Announcements:

- Discord server: email me if you want to be added
- H/w 1 graded (1 week for regrade requests)
- H/w 3 will be posted later today
- Midterm 1: Wed. 9/20 7:00-8:30pm
  (Noyes Lab. 217)
- Quiz 1: Fri. 9/15 (in class)

Thm 1.2.26 [Euler]:

G has an Eulerian circuit $\Longleftrightarrow$ a) G has ≤ 1 "nontrivial" connected component AND b) G is even

containing edges

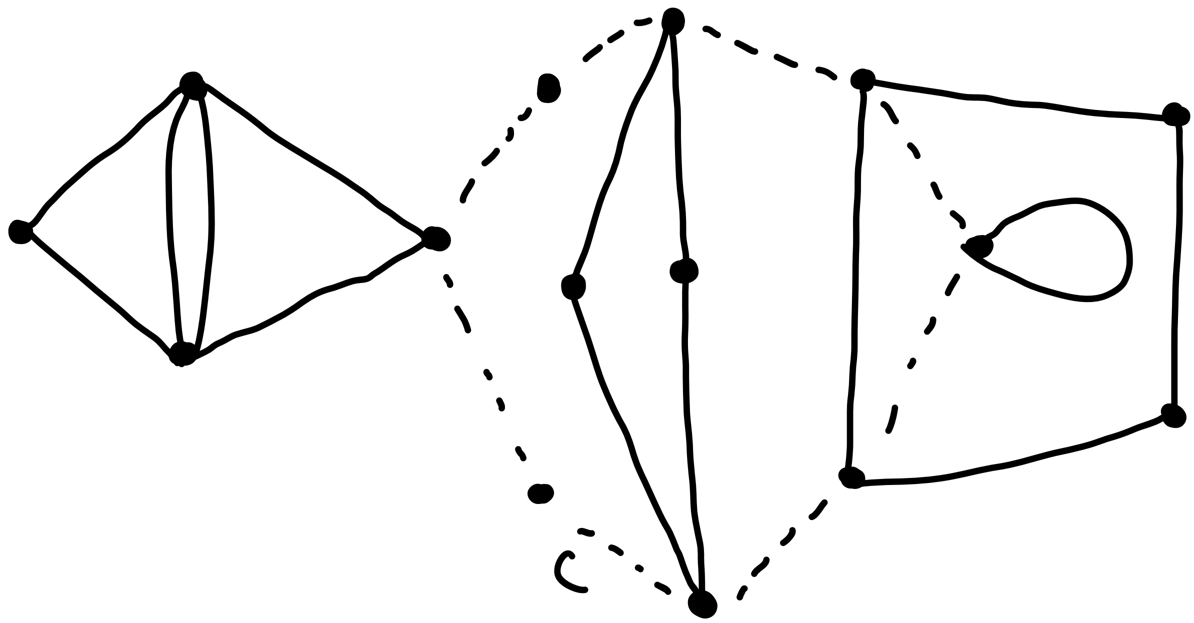
Pf: $\Rightarrow$) Done last time

$\Leftarrow$) Induction on $m := |E(G)|$

Base case: $m = 0$. The circuit $\underbrace{v}_{\text{walk}}$ for any $v \in V(G)$ is Eulerian.

Inductive step: Let H be the nontriv. component of G. Every vertex $v \in V(H)$ has degree ≥ 2, and so by Lemma 1.2.25, H has a cycle C. Let $G' = G \setminus E(C)$. Since C has 0 or 2 edges incident to each vertex of G, G' is also even, so each component of G' has an Eulerian circuit by the inductive hypothesis.

