Duffing Oscillator

Ph235 Spring 2019

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Problem statement

- Initial: Model chaotic motion of a magnetoelastic steel beam due to magnetic forces and elasticity
- **Modified** → examine and identify points of bifurcation through visuals for non-obvious duffing oscillator system

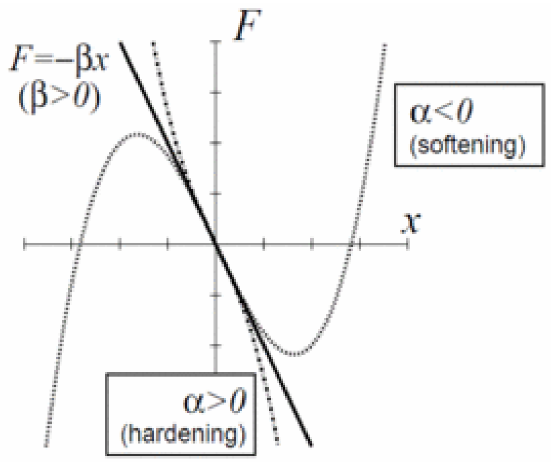


Figure 1. For β>0: a forced oscillator model with a spring whose restoring force is written as F=−βx−αx3

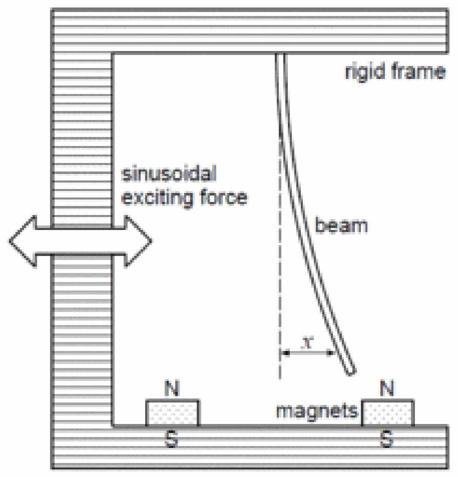
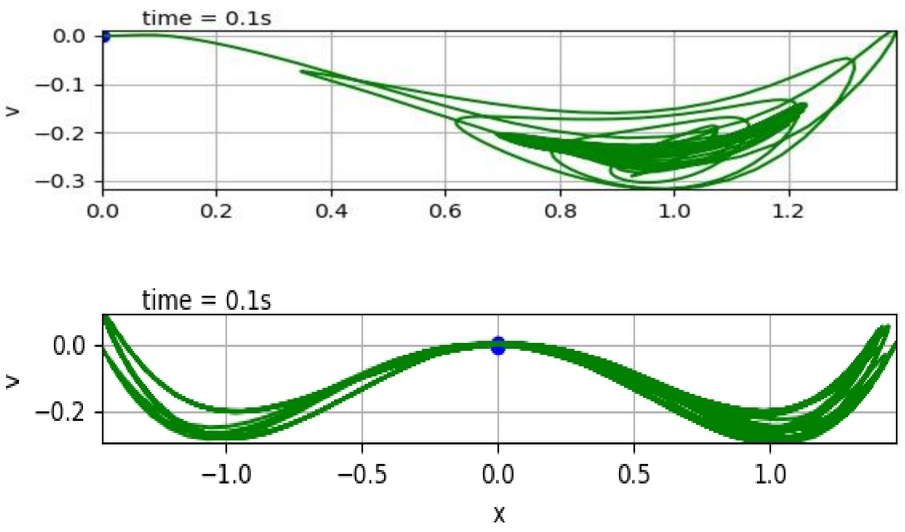


Figure 2. For β<0 a model of a periodically forced steel beam which is deflected toward the two magnets.
(Moon and Holmes, 1979; Guckenheimer and Holmes, 1983; Ott, 2002)

Duffing's Equation

- Nonlinear 2nd order ODE used to model damped driven systems

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \Upsilon\cos(\omega t)$$



where $x = displacement\ at\ time\ t,$

$\delta = damping\ coefficient,$

α = linear stiffness coefficient

β = nonlinearity in the restoring force

ϒ = amplitude of *excitation*

ω = angular frequency of excitation

Stability Check

- Eigenvalues of Jacobian Matrix of undamped, unforced system

$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \Upsilon\cos(\omega t)$ … (1)

$x\,(\alpha + \beta x^2) = 0$ … (2)

$\rightarrow x = 0;\;\; x = \pm\sqrt{\frac{-\alpha}{\beta}}$

For δ=0, one 2D → two 1D equations

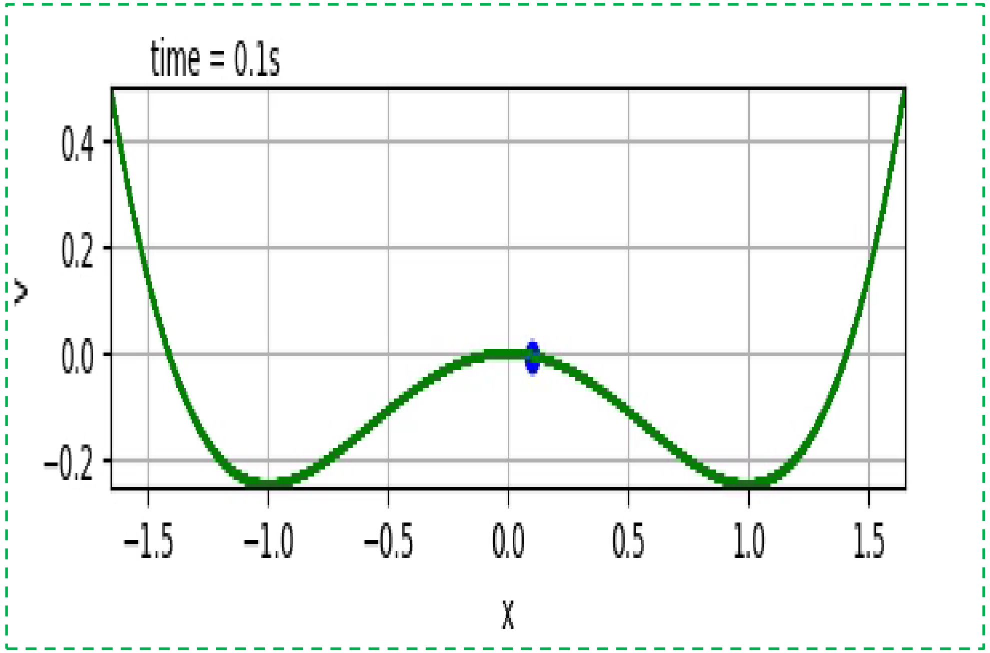
$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\delta\dot{x} - \alpha x - \beta x^3 \end{pmatrix}$$

$$J(x) = \begin{pmatrix} 0 & 1 \\ -\alpha - 3\beta x^2 & -\delta \end{pmatrix}$$

$$\lambda_{eig} = \frac{-\delta \pm \sqrt{\delta^2 - 4\alpha}}{2} \qquad \lambda_{eig} = \frac{-\delta \pm \sqrt{\delta^2 + 8\alpha}}{2}$$

$$x = 0 \qquad x = \pm\sqrt{\frac{-\alpha}{\beta}}$$

Stability Check



$$\lambda_{eig} = \frac{-\delta \pm \sqrt{\delta^2 - 4\alpha}}{2} \qquad \lambda_{eig} = \frac{-\delta \pm \sqrt{\delta^2 + 8\alpha}}{2}$$

