

Announcements

Midterm 3: tonight 7:00-8:30 Sidney Lu 1043

See policy email for details

HW10 will be posted today (due next Wed.)

Midterm 3 review

Integral domains, poly. rings, irreducibility

Basic tools: irreducibility, field ext's, degrees, splitting fields, min'l polys., tower law

Constructibility: 4 classical problems, type of ext's allowed

Separability: derivative criterion, irreds. over char 0 or fin. field

Galois theory:

- Compute automorphisms, fixed fields
- Characterization of Galois ext'n (autom. gp. size, poly. splitting)
- Galois correspondence (inc. properties e.g. normal subgps.)
- trace, norm, and sym. funcs. (lie in base field)

Important cases:

- finite fields
- cyclotomic extns

Compute Galois gps.

- discriminant (def and A_n criterion)
- Compute Gal. gp. for deg 2, 3
- gens. and relns and/or cycle type
(find some automs. and determine the gp. they gen.)

Solvability by radicals:

- Solvable gps and solvability criterion (Galois' thm)
- Cardano's formula (don't need to memorize)
- Prove that a poly. is/isn't solvable by radicals

Algebraic geometry

- Basic defs: ideal, variety
- Compute $\mathcal{I}(V)$, $V(\mathcal{I})$, $\sqrt{\mathcal{I}}$
- Basic properties from propositions

14.2.3) Determine the Galois gp of $f = (x^2 - 2)(x^2 - 3)(x^2 - 5)$
Determine all subfields of $S_p f$

$$K := S_{p_{\mathbb{Q}}} f = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$$

$$[K:\mathbb{Q}] = 8 \text{ by D&F \#14.4.2 (on HW 7)}$$

Since K is a splitting field / \mathbb{Q} , K/\mathbb{Q} is Galois,

so $G := \text{Gal}(K/\mathbb{Q})$ has order 8

$\sigma \in G$ must send $\sqrt{2} \mapsto \pm\sqrt{2}$
 $\sqrt{3} \mapsto \pm\sqrt{3}$
 $\sqrt{5} \mapsto \pm\sqrt{5}$ } 8 possibilities, so
all must be valid automorphisms

$$\text{Let } \sigma_{ijk} : \begin{cases} \sqrt{2} \mapsto (-1)^i \sqrt{2} \\ \sqrt{3} \mapsto (-1)^j \sqrt{3} \\ \sqrt{5} \mapsto (-1)^k \sqrt{5} \end{cases} \text{ for } i, j, k \in \{0, 1\}$$

$$\sigma_{ijk} \sigma_{abc} = \sigma_{i+a, j+b, k+c} \text{ taken mod 2}$$

$$\text{Let } \sigma := \sigma_{100}, \tau := \sigma_{010}, \rho := \sigma_{001}$$

$$\text{So } G = \{\sigma, \tau, \rho \mid \sigma^2 = \tau^2 = \rho^2 = 1, \sigma\tau = \tau\sigma, \\ \sigma\rho = \rho\sigma, \tau\rho = \rho\tau\} \cong C_2 \times C_2 \times C_2$$