Since every nontriv. component of G' has at least one vertex in $V(C)$, taking C union Eulerian circuits of every component gives an Eulerian circuit of G. We traverse C, but when a component of G' is entered for the first time, we detour along an Eulerian circuit of that component $\square$

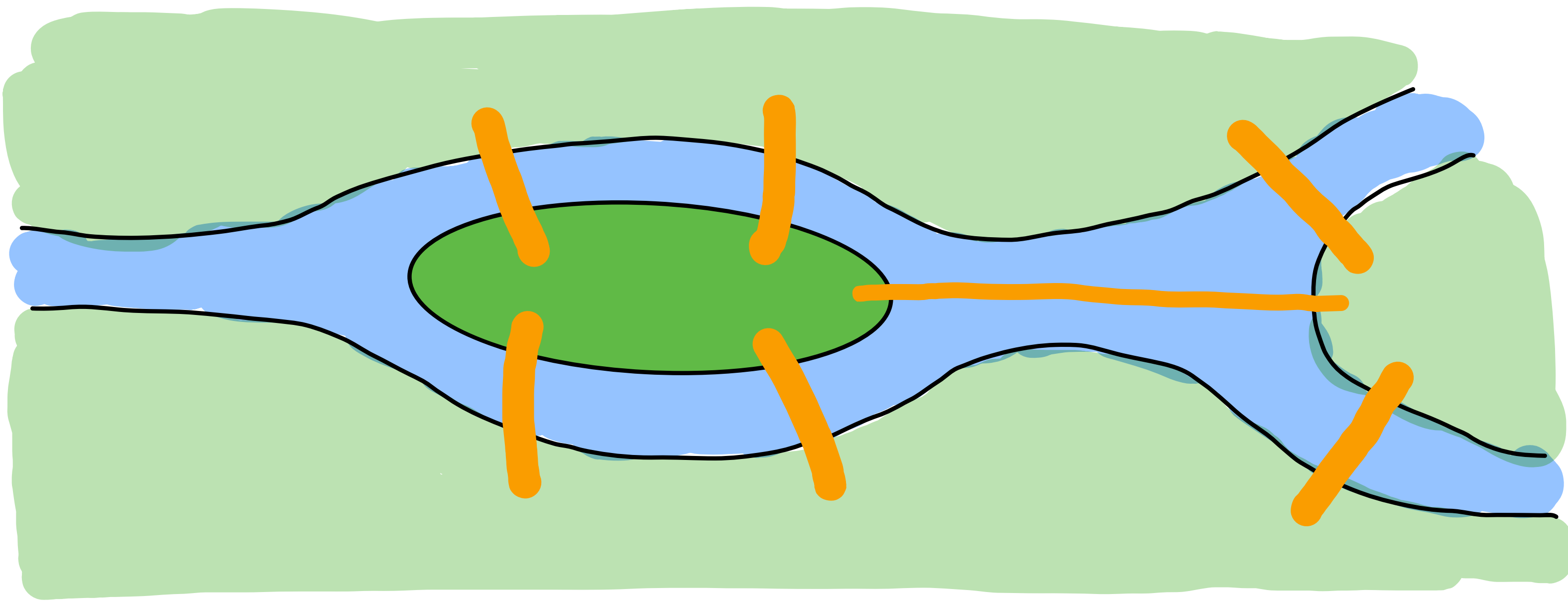


Def 1.1.32: A decomposition of G is a list of subgraphs s.t. each edge appears in exactly one subgraph from the list

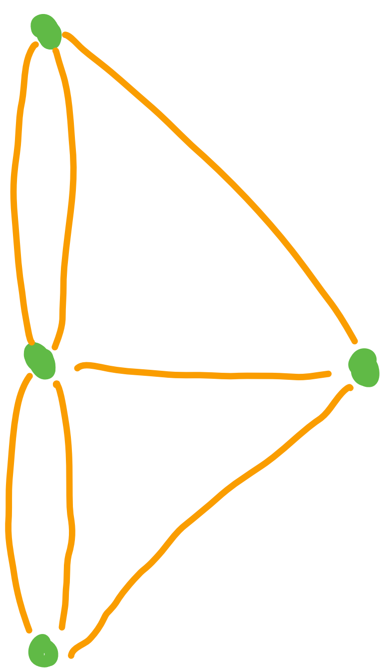
Corollary (Prop 1.2.27): Every even graph decomposes into cycles.