$\delta = 0$, $\alpha = -1$:

at $x = 0$, $\lambda_{eig} = 1$

at $x = \pm 1$, $\lambda_{eig} \approx 0$

Numerical Approximation

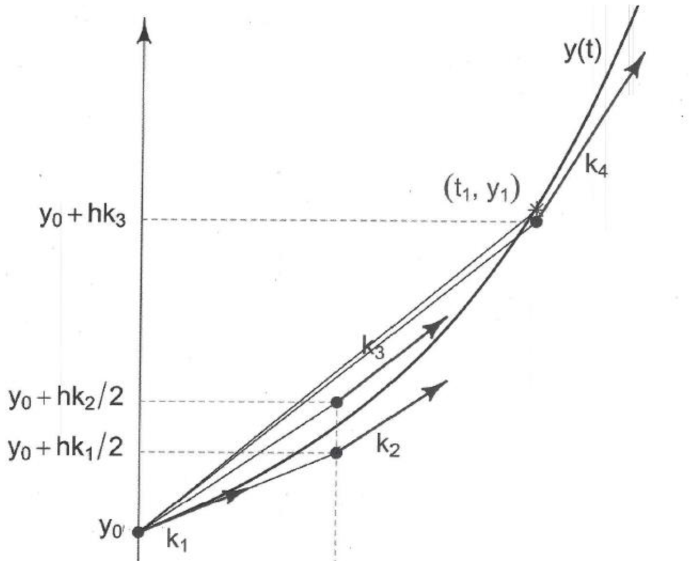


Figure 3. RK4 Method

- Fourier Series, Frobenius method, Homotopy analysis method (HAM)
- **4th order Runge-Kutta Method**

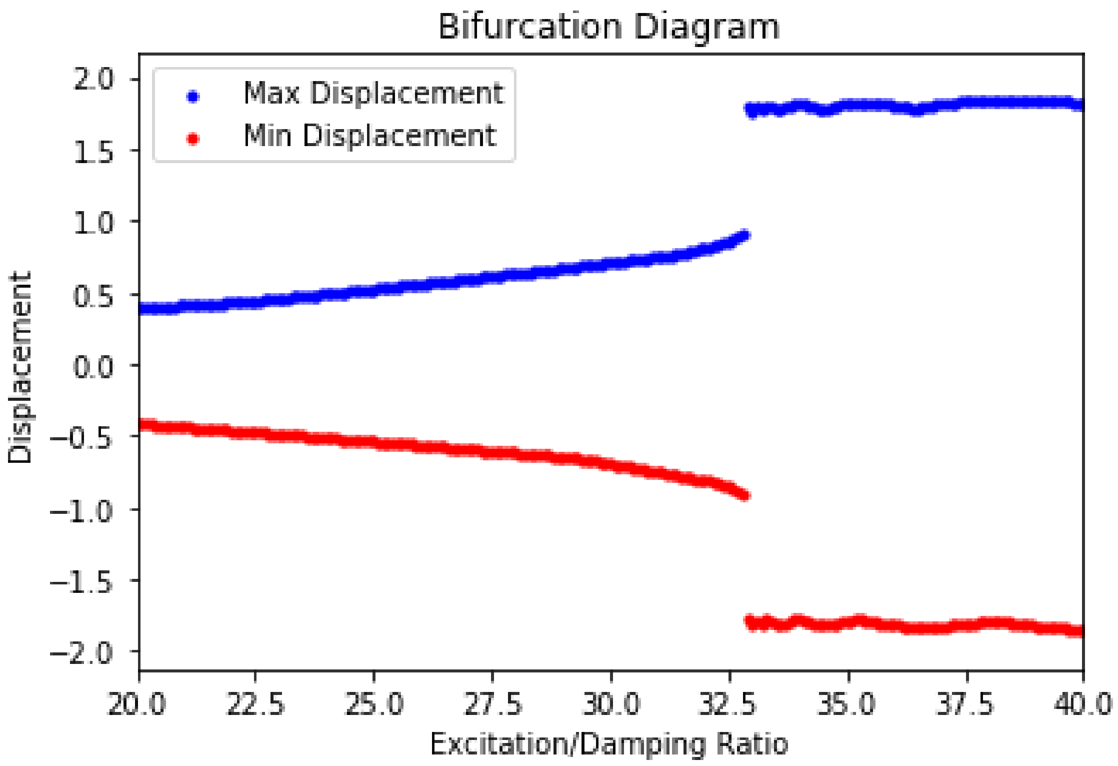
$$\mathrm{m}\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \beta x^3 = F(t)$$

- application:

$$\dot{x} = v$$
$$\dot{v} = -\delta v - \alpha x - \beta x^3$$
$$\dot{\theta} = w$$

TABLE 1 FOURTH-ORDER RUNGE-KUTTA IMPLEMENTATION VERIFICATION[1]

Step size (h)	Approximation of x at t = 0.1, x_{approx}	Final Global Error(F.G.E) at t = 0.1 (x_{approx} - x_{true})	Error Ratio
0.1	-0.001892171520724390	-0.0000000076565361900	-
0.05	-0.001892164298821270	-0.00000000043463307 00	17.6
0.025	-0.001892163889983960	-0.0000000000257957601	16.8
0.0125	-0.001892163865757750	-0.0000000000015695500	16.4
0.0001	-0.001892163864188200 (=x_{true})	(assume true value)	-



Bifurcation

- Def: Point where the change in the parameter values results in a change in the stability at the equilibrium point(s).

