Rational lower/upper bounds on log

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The logarithm function has the following integral representation

$$\log(x) = \int_0^1 f(t,x)dt \tag{1}$$

where

$$f(t,x) = \frac{x-1}{t(x-1)+1}.$$

To get a rational approximation of log, we can discretize the integral (1) with nodes $t_1, \ldots, t_m \in [0,1]$ and weights $w_1, \ldots, w_m > 0$ to get

$$r(x) = \sum_{i=1}^m w_i f(t_i, x).$$

Any such approximation is automatically (operator) concave, because $f(t,x)$ is (operator) concave for any fixed $t \in [0,1]$. In [1] we used Gaussian quadrature: the resulting function $r(x) = r_m(x)$ happens to be the diagonal (m,m) Padé approximant of log. One disadvantage of this approximation is that it is neither an upper nor a lower bound on log; indeed we have $r_m(x) \leq \log(x)$ for $x \geq 1$, and $r_m(x) \geq \log(x)$ for $x \leq 1$.

In this short document we show how one can use a different quadrature rule on (1) to get rational approximations that are also upper/lower bounds on log. More precisely, we show that if we use Gauss-Radau quadrature then we get rational functions that are global bounds on log. Gauss-Radau quadrature is a variant of Gaussian quadrature where one of the endpoints of the integral interval is required to be a node of the quadrature formula.

It is useful to recall the following facts. For any $t \in (0,1]$ we have the following Taylor expansion of $f(t,x)$ around $x = 1$ (valid for $|t(x-1)| < 1$):

$$f(t,x) = (x-1)\sum_{k=0}^{\infty} (-1)^k (x-1)^k t^k.$$

If we let $\nu_m = \sum_{i=1}^m w_i \delta_{t_i}$ be the discrete measure used to approximate the integral (1) then we get

$$\log(x) - r(x) = (x-1)\sum_{k=0}^{\infty}(-1)^k(x-1)^k \left(\int_0^1 t^k dt - \int_0^1 t^k d\nu_m(t)\right). \tag{2}$$

With the m-point Gaussian quadrature, the first $2m-1$ moments of ν_m agree with those of $\lambda|_{[0,1]}$ (the uniform measure on $[0,1]$) and so we get that $\log(x) - r_m(x) = O((x-1)^{2m+1})$ (for $x \to 1$).

Rational lower bound for the logarithm Consider an m-point Gauss-Radau quadrature of the form:

$$\int_0^1 g(t)dt \approx \sum_{i=1}^{m-1} w_i g(t_i) + w_m g(1) \tag{3}$$

that is satisfied with equality for all polynomials g of degree up to $2m-2$. If we use this quadrature formula in (1) we get the following rational approximation of log:

$$r_m^1(x) = \sum_{i=1}^{m-1} w_i f(t_i, x) + w_m f(1,x) = \sum_{i=1}^{m-1} w_i f(t_i,x) + w_m \frac{x-1}{x}.$$

Proposition 1. *$r_m^1(x)$ is a (m,m) rational function, that satisfies the following: (a) $\log(x) - r_m^1(x) = O((x-1)^{2m})$ for $x \to 1$, and (b) $\log(x) \geq r_m^1(x)$ for all $x > 0$.*

Proof. (a) The fact that the Gauss-Radau formula (3) is exact for all polynomials of degree up to $2m-2$ means that the terms $k = 0, \dots, 2m-2$ in (2) are equal to 0. Thus this gives $\log(x) - r_m^1(x) = O((x-1)^{2m})$.

(b) Since $f(1,x) = (x-1)/x$ note that 0 is a pole of $r_m^1(x)$. This fact, combined with $\log(x) - r_m^1(x) = O((x-1)^{2m})$ allows us to show that $\log(x) \geq r_m^1(x)$. Indeed, write $r_m^1(x) = F(x)/G(x)$ where F and G are polynomials of degree m and with $G(0) = 0$. Let $\delta(x) = \log(x) - r_m^1(x)$. Differentiating the identity $\delta(x) = O((x-1)^{2m})$ we get that $\delta'(x) = \frac{1}{x} - \frac{F'G - FG'}{G^2} = O((x-1)^{2m-1})$, i.e., $G^2 - x(F'G - FG') = O((x-1)^{2m-1})$. Since $G(0) = 0$ we can write $G(x) = xH(x)$ which gives $xH^2 - (F'G - FG') = O((x-1)^{2m-1})$. Since $xH^2 - (F'G - FG')$ has degree at most $2m-1$ it must be that it is $a(x-1)^{2m-1}$ for some constant a. This means that $\delta'(x) = \frac{a(x-1)^{2m-1}}{xH^2}$. With $\delta(0) = 0$ this means that $\delta(x)$ has constant sign for all $x > 0$, and is same sign as a. We need to show that $a > 0$. Since $r_m^1(x)$ is a (m,m) rational function we know that $\lim_{x\to\infty} r_m^1(x) < \infty$ and so $\delta(x) > 0$ for $x \to \infty$. This means that $a > 0$, and $\log(x) \geq r_m^1(x)$ for all $x > 0$, as desired. $\square$

Remark 1. Since $f(t,x)$ is decreasing with t, one also has the bound $r_m^1(x) \geq 1 - 1/x$.

