Announcements:

- H/W 1 posted (due 9am Wed. 8/30 via Gradescope)
- Midterm etc. times posted to course website

Last time: Def'n of graph, chromatic #, path/cycle, etc.

Today: Isomorphism classes, special graphs

Adjacency Matrix

Let G be a loopless graph

Write $V(G) = \{v_1, \dots, v_n\}$

$E(G) = \{e_1, \dots, e_m\}$

Def 1.1.17

a) $v \in V(G)$ and $e \in E(G)$ are <u>incident</u> if v is an endpoint of e

b) The <u>degree</u> of $v \in V(G)$ is the number of edges incident to v

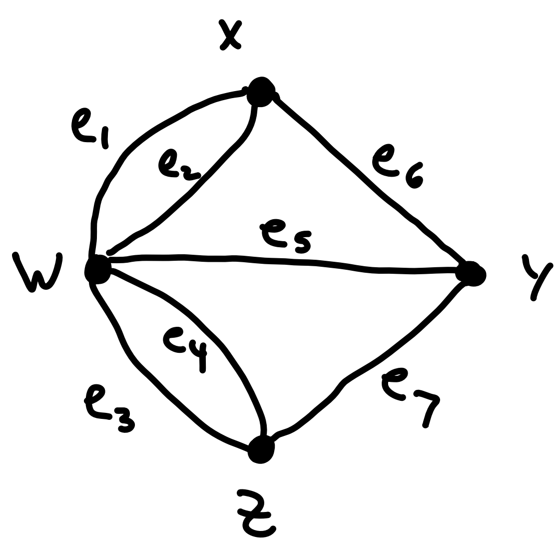
c) The <u>adjacency matrix</u> $A(G)$ is the $n \times n$ matrix where

$$a_{ij} = \text{number of edges w/ endpoints } v_i \text{ and } v_j$$

d) The incidence matrix $M(G)$ is the $n \times m$ matrix where

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is an endpoint of } e_j \\ 0 & \text{otherwise} \end{cases}$$

G:



$$A(G) = \begin{array}{c|c} & \begin{matrix} w & x & y & z \end{matrix} \\ \hline \begin{matrix} w \\ x \\ y \\ z \end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

$$M(G) = \begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{matrix} \\ \hline \begin{matrix} w \\ x \\ y \\ z \end{matrix} & \begin{bmatrix} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{bmatrix} \end{array}$$

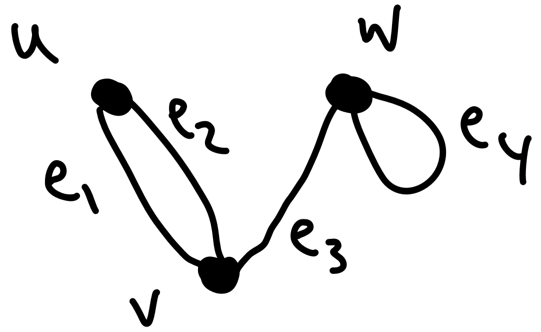
Def 1.1.20: An <u>isomorphism</u> from a graph G to a graph H consists of bijections

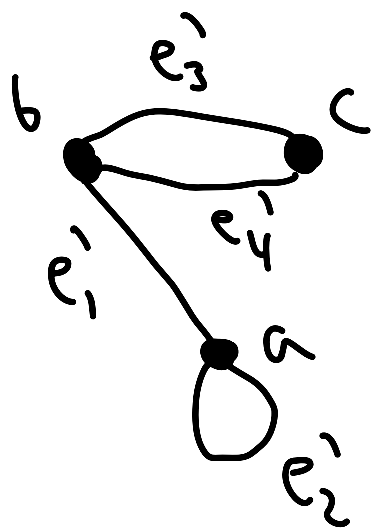
$$f: V(G) \to V(H)$$

$$g: E(G) \to E(H)$$

such that if $e \in E(G)$ has endpoints u and v, $g(e) \in E(H)$ has endpoints $f(u)$ and $f(v)$. We write $G \cong H$.

