Section 8.5 – Integration of Rational Functions by Partial Fractions

Integral of the form: $\int \frac{p(x)}{q(x)}\,dx$ where $p(x)$ and $q(x)$ are polynomials

① If $\deg(p(x)) < \deg(q(x)) \Rightarrow$ proper fraction (Use partial fractions)

② If $\deg(p(x)) \geq \deg(q(x)) \Rightarrow$ improper fraction (Use Euclidean division)

① Proper fractions:

Classical method of partial fractions:

Example: $I = \int \frac{5x-3}{x^2-2x-3}\,dx$

Factorize: $x^2-2x-3 = a(x-x_1)(x-x_2)$

$$= \underbrace{(x+1)(x-3)}_{\text{linear factors of degree 1.}}$$

Write: $\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

Use identification to find A and B

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$$

$$\Rightarrow 5x-3 = Ax+Bx-3A+B$$

$$5x-3 = (A+B)x-3A+B$$

$$\begin{cases} A+B = 5 & ① \\ -3A+B = -3 & ② \end{cases} \Longrightarrow ① - ② \Rightarrow 4A = 8 \Rightarrow \boxed{A = 2}$$

$$\boxed{B = 3}$$

$$\Rightarrow \int \frac{5x-3}{(x+1)(x-3)}\,dx = \int \left(\frac{2}{x+1} + \frac{3}{x-3}\right) dx$$

$$= 2\int \frac{1}{x+1}\,dx + 3\int \frac{1}{x-3}\,dx = 2\ln|x+1| + 3\ln|x-3| + C$$

General Description of the method of partial fractions:

Step 1: Factorize $q(x)$

Step 2: * let $(x-r)$ be a linear factor of $q(x)$.

Assume $(x-r)^m$ is the highest power of $(x-r)$ that divides $q(x)$.

Corresponding to this factor, we assign the following partial fractions:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$$

* Let (x^2+sx+t) be a quadratic factor of $q(x)$. (unfactorizable)

Assume $(x^2+sx+t)^n$ is the highest power of this factor that divides $q(x)$. We assign the following partial fractions:

$$\frac{B_1x+C_1}{x^2+sx+t} + \frac{B_2x+C_2}{(x^2+sx+t)^2} + \cdots + \frac{B_nx+C_n}{(x^2+sx+t)^n}$$

Step 3: $\frac{p(x)}{q(x)}$ = sum of all partial fractions.

Solve for $A_1, A_2, \dots, A_m, B_1, C_1, B_2, C_2, \dots, B_n, C_n$.

Example: $\int \frac{dx}{x(x^2+1)^2}$

$$\frac{1}{x(x^2+1)^2} = \underbrace{\frac{A}{x}}_{\text{linear}} + \underbrace{\frac{Bx+C}{x^2+1}}_{\text{quadratic (unfactorizable)}} + \frac{Dx+E}{(x^2+1)^2}$$

$$\frac{1}{x(x^2+1)^2} = \frac{A(x^2+1)^2 + (Bx+C)(x^2+1)x + (Dx+E)x}{x(x^2+1)^2}$$

$$1 = A(x^4+2x^2+1) + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex$$

$$1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$$\begin{cases} A+B=0 \\ \boxed{C=0} \\ 2A+B+D=0 \\ C+E=0 \\ \boxed{A=1} \end{cases} \Rightarrow \begin{array}{l} A+B=0 \Rightarrow \boxed{B=-1} \\ C+E=0 \Rightarrow \boxed{E=0} \\ D=-2A-B \Rightarrow \boxed{D=-1} \end{array}$$

$$I = \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2} \right] dx$$

$$= \int \left(\frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2} \right) dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + \frac{1}{2(x^2+1)} + C$$

The Heavyside Cover-up method for linear factors

Examples:

1) $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} dx$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

* To find A, multiply both sides by $(x-1)$ and let $x=1$.

$$\frac{x^2+4x+1}{(x+1)(x+3)} = A + \frac{B(x-1)}{x+1} + \frac{C(x-1)}{x+3}$$

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$$\Longrightarrow A = \frac{1^2 + 4(1) + 1}{(1+1)(1+3)} = \frac{6}{+8} = \frac{3}{+4}$$

* To find B, multiply both sides by $(x+1)$ and let $x = -1$.