Rational upper bound on the logarithm To get an upper bound on log, we consider an m-point Gauss-Radau quadrature of the form:

$$\int_0^1 g(t)dt \approx w_1 g(0) + \sum_{i=2}^{m} w_i g(t_i) \tag{4}$$

that is satisfied with equality for all polynomials g of degree up to $2m-2$. If we use this quadrature formula in (1) we get the following rational approximation of log:

$$r_m^0(x) = w_1 f(1,x) + \sum_{i=2}^{m} w_i f(t_i, x) = w_1(x-1) + \sum_{i=1}^{m-1} w_i f(t_i, x).$$

Proposition 2. *$r_m^0(x)$ is a $(m, m-1)$ rational function that satisfies the following: (a) $\log(x) - r_m^0(x) = O((x-1)^{2m})$ for $x \to 1$, and (b) $\log(x) \leq r_m^0(x)$ for all $x > 0$.*

Proof. (a) Same proof as Proposition 1

(b) Write $r_m^0(x) = F(x)/G(x)$ where $\deg F = m$ and $\deg G = m-1$ are polynomials. Let $\delta(x) = \log(x) - r_m^0(x)$. Differentiating the identity $\delta(x) = O((x-1)^{2m})$ we get that $\delta'(x) = \frac{1}{x} - \frac{F'G - FG'}{G^2} = O((x-1)^{2m-1})$, i.e., $G^2 - x(F'G - FG') = O((x-1)^{2m-1})$. Since $G^2 - x(F'G - FG')$ has degree at most $2m-1$ it must be that it is $a(x-1)^{2m-1}$ for some constant a. This means that $\delta'(x) = \frac{a(x-1)^{2m-1}}{G^2}$. With $\delta(0) = 0$ this means that $\delta(x)$ has constant sign for all $x > 0$, and is same sign as a. We need to show that $a < 0$. Since $r_m^0(x)$ has no poles in $[0, \infty)$ we know that $r_m^0(0) > -\infty$. This means that $a < 0$ and $r_m^0(x) > \log(x)$ for all $x > 0$. $\square$

Remark 2. Note that r_m^0 is the $(m, m-1)$ Padé approximant of log, and several properties about this rational function are derived in [2]. For example from [2, Eq. (27)] it follows that $r_{m+1}^0(x) \leq r_m^0(x)$. Also note that r_m^1 and r_m^0 are related by

$$r_m^0(x^{-1}) = -r_m^1(x)$$

since $f(t, x^{-1}) = -f(1-t, x)$. As such, properties derived for r_m^0 can be transferred to r_m^1 and vice-versa. Recent works that use these rational approximations are [5] and [6]. Using similar ideas as in [1, Prop. 6], one can show that these rational functions converge uniformly to log on any compact segment $[\epsilon, \epsilon^{-1}] \subset (0, \infty)$ at a rate $\approx (1 - \sqrt{\epsilon})^{2m}$.

Computing nodes/weights for Gauss-Radau quadrature Gauss-Radau quadrature nodes and weights can be obtained from Gaussian quadrature rather easily. Indeed it is straightforward to show the following.

Proposition 3. *Let $\beta_1, \ldots, \beta_{m-1} > 0$ and $t_1, \ldots, t_{m-1} \in (0,1)$ be the $(m-1)$-point Gaussian quadrature rule on $(0,1)$ for the weight function $\omega(t) = 1-t$. Then the Gauss-Radau quadrature on $[0,1]$ with endpoint at 1 (Equation (3)) is given by the nodes $t_1, \ldots, t_{m-1}, 1$ and weights $w_i = \beta_i/(1-t_i)$ for $i = 1, \ldots, m-1$ and $w_m = 1 - \sum_{i=1}^{m-1} w_i$.*

Proof. Let g be a polynomial of degree $2m-2$. We can write $g = (1-t)q + g(1)$ where $\deg q \le 2m-3$. Then

$$\int_0^1 g(t)dt = \int_0^1 q(t)(1-t)dt + g(1) = \sum_{i=1}^{m-1} \beta_i q(t_i) + g(1) = \sum_{i=1}^{m-1} \beta_i \frac{g(t_i) - g(1)}{1-t_i} + g(1) = \sum_{i=1}^{m-1} w_i g(t_i) + w_m g(1).$$

□

There are other methods to compute the nodes/weights for Gauss-Radau quadrature e.g., by computing the eigenvalues/eigenvectors of a modified Jacobi matrix, see [3, Section 7] or [4, Section 1]. See also [7, p. 103] for explicit formulae involving Legendre polynomials.

References

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