Section 8.3 - Trigonometric Integrals

Product of Powers of Sines and Cosines:

$$\int \sin^m x \cos^n x \, dx \qquad (m \text{ and } n \text{ are integers}).$$

① If m or n is odd (or both), then solve as follows:

- If m is odd, save one factor of $\sin x$, and let $u = \cos x$.
- If n is odd, save one factor of $\cos x$, and let $u = \sin x$.
- If both are odd, apply any one of the 2 previous points.

Examples:

1) $\int \sin^6 x \cos^5 x \, dx = \int \sin^6 x \cos^4 x \cos x \, dx$

$= \int \sin^6 x (1-\sin^2 x)^2 \cos x \, dx$ $\qquad \boxed{u = \sin x \qquad du = \cos x \, dx}$

$= \int u^6 (1-u^2)^2 \, du = \int u^6 (1 - 2u^2 + u^4) \, du = \int (u^6 - 2u^8 + u^{10}) \, du$

$= \frac{u^7}{7} - \frac{2u^9}{9} + \frac{u^{11}}{11} + c = \frac{\sin^7 x}{7} - \frac{2\sin^9 x}{9} + \frac{\sin^{11} x}{11} + c$

2) $\int \cos^5 x \sin^3 x \, dx = \int \cos^5 x \sin^2 x \sin x \, dx$

$= \int \cos^5 x (1 - \cos^2 x) \sin x \, dx$ $\qquad \boxed{u = \cos x \qquad du = -\sin x \, dx}$

$= \int u^5 (1 - u^2)(-du) = -\int (u^5 - u^7) \, du = -\frac{u^6}{6} + \frac{u^8}{8} + c$

$= -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + c$

② If m and n are both even, then:

$$\text{use } \begin{cases} \cos^2 x = \dfrac{1 + \cos 2x}{2} \\[2ex] \sin^2 x = \dfrac{1 - \cos 2x}{2} \end{cases}$$

Example:

$$I=\int \sin^2 x \cos^4 x \, dx = \int \frac{1-\cos(2x)}{2} \cdot \left[\frac{1+\cos(2x)}{2}\right]^2 dx$$

$$= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1+ 2\cos 2x + \cos^2 2x - \cos 2x - 2\cos^2 2x - \cos^3 2x)dx$$

$$= \frac{1}{8}\int (1+\cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\int\left(1+\cos 2x - \frac{1+\cos 4x}{2}\right)dx - \frac{1}{8}\int \cos^3 2x \, dx$$

$$= \frac{1}{16}\int (2+2\cos 2x - 1 - \cos 4x)\, dx - \frac{1}{8}\int \cos^3 2x \, dx$$

$$= \frac{1}{16}\int (1+2\cos 2x - \cos 4x)\, dx - \frac{1}{8}\int \cos^3 2x \, dx$$

$$\int \cos^3 2x \, dx = \int \cos^2 2x \cos 2x \, dx = \int (1-\sin^2 2x)\cos 2x \, dx$$

$u = \sin 2x$

$du = 2\cos 2x \, dx$

$$= \int (1-u^2)\frac{du}{2} = \frac{1}{2}\int (1-u^2)du$$

$$= \frac{1}{2}\left(u - \frac{u^3}{3}\right) = \frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right)$$

$$\Rightarrow I = \frac{1}{16}\left(x + \sin 2x - \frac{1}{4}\sin 4x\right) - \frac{1}{8}\frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right) + C$$

$$= \frac{1}{16}x + \cancel{\frac{1}{16}\sin 2x} - \frac{1}{64}\sin 4x - \cancel{\frac{1}{16}\sin 2x} + \frac{1}{48}\sin^3 2x + C$$

$$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$$

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Eliminating Square Roots:

Example: $\displaystyle\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx$

$$\boxed{\begin{aligned} 0 \le x \le \tfrac{\pi}{4} \\ 0 \le 2x \le \tfrac{\pi}{2} \\ \Rightarrow \cos 2x \ge 0 \end{aligned}}$$

$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx = \frac{\sqrt{2}}{2}\left[\sin 2x\right]_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{\sqrt{2}}{2}(1-0) = \frac{\sqrt{2}}{2}$$

Products of Sines and Cosines:

Use : $\begin{cases} \sin(mx)\sin(nx) = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right] \\ \sin(mx)\cos(nx) = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right] \\ \cos(mx)\cos(nx) = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right] \end{cases}$

