Math 412, Fall 2023 – Homework 4

Due: Wednesday, September 27th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should choose four of the following five problems to solve and turn in–if you do all five, only the first four will be graded. Graduate students taking the course for four credits should solve all five. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. Let G be a digraph with vertices x and y such that $d^+(x) - d^-(x) = d^-(y) - d^+(y) = 4$, and for all other vertices v, $d^+(v) = d^-(v)$. Prove that there exist paths P_1, P_2, P_3, P_4 from x to y in G, such that none of these paths share an edge.

   Proof. By the degree-sum formula for digraphs, x and y must be in the same connected component of G; call this component D.

   Let D' be obtained from D by adding 4 edges $e_1, \ldots, e_4$ each of which has tail y and head x. Then in D' $d^+(x) = d^-(x)$ and $d^+(y) = d^-(y)$, so every vertex in D' has the same in-degree and out-degree. By the Euler Theorem for digraphs, D' has an Eulerian circuit, say C. Order the edges of C as $f_1, \ldots, f_{e(D')}$ so that $f_1 = e_1$. Suppose that $e_i = f_{j_i}$ for $i = 1, \ldots, 4$. We can rename $e_1, \ldots, e_4$ so that $f_{j_1} < f_{j_2} < \ldots < f_{j_4}$. For $i = 1, \ldots, 4$, let C_i denote the part of C from f_{j_i+1} to $f_{j_{i+1}-1}$ (considering 1 as $k+1$). Then $C_1, \ldots, C_4$ are edge-disjoint x,y-trails in D, and none of them share as edge since they are part of the same Eulerian circuit. Since each x,y-trail contains an x,y-path, we find 4 edge-disjoint x,y-paths $P_1, \ldots, P_4$ in D such that $E(P_i) \subseteq E(C_i)$ for all i. □

2. Let T be a tournament having no vertex with indegree 0. Prove that T has at least three kings. *[Hint: Prove that if x is a king in T, then T has another king in $N^-(x)$, the in-neighborhood of x.]*

   Proof. Let x be a king in T and let $T_x = T[N^-(x)]$ be the subgraph of T induced by $N^-(x)$ (since x doesn't have in-degree zero, T_x is not empty). By Proposition 1.4.30, T_x has a king, say y. We claim that y is a king in T, as well. Indeed, let z be any vertex in $T \setminus y$. If $z \in N^-(x)$ then T_x contains a y,z-path of length at most two, since y is a king in T_x. Since $yx \in E(T)$, for every $z \in \{x\} \cup N^+(x)$, T contains a y,z-path of length at most two. Since T is a tournament i.e. an orientation of a complete graph, every vertex in in one of $\{x\}, N^-(x)$, and $N^+(x)$, so there is a path of length at most 2 from z to every other vertex, so z is a king.

By Proposition 1.4.30, T has at least one king, say x. By the above, T has another king, say z in $N^-(x)$, and applying this again, T has another king, y, in $N^-(z)$. Clearly, $x \neq z$ and $z \neq y$. Furthermore, $x \neq y$ since then x would be in both the in-neighborhood and out-neighborhood of z, which is not possible for a tournament, so T has at least three kings. $\square$

3. Prove that for each simple graph G the following are equivalent.

   (a) G is a forest (i.e. a simple graph with no cycles).

   (b) Every induced subgraph of G has a vertex of degree at most 1.

   (c) The intersection of any two intersecting paths in G is either empty or a path.

   (d) The number of connected components of G is the number of vertices minus the number of edges.

   Proof. We prove that (a) is equivalent to the other three conditions.

   (a) ⇔ (b): We use contrapositives. If G has a cycle, say C, then the minimum degree of the subgraph $G[V(C)]$ is at least two. This proves (b) ⇒ (a). On the other hand, if G has a subgraph G' with minimum degree at least two, then by Lemma 1.2.25, G' has a cycle. This proves (a) ⇒ (b).

   (a) ⇔ (c): If G has a cycle $C = (v_1, \ldots, v_k, v_1)$, then the path $P_1 = (v_1, \ldots, v_k)$ and the 2-vertex path (v_1, v_k) share only vertices v_1 and v_k and no edges. This proves that (c) ⇒ (a). Suppose now that G is a forest and P_1 and P_2 are paths in G whose intersection is nonempty. Then both P_1 and P_2 are in the same component T of G which is a tree, by definition. Let x be the first and y be the last vertex on P_1 that also belongs to P_2. By Theorem 2.1.4, T contains exactly one x, y-path, say P_0. This means that P_0 must be a subgraph of both P_1 and P_2 since both paths contain x and y, so $P_0 \subset P_1 \cap P_2$. On the other hand, $P_1 \cap P_2 \subset P_0$ since the first and last vertex on P_1 where it intersects P_2 are both part of P_0. Therefore, $P_1 \cap P_2$ is a path, so (a) ⇒ (c).

   (a) ⇔ (d): Let $G_1, \ldots, G_t$ be the connected components of G and $n_i := |V(G_i)|$ for $1 \leq i \leq t$. If G is a forest, then each G_i is a tree and by Theorem 2.1.4 has $n_i - 1$ edges. This yields (a) ⇒ (d). Suppose now that (d) holds. Let $1 \leq i \leq t$. Since G_i is connected, it has at least $n_i - 1$ edges. Moreover, if G_i is not a tree, then by Theorem 2.1.4, it has more than $n_i - 1$ edges, and so

   $$\sum_{i=1}^{t} |E(G_i)| > \sum_{i=1}^{t} (n_i - 1) = n - t,$$

   a contradiction to (d). $\square$

4. For $n \geq 3$, prove that if an n-vertex graph G has three vertices v_1, v_2, v_3 such that the subgraphs $G \setminus v_1, G \setminus v_2$ and $G \setminus v_3$ are acyclic, then G has at most one cycle.

Proof. Suppose for a contradiction that a graph G has two distinct cycles C_1 and C_2, but also has three vertices v_1, v_2, v_3 such that the subgraphs $G \setminus v_1, G \setminus v_2$ and $G \setminus v_3$ are acyclic. Then $\{v_1, v_2, v_3\} \subseteq V(C_1) \cap V(C_2)$. Since C_1 and C_2 are distinct, C_2 contains a path P connecting two vertices, say x and y, of C_1 such that the internal vertices of P do not belong to C_1. At least one of v_1, v_2, v_3 is not in $\{x, y\}$, say v_3 without loss of generality. Now, P is in $G \setminus v_3$ since v_3 is not in P. In addition, $G \setminus v_3$ contains another path from x to y formed by traversing C_1 in the direction not involving v_3. These paths are disjoint except for their endpoints by assumption, so together they form a cycle in $G \setminus v_3$, a contradiction to the assumption that $G \setminus v_3$ is acyclic. □

5. Let T be a tree, and let $B \subseteq V(T)$ be the set of vertices of T with degree at least 3. Prove that the number of leaves ℓ of T equals

$$2 + \sum_{v \in B} (d(v) - 2).$$

Proof. Let n be the order of T. Since T is a tree, it has $n-1$ edges, so the degree-sum formula says

$$2(n-1) = \sum_{v \in V(T)} d(v) = \ell + 2(\text{number of degree-2 vertices of } T) + \sum_{v \in B} d(v).$$

Subtracting 2 from the degree of each non-leaf gives

$$2(n-1) - 2(n-\ell) = \ell + 2\sum_{v \in B} (d(v) - 2),$$

since the degree-2 vertices disappear from the right side. Rearranging and solving for ℓ, we obtain

$$\ell = 2 + \sum_{v \in B} (d(v) - 2),$$

as desired. □