Math 412, Fall 2023 – Homework 3

Due: Wednesday, September 13th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should choose four of the following five problems to solve and turn in–if you do all five, only the first four will be graded. Graduate students taking the course for four credits should solve all five. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. Let G be an n-vertex simple graph, with $n \geq 2$. Determine the maximum possible number of edges in G for each of the following conditions:

   (a) G has an independent set of size a.
   Solution: The maximum possible size of $E(G)$ is $\binom{n}{2} - \binom{a}{2}$.

   Proof. We start with K_n, which has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges, and remove as few edges as possible until we obtain the desired graph. In this case, for any set S of a vertices, there are $\binom{a}{2}$ edges with both endpoints in S; in order for S to be an independent set, we must remove these edges from K_n. Therefore, G must have at most $\binom{n}{2} - \binom{a}{2}$ edges. $\square$

   (b) G has exactly k connected components.
   Solution: The maximum possible size of $E(G)$ is $\binom{n-k+1}{2}$.

   Proof. Suppose G has k connected components, $H_1, \ldots, H_k$, with orders $n_1, \ldots, n_k$, and without loss of generality, assume $n_1 \geq n_2 \geq \cdots \geq n_k$.

   Suppose G has maximum possible size of $E(G)$ (given n vertices and k connected components). Then each H_j is a complete graph; otherwise we could add another edge to H_j, contradicting maximality. This means that $|E(G)| = \sum_{j=1}^{k} |E(H_j)| = \sum_{j=1}^{k} \binom{n_j}{2}$.

   If $n_2 \geq 2$, let G' be the graph obtained from G by moving one vertex v from H_2 to H_1. We lose $n_2 - 1$ edges since in G, v was adjacent to every other vertex in H_2, but we gain n_1 edges since in G', v is now adjacent to every vertex that was originally in H_1. G' still has n vertices and k connected components, but since $n_1 \geq n_2$, G' has more edges than G, contradicting the maximality of G.

   Therefore, we have $n_2 = n_3 = \cdots = n_k = 1$, so G is a complete graph on $n-k+1$ vertices, plus $k-1$ isolated vertices. $|E(G)|$ is therefore $\binom{n-k+1}{2}$. $\square$

(c) G is disconnected.

Solution: The maximum possible size of $E(G)$ is $\binom{n-1}{2}$.

Proof. This follows from the previous problem. For a given k, the maximum number of edges of G is $\binom{n-k+1}{2}$. If G is disconnected, $k \geq 2$, and the largest value of $\binom{n-k+1}{2}$ for any value $k \geq 2$ is $\binom{n-1}{2}$, when $k = 2$. □

2. Determine for which values of n there exists an n-vertex 5-regular simple connected graph.

Solution: There exists such a graph G precisely when $n \geq 6$ and n is even.

Proof. If $n < 6$, then since G is simple, no vertex of G can have degree 5. If n is odd, then the degree sum of G is $5n$, which is odd, contradicting the degree-sum formula.

Now, if $n \geq 6$ is even, let $V(G) = \{v_1, \ldots, v_n\}$. Consider these values modulo n, so for instance $v_1 = v_{n+1}$, and so on. Since n is even, it is well-defined to consider the parity (even vs. odd) of integers modulo n.

Draw edges as follows: if k is even, v_k is adjacent to $v_{k-2}, v_{k-1}, v_{k+1}, v_{k+2}$, and v_{k+3}. If k is odd, v_k is adjacent to $v_{k-3}, v_{k-2}, v_{k-1}, v_{k+1}$, and v_{k+2}. Since $n \geq 6$, these five vertices are all distinct, and none of them equals v_k.