Pf: In the previous proof, G decomposes into G' and C; use induction on $|E(G)|$. $\square$

Bridges of Königsberg (redux)



Question: Can we cross each bridge exactly once?



Answer: No, since the corresponding graph is not even (in fact, it's odd).

Cor:

G has an Eulerian ~~circuit~~ trail $\iff$ a) G has ≤ 1 "nontrivial" connected component

AND

b) ~~G is even~~ G has at most two $\underbrace{\text{odd vertices}}_{\text{vertices of odd degree}}$

Pf: $\Rightarrow$) If the trail is closed, it's a circuit. Otherwise, the starting and ending vertices have odd degree; add an edge between them and apply Thm. 1.2.26.

$\Leftarrow$) If G has no odd vertices, by Thm. 1.2.26 it has an Euler circuit. Otherwise, add an edge between the two odd vertices, and the resulting graph has an Euler circuit (again, by Thm. 1.2.26). Remove the edge you just added, and it becomes an Euler trail. $\square$

Cor: The Königsberg bridge graph doesn't have an Euler trail. $\square$

§1.3: Vertex Degrees and Counting

Def 1.3.1:

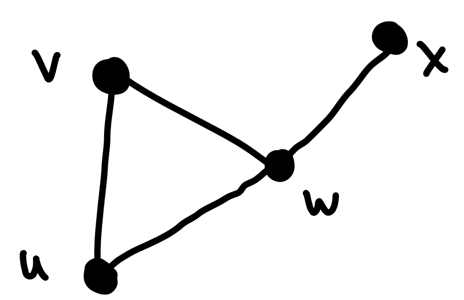
a) Max. degree $= \Delta(G)$

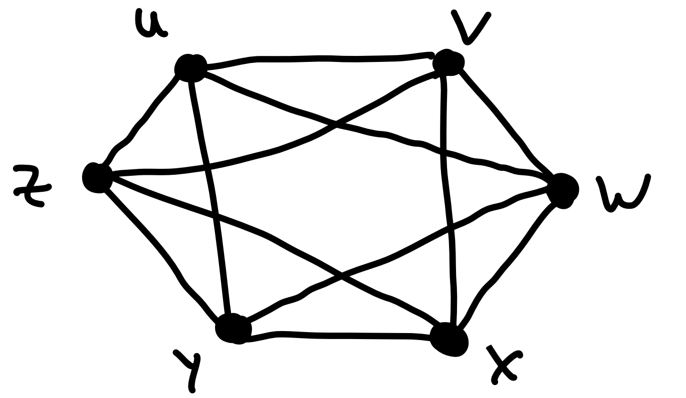
b) Min. degree $= \delta(G)$

c) If $\Delta(G) = \delta(G) = k$, G is <u>k-regular</u>

d) $N_G(v) = N(v) = \{\text{vertices adjacent to } v\}$
"neighborhood"

Class activity:





$\Delta(G) = 3$

$\delta(G) = 1$

regular? No!

$N(u) = \{v, w\}$

$\Delta(G) = 4$

$\delta(G) = 4$

regular? Yes! 4-regular

$N(u) = \{v, w, y, z\}$

Def 1.3.2:

a) $n(G) = |V(G)|$ "order"

b) $e(G) = |E(G)|$ "size"

Important idea:

We can prove a lot about a graph using simple counting arguments

Prop (1.3.3 – 1.3.6):

a) (degree sum formula):

$$\sum_{v \in V(G)} \underbrace{d(v)}_{\text{degree}} = 2e(G)$$

(Pf: each edge is incident to 2 vertices)

b) $\delta(G) \leq \dfrac{2e(G)}{n(G)} \leq \Delta(G)$

(Pf: middle quantity equals avg. degree)

