Duffing Oscillator

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Equation and Parameters

- Nonlinear 2nd order ODE used to model systems with damping and driving forc

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \Upsilon\cos(\omega t)$$

where $\ddot{x} = 2nd\ time\ derivative,$

$\dot{x} = 1st\ time\ derivative,$

$x = displacement\ at\ time\ t,$

$\delta = damping\ coefficient,$

α = linear stiffness coefficient

β = nonlinearity in the restoring force; (if =0, simple harmonic oscillator)

ϒ = amplitude of the periodic driving force; (if =0, no driving force present)

ω = angular frequency of the periodic driving force

Approximation for Solution

- Fourier Series
- Treat the cubic term (Duffing term) as a perturbed simple harmonic oscillator
- complex: Frobenius method
- Numerical methods: Euler’s / Runge-Kutta (4^{th} order)
- Homotopy analysis method (HAM) for strong nonlinear systems

Frequency Response

- β = 0: linear oscillator; else: nonlinear

$Substitute\ \boldsymbol{z^2} = \boldsymbol{a^2} + \boldsymbol{b^2}\ and\ \boldsymbol{tan\Phi} = \frac{\boldsymbol{b}}{\boldsymbol{a}}\ into\ \boldsymbol{x} = \mathbf{a\ cos(\omega t) + b\ sin(\omega t) = z\ cos(\omega t - \Phi)}$

$(-\omega^2 a + \omega\delta b + \alpha a + \frac{3}{4}\beta a^3 + \frac{3}{4}\beta a b^2 - \Upsilon)\cos(\omega t) + (-\omega^2 b + \omega\delta a + \frac{3}{4}\beta b^3 + \alpha b + \frac{3}{4}\beta b a^2)\sin(\omega t) +$
$+ (\frac{1}{4}\beta a^3 - \frac{3}{4}\beta a b^2)\cos(3\omega t) + (\frac{3}{4}\beta b a^2 - \frac{1}{4}\beta b^3) sin(3\omega t) = 0.$

Neglecting superharmonics at 3ω, and considering cos(ωt) and sin(ωt) terms need to be zero,

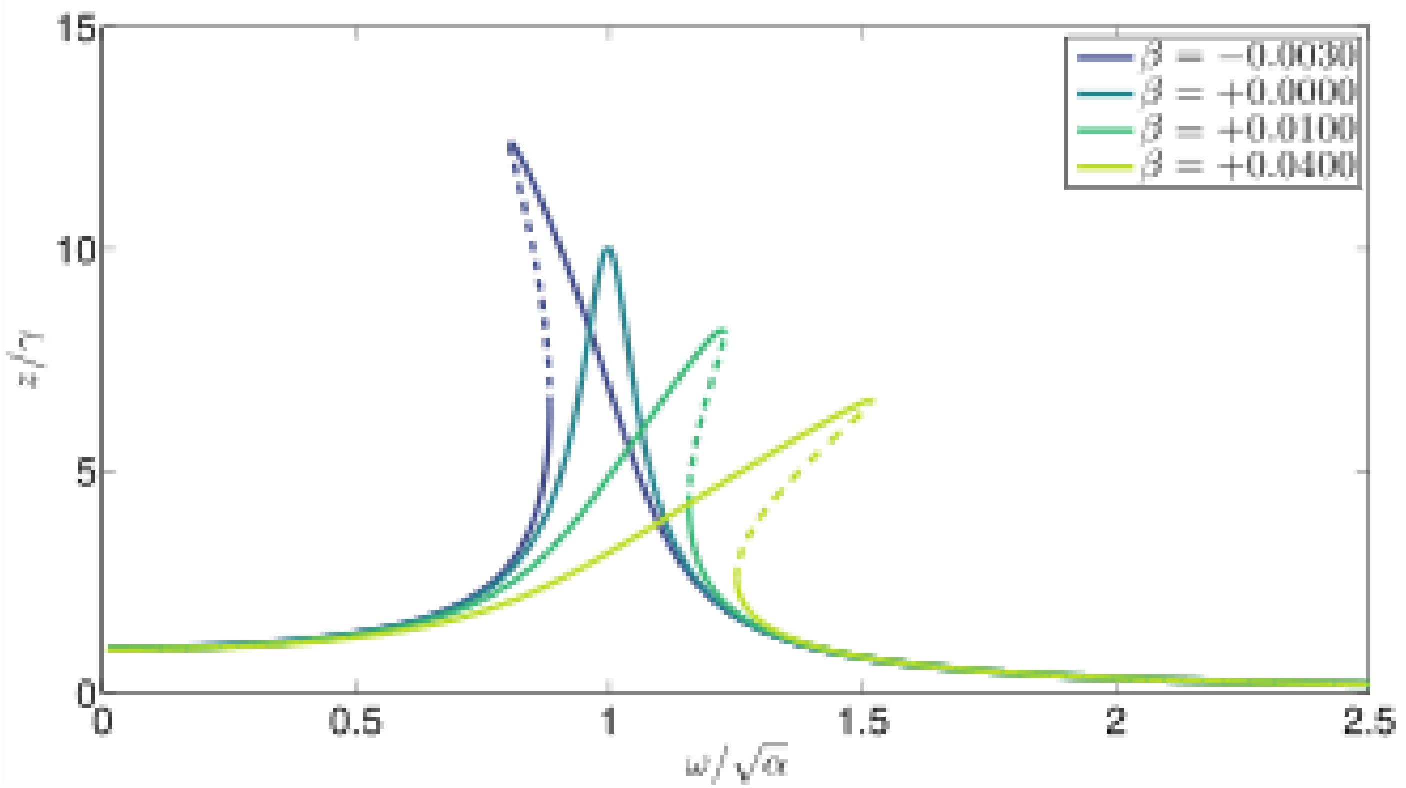
$(-\omega^2 a + \omega\delta b + \alpha a + \frac{3}{4}\beta a^3 + \frac{3}{4}\beta a b^2 - \Upsilon) = -\omega^2 b + \omega\delta a + \frac{3}{4}\beta b^3 + \alpha b + \frac{3}{4}\beta b a^2 = 0$