- Divergence at $\frac{\Upsilon}{\delta} \sim 33.0$

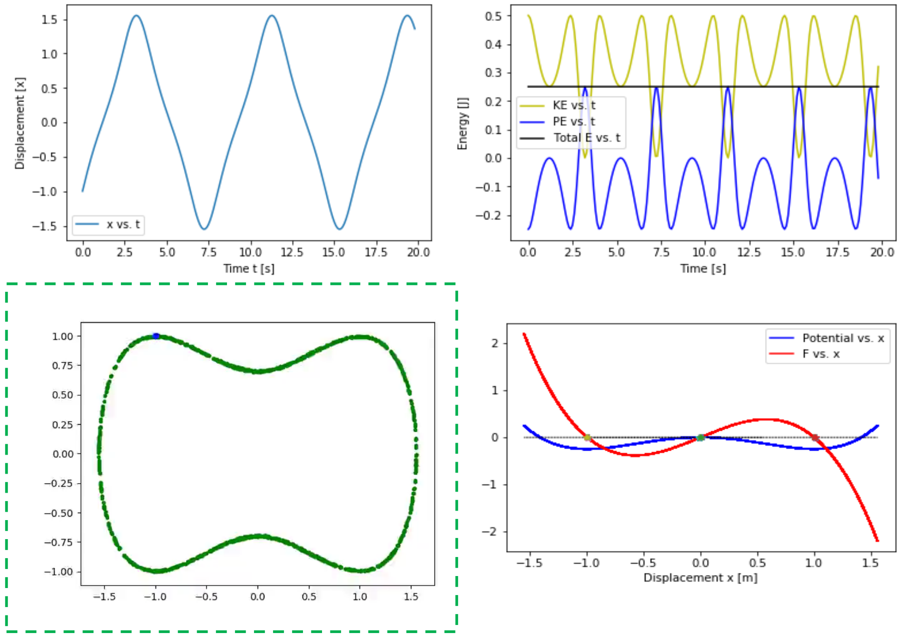
Initial conditions : x=0, v=0

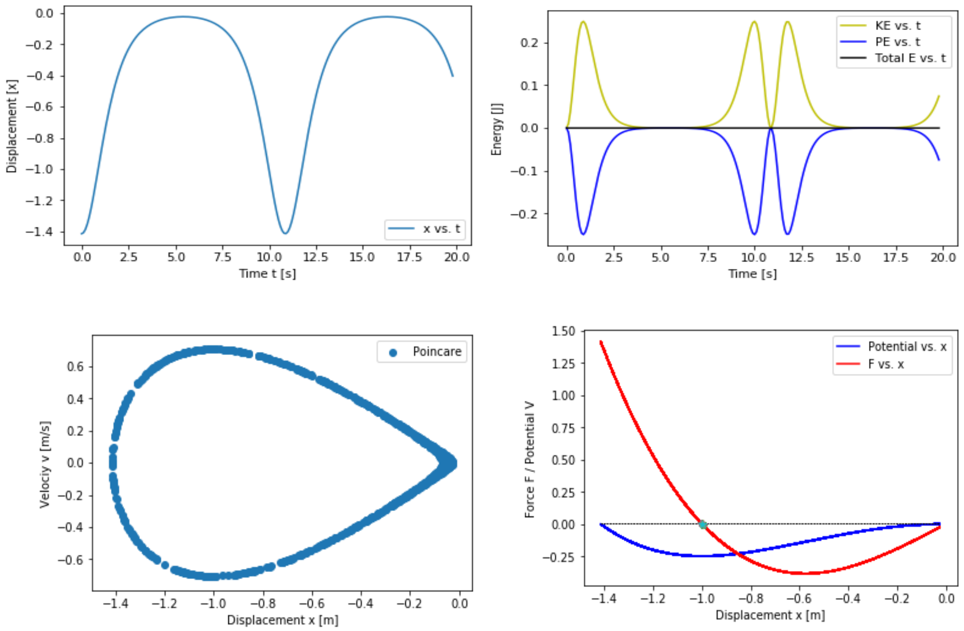
$\delta = 0.01$, $\alpha = 1$, $\beta = 1$, w = 1.414, h = 0.01

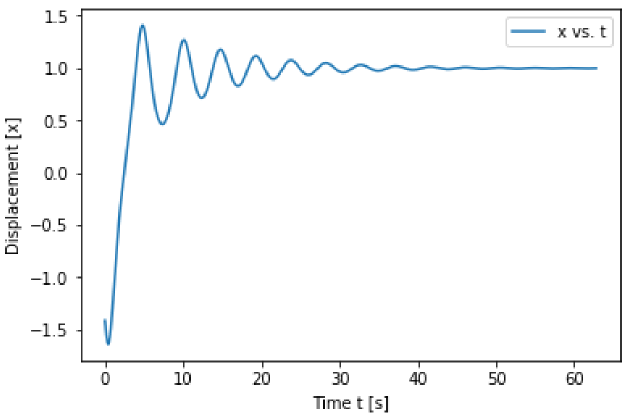
Free motion

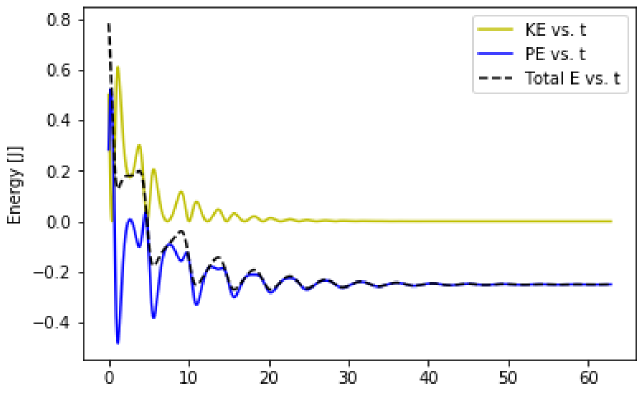
$\delta = 0, \alpha = -1, \beta = 1, \gamma = 0,$
$\mathrm{x} = \mathbf{-1}, \mathrm{v} = \mathbf{1}$

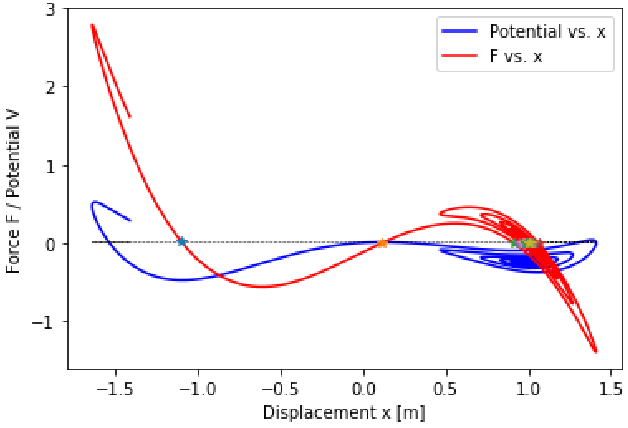
$\delta = 0, \alpha = -1, \beta = 1, \gamma = 0,$
$\mathrm{x} = \mathbf{-1.414}, \mathrm{v} = \mathbf{0}$