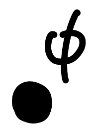
c) G has an even number of vertices of odd degree (pf: sum of degrees must be even)

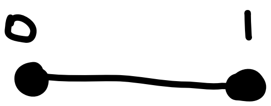
d) A k-regular graph of order n has $nk/2$ edges (Pf: degree-sum formula)

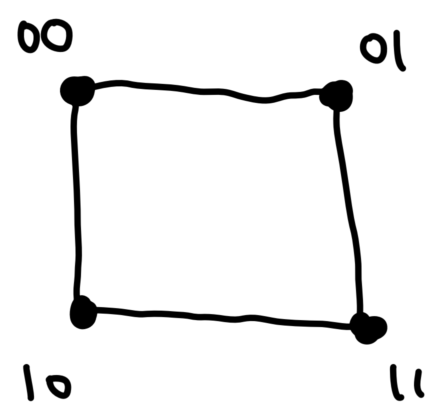
Example 1.3.8: The k-hypercube Q_k

Q_k has vertices labelled by length-k strings of 0's and 1's 010 ← length 3

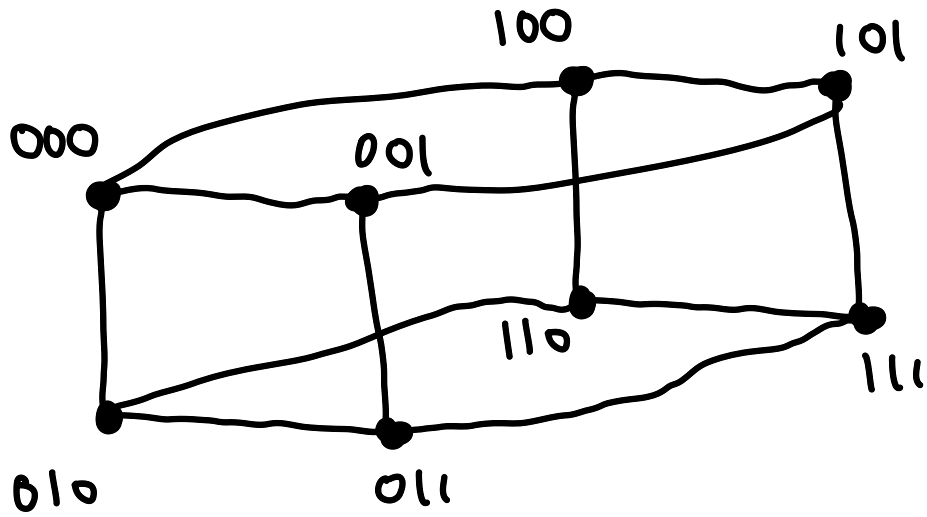
Two vertices are adjacent iff their labels differ in exactly one position

Q_0:


Q_1:


Q_2:


Q_3:



Facts:

(i) Q_k is k-regular

(ii) Q_k is bipartite

(iii) $n(Q_k) = 2^k$

(iv) $e(Q_k) = k2^{k-1}$

(v) If $j \le k$, Q_k has $\binom{k}{j} 2^{k-j}$ subgraphs isomorphic to Q_j.

Extremal Problems

Questions involving the word "minimum" or "maximum"

Q: What is the maximum number of edges in a simple graph w/ n vertices?

A: $\binom{n}{2} = \frac{n(n-1)}{2}$

Q: What is the minimum number of edges in a simple graph w/ n vertices

A: zero

Q: What is the minimum number of edges in a connected simple graph w/ n vertices

A: $n-1$

Prop 1.3.15: If G is simple of order n, and $\delta(G) \geq \frac{n-1}{2}$, then G is connected

Can rephrase in terms of extremality:

Q: What is the minimum value d such that all simple n-vertex graphs with $\delta(G) \geq d$ are connected

A: We don't necessarily know!

However, this prop. says that $d \leq \frac{n-1}{2}$

Pf: Next time