Math and Proofs Class 5

October 24th, 2017

Recap of Last Class

- We looked at equivalence relations, functions, and bijections.
- At the very end, we started to talk about cardinality.
- This class: more about cardinality

Recap: Functions and Bijections

Let A and B be sets

- A *function* from A to B is a set of ordered pairs where the first element in each pair is in A and the second is in B AND each input element appears exactly once.
- A *bijection* is a function where each output element appears exactly once too.

Cardinality

- Two sets A and B are equivalent if there's a bijection between them.
- Remember what this means: A and B are equivalent if they have the same **number** of elements

Application of Cardinality for Finite Sets: Pigeonhole Principle

Pigeonhole Principle: If you put n pigeons in m pigeonholes and $n > m$, then there must be a hole with more than one pigeon. Examples:

- Prove that in any room with at least 8 people, at least two of them were born on the same day of the week.
- Minneapolis has 413,000 people. Humans have no more than 300,000 hairs of their heads. Prove that there are (at least) two people in Minneapolis with the same number of hairs on their heads.
- Suppose 5 points are chosen in (or on) the equilateral triangle of side length 1 inch. Prove that there are two points in the triangle that are no farther than $\frac{1}{2}$ inch apart.
- Video: https://www.youtube.com/watch?v=ROnetLvbI6M

Schroder-Bernstein Theorem

- An *injection* is a function where each output appears AT MOST once (could be zero times). We say a function is *injective* if it is an injection.
- A bijection, then, is an injection where every output is hit.
- Examples:
  1. $A = \{1, 2\}, B = \{1, 2, 3\}, f(1) = 2, f(2) = 1$ is injective
  2. $A = \{1, 2\}, B = \{1, 2, 3\}, f(1) = 2, f(2) = 2$ is NOT injective.
  3. $A = \{1, 2, 3\}, B = \{1, 2\}$, if $f : A \to B$, then f cannot be injective.
- So if there is an injection from A to B, this means that $|A| \leq |B|$.
- Schroder-Bernstein: If there are injections $f : A \to B$ and $g : B \to A$, then there a bijection from A to B.

Bijections Between Infinite Sets

- $|\{\text{even integers}\}| = |\{\text{odd integers}\}|$
- $|\{\text{even integers}\}| = |\{\text{integers}\}|$
- $|\{\text{odd integers}\}| = |\{\text{integers}\}|$
- $|\{\text{integers}\}| = |\{\text{rational numbers}\}|$

Do All Infinite Sets Have the Same Cardinality?

- $|\{\text{integers}\}| < |\{\text{real numbers}\}|$
- If A is a set, then $|A| < |P(A)|$.

Next Time

- Perhaps some more about cardinality (otherwise, it'll still come up in what's to come)
- Some "controversial" axioms