Section 8.2 - Integration By Parts.

Integration By parts:

$$\int u\,dv = uv - \int v\,du$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du.$$

Examples:

1) $\int x \sin x\,dx = -x\cos x - \int -\cos x\,dx$

$u = x$	$dv = \sin x\,dx$
$du = 1\,dx$	$v = -\cos x$

$= -x\cos x + \int \cos x\,dx$

$= -x\cos x + \sin x + C.$

2) $\int \ln x\,dx = x\ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - \int 1\,dx$

$= x\ln x - x + C.$

$u = \ln x$	$dv = 1\,dx$
$du = \frac{1}{x}\,dx$	$v = x$

3) $\int_0^4 xe^{-x}\,dx = [-xe^{-x}]_0^4 - \int_0^4 -e^{-x}\,dx = [-xe^{-x}]_0^4 + \int_0^4 e^{-x}\,dx$

$u = x$	$dv = e^{-x}\,dx$
$du = 1\,dx$	$v = -e^{-x}$

$= [-xe^{-x}]_0^4 - [e^{-x}]_0^4$

$= [-4e^{-4} + 0] - [e^{-4} - e^{0}]$

$= -4e^{-4} - e^{-4} + 1 = -5e^{-4} + 1$

4) $I = \int x^2 e^x \, dx \quad = \quad x^2 e^x - \int 2x e^x \, dx$

$u = x^2$	$dv = e^x dx$
$du = 2x \, dx$	$v = e^x dx$

$I = x^2 e^x - \int 2x e^x \, dx$

To solve $\int 2x e^x \, dx = 2x e^x - \int 2 e^x \, dx \quad = \quad 2x e^x - 2e^x + C$

$u = 2x$	$dv = e^x dx$
$du = 2dx$	$v = e^x$

So, $I = x^2 e^x - \left(2x e^x - 2e^x + C\right) = x^2 e^x - 2x e^x + 2e^x + C$

Tabular Integration:

It is used when you have an integral of the form:

$$\int \underbrace{P_n(x)}_{\text{polynomial of degree } n} \cdot \underbrace{f(x)\, dx}_{\text{function that can be integrated } (n+1) \text{ times.}}$$

Examples:

1) $\int x^2 e^x \, dx \qquad$ (We take $P_n(x) = x^2$ and $f(x) = e^x$)

Differentiate $P_n(x)$		Integrate $f(x)$
		e^x
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		

So, $\int x^2 e^x \, dx = + x^2 e^x - 2x e^x + 2e^x + C$

2) $J = \int x^3 \sin(3x)\, dx$ $\quad \left(P_n(x) = x^3 \text{ and } f(x) = \sin(3x) \right)$

Differentiate $P_n(x)$		Integrate $f(x)$
x^3	$+$	$\sin(3x)$
$3x^2$	$-$	$-\frac{1}{3}\cos(3x).$
$6x$	$+$	$-\frac{1}{9}\sin(3x)$
6	$-$	$\frac{1}{27}\cos(3x).$
0		$+\frac{1}{81}\sin(3x)$

$$J = \int x^3 \sin(3x)\, dx = \frac{-x^3}{3}\cos(3x) + \frac{3x^2}{9}\sin(3x) + \frac{6x}{27}\cos(3x).$$

$$- \frac{6}{81}\sin(3x).$$

$$= \frac{-x^3}{3}\cos(3x) + \frac{x^2}{3}\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{27}\sin(3x) + C$$

Solving For the Unknown Integral:

$$I = \int e^x \cos x \, dx$$

Using integration by parts: $u = e^x$ $\quad dv = \cos x\, dx$

$du = e^x dx$ $\quad V = \sin x$

$$I = e^x \sin x - \int e^x \sin x \, dx$$

To solve $\int e^x \sin x\, dx$, use integration by parts again.