Square both sides and isolate for ϒ:

$\rightarrow \boldsymbol{\Upsilon^2 = [(\omega^2 - \alpha - \frac{3}{4}\beta z^2)^2 + (\delta\omega^2)^2]\, z^2}$

→ Graph

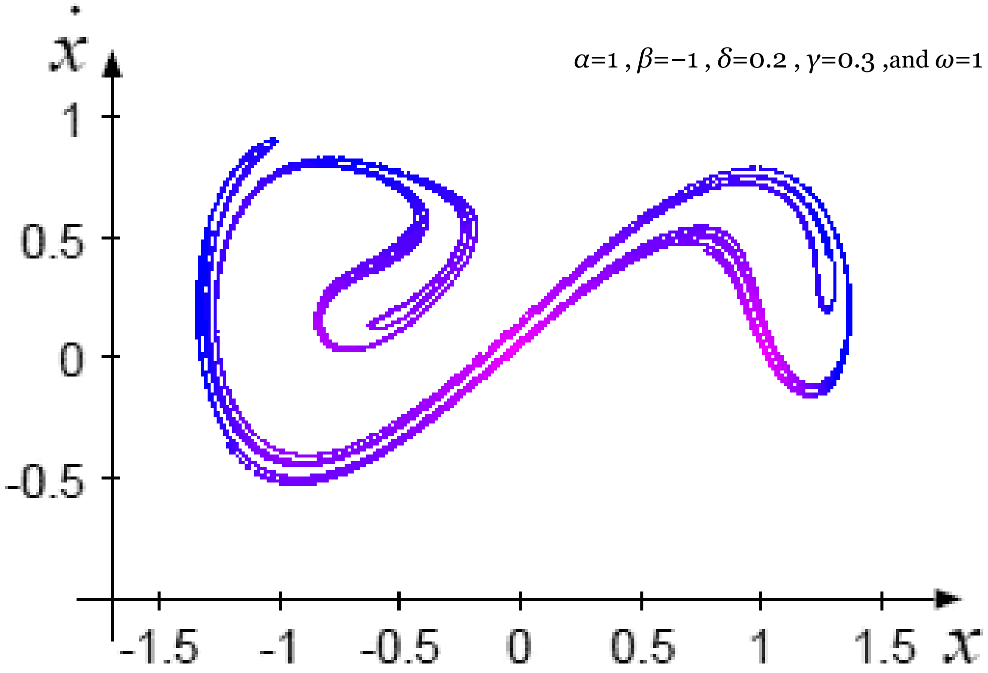
Frequency Response



Z / ϒ vs. w/ sqrt(α)

Image source: Wikipedia : Duffing Equation

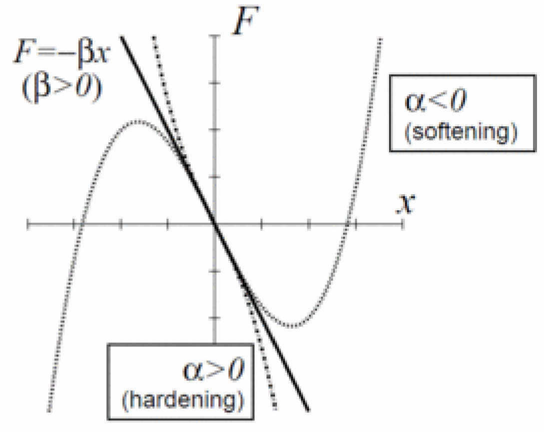
Periodic change of the chaotic attractor