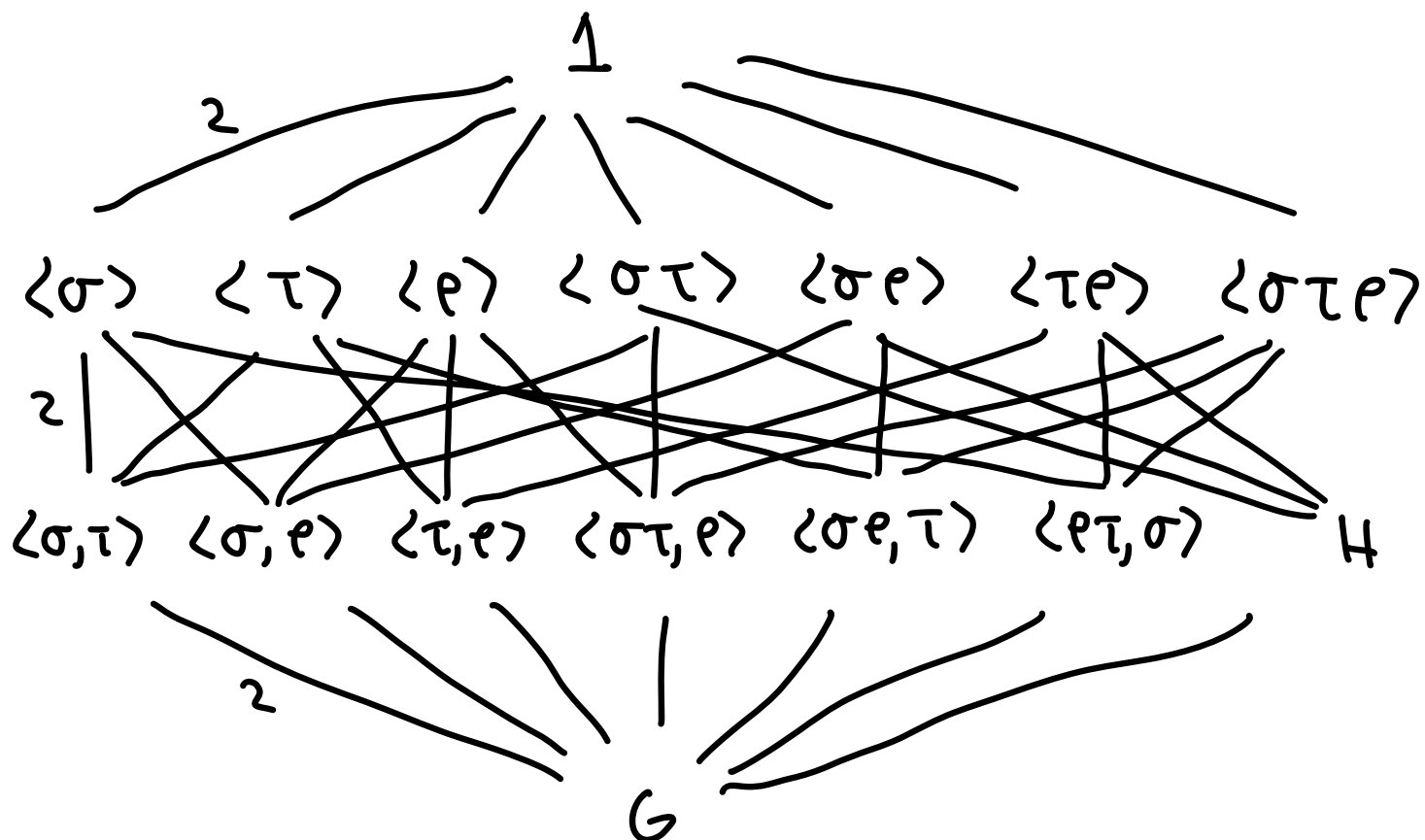
Sub gpr:

Order	Sub gp.
1	$\{1\}$
2	$\langle \sigma \rangle \quad \forall \sigma \in G \setminus \{1\}$
4	$\{\sigma_{ijk} \mid i=0\} = \langle \tau, e \rangle$ $\{\sigma_{ijk} \mid j=0\} = \langle \sigma, e \rangle$ $\{\sigma_{ijk} \mid k=0\} = \langle \sigma, \tau \rangle$ $\{\sigma_{ijk} \mid i=j\} = \langle \sigma\tau, e \rangle$ $\{\sigma_{ijk} \mid i=k\} = \langle \sigma e, \tau \rangle$ $\{\sigma_{ijk} \mid j=k\} = \langle \tau e, \sigma \rangle$ $H := \{\sigma_{ijk} \mid i+j+k=0\} = \langle \sigma\tau, \sigma e, \tau e \rangle$

