Math 418, Spring 2024 – Practice Problems 2

13.2.6 *Prove directly from the definitions that the field* $F(a_1, \dots, a_n)$ *is the composite of the fields* $F(a_1), F(a_2), \dots, F(a_n)$.

13.3.1 *Prove that it is impossible to construct the regular 9-gon.*

13.4.4 *Determine the splitting field and its degree over* $\mathbb{Q}$ *for* $f(x) = x^6 - 4$.

13.5.2 *Find all irreducible polynomials of degrees 1, 2 and 4 over* $\mathbb{F}_2$ *and prove that their product is* $x^{16} - x$.

13.5.4 *Let* $a > 1$ *be an integer. Prove for any positive integers* n, d *that* d *divides* n *if and only if* $a^d - 1$ *divides* $a^n - 1$. *Conclude in particular that* $\mathbb{F}_{p^d} \subseteq \mathbb{F}_{p^n}$ *if and only if* d *divides* n.

13.6.6 *Prove that for* n *odd,* $n > 1$ *that* $\Phi_{2n}(x) = \Phi_n(-x)$

13.6.10 *Let* ϕ *denote the Frobenius map* $\mathbb{F}_{p^n}$ $\cdot$ *Prove that* ϕ *gives an automorphism of order* n

14.1.1 (a) *Show that if the field* K *is generated over* F *by the elements* $a_1, ..., a_n$ *then an automorphism* a *of* K *fixing* F *is uniquely determined by* $\sigma(a_1), \dots, \sigma(a_n)$. *In particular, show that an automorphism fixes* K *if and only if it fixes a set of generators for* K.

(b) *Let* $G \leq Gal(K/F)$ *be a subgroup of the Galois group of the extension* K/F *and suppose* $\sigma_1, \dots, \sigma_k$ *are generators for* G. *Show that the subfield* E *of* K *containing* F *is fixed by* G *if and only if it is fixed by the generators* $\sigma_1, \dots, \sigma_k$.

14.1.9 *Determine the fixed field of the automorphism* $\phi : t \mapsto t + 1$ *of* $k(t)$

14.1.10 *Let* K *be an extension of the field* F. *Let* $\phi : K \to K'$ *be an isomorphism of* K *with a field* K' *which maps* F *to the subfield* F' *of* K'. *Prove that the map* $\sigma \mapsto \phi\sigma\phi^{-1}$ *defines a group isomorphism* $Aut(K/F) \to Aut(K'/F)$.