$u = e^x$ $\quad dv = \sin x\, dx$

$du = e^x dx$ $\quad V = -\cos x$

$$\int e^x \sin x \, dx = -e^x \cos x - \int -e^x \cos x\, dx = -e^x \cos x + \int e^x \cos x\, dx$$

We get $I = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\, dx\right)$

So, $I = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$

$$I = e^x \sin x + e^x \cos x - I$$

$$2I = e^x \sin x + e^x \cos x$$

$$I = \frac{e^x \sin x + e^x \cos x}{2} + c$$

Reduction Formula:

Example: Obtain a reduction formula for $\int \cos^n x \, dx$. Use this to evaluate $\int \cos^5 x \, dx$.

$$\int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx$$

Using integration by parts, $u = \cos^{n-1} x$

$du = (n-1)\cos^{n-2} x \, (-\sin x) \, dx$

$dv = \cos x \, dx$

$v = \sin x$

$$\int \cos^n x \, dx = \cos^{n-1} x \sin x - \int \sin x \, (n-1)\cos^{n-2} x \, (-\sin x) \, dx$$

$$I = \cos^{n-1} x \sin x + \int (n-1) \sin^2 x \cos^{n-2} x \, dx$$

$$I = \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx$$

$$I = \cos^{n-1} x \sin x + (n-1) \int (\cos^{n-2} x - \cos^n x) \, dx$$

$$I = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$$I = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) I$$

$$I + (n-1) I = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$$

$$nI = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$$

$$I = \frac{\cos^{n-1}x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

$$\Rightarrow \int \cos^n x\,dx = \frac{\cos^{n-1}x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

For $n=5$: $\displaystyle\int \cos^5 x\,dx = \frac{\cos^4 x \sin x}{5} + \frac{4}{5}\int \cos^3 x\,dx$

To calculate $\int \cos^3 x\,dx$, take $n=3$ to get:

$$\int \cos^3 x\,dx = \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx$$

$$= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x$$

Therefore, $\displaystyle\int \cos^5 x\,dx = \frac{\cos^4 x \sin x}{5} + \frac{4}{5}\left(\frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x\right)$

$$= \frac{\cos^4 x \sin x}{5} + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + C$$

Exercises:

5) $\displaystyle\int_1^2 x \ln x\,dx$

Using integration by parts:	
$u = \ln x$	$dv = x\,dx$
$du = \frac{1}{x}\,dx$	$V = \frac{x^2}{2}$

$$= \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x^2}{2}\cdot\frac{1}{x}\,dx$$

$$= \left[\frac{x^2}{2}\ln x\right]_1^2 - \frac{1}{2}\int_1^2 x\,dx = \left[\frac{x^2}{2}\ln x\right]_1^2 - \left[\frac{x^2}{4}\right]_1^2$$

$$= \frac{4}{2}\ln 2 - \frac{1}{2}\ln 1 - \left(\frac{4}{4} - \frac{1}{4}\right) = 2\ln 2 - \frac{3}{4}$$

20) $\int t^2 e^{4t}\,dt$

$= \frac{t^2}{4}e^{4t} - \frac{2t}{16}e^{4t} + \frac{2}{64}e^{4t} + C$

$= \frac{t^2}{4}e^{4t} - \frac{t}{8}e^{4t} + \frac{1}{32}e^{4t} + C$

Using Tabular Integration:

t^2	$+$	e^{4t}
$2t$	$-$	$\frac{1}{4}e^{4t}$
2	$+$	$\frac{1}{16}e^{4t}$
0		$\frac{1}{64}e^{4t}$

22) $I = \int e^{-y}\cos y\,dy$ (Integration by parts)

$u = e^{-y}$ $\quad dv = \cos y\,dy$

$du = -e^{-y}dy$ $\quad v = \sin y$

$I = e^{-y}\sin y + \int e^{-y}\sin y\,dy$

To solve, $\int e^{-y}\sin y\,dy$, use integration by parts

$u = e^{-y}$ $\quad dv = \sin y\,dy$

$du = -e^{-y}$ $\quad V = -\cos y$

$\int e^{-y}\sin y\,dy = -e^{-y}\cos y - \int e^{-y}\cos y\,dy$

So, $I = e^{-y}\sin y - e^{-y}\cos y - \int e^{-y}\cos y\,dy$

$I = e^{-y}\sin y - e^{-y}\cos y - I$

$2I = e^{-y}\sin y - e^{-y}\cos y$

$I = \frac{e^{-y}\sin y - e^{-y}\cos y}{2} + C$

25) $\int e^{\sqrt{3x+9}}\,dx$

$(\sqrt{u})' = \frac{u'}{2\sqrt{u}}$

let $w = \sqrt{3x+9}$

$dw = \frac{3}{2\sqrt{3x+9}}\,dx$

$\Rightarrow dw = \frac{3}{2w}\,dx \Rightarrow \frac{2w}{3}\,dw = dx$

So, $\int e^{\sqrt{3x+9}}\,dx = \int e^{w}\,\frac{2w}{3}\,dw = \frac{2}{3}\int e^{w}\,w\,dw$