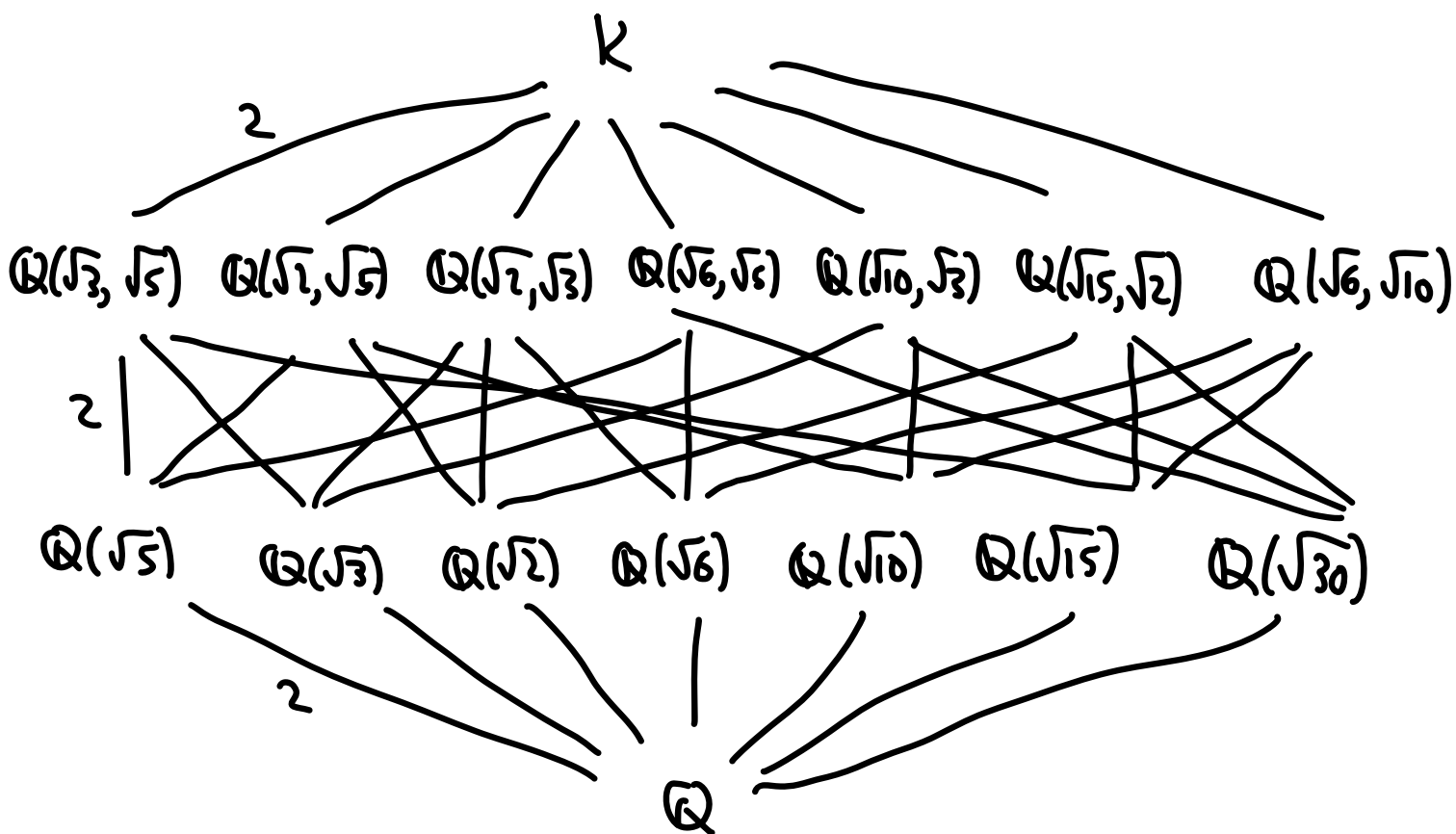
8

G

Subgp lattice



Int fields



14.4.4) Let $f(x) \in F[x]$ be an irred. poly. of deg n over F . Let $L = S_{p_F} f$, and let α be a root of f in L . If K is any Galois ext'n of F , show that

$$f(x) = \underbrace{p_1(x) \cdots p_m(x)}_{\substack{\text{irred. of} \\ \text{deg } d}} \in K[x]$$

where $d = [K(\alpha) : K] = [L \cap K(\alpha) : L \cap K]$ and

$$m = n/d = [F(\alpha) \cap K : F]$$

* Let's assume L/F is also Galois

Pf: Every factor of f lies in $L[x]$, so the irred.

factorization of f in $K[x]$ equals the irred.

factorization of f in $(K \cap L)[x]$, so the two

def'n's of d are the same. We also have

$$n = [F(\alpha) : F] = \underbrace{[F(\alpha) : F(\alpha) \cap K]}_d \underbrace{[F(\alpha) \cap K : F]}_m, \text{ so}$$

the def of m is consistent too.

Let $H \leq \text{Gal}(L/F)$ correspond to the int field $L \cap K$. By our construction of min'l polys, for any root α of f in L ,

$$m_{\alpha, L \cap K}(x) = \prod_{\beta \in H\alpha} (x - \beta)$$

Thus, the degs. of the irred. factors of $f(x)$ over $L \cap K$ equal the sizes of the H -orbits of $S := \{\text{roots of } f\}$

Since K/F is Galois, by prop. 4 of the Fun. Thm., $H \trianglelefteq G$. By Dummit & Foote Ex. 4.9a, since G acts transitively on S and H is normal, the H -orbits must be the same size.

(pf: transitivity $\Rightarrow S = \{g\alpha \mid g \in G\}$. Orbits are $Hg\alpha = gH\alpha$, which has order $|H\alpha|$)

Thus, all the p_i have the same degree, and this degree equals

$$\deg m_{\alpha, L \wedge K}(x) = [(L \wedge K)(\alpha) : L \wedge K]$$

□