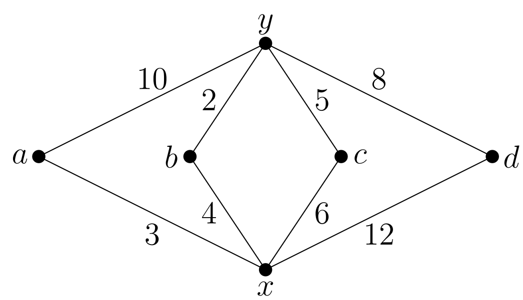
Math 412, Fall 2023 – Homework 6

Due: Wednesday, October 11th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should choose four of the following five problems to solve and turn in–if you do all five, only the first four will be graded. Graduate students taking the course for four credits should solve all five. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. Consider the following graph.



(a) Use Kruskal's algorithm to construct a minimal spanning tree, and find its weight. Show your work step by step: which edge is considered at each step of the algorithm, is it accepted or rejected, and why?

Solution.
Step 1: consider yb, weight 2; add it to T
Step 2: consider xa, weight 3; add it to T
Step 3: consider xb, weight 4; add it to T
Step 4: consider yc, weight 5; add it to T
Step 5: consider xc, weight 2; do not add it to T since this would create a cycle
Step 6: consider yd, weight 8; add it to T
Algorithm terminates since T is now connected (fine to consider the last two edges).

The resulting tree T has edges yb, xa, xb, yc, and yd, for a total weight of 22.

(b) Use Dijkstra's algorithm to find the distance from y to every vertex. Again, show your work step by step, including the set S and the values of the function t at each step.

Solution.
Start: S $= \{y\}, t(y) = 0, t(a) = 10, t(b) = 2, t(c) = 5, t(d) = 8, t(x) = \infty$
Step 1: S $= \{y, b\}, t(y) = 0, t(a) = 10, t(b) = 2, t(c) = 5, t(d) = 8, t(x) = 6$
Step 2: S $= \{y, b, c\}, t(y) = 0, t(a) = 10, t(b) = 2, t(c) = 5, t(d) = 8, t(x) = 6$
Step 3: S $= \{y, b, c, x\}, t(y) = 0, t(a) = 9, t(b) = 2, t(c) = 5, t(d) = 8, t(x) = 6$
Step 4: S $= \{y, b, c, x, d\}, t(y) = 0, t(a) = 9, t(b) = 2, t(c) = 5, t(d) = 8, t(x) = 6$
Step 5: S $= \{y, b, c, x, d, a\}, t(y) = 0, t(a) = 9, t(b) = 2, t(c) = 5, t(d) = 8, t(x) = 6$

Therefore, $d(y, y) = 0, d(y, a) = 9, d(y, b) = 2, d(y, c) = 5, d(y, d) = 8, d(y, x) = 6$.

2. For any spanning tree T in a weighted graph G, let

$$m(T) = \max_{e \in E(T)} \mathrm{wt}(e).$$

Further, let

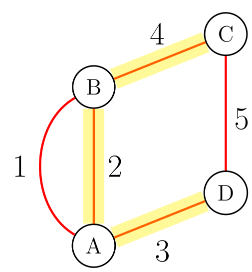
$$x(G) = \min_{T} m(T),$$

where the minimum is over all spanning trees of G.

(a) If T is a minimal spanning tree of G, prove that $m(T) = x(G)$.

Proof. Let T be a minimal spanning tree of G and T^* is any spanning tree of G with $m(T^*) = x(G)$, then for any edge $e \in E(T)$, by Proposition 2.1.7, there exists an edge $e' \in E(T^*)$ such that $T' := (T \setminus e) \cup e'$ is a spanning tree of G. We have $\mathrm{wt}(T') = \mathrm{wt}(T) + \mathrm{wt}(e') - \mathrm{wt}(e)$, so since T is minimal, we must have $\mathrm{wt}(e) \le \mathrm{wt}(e') \le m(T^*) = x(G)$. $\square$

(b) Give an example of a graph G and a *non-minimal* spanning tree T such that $m(T) = x(G)$.
Solution.



The highlighted spanning tree is not minimal, since it using the other edge with endpoints A and B would result in a lower weight. However, every spanning tree of this graph must contain either BC or CD, so the maximal edge here is minimal.

3. Let G be an X, Y-bigraph, i.e. a bipartite graph whose partite sets are X and Y. If $|X| = |Y|$, prove that there exists a subset $S \subseteq X$ with $|N(S)| < |S|$ if and only if there exists a subset $T \subseteq Y$ with $|N(T)| < |T|$.

Proof. This is a simple consequence of Hall's Theorem. Since G is bipartite, every edge has one endpoint in X and the other in Y, so a matching of G saturates X if and only if it saturates Y. Applying Hall's Theorem to both sides of this, $|N(S)| \geq |X|$ for all $S \subseteq X$ if and only if $|N(T)| \geq |Y|$ for all $T \subseteq Y$, and the contrapositive of this is the statement to prove. □

4. Let G be a graph. Prove that the number of edges in every maximal matching in G is at least half the number of edges of a maximum matching of G.

   Proof. Let L be a maximal matching of G and let M be a maximum matching. Assume for a contradiction that $|E(L)| < |E(M)|/2$. Since L is maximal, no edge of M that is not already in L can be added to L such that the resulting graph is a matching. This means that every edge e of M has at least one endpoint that is saturated in L, since otherwise $L \cup e$ would still be a matching. However, there are $|E(M)|$ edges in M, but only $2|E(L)|$ vertices of saturated in L, and since by assumption $2|E(L)| < |E(M)|$, there must be an edge of M for which neither endpoint is saturated in L, a contradiction. □

5. Let D be a digraph. Prove that there exist pairwise disjoint cycles in D such that each vertex of D lies in exactly one of the cycles if and only if

$$|N^+(S)| \geq |S| \text{ for all } S \subseteq V(D).$$

   Proof. **Only if:** If $|N^+(S)| < |S|$ for some $S \subseteq V(D)$, then there are not enough distinct vertices in $N^+(S)$ such that every vertex in S has an edge coming into it from a distinct vertex.

   If: We construct a bipartite graph G from D as follows: replace each vertex v of D with two vertices, v^+ and v^-, and replace each edge $xy \in E(D)$ (tail x and head y) with edge x^+y^- in G. The graph G is bipartite with partite sets $X := \{v^+ : v \in V(D)\}$ and $Y := \{v^- : v \in V(D)\}$. The condition $N^+(S)| \geq |S|$ for all $S \subseteq V(D)$ yields that

$$|N_G(S)| \geq |S| \text{ for all } S \subseteq X.$$

   So by Hall's Theorem, G has a matching M saturating X. Since $|X| = |Y|$, M is a perfect matching. Then the edges of D corresponding to the edges of M, by definition, form a set M' such that for each $v \in V(D)$ exactly one edge of M' leaves v and exactly one edge of M' enters v. This is exactly what we need. □