$$B = \frac{(-1)^2 + 4(-1) + 1}{(-1-1)(-1+3)} = \frac{-2}{-4} = \frac{1}{2}$$

* Similarly, $C = \dfrac{(-3)^2 + 4(-3) + 1}{(-3-1)(-3+1)} = \dfrac{-2}{8} = \dfrac{-1}{4}$

$$\Longrightarrow \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \int\left(\frac{3/4}{x-1} + \frac{1/2}{x+1} + \frac{-1/4}{x+3}\right)dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C$$

2) $J = \displaystyle\int \frac{2x+5}{(x+1)^2(x-2)}\,dx$

$$\frac{2x+5}{(x+1)^2(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

* To find C, cover up : $C = \dfrac{2(2)+5}{(2+1)^2} = \dfrac{9}{9} = 1$

* Find B, cover up: $B = \dfrac{2(-1)+5}{-1-2} = \dfrac{3}{-3} = -1$
multiply by $(x+1)^2$

* To find A, (cover up doesn't work) take $x = 0$:

$$\frac{5}{1(0-2)} = \frac{A}{1} + \frac{-1}{1} + \frac{1}{-2}$$

$$\frac{-5}{2} = A - 1 - \frac{1}{2} \Longrightarrow A = \frac{-5}{2} + \frac{3}{2} = -1$$

$$\Rightarrow J = \int \left(\frac{-1}{x+1} + \frac{-1}{(x+1)^2} + \frac{1}{x-2} \right) dx$$

$$= -\ln|x+1| + \frac{1}{x+1} + \ln|x-2| + C$$

$$\int \frac{-1}{(x+1)^2}\, dx$$

$$\int -\frac{du}{u^2} = \frac{1}{u}$$

where $u = x+1$

3) $I = \int \frac{x-1}{(x+1)^3}\, dx$

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

* To find C, cover up: $C = -1-1 = -2$

* To find A and B:

Take $x=0$: $\frac{-1}{1} = \frac{A}{1} + \frac{B}{1} - \frac{2}{1} \Rightarrow \boxed{A+B=1}$

Take $x=1$: $0 = \frac{A}{2} + \frac{B}{4} - \frac{2}{8} \Rightarrow \boxed{2A+B=1}$

$$\begin{cases} A+B=1 & (1) \\ 2A-B=1 & (2) \end{cases} \Rightarrow \boxed{A=0} \text{ and } \boxed{B=1}$$

$$\Rightarrow I = \int \left(\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3} \right) dx$$

$$= \frac{-1}{x+1} + \frac{1}{(x+1)^2} + C$$

$$\int \frac{-2}{(x+1)^3}\, dx$$

$u = x+1$

$du = dx$

$$\int \frac{-2\,du}{u^3} = \int -2u^{-3}\, du$$

$$= -\frac{2u^{-2}}{-2} = \frac{1}{u^2}$$

$$= \frac{1}{(x+1)^2}$$

② Improper Fractions:

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx \Longrightarrow$$

Use Euclidean division to write as:

$$\text{quotient} + \frac{\text{Remainder}}{\text{divisor}}$$

$$\begin{array}{r|l} & \text{quotient} \\ \hline \text{divisor} & \vdots \\ & \overline{\text{remainder}} \end{array}$$

$$\begin{array}{r|l} & 2x \\ \hline x^2-2x-3 & 2x^3-4x^2-x-3 \\ & \ominus\ 2x^3-4x^2-6x \\ \hline & \qquad 5x-3 \end{array}$$

$$\Rightarrow I = \int\left(2x + \frac{5x-3}{x^2-2x-3}\right)dx = \int\left(2x + \frac{5x-3}{(x+1)(x-3)}\right)dx$$

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

* Cover up to find A: $A = \frac{-5-3}{-4} = \frac{-8}{-4} = 2$

* Cover up to find B: $B = \frac{15-3}{4} = \frac{12}{4} = 3$

$$\Rightarrow I = \int\left(2x + \frac{2}{x+1} + \frac{3}{x-3}\right)dx$$

$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C$$

Exercises:

