Gelfand-Tsetlin Patterns and Polytope Combinatorics
Information Sheet

1 Outline

This document contains some background resources for students interested in the 2021 Polymath Jr. project on Gelfand-Tsetlin patterns and polytope combinatorics, mentored by Ben Brubaker, with TA Andy Hardt. The project description can be found HERE.

Please note that none of this background material is required to start work on the project. One of our goals during the first few weeks is to learn and assimilate the background material. So this document serves only to highlight material that you can check out if you are interested.

2 Resources for Gelfand-Tsetlin patterns

This is the most hands-on topic, and the one most likely to directly come up in the project.

Topics that might be useful for us to know right away are:

- Definition and weight;
- Strict vs. non-strict;
- Schützenberger involution. An involution is a map that's its own inverse, so if you do it twice, you get the identity. This means that the Schützenberger involution is in a sense a way to "pair up" GT patterns.
- Bijections with other combinatorial objects, such as semistandard Young tableaux, or states of a lattice model.

Here are a few sources that could be helpful. Links are clickable.

- The first orange section ("Gelfand-Tsetlin patterns") of THIS WEBPAGE. Careful: these conventions for the indices in the triangular array are different than ours; our top row has subscript 0
- THIS WEBPAGE
- THIS DOCUMENT explains bijections between strict GT patterns and some other combinatorial objects.
- If you're *really* interested in GT patterns, and want to learn about them beyond the likely scope of this project, THIS PAPER [5] by Berenstein and Kirillov is a very nice (advanced!) paper.

You can play with GT patterns on Sage. See HERE for documentation and HERE for a website where you can program in Sage for free. It's not necessary to know how to program for this project, but it's a good skill for a mathematician to have to be able to program in a language like Sage or Mathematica. Plus, Sage is based on the oft-used programming Python.

3 Our main source

Our math source will be THIS BOOK, [3], "Weyl Group Multiple Dirichlet Series: Type A Combinatorial Theory" by Brubaker, Bump, and Friedberg. Eventually, we will explore most of the first half of that book, and perhaps the second half as well.

One of the first things we'll learn how to do in the first couple weeks of the project is computations like the ones on pages 3-12 of [3]. In particular, $G_\Gamma(\mathfrak{T})$ and $G_\Delta(\mathfrak{T})$ are defined on Page 12. The relevant definitions are given in the previous pages, as is the Schützenberger involution, which we'll call Sch.

The following sentence is the starting point for our investigations:

$$\text{If } \mathfrak{T} \text{ and } \mathrm{Sch}(\mathfrak{T})) \text{ are strict GT patterns, then } G_\Gamma(\mathfrak{T}) = G_\Delta(\mathrm{Sch}(\mathfrak{T})). \tag{1}$$

You've understood the relevant material when you can check that this equation is correct for the following pair of GT patterns

$$\mathfrak{T} = \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \qquad Sch(\mathfrak{T}) = \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array}$$

and also check that the equation does not hold for the following GT patterns

$$\mathfrak{T} = \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{array} \qquad Sch(\mathfrak{T}) = \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} .$$

The first pair of patterns are strict, while in the second pair $\mathfrak{T}$ is nonstrict (why?). One of the main goals of this project is to find a replacement for (1) for nonstrict GT patterns.

4 Colored GT patterns

The main new tool we will use is the concept of *colored Gelfand-Tsetlin patterns*. Each entry is colored according to specific rules. There aren't a lot of resources that discuss colored GT patterns directly, since they're so new. Hopefully this will be to our advantage since no one has approached our problem in quite this way!

We'll explain how colored GT patterns work at the start of the project. In the meantime, you can try to figure them out yourself once you understand the bijection between lattice model states. The bijection is the same between colored lattice model states and colored GT patterns.

See THIS PAPER [1] for the colored lattice models that correspond to the colored GT patterns. Specifically, Figures 10, 14, and the top of Figure 16. There is exactly one way to color a nonstrict GT pattern, but multiple ways to color a strict GT pattern.

5 Number theoretic motivation: metaplectic Whittaker functions

This section consists of several advanced papers. Most likely, none of them will be essential for us at *any* point of the project. However, this section is here for those of you who are interested in our motivation for studying these functions.

The original purpose of GT patterns was to study representations of $GL_n(\mathbb{C})$. A representation is a special sort of vector space, and understanding the representations of a given group is a very important field of mathematics. Each representation of $GL_n(\mathbb{C})$ is indexed by an integer partition. This partition is essentially the top row of the GT pattern, and the GT patterns with that top row form a basis of that representation. Even better, when we lop off the top row of a GT pattern, we get a smaller GT pattern. The new top row tells us what representation of GL_{n-1} we are now in.

We can put a function on GT patterns that helps us evaluate the *character* of these representations. These functions are called *Schur polynomials*.

THIS PAPER [6] gives more information on this perspective, plus similar constructions for other "classical Lie groups".

The functions H_Γ and H_Δ introduced in [3, p. 4] are called *metaplectic Whittaker functions*. They are essentially souped up Schur functions, with extra number-theoretic information. Instead of representations of $GL_n(\mathbb{C})$, Whittaker functions arise when we study representations of $GL_n(\mathbb{Q}_p)$, where $\mathbb{Q}_p$ is a p-adic number field. THIS TALK is a fun introduction to the p-adic numbers. See THIS TALK for a short introduction to Whittaker functions, as well as some additional sources.

The word "metaplectic" refers to a "cover" of $GL_n(\mathbb{Q}_p)$, so these are quite sophisticated objects indeed! If the want the full story, THIS PAPER [2] is probably the best source to include pieces from everything in this section.

Studying groups like $GL_n(\mathbb{Q}_p)$ is a part of the larger landscape of number theory, and special periodic functions called *automorphic forms*. For instance, the famous Riemann zeta function is an automorphic form. If you're looking to understand automorphic forms more generally, and how they fit into number theory, THIS ARTICLE [4] gives a good exposition that makes its way towards Whittaker functions in the second half.

References

[1] B. Brubaker, V. Buciumas, D. Bump, and H. P. A. Gustafsson, *Colored vertex models and iwahori whittaker functions*, 2020.

[2] B. Brubaker, D. Bump, and S. Friedberg, *Eisenstein series, crystals, and ice*, Notices Amer. Math. Soc., 58 (2011), pp. 1563–1571.

[3] B. Brubaker, D. Bump, and S. Friedberg, *Weyl group multiple Dirichlet series: type A combinatorial theory*, vol. 175 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 2011.

[4] S. S. Gelbart and S. D. Miller, *Riemann's zeta function and beyond*, Bull. Amer. Math. Soc. (N.S.), 41 (2004), pp. 59–112.

[5] A. N. Kirillov and A. D. Berenstein, *Groups generated by involutions, Gelfand-Tsetlin patterns, and combinatorics of Young tableaux*, Algebra i Analiz, 7 (1995), pp. 92–152.

[6] A. I. Molev, *Gelfand-Tsetlin bases for classical Lie algebras*, in Handbook of algebra. Vol. 4, vol. 4 of Handb. Algebr., Elsevier/North-Holland, Amsterdam, 2006, pp. 109–170.