Announcements:

- Quiz today!
- Midterm 2 next Wed.

  Wed. 10/18 7:00pm - 8:30pm in 217 Noyes Lab.

  See email for policies

Recall: Tutte's Thm.

$o(G) :=$ # odd order components of G

G has a perfect matching $\iff o(G \setminus S) \leq |S| \;\forall\, S \subseteq V(G)$

Cor 3.3.7 [Berge-Tutte Formula]:

The number of vertices u unsaturated by a maximum matching of G is

$$d := \max_{S \subseteq V(G)} \{ o(G \setminus S) - |S| \}$$

Pf: For any $S \subseteq V(G)$, at most $|S|$ edges can match vertices of S to vertices in odd components of $G \setminus S$. Any extra odd components will have a vertex left over, so every matching has $\geq o(G \setminus S) - |S|$ unsaturated vertices, and so

$$u \geq \max_{S \subseteq V(G)} \{ o(G \setminus S) - |S| \} = d$$

We know $d \geq 0$ since $o(G \setminus \phi) - |\phi| \geq 0$.

Define G' as:

$$V(G') = V(G) \cup V(K_d)$$

"join of G and K_d"

$$E(G') = E(G) \cup E(K_d) \cup \{uv \mid u \in V(G), v \in V(K_d)\}$$

If G' has a perfect matching, then G has a matching w/ $\leq d$ unsaturated vertices, since deleting the d added vertices eliminates edges that saturate at most d vertices of G, so we'll have $u \le d$.

For any S, $n(G \setminus S) \equiv o(G \setminus S) \pmod{2}$

$$n(G) - |S| \equiv o(G \setminus S) \pmod{2}$$

$$n(G) \equiv o(G \setminus S) \pmod{2}$$

$$n(G) \equiv d \pmod{2}$$

So $n(G') = n(G) + d$ is even

Evaluate Tutte's condition on G':

Let $S' \subseteq V(G')$. WTS: $o(G' \setminus S') \le |S'|$

a) $S' = \emptyset$ ✓

b) $S' \ne \emptyset$ but $V(K_d) \not\subseteq S'$: $G' \setminus S'$ has 1 component, so $o(G' \setminus S') \le 1 \le |S'|$

c) $V(K_d) \subseteq S'$: Let $S = S' \setminus V(K_d)$.

Then $G' \setminus S' = G \setminus S$, so

$$o(G' \setminus S') = o(G \setminus S) \underset{\substack{\text{by def'n} \\ \text{of } d}}{\leq} |S| + d = |S'| \qquad \square$$

Cor 3.3.8 [Petersen, 1891]: Every 3-regular graph w/ no cut-edge has a perfect matching

Pf:

Def 3.3.1: A k-factor is a spanning k-regular subgraph

Special case: perfect matching $\equiv$ 1-factor

Cor 3.1.13: If $k > 0$, every k-regular bipartite graph has a perfect matching

Pf sketch.

Thm 3.3.9 [Petersen, 1891]: Every regular graph of even degree has a 2-factor

Pf:

A related idea allowed Tutte to find a necessary and sufficient condition for G to have a k-factor for any k, or, even more generally, a subgraph w/ <u>any</u> degree sequence (see optional subsection)