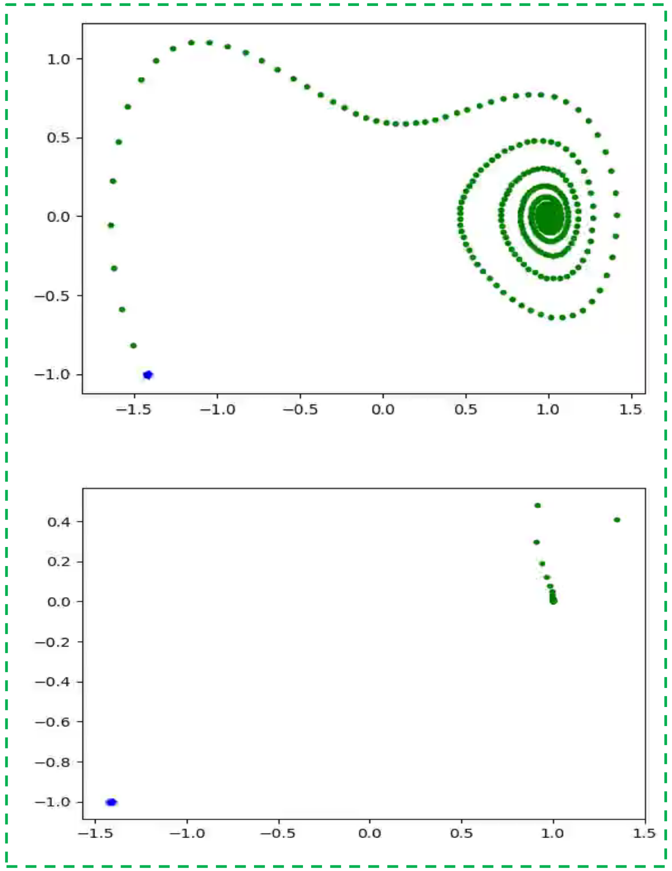












Damped Motion

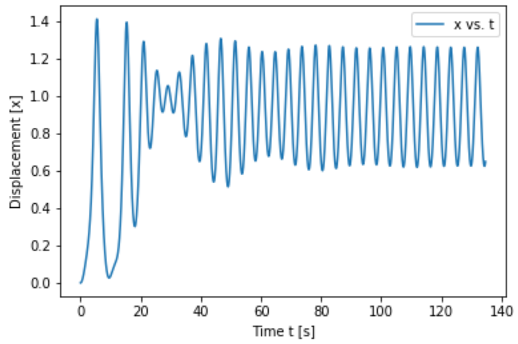
$\delta = \mathbf{0.2}, \alpha = -1, \beta = 1, \gamma = 0,$

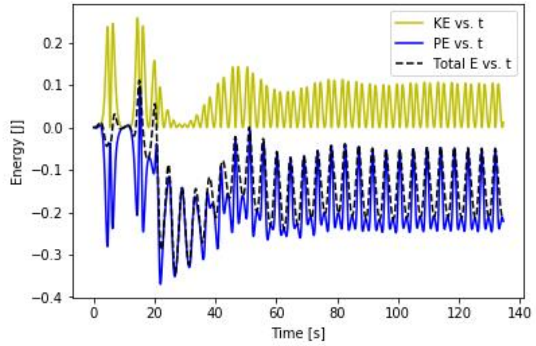
$\mathrm{x} = -1.414, \mathrm{v} = -1$

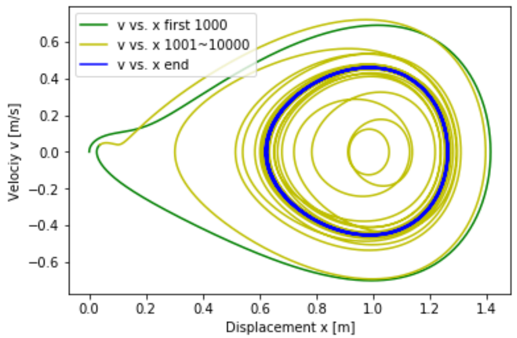
Damped Driven Motion

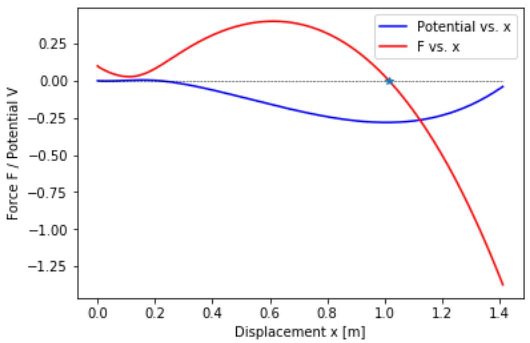
$\delta = 0.1, \alpha = -1, \beta = 1, \gamma = \mathbf{0.1}, \omega = 1.4$