Example:





$f(u) =$

$f(v) =$

$f(w) =$

$g(e_1) =$

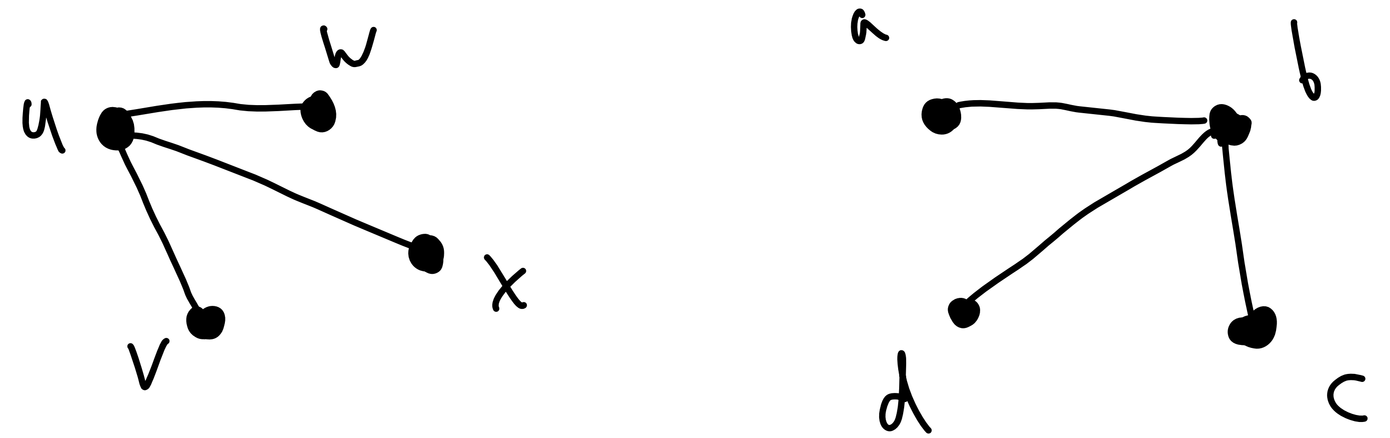
$g(e_2) =$

$g(e_3) =$

$g(e_4) =$

When we have a simple graph, the map g is implied

Ex:



$f(u) = b$

$f(v) = a$

$f(w) = c$

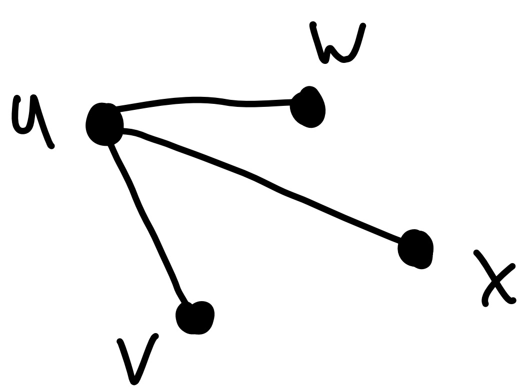
$f(x) = d$

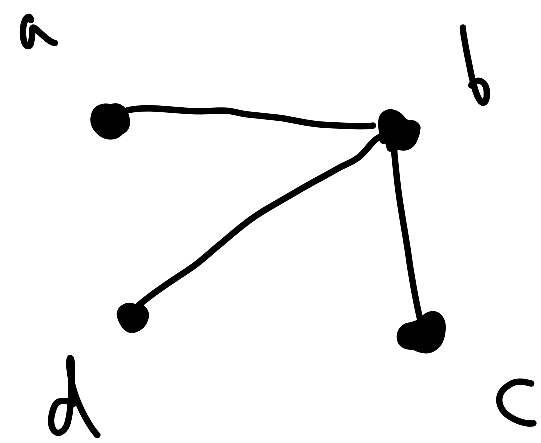
so $g(uv) = g(u)\,g(v) = ba$

etc.

Remark: $G \cong H$ if and only if there exists a permutation σ such that applying σ to both the rows and columns of $A(G)$ gives $A(H)$

Ex (cont.)





$$\begin{array}{c|cccc} & u & v & w & x \\ \hline u & 0 & 1 & 1 & 1 \\ v & 1 & 0 & 0 & 0 \\ w & 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 1 & 0 & 1 & 1 \\ c & 0 & 1 & 0 & 0 \\ d & 0 & 1 & 0 & 0 \end{array}$$

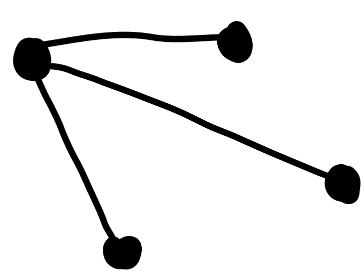
Swap u & v rows and columns:

$$\begin{array}{c|cccc} & v & u & w & x \\ \hline v & & & & \\ u & & \smash{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} & & \\ w & & & & \\ x & & & & \end{array}$$

Prop 1.1.24: Isomorphism is an equivalence rel'n on (simple) graphs.