Use tabular integration to solve $\int e^{w}\,w\,dw$

$$\begin{array}{ccc} w & \xrightarrow{+} & e^{w} \\ 1 & \xrightarrow{-} & e^{w} \\ 0 & & e^{w} \end{array}$$

$\Rightarrow \int e^{\sqrt{3x+9}}\,dx = \frac{2}{3}\left(we^{w} - e^{w}\right) + C = \frac{2}{3}we^{w} - \frac{2}{3}e^{w} + C$

$= \frac{2}{3}\sqrt{3x+9}\,e^{\sqrt{3x+9}} - \frac{2}{3}e^{\sqrt{3x+9}} + C$

26) $\int_0^1 x\sqrt{1-x}\,dx$

Let $w = \sqrt{1-x}$

$w^2 = 1-x$

$x = 1-w^2$

$dx = -2w\,dw$

$x=0 \Rightarrow w = \sqrt{1} = 1$

$x=1 \Rightarrow w = 0$

$\int_0^1 x\sqrt{1-x}\,dx = \int_1^0 (1-w^2)\,w\,(-2w)\,dw = -2\int_1^0 w^2(1-w^2)\,dw$

$= -2\int_1^0 (w^2 - w^4)\,dw = -2\left[\frac{w^3}{3} - \frac{w^5}{5}\right]_1^0$

$= -2\left(0 - 0 - \frac{1}{3} + \frac{1}{5}\right) = -2\left(\frac{-2}{15}\right) = \frac{4}{15}$

30) $\int z(\ln z)^2\,dz$

let $w = \ln z$

$\Rightarrow z = e^w$

$dz = e^w\,dw$

So $\int z(\ln z)^2\,dz = \int e^w\, w^2\, e^w\,dw = \int e^{2w}\, w^2\,dw$

Using Tabular integration. to solve $\int e^{2w} w^2\,dw$:

$$\begin{array}{lcl} w^2 & \oplus & e^{2w} \\ 2w & \ominus & \frac{1}{2}e^{2w} \\ 2 & \oplus & \frac{1}{4}e^{2w} \\ 0 & & \frac{1}{8}e^{2w} \end{array}$$

$\Rightarrow \int z(\ln z)^2\,dz = \frac{w^2}{2}e^{2w} - \frac{2w}{4}e^{2w} + \frac{2}{8}e^{2w} + C$

$= \frac{w^2}{2}e^{2w} - \frac{w}{2}e^{2w} + \frac{1}{4}e^{2w} + C$

$= \frac{(\ln z)^2}{2}e^{2\ln z} - \frac{\ln z}{2}e^{2\ln z} + \frac{1}{4}e^{2\ln z} + C$

$= \frac{(\ln z)^2}{2}z^2 - \frac{\ln z}{2}z^2 + \frac{1}{4}z^2 + C.$

70) Establish the reduction formula.

$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$

$u = (\ln x)^n$ $\qquad$ $dv = dx$

$du = n(\ln x)^{n-1}\frac{1}{x}\,dx$ $\qquad$ $v = x$

$\Rightarrow \int (\ln x)^n\,dx = x(\ln x)^n - \int x\, n\,(\ln x)^{n-1}\frac{1}{x}\,dx$

$= x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$

78) $\int \tan^{-1}x \, dx$

$u = \tan^{-1}x$ $\quad\quad$ $dv = dx$

$du = \frac{1}{1+x^2}\,dx$ $\quad\quad$ $v = x$

$$\Rightarrow \int \tan^{-1}x \, dx = x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx.$$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C.$$