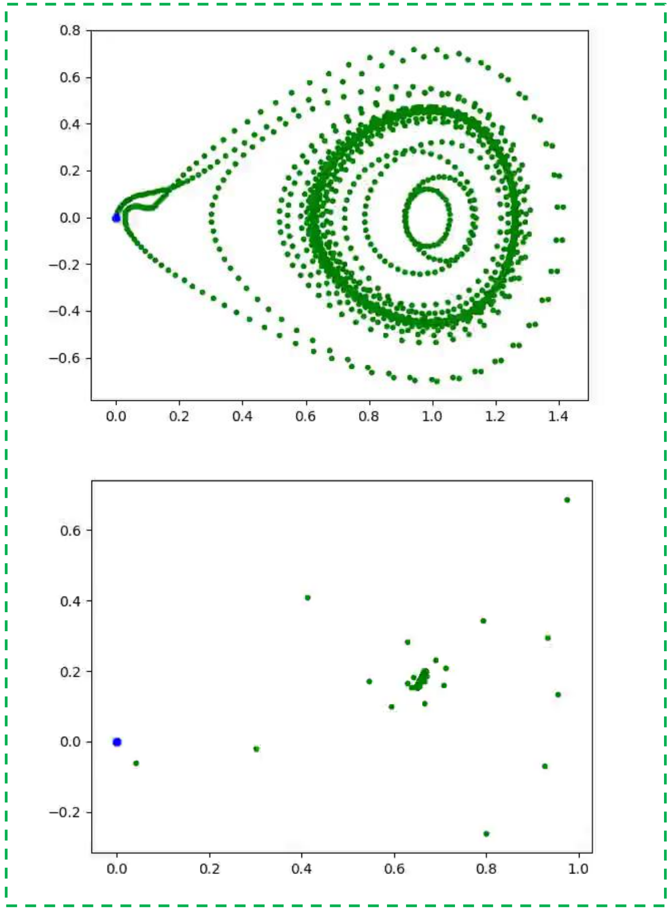
$\mathrm{x} = 0, \mathrm{v} = 0$







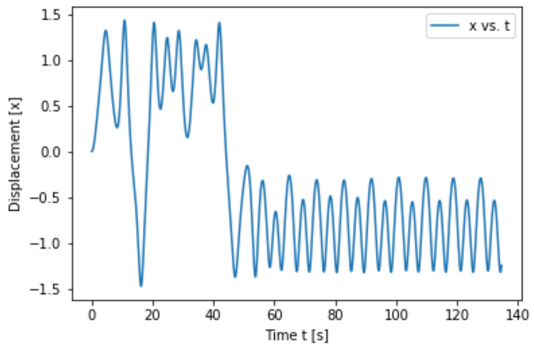


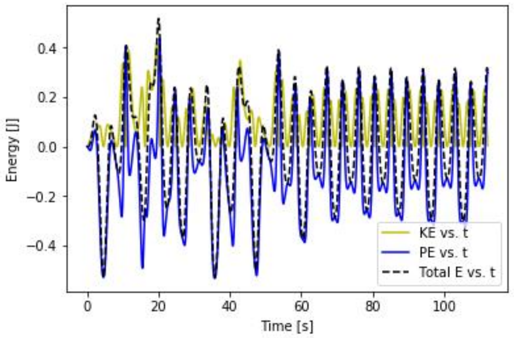


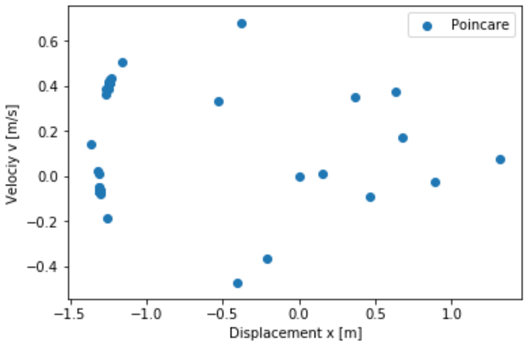
Damped Driven Motion2

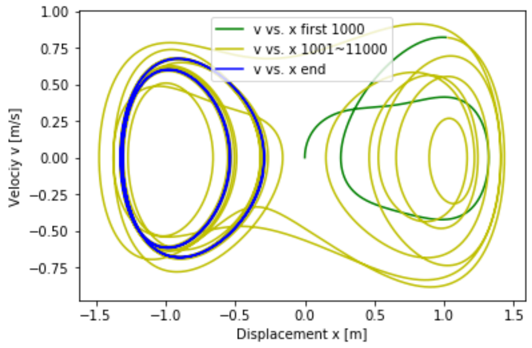
$\delta = 0.1, \alpha = -1, \beta = 1, \gamma = \mathbf{0.32}, \omega = 1.4$

