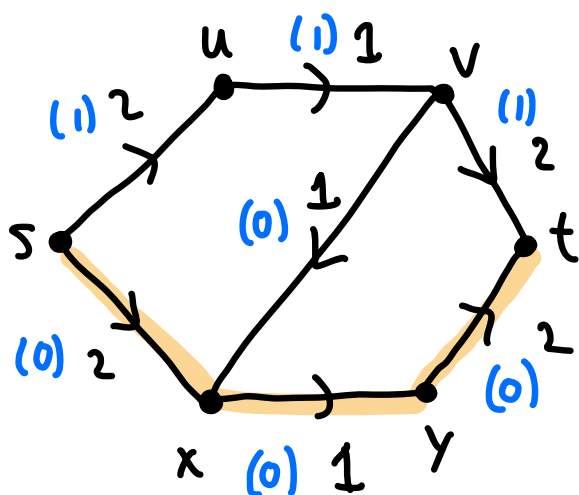
- If P follows e forwards, then $f(e) < c(e)$
- If P follows e backwards, then $f(e) > 0$

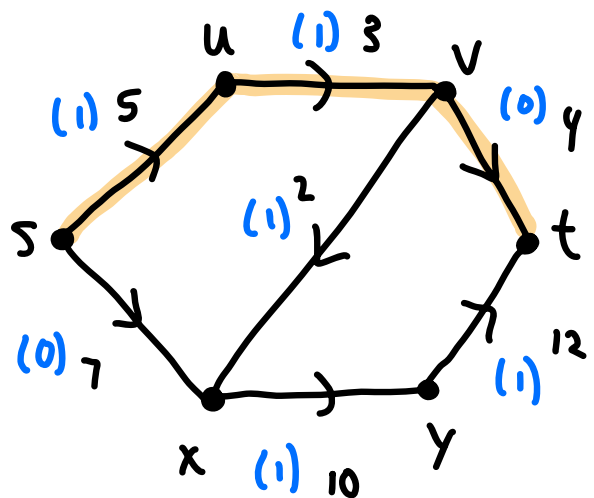
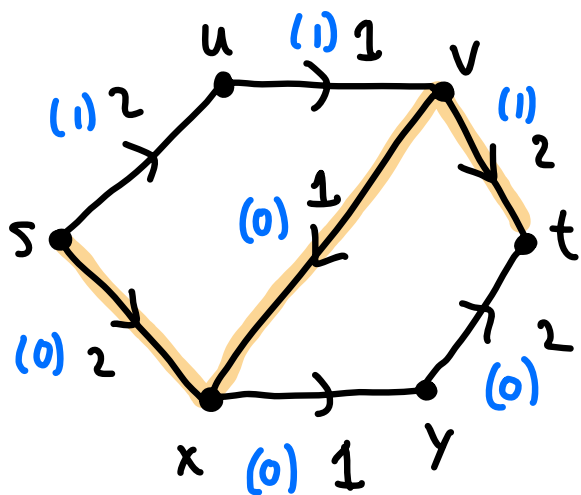
b) If $e \in E(N)$, then

$$\varepsilon(e) = \begin{cases} c(e) - f(e), & \text{if } P \text{ follows } e \text{ forwards} \\ f(e) - 0, & \text{if } P \text{ follows } e \text{ backwards} \end{cases}$$

c) The tolerance of P is $\min_{e \in E(P)} \varepsilon(e)$.

Class activity: find the tolerances





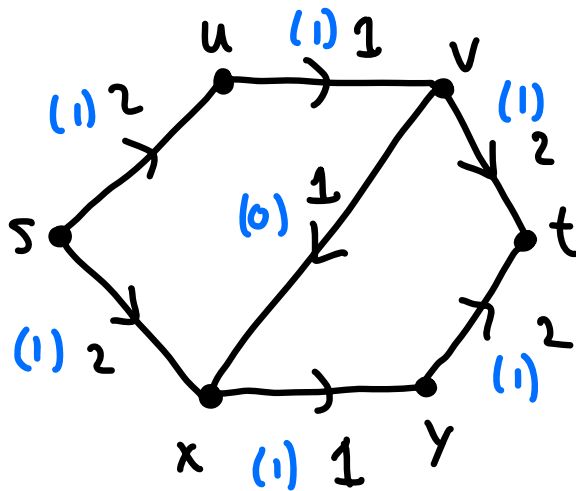
Lemma 4.3.5: If P is an f -augmenting path w/ tolerance z , then the flow

$$f'(e) = \begin{cases} f(e) + z, & \text{if } P \text{ follows } e \text{ forwards} \\ f(e) - z, & \text{if } P \text{ follows } e \text{ backwards} \\ f(e), & \text{if } e \notin P \end{cases}$$

is feasible with value $\text{val}(f') = \text{val}(f) + z$

Pf:

Now, how can we show a flow is maximum?



There's a "bottleneck" from the left half to the right half.

Def 4.3.6:

a) A source/sink cut $[S, T]$ is an edge cut where $s \in S$, $t \in T$, and $\overline{S} = T$.
all edges from S to T

b) The capacity of $[S, T]$ is

$$\text{cap}(S, T) := \sum_{e \in [S, T]} c(e)$$

c) If $U \subseteq V(N)$, let

$$f^+(U) = \sum_{\substack{\bullet \rightarrow \bullet \\ \uparrow U}} f(e) = \sum_{v \in U} f^+(v)$$

$$f^-(U) = \sum_{\substack{\bullet \rightarrow \bullet \\ \downarrow U}} f(e) = \sum_{v \in U} f^-(v)$$

The net flow out of U is $f^+(U) - f^-(U) = \sum_{v \in U} [f^+(v) - f^-(v)]$

Note: if f is feasible, by the conservation constraint,

$$f^+(v) - f^-(v) = \begin{cases} 0, & \text{if } v \neq s, v \neq t \\ \text{val}(f), & \text{if } v = s \\ -\text{val}(f), & \text{if } v = t \end{cases}$$

So for a source/sink cut $[S, T]$,

$$\text{net flow out of } S = -(\text{net flow out of } T) = \text{val}(f)$$

Cor 4.3-8: If f is feasible, and $[S, T]$ is source/sink cut,
then $\text{val}(f) \leq \text{cap}(S, T)$.

Pf: