Announcements

HW4 posted (due Wed. 9/27)

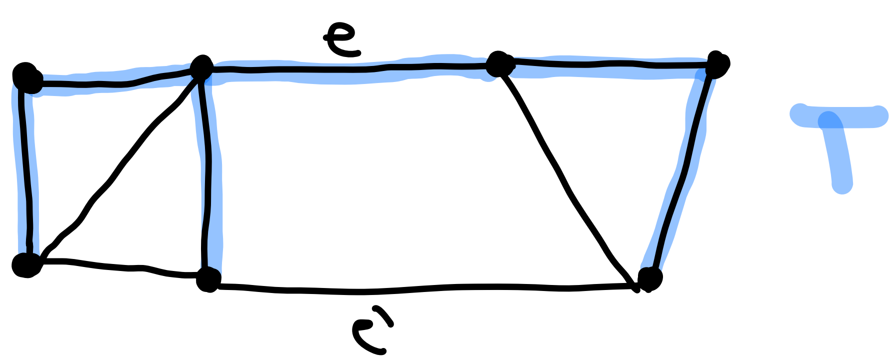
Prop (2.1.6/2.1.7): Let G be a graph w/ spanning trees T, T'.

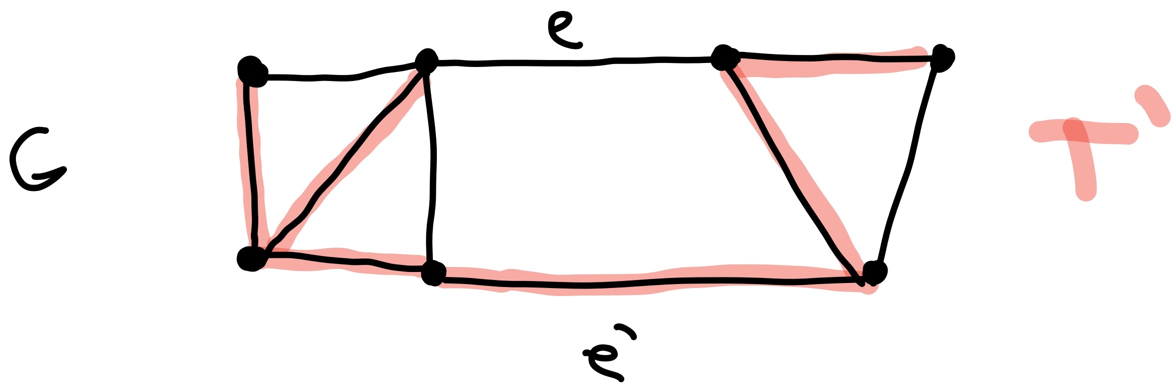
a) For all $e \in E(T)$, $\exists e' \in E(T')$ s.t. $(T \cup e') \setminus e$ is a spanning tree of G.

b) For all $e' \in E(T')$, $\exists e \in E(T)$ s.t. $(T \cup e') \setminus e$ is a spanning tree of G.

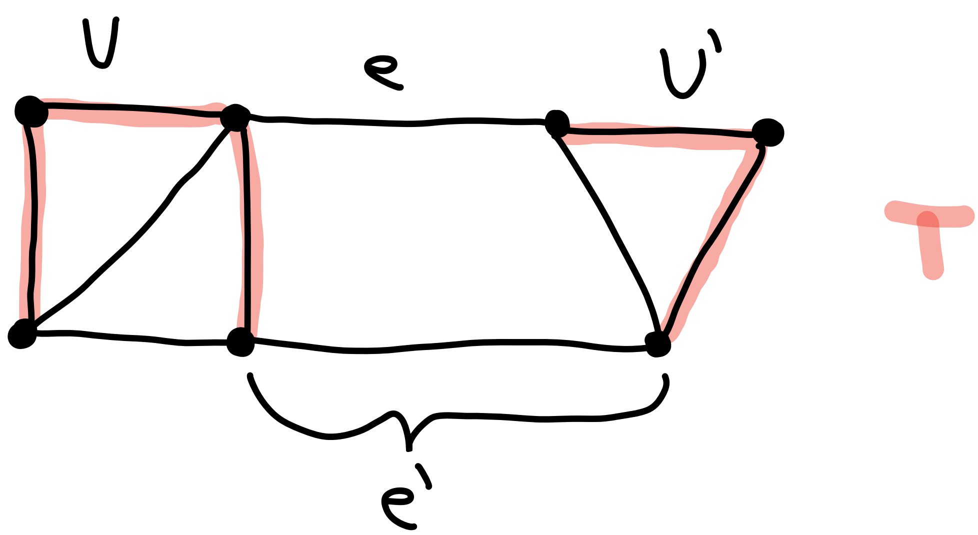
If you tell me which edge to remove, I'll tell you which edge to add
If you tell me which edge to add, I'll tell you which edge to remove

G



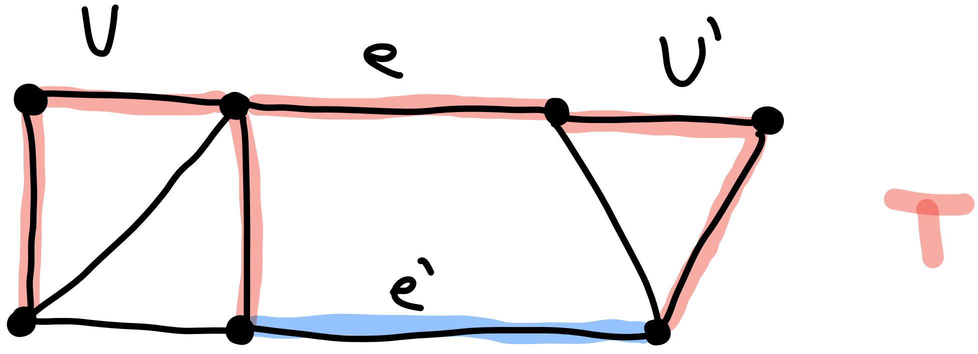


Pf: a)



b)

[Note the choice
for e]



Def 2.1.9/12:

a) The distance $d(u,v)$ from u to v is the shortest length of a u,v-path (∞ if no path)

b) The diameter $\operatorname{diam} G$ is the maximum distance btwn. any two vertices in G (∞ if disconn.)

$$\operatorname{diam} G = \max_{u,v \in V(G)} d(u,v)$$

c) The eccentricity of a vertex u is

$$\varepsilon(u) := \max_{v \in V(G)} d(u,v)$$

(i.e. $\operatorname{diam} G = \max_{u \in V(G)} \varepsilon(u)$)

d) The radius of G is

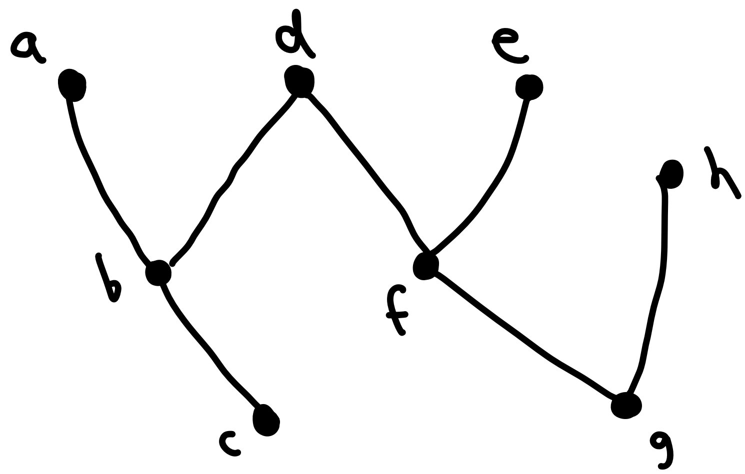
$$\operatorname{rad} G := \min_{u \in V(G)} \varepsilon(u)$$

e) The center of G is the induced subgraph

$$G\left[\{u \in V(G) \mid \varepsilon(u) = \operatorname{rad}(u)\}\right]$$

Class activity:

Find diam G, rad G, the eccentricity of each vertex, and the center



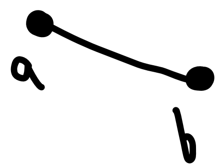
Jordan Tree Theorem: The center of a tree is $\bullet$ or $\bullet\!\!-\!\!\!-\!\!\bullet$

Pf sketch: induction on $n := n(T)$

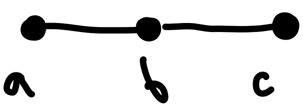
How many (labelled) n-vertex trees are there?

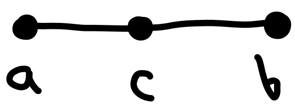
$n=1$:

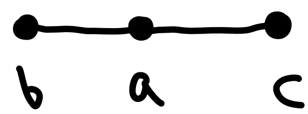
$n=2$:



$n=3$:



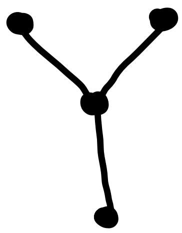




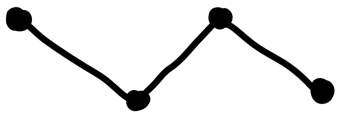
$$\begin{bmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{bmatrix} \qquad \begin{bmatrix} & & 1 \\ & & 1 \\ 1 & 1 & \end{bmatrix} \qquad \begin{bmatrix} & 1 & 1 \\ 1 & & \\ 1 & & \end{bmatrix}$$

n = 4:

Total: 16

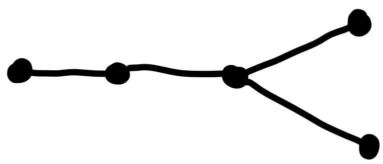


x4

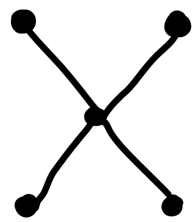


x12

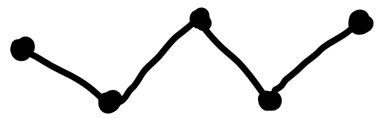
n = 5:



x60



x5



x60

Total: 125

Pattern?

Cayley's Formula (Thm 2.2.3): There are n^{n-2} labelled trees with n vertices

[For technical reasons, label set is some $S \subseteq \mathbb{N}$]

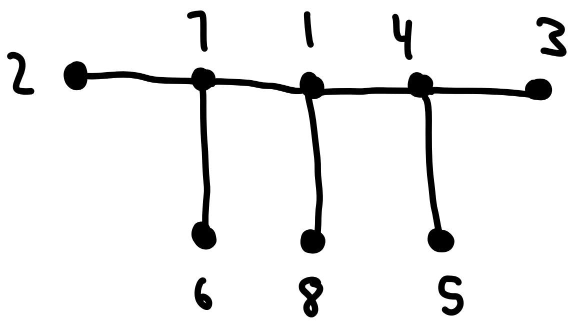
Pf idea:

Def: The <u>Prüfer Code</u> $f(T) = (a_1, \ldots, a_{n-2})$ of T is given by the following algorithm:

At step i:

- delete the leaf w/ the smallest label
- a_i is the label for the (unique) neighbor of the leaf

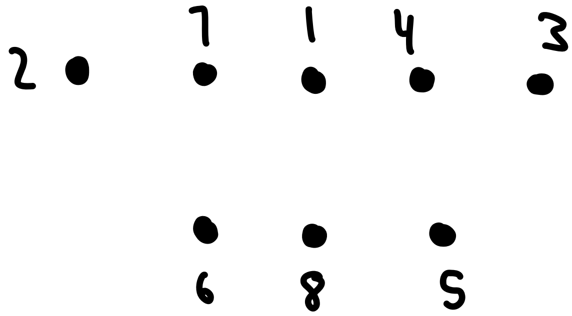
Ex:



$\mathrm{Pru}(T) = 744171$

Can go backwards:

$\text{Pru}(T) = 744171$

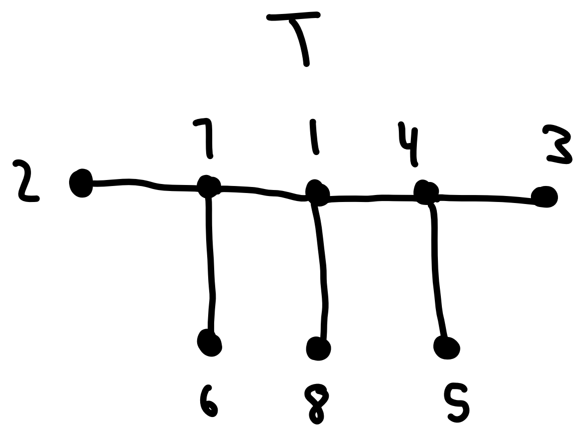


Pf of Cayley's Formula: $n=1$ good

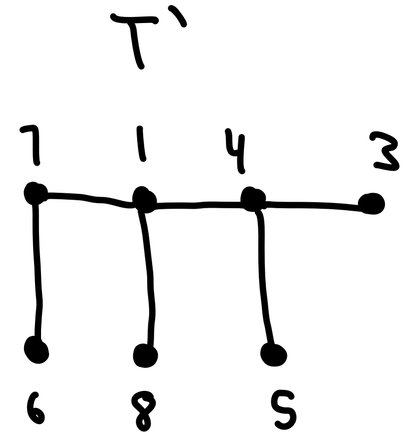
We prove that for $n \geq 2$

$$T \longleftrightarrow \text{Pru}(T)$$

is a bijection.



$\mathrm{Prü}(T) = 744171$



$\mathrm{Prü}(T') = 44171$

Cor 2.2.4: Let $d_1, \dots, d_n \in \mathbb{Z}_{\geq 1}$ s.t. $d_1 + \dots + d_n = 2n-2$. Then the number of trees w/ label set $\{1, \dots, n\}$ s.t. vertex i has degree d_i is $\dfrac{(n-2)!}{\prod (d_i - 1)!}$