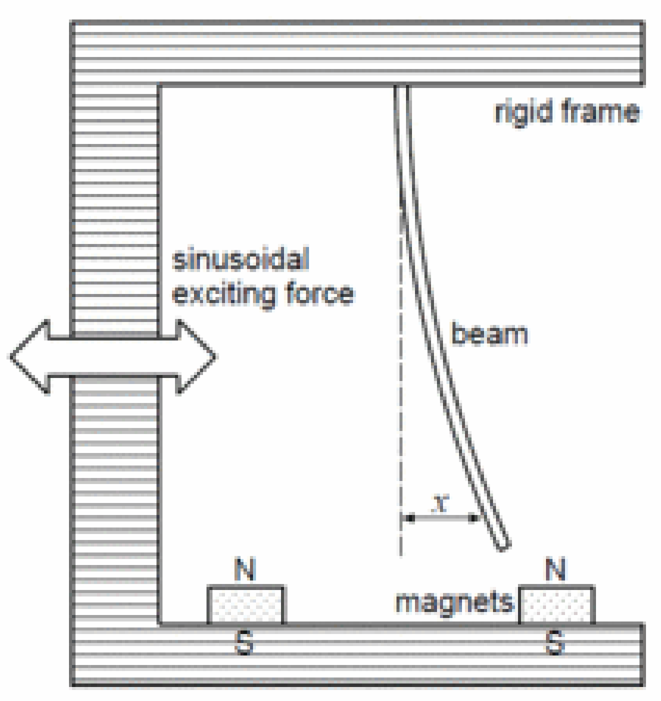
Velocity vs. displacement

Image source: Wikipedia : Duffing Equation

Forced System



For $\beta>0$ → a forced oscillator model with a spring whose restoring force is written as $F=-\beta x-\alpha x_3$



For $\beta<0$ → a model of a periodically forced steel beam which is deflected toward the two magnets.

(Moon and Holmes, 1979; Guckenheimer and Holmes, 1983; Ott, 2002)

Image source: Wikipedia : Duffing Equation

Unforced System (γ=0)

Unforced system (γ=0)
(i) No damping (δ=0) → $E(t) = 0.5\dot{x}^2+0.5\beta x^2+0.25\alpha x^4$ = constant
(ii) $\delta>0$, E(t) $dE(t)/dt=-\delta\dot{x}^2 \le 0$

$E(t)$ for $\alpha>0$ (no storing force), $\delta = 0$ (no damping)→
$\beta>0$ → single-well potential
$\beta<0$ → double-well potential
* trajectory $(x, \dot{x})$ moves on the surface of $E(t)$ keeping $E(t)$ constant.

$E(t)$ for $\alpha>0$ (no storing force), $\delta > 0$ →

$\alpha>0$, $\beta>0$, and $\delta>0$ → 1 equilibrium : (x,v) = (0,0)
$\alpha>0$, $\beta<0$, and $\delta>0$ → 3 equilibria: 1 top, 2 bottom

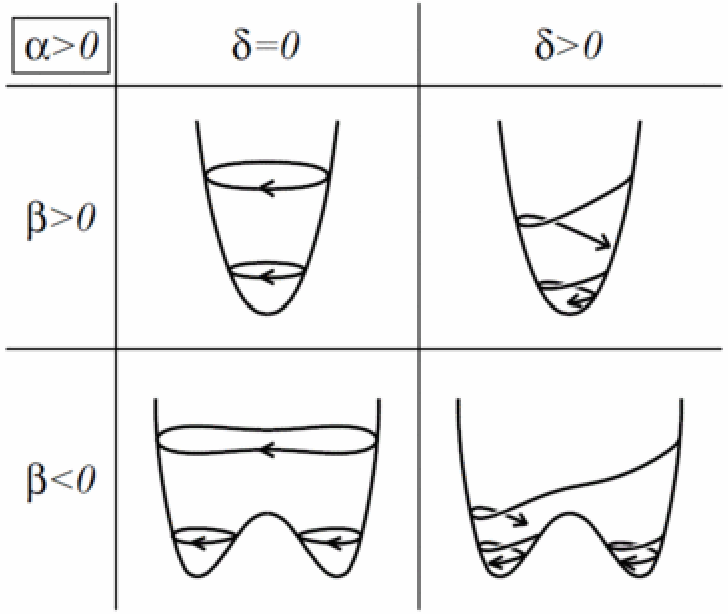


Image source: Wikipedia : Duffing Equation

Deliverables

- Initial conditions
  - identify stable/unstable fixed points
  - examine transient/long-term behavior
    - (i) undamped, undriven (γ=0, δ =0)
    - (ii) damped, undriven(γ=0, δ>0)
    - (iii) damped, driven(γ=0, δ>0) with α>=0 , β>=0

- phase-space projections (x, $\dot{x}$)
- time-series trajectories
- energy / power spectrum
- Poincaré sections
- RK4 error estimate (table of data from varying step sizes)