Pf (in simple case): see textbook

Def: An <u>unlabelled graph</u> is an isomorphism class of graphs



Special (unlabelled, simple) graphs:

P_n: path on n vertices

C_n: cycle on n vertices

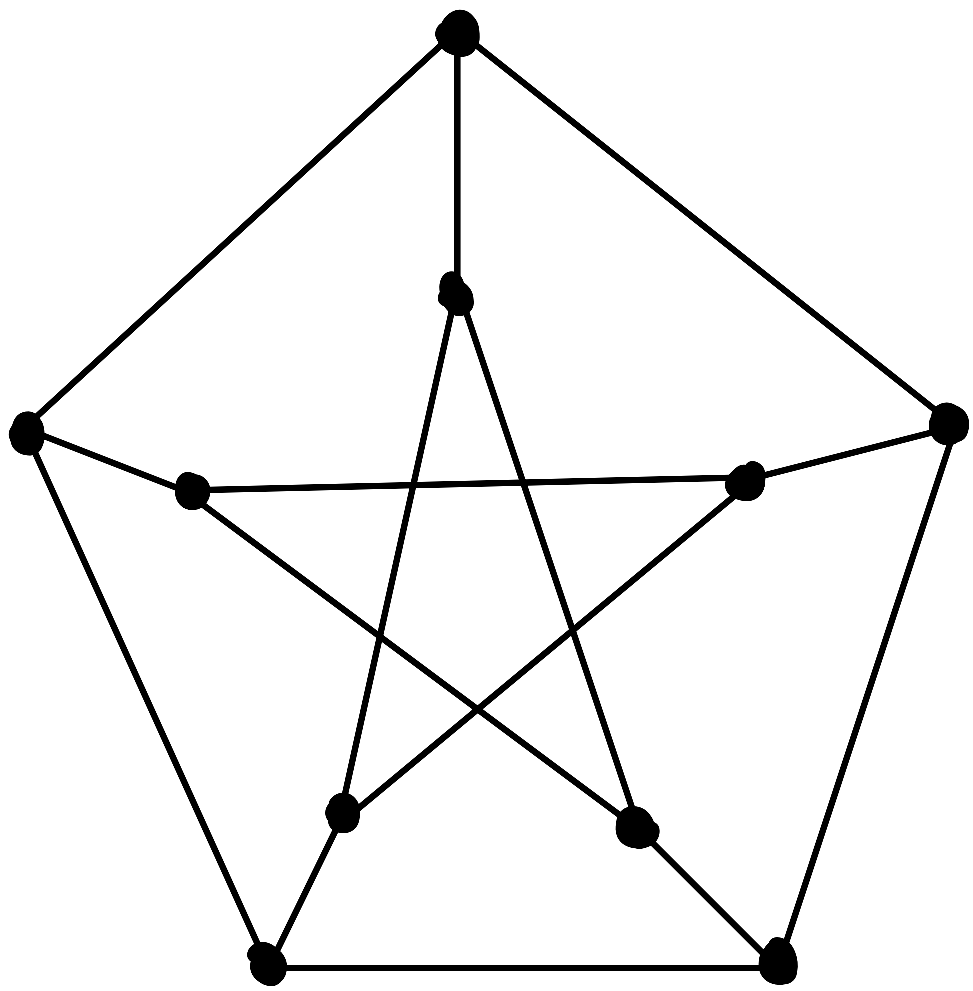
K_n: complete graph on n vertices
(every vertex is adjacent to every other vertex)

$K_{r,s}$: complete bipartite graph with parts of size r and s $(= K_{s,r})$

(all vertices in opposite parts are adjacent)

Note: $K_{r,s}$ is <u>not</u> a complete graph

Petersen graph:



Idea for thought:

How can we describe this graph using subsets of a 5-element set?

(Book has the answer)

Next week: Königsberg bridge problem