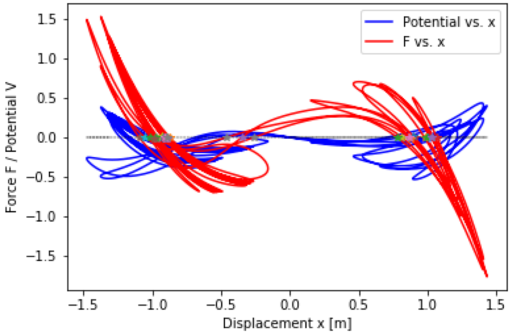
$\mathrm{x} = 0, \mathrm{v} = 0$

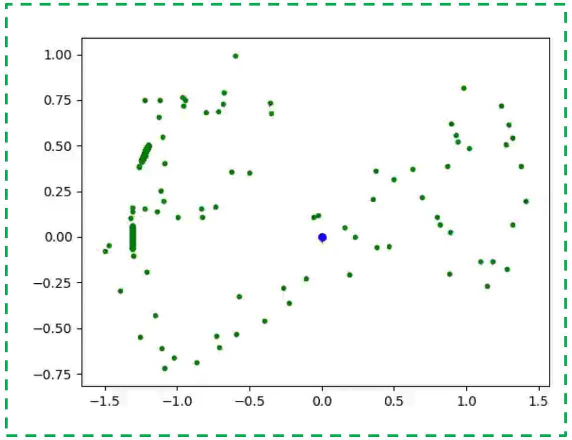








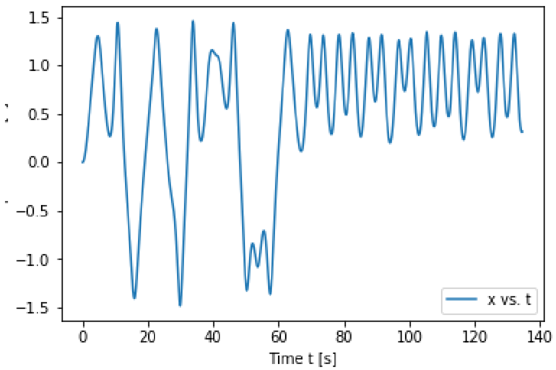


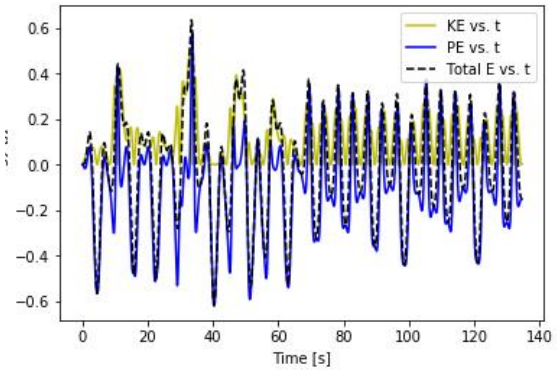


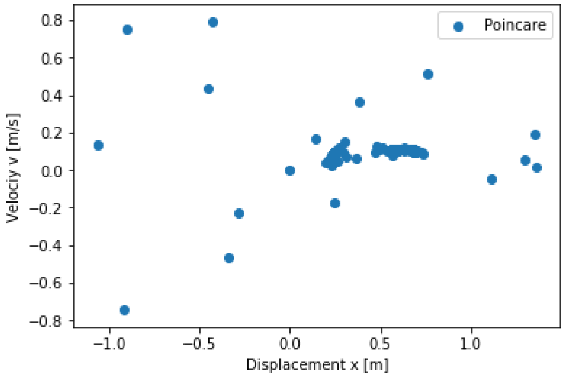
Damped Driven Motion3

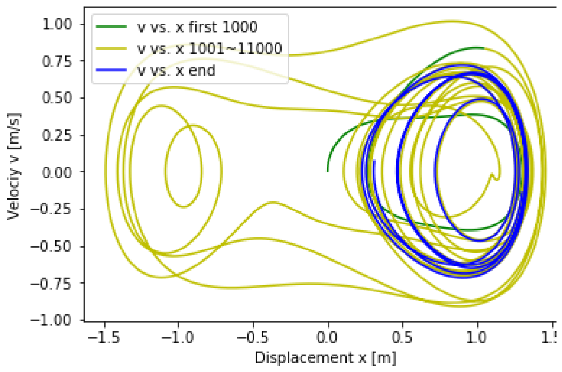
$\delta = 0.1, \alpha = -1, \beta = 1, \gamma = \mathbf{0.34}, \omega = 1.4$