11) $\int \frac{x+4}{x^2+5x-6}\,dx = \int \frac{x+4}{(x-1)(x+6)}\,dx$

$$\frac{x+4}{(x-1)(x+6)} = \frac{A}{x-1} + \frac{B}{x+6}$$

Cover up to find A: $A = \frac{5}{7}$

Cover up to find B: $B = \frac{-6+4}{-6-1} = \frac{-2}{-7} = \frac{2}{7}$

$$\Rightarrow \int \frac{x+4}{(x-1)(x+6)}\,dx = \int \left(\frac{5/7}{x-1} + \frac{2/7}{x+6} \right) dx$$

$$= \frac{5}{7}\ln|x-1| + \frac{2}{7}\ln|x+6| + C$$

16) $\int \frac{x+3}{2x^3-8x}\,dx = \int \frac{x+3}{2x(x^2-4)}\,dx = \int \frac{x+3}{2x(x-2)(x+2)}\,dx$

$$\frac{x+3}{2x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

Cover up to find A: $A = \frac{3}{2(-2)2} = \frac{-3}{8}$

Cover up to find B: $B = \frac{5}{16}$

Cover up to find C: $C = \frac{1}{16}$

$$\Rightarrow \int \left(\frac{-3/8}{x} + \frac{5/16}{x-2} + \frac{1/16}{x+2} \right) dx$$

$$= \frac{-3}{8}\ln|x| + \frac{5}{16}\ln|x-2| + \frac{1}{16}\ln|x+2| + C$$

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21) $\int_0^1 \frac{dx}{(x+1)(x^2+1)}$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

Cover up to find A: $A = \frac{1}{2}$

$$\frac{1}{(x+1)(x^2+1)} = \frac{\frac{1}{2}(x^2+1) + (Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$$\Rightarrow 1 = \frac{1}{2}x^2 + \frac{1}{2} + Bx^2 + Bx + Cx + C$$

$$1 = \left(\frac{1}{2} + B\right)x^2 + (B+C)x + C + \frac{1}{2}$$

$$B + \frac{1}{2} = 0 \Rightarrow \boxed{B = -\frac{1}{2}}$$

$$B + C = 0 \Rightarrow \boxed{C = \frac{1}{2}}$$

$$\Rightarrow \int_0^1 \frac{dx}{(x+1)(x^2+1)} = \int_0^1 \left(\frac{1/2}{x+1} + \frac{-1/2\,x + 1/2}{x^2+1}\right) dx$$

$$= \frac{1}{2}\int_0^1 \frac{1}{x+1}\,dx - \frac{1}{2}\int_0^1 \frac{x-1}{x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int_0^1 \frac{x}{x^2+1}\,dx + \frac{1}{2}\int_0^1 \frac{1}{x^2+1}\,dx$$

$$= \left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\tan^{-1}x\right]_0^1$$

$$= \frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 + \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\ln 1 + \frac{1}{4}\ln 1 - \frac{1}{2}\cdot 0$$

$$= \frac{1}{4}\ln 2 + \frac{\pi}{8}$$

36) $$\int \frac{16x^3}{4x^2-4x+1}\,dx$$

$$\begin{array}{r|l} & 4x+4 \\ \hline 4x^2-4x+1 & 16x^3 \\ & \ominus\ 16x^3-16x^2+4x \\ \hline & 16x^2-4x \\ & \ominus\ 16x^2-16x+4 \\ \hline & 12x-4 \end{array}$$

$$\int \frac{16x^3}{4x^2-4x+1}\,dx = \int\left(4x+4+\frac{12x-4}{4x^2-4x+1}\right)dx$$

$$\frac{12x-4}{4x^2-4x+1} = \frac{12x-4}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2}$$

cover upto find B: $B = \frac{6-4}{1} = 2.$

take $x=0$ to find A: $\frac{-4}{1} = \frac{A}{-1} + \frac{2}{1} \Rightarrow -A=-6$ $\boxed{A=6}$

$$\Rightarrow \int\left(4x+4+\frac{6}{2x-1}+\frac{2}{(2x-1)^2}\right)dx$$

$$= 2x^2+4x+3\ln|2x-1| - \frac{1}{(2x-1)} + C$$

42) $\displaystyle\int \frac{\sin\theta \, d\theta}{\cos^2\theta + \cos\theta - 2}$

let $x = \cos\theta$

$dx = -\sin\theta \, d\theta$

$$\Rightarrow I = \int \frac{\sin\theta \, d\theta}{\cos^2\theta + \cos\theta - 2} = \int \frac{-dx}{x^2 + x - 2}$$

$$\frac{-1}{x^2+x-2} = \frac{-1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

cover up : $A = \frac{-1}{3}$ and $B = \frac{1}{3}$

$$\Rightarrow I = \int \left(\frac{-1/3}{x-1} + \frac{1/3}{x+2} \right) dx$$

$$= \frac{-1}{3} \ln|x-1| + \frac{1}{3} \ln|x+2| + C$$

$$= \frac{-1}{3} \ln|\cos\theta - 1| + \frac{1}{3} \ln|\cos\theta + 2| + C$$