Math 418, Spring 2025 – Homework 5

Due: Wednesday, March 5th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition.* All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 1 bonus point per assignment.

1. **Dummit and Foote #13.4.1:** *Determine the splitting field and its degree over $\mathbb{Q}$ for $f(x) = x^4 - 2$.*

   Solution. Let K be the desired splitting field. As usual, let $\sqrt[4]{2}$ be the positive real fourth root of 2. Then, using polar coordinates, the roots for $f(x)$ are $e^{2\pi i a/4} \cdot \sqrt[4]{2}, 0 \leq a < 4$ i.e. $\pm\sqrt[4]{2}, \pm i\sqrt[4]{2}$. This means that $i = \frac{i\sqrt[4]{2}}{\sqrt[4]{2}} \in K$, and conversely, $i\sqrt[4]{2} \in \mathbb{Q}(\sqrt[4]{2}, i)$. Therefore, $K = \mathbb{Q}(\sqrt[4]{2}, i)$.

   Using the tower law,
   $$[K : \mathbb{Q}] = [K : \mathbb{Q}(\sqrt[4]{2})][\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}].$$
   The latter factor is 4 since $x^4 - 2$ is irreducible. The former factor is ≤ 2 since the minimal polynomial for i over $\mathbb{Q}$ is $x^2 + 1$, so the minimal polynomial for i over $\mathbb{Q}(\sqrt[4]{2})$ must divide this. However, $i \notin \mathbb{Q}(\sqrt[4]{2})$ since $\mathbb{Q}(\sqrt[4]{2}) \subseteq \mathbb{R}$, so the degree must be 2. Therefore, $[K : \mathbb{Q}] = 8$.

2. **Dummit and Foote #13.4.2:** *Determine the splitting field and its degree over $\mathbb{Q}$ for $f(x) = x^4 + 2$.*

   Solution. Let K be the desired splitting field. Surprisingly, $K = \mathbb{Q}(\sqrt[4]{2}, i)$, so the answer is the same as the previous problem.

   Let ζ be a primitive 8th root of unity (we could make the process go faster by using coordinates, but I want to emphasize that knowledge of the roots isn't always necessary). Then $\zeta\sqrt[8]{4} = \zeta\sqrt[4]{2}$ is a root of $x^8 - 4 = (x^4 + 2)(x^4 - 2)$, but $\zeta^4 = -1$ (since by primitivity it can't equal 1), so $\zeta\sqrt[4]{2}$ must be a root of $x^4 + 2$. Thus, the four roots of this polynomial are $\zeta^a\sqrt[4]{2}$, $a = 1, 3, 5, 7$ since ζ^a, $a = 1, 3, 5, 7$ are the primitive 8th roots of unity.

   To show that $K = \mathbb{Q}(\sqrt[4]{2}, i)$, note that $\frac{\zeta^3\sqrt[4]{2}}{\zeta\sqrt[4]{2}} = \zeta^2 = \pm i$, so $i \in K$, and $\sqrt{2} = \pm i \cdot (\zeta\sqrt[4]{2})^2 \in K$ as well. Also, $(\zeta + \zeta^7)^4 = \zeta^4 + 4\zeta^{10} + 6\zeta^{16} + 4\zeta^{22} + \zeta^{28} = 4$, so
   $$\frac{1}{\sqrt{2}}(\zeta\sqrt[4]{2} + \zeta^7\sqrt[4]{2}) = \frac{\zeta}{\sqrt[4]{2}} + \frac{\zeta^7}{\sqrt[4]{2}}$$

is a 4th root of 2. After multiplication by a power of ζ^2, we see that $\sqrt[4]{2} \in K$.

On the other hand, one can check directly that $\sqrt[4]{2} + i\sqrt[4]{2}$ is a primitive 8th root of unity, and by multiplying by powers of i, $\mathbb{Q}(\sqrt[4]{2})$ contains all four primitive 8th roots of unity, so $K = \mathbb{Q}(\sqrt[4]{2}, i)$.

3. **Dummit and Foote #13.4.3:** *Determine the splitting field and its degree over $\mathbb{Q}$ for $f(x) = x^4 + x^2 + 1$.*

   Solution. Let $g(x) = x^2 + x + 1$. Then $f(x) = g(x^2)$. Since $g(x) = \frac{x^3-1}{x-1}$, its roots are the nonreal cube roots of unity i.e. $e^{\pm 2\pi i/3}$, so the roots of f are sixth roots of unity that square to these i.e. the roots of f are $e^{\pm 2\pi i/3}$ and $e^{\pm 2\pi i/6}$. Noting that some of these roots are primitive, the splitting field for f is the cyclotomic extension $\mathbb{Q}(\zeta_6) = \mathbb{Q}(e^{2\pi i/6})$. The minimal polynomial for $e^{2\pi i/6}$ is the cyclotomic polynomial $\Phi_6(x) = x^2 - x + 1$, so the extension is degree 2. (See Dummit and Foote for this polynomial, or use Euler's formula to compute $e^{2\pi i/6} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$, $(e^{2\pi i/6})^2 = e^{2\pi i/3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$).

4. **Dummit and Foote #13.4.6:** *Let K_1 and K_2 be finite extensions of F contained in the field K, and assume both are splitting fields over F.*

   a. *Prove that their composite K_1K_2 is a splitting field over F.*

      Solution. If K_1 is a splitting field (over F) for f_1 and K_2 is a splitting field for f_2, then the splitting field E for f_1f_2 is the intersection of all fields containing F and all the roots of both f_1 and f_2. E must contain K_1 since it is the intersection of all fields containing F and the roots of f_1, and similarly for K_2. On the other hand, f_1f_2 splits over any field containing both K_1 and K_2, so E is the smallest field containing K_1 and K_2, which by definition is the composite K_1K_2.

   b. *Prove that $K_1 \cap K_2$ is a splitting field over F.*

      Solution. If you look at this problem in Dummit & Foote, there is a hint, using a previous exercise. Let $g(x)$ be an irreducible polynomial in $F[x]$ with a root in $K_1 \cap K_2$. However, I failed to copy this hint over with the problem. Kudos to you if you looked at Dummit & Foote and saw the hint, or found another approach.

      We will show that $g(x)$ splits over $K_1 \cap K_2$, so by Dummit & Foote, Problem 13.4.5, $K_1 \cap K_2$ is a splitting field. Since K_1 and K_2 are splitting fields containing a root of $g(x)$, it must be the case that $g(x)$ splits in both K_1 and K_2. But since $K[x]$ is a UFD, these factorizations must be identical (up to units and order), so every factor must be contained in $(K_1 \cap K_2)[x]$, so g splits over $K_1 \cap K_2$.