All that remains is to show that G is well-defined; that is, that the procedure above designates v_k to be adjacent to v_j precisely when it designates v_j to be adjacent to v_k. If j and k are at most 2 units apart (but not equal), then regardless of parity, v_j and v_k are adjacent, while if j and k are more than 3 units apart, they are not adjacent. If they are 3 units apart, say without loss of generality that $k = j + 3$, then if k is odd it is adjacent to $v_{k-3} = v_j$. In this case, j is even, and it is adjacent to $v_{k+3} = v_j$, which is consistent. On the other hand, if k is even it is not adjacent to $v_{k-3} = v_j$, and in this case, j is odd, and it is not adjacent to $v_{k+3} = v_j$, which is again consistent. Therefore, G is a simple 5-regular graph of order n. □

3. Consider the n-dimensional hypercube Q_n. Let C be a cycle of length $2r$ in Q_n for some $r \leq n$. Prove that C is contained in an r-dimensional hypercube $Q_r \subseteq Q_n$.

Proof. The vertices $V(C)$ are labelled by the set of binary strings of length k. Let S be the set of coordinates that change at some step while traversing the strings in $V(C)$. In order to return to the first string, each position must flip between 0 and 1 an even number of times. Thus traversing C changes each coordinate in S at least twice, but only one coordinate changes with each edge. Hence $2|S| \leq 2r$, or $|S| \leq r$. Outside the coordinates of S, the strings of $V(C)$ all agree. Hence $V(C)$ is contained in some $|S|$-dimensional subcube. □

4. Using the type of argument that we used to prove Mantel's Theorem in Friday's lecture (see posted lecture notes), prove that for $n \geq 1$ the only n-vertex triangle free simple graph with the maximum possible number of edges is $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

   Proof. We follow the proof technique for Mantel's Theorem from class. Let G be an n-vertex triangle-free simple graph. Let $x \in V(G)$ with $d(x) = \Delta(G) =: k$. Since G is triangle-free, $N(x)$ is an independent set, so every edge has at least one endpoint not in $N(x)$. Therefore, we have

   $$e(G) \leq \sum_{v \notin N(x)} d(v) \leq k(n-k) \leq \left\lfloor \frac{n^2}{4} \right\rfloor,$$

   and the only way for G to have the maximum number of $\lfloor \frac{n^2}{4} \rfloor$ edges is if all these inequalities are equalities. Starting from the left, $e(G) = \sum_{v \notin N(x)} d(v)$ implies that every edge has precisely one endpoint not in $N(x)$, so $V(G) \setminus N(x)$ is also an independent set, and so G is bipartite. $\sum_{v \notin N(x)} d(v) = k(n-k)$ implies that every vertex not in $N(x)$ has degree k, which is precisely the size of $N(x)$; thus G is complete bipartite. Finally, $k(n-k) \leq \lfloor \frac{n^2}{4} \rfloor$ implies that $k(n-k) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, and this can only hold if $k = \lfloor \frac{n}{2} \rfloor$ and $n - k = \lceil \frac{n}{2} \rceil$ or vice-versa, and in either case, G is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. □

5. Using Problem 4, determine the minimum number of edges for any n-vertex connected graph with no independent set of size 3 or larger.

   Solution: We claim that this minimum is $h(n) := \binom{\lceil n/2 \rceil}{2} + \binom{\lfloor n/2 \rfloor}{2} + 1 = \binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor + 1$.

   Proof. To prove the upper bound, consider the graph H_n obtained from disjoint copies of $K_{\lceil n/2 \rceil}$ and $K_{\lfloor n/2 \rfloor}$ by connecting them with one edge. Then H_n is connected, has largest independent set size 2, and has exactly $h(n)$ edges.

   Conversely, let G be a simple connected graph with no independent set of size 3 or larger, consider the complement, $\overline{G}$, of G. Then $\overline{G}$ is triangle-free since a triangle in $\overline{G}$ corresponds to a size-3 independent set in G. By Problem 4, either $\overline{G} = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ or $\overline{G}$ has at most $\left\lfloor \frac{n^2}{4} \right\rfloor - 1$ edges. Since the complement of $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is not connected, G can't be that graph, and so

   $$|E(G)| = \binom{n}{2} - |E(\overline{G})| \geq \binom{n}{2} - \left(\left\lfloor \frac{n^2}{4} \right\rfloor - 1 \right) = h(n),$$

   as claimed. □