Announcements

Quiz 2 today!

Midterm 1 next Wed in-class (50 minutes)

- Reference sheet allowed (one A4 sheet, both sides)
- No other resources
- Sections covered: 1.1-1.4, 2.1-3, 3.1-2, 5.1-2, 6.1-2
- Practice problems posted
- See policy email for more

we'll see about §6.2

Sub this Friday

My Friday office hour will be moved to next Tuesday @ 10:30 via Zoom (just for this week!)

Strong induction (cont.)

Ex 3: Consider the following game: Two piles of n matches

The players take turns removing ≥ 1 matches from one of the piles. The players who takes the last match wins.

Show that Player 2 can always guarantee a win.

Class activity: play this game, and try to figure out a strategy.

Pf: We use strong induction. Let $P(n)$ be "Player 2 can win whenever there are initially n matches in each pile"

Base case: If $n=1$, Player 1 must remove the 1 match from one of the piles. Player 2 takes the match from the other pile and wins.

Inductive step: Suppose $k \geq 1$ and $P(1), \ldots, P(k)$ are true. For $k+1$ matches per pile, suppose Player 1 takes r matches from the first pile. Then Player 2 can take r matches from the other pile. If $r = k+1$, Player 2 wins. If $1 \leq r < k+1$, then each pile has $k+1-r$

matches remaining, and it is Player 1's turn again. Since $1 \leq k+1-r \leq k$, $P(k+1-r)$ is true, so Player 2 can now guarantee a win. Thus, $P(k+1)$ is true, so by strong induction, $P(n)$ is true for all n. $\square$

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§6.1: Counting

Counting problem: determine the cardinality of a set
"combinatorics"

Product rule: Suppose that a procedure can be broken down as a sequence of two tasks. If there are

m ways to do the first task
n ways to do the second task*,

then there are mn ways to do the procedure

* For any of the m choices for the first task

Ex 2: How many ways are there to write a letter followed by a digit? (e.g. A0, C8, Y2)

Ans: 26 letters · 10 digits = 260 ways

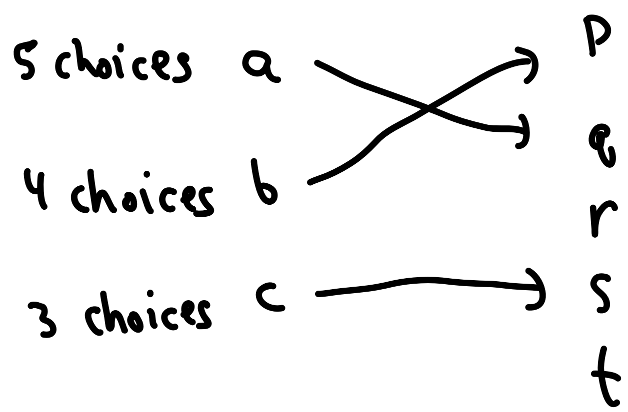
Ex 1: 2 employees, 12 offices. How many ways are there to assign each employee an office?

Ans: 12 choices for Employee 1, then 11 (remaining!) choices for Employee 2.

$12 \cdot 11 = 132$ ways

Ex 7: How many one-to-one functions are there from a set with m elts. to one with n elts.

e.g. $m = 3$ $\quad$ $n = 5$



Ans: $n(n-1)(n-2)\cdots(n-m+1)$ (If $m > n$, this is 0)

Sum rule: If a task can be done either in one of m ways or one of n ways, with no overlap, then there are $m+n$ ways to do the task.

Ex: How many length-2 "words" are there, where the first letter is capital or lower-case, and the second is lower-case?

First letter: $26 + 26 = 52$ choices (sum rule)

Second letter: 26 choices

Total: $52 \cdot 26 = 1352$ "words"
(product rule)

Ex 16: How many passwords are there satisfying:

a) Length 6, 7, or 8

b) Made up of digits and uppercase letters

c) At least one digit

Length 6:

$26 + 10 = 36$ choices for each digit

Total passwords satisfying b):

$$\underbrace{36}_{\substack{\text{1st}\\ \text{digit}}} \cdot \underbrace{36}_{\substack{\text{2nd}\\ \text{digit}}} \cdot 36 \cdot 36 \cdot 36 \cdot 36 = 36^6$$

Passwords containing only letters (i.e. violating c)):

$$26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^6$$

Length-6 valid passwords: $36^6 - 26^6 = 1{,}867{,}866{,}560$

Length-7 valid passwords: $36^7 - 26^7$

Length-8 valid passwords: $36^8 - 26^8$

Total: $36^6 - 26^6 + 36^7 - 26^7 + 36^8 - 26^8 = 2{,}684{,}483{,}063{,}360$

Subtraction rule: If a task can be done either in one of m ways or one of n ways, with overlap of k, then there are m+n-k ways to do the task.

$$|A \cup B| = \underset{m}{|A|} + \underset{n}{|B|} - \underset{k}{|A \cap B|}$$

Ex 18: How many 01-strings of length 8 either start w/ 1 or end w/ 00?

Start w/ 1:

1 * * * * * * *

$1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 128$ choices

End w/ 00

* * * * * * 0 0

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 64$ choices

start w/ 1 AND end w/ 00:

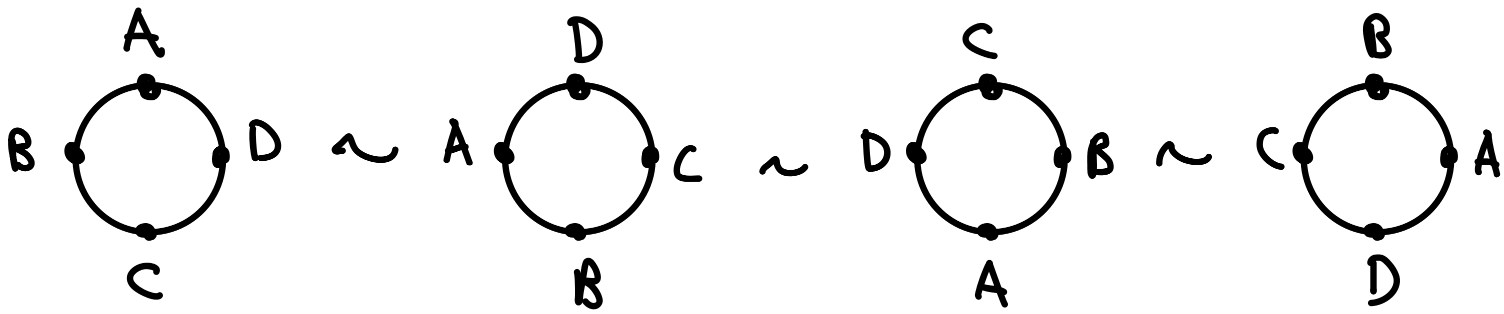
1 * * * * * 0 0

$1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 32$ choices

Ans: $128 + 64 - 32 = 160$ strings

Division rule: If there are n ways to do a task, and groups of d of these ways are equivalent, then there are n/d ways up to equivalence.

Ex 20: How many different ways are there to seat 4 people around a circular table, where two seatings are considered equivalent if they are rotations of each other?



4 rotations of each seating arrangement

$4 \cdot 3 \cdot 2 \cdot 1 = 24$ seating arrangements

$\frac{24}{4} = 6$ nonequivalent seating arrangements