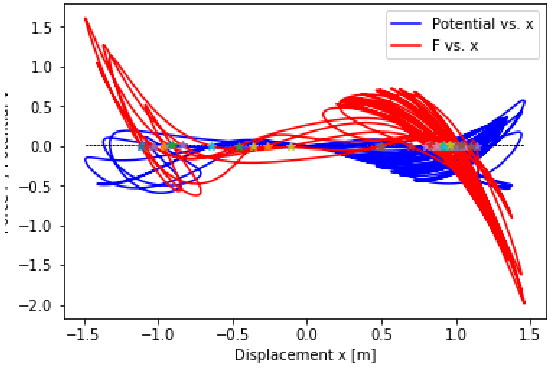
$x = 0, v = 0$







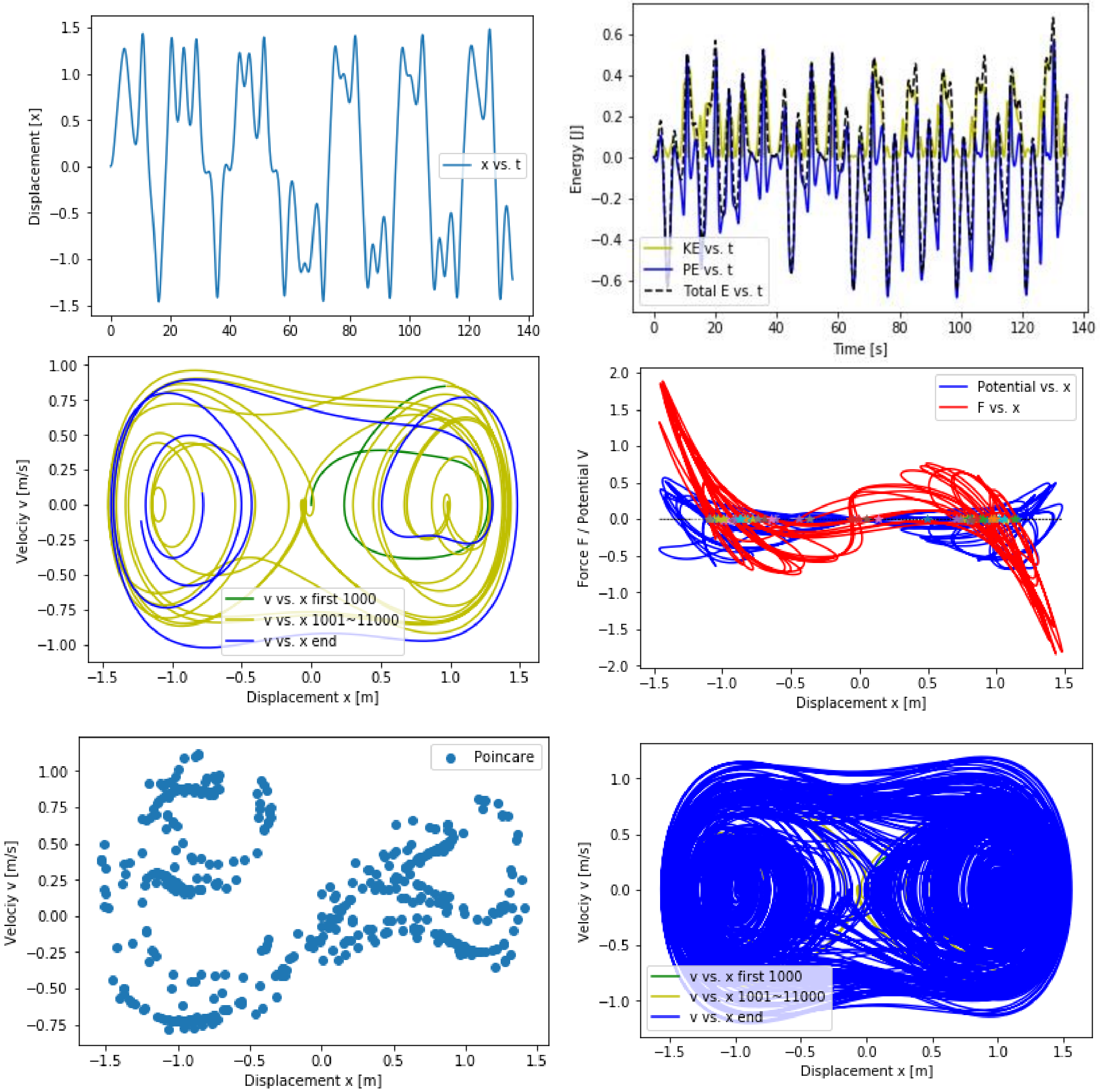


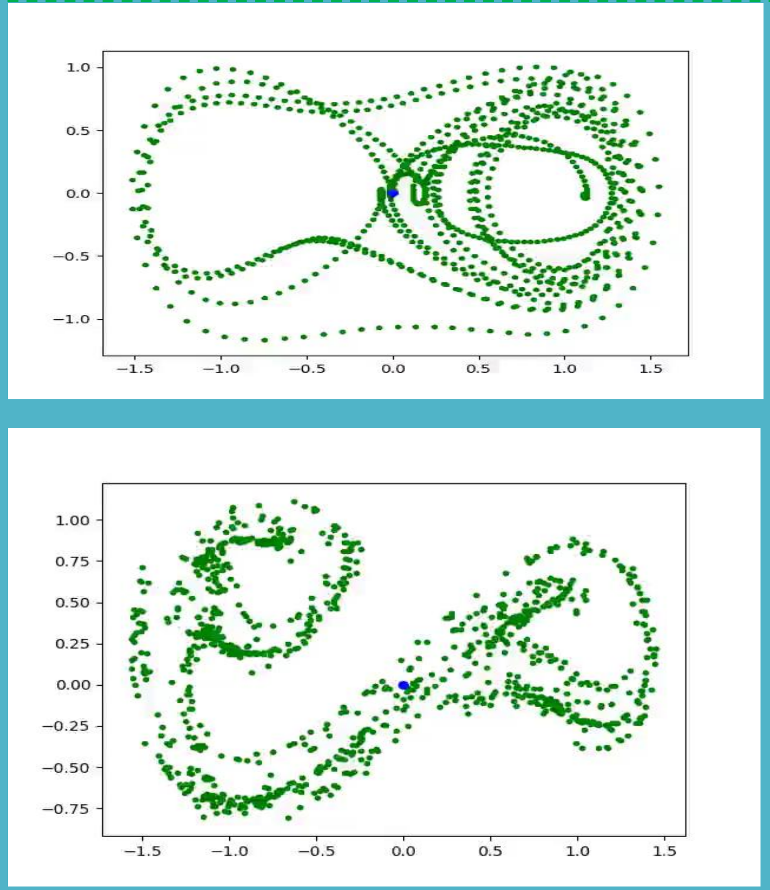


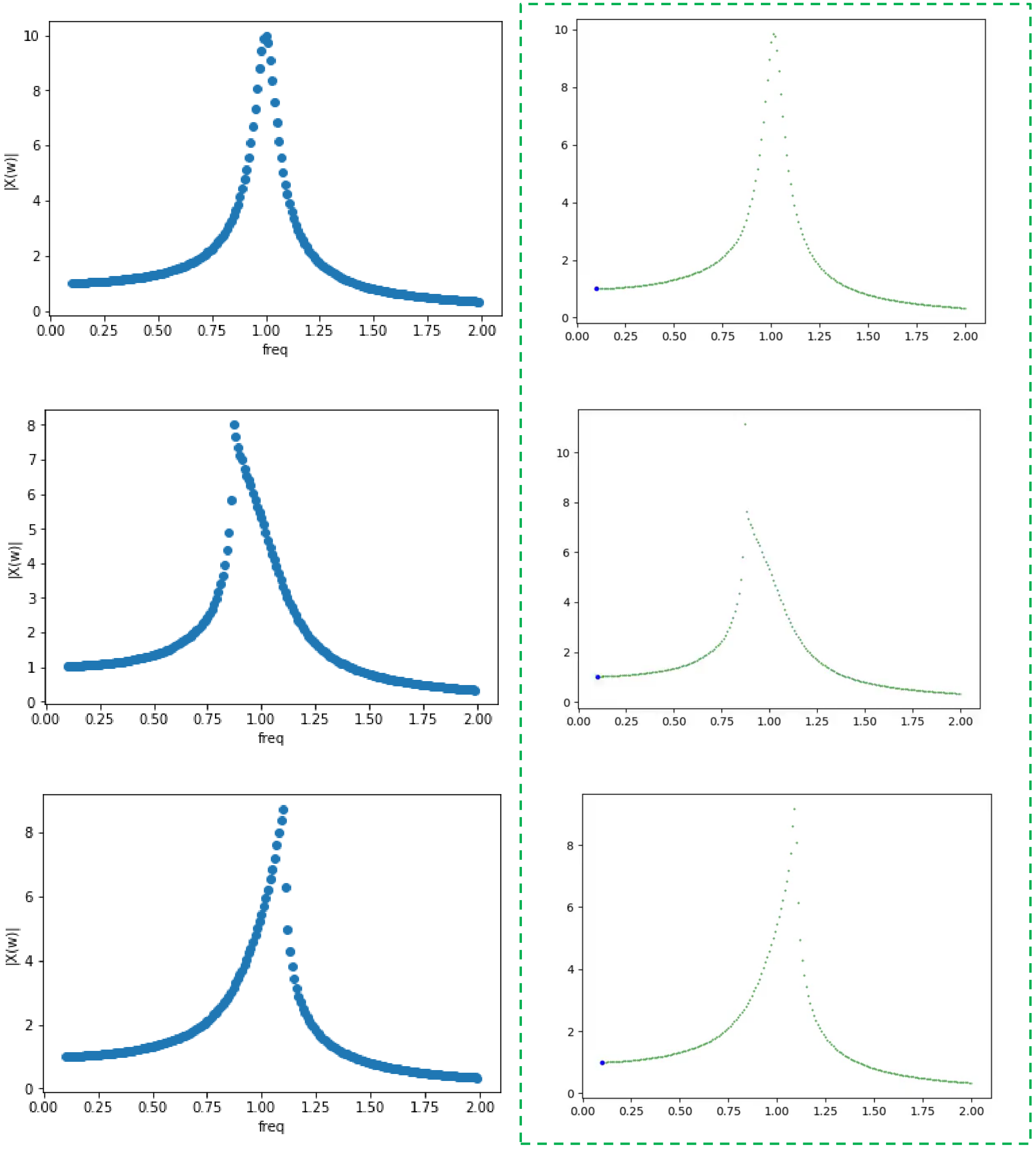
Damped Driven Motion$_4$: Chaos!

$\delta = 0.1, \alpha = -1, \beta = 1,$

$\gamma = \mathbf{0.38}, \omega = 1.4,\ x = 0, v = 0$







Frequency Response

* harmonic balance method

(i) Linear System (point of bifurcation)

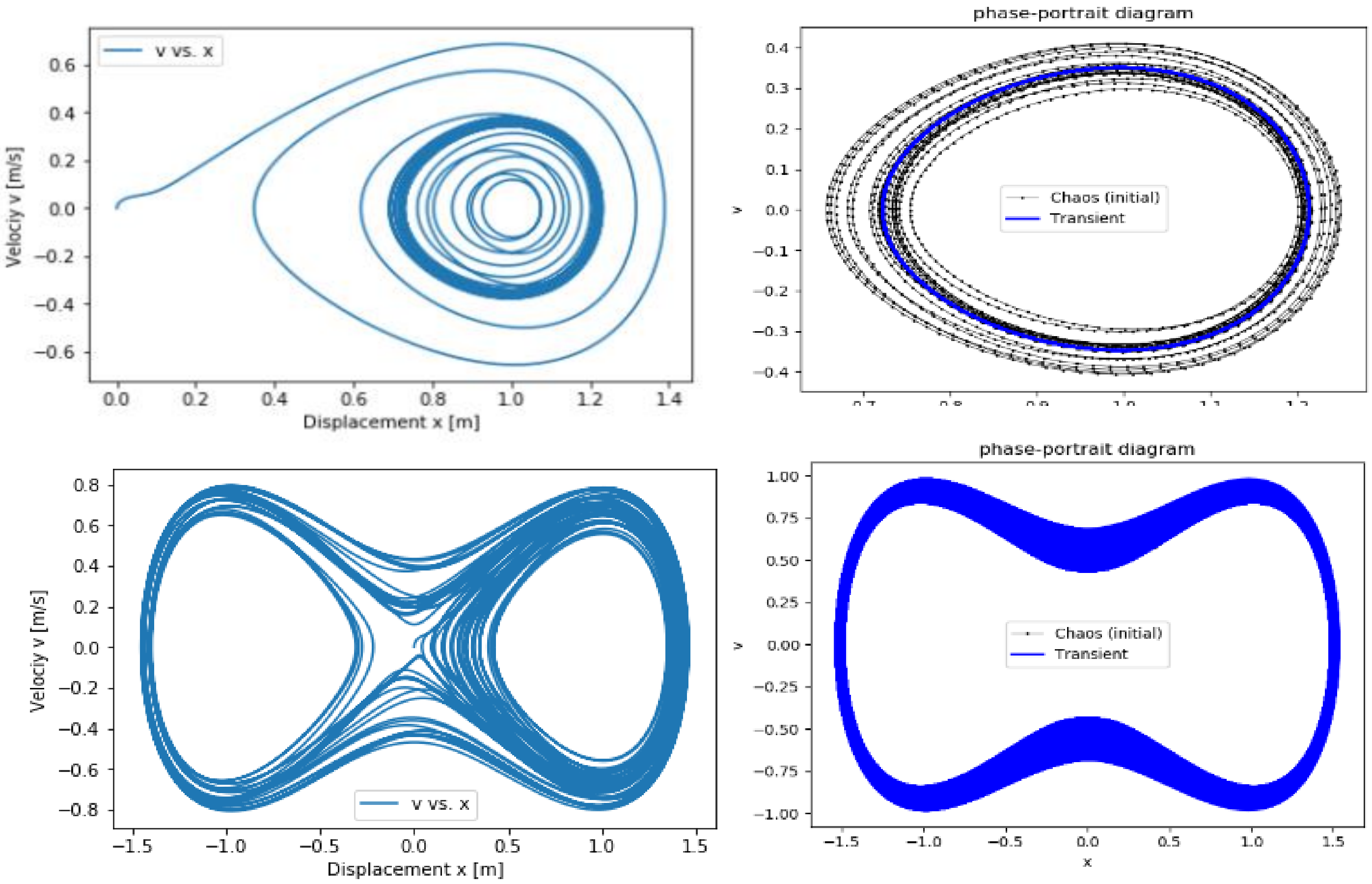
(ii) Softening System

(iii) Hardening system

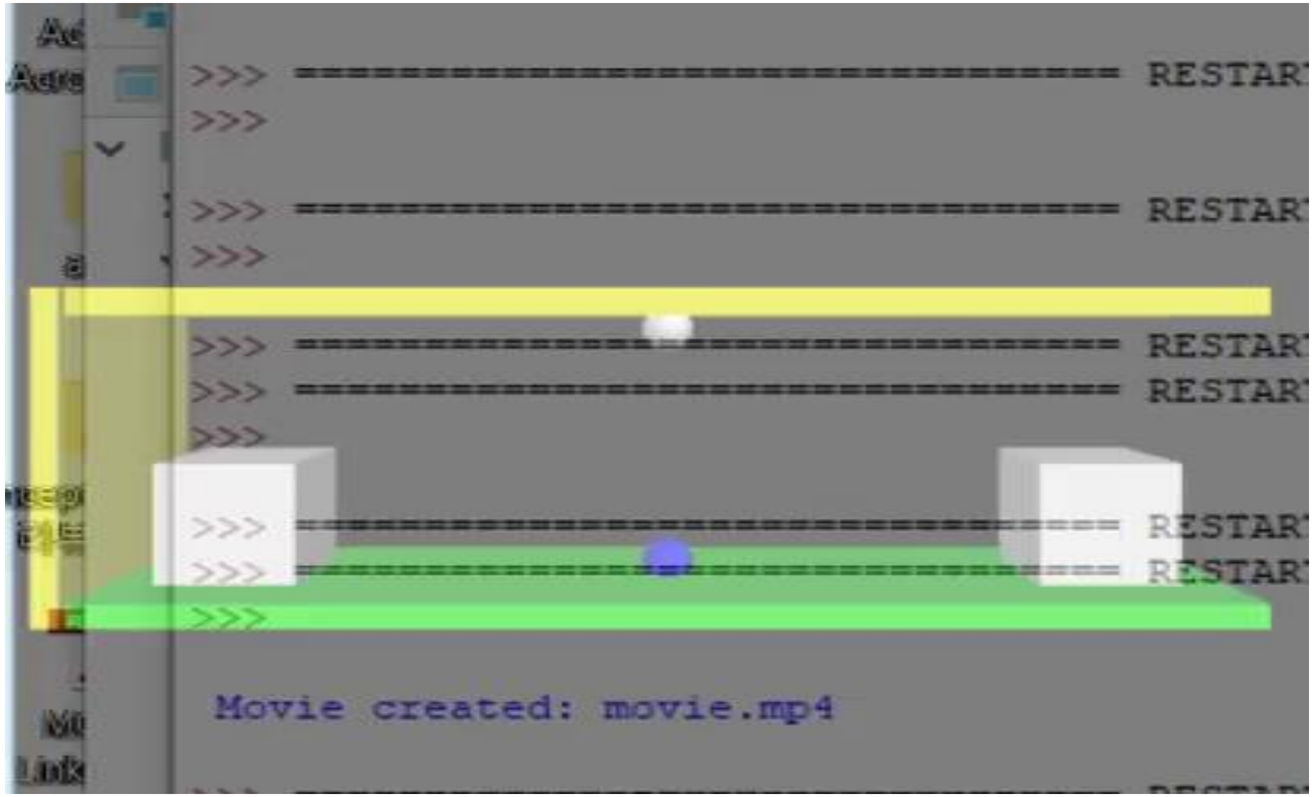
Built-in ‘Odeint’

(i) Single-Well

(ii) Double-Well



Visual (1D)



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