Example: $\displaystyle\int \sin(3x)\cos(5x)\, dx = \int \frac{1}{2}\left[\sin(3-5)x + \sin(3+5)x\right] dx$

$$= \frac{1}{2}\int\left[\sin(-2x) + \sin 8x\right] dx = \frac{1}{2}\int\left(-\sin 2x + \sin 8x\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\cos 2x - \frac{1}{8}\cos 8x\right) + c = \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + c$$

Powers of $\tan x$ and $\sec x$:

Examples:

1) $\displaystyle\int \tan^2 x\, dx = \int(\sec^2 x - 1)\, dx = \tan x - x + C$

$$\begin{aligned} \tan^2 x + 1 &= \sec^2 x \\ \cot^2 x + 1 &= \csc^2 x \end{aligned}$$

2) $\int \tan^4 x \, dx = \int \tan^2 x \tan^2 x \, dx = \int \tan^2 x \left(\sec^2 x - 1\right) dx$

$= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$

$\Downarrow$

$$\boxed{\begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \end{array}}$$

$= \int u^2 \, du - \left(\tan x - x\right) = \frac{u^3}{3} - \tan x + x + C$

$= \frac{\tan^3 x}{3} - \tan x + x + C$

3) $I = \int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$

by parts

$$\boxed{\begin{array}{ll} u = \sec x & dv = \sec^2 x \, dx \\ du = \sec x \tan x & v = \tan x \end{array}}$$

$= \sec x \tan x - \int \sec x \tan^2 x \, dx$

$= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$\Rightarrow I = \sec x \tan x - I + \ln\left|\sec x + \tan x\right| + C$

$2I = \sec x \tan x + \ln\left|\sec x + \tan x\right| + C$

$I = \frac{\sec x \tan x + \ln\left|\sec x + \tan x\right|}{2} + C$

Exercises:

9) $\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int (1-\sin^2 x) \cos x \, dx$

$= \int (1-u^2) du = u - \frac{u^3}{3} + c$

$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array}}$

$= \sin x - \frac{\sin^3 x}{3} + c$

18) $\int 8 \cos^4 2\pi x \, dx = 8 \int \left(\frac{1+\cos 4\pi x}{2}\right)^2 dx$

$= 2 \int (1 + 2\cos 4\pi x + \cos^2 4\pi x) dx$

$= 2 \int \left(1 + 2\cos 4\pi x + \frac{1+\cos 8\pi x}{2}\right) dx = \int (2 + 4\cos 4\pi x + 1 + \cos 8\pi x) dx$

$= \int (3 + 4\cos 4\pi x + \cos 8\pi x) dx = 3x + \frac{\sin 4\pi x}{\pi} + \frac{\sin 8\pi x}{8\pi} + c$

22) $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta = \int_0^{\pi/2} \sin^2 2\theta \cos^2 2\theta \cos 2\theta \, d\theta$

$= \int_0^{\pi/2} \sin^2 2\theta (1 - \sin^2 2\theta) \cos 2\theta \, d\theta$

$\boxed{\begin{array}{l} u = \sin 2\theta \\ du = 2\cos 2\theta \, d\theta \\ \theta = 0 \Rightarrow u = 0 \\ \theta = \frac{\pi}{2} \Rightarrow u = 0 \end{array}}$

$= \int_0^0 u^2 (1-u^2) \frac{du}{2} = 0$

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Remark: If you are evaluating an integral of the form

$\int \tan^m x \sec^n x \, dx$, you might encounter one of the cases:

Case 1: If n is even, save a factor of $\sec^2 x$ and use

$\sec^2 x = \tan^2 x + 1$ to express the remaining factors, then

let $u = \tan x$.

Case 2: If m is odd, save a factor of $\tan x \sec x$, and

use $\tan^2 x = \sec^2 x - 1$, then let $u = \sec x$.

36) $\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \tan^2 x \sec x \tan x \, dx$

$= \int \sec^2 x (\sec^2 x - 1) \sec x \tan x \, dx$

$$\boxed{\begin{array}{l} u = \sec x \\ du = \sec x \tan x \, dx \end{array}}$$

$= \int u^2 (u^2 - 1) \, du = \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + c$

$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c$

38) $\int \sec^4 x \tan^2 x \, dx = \int \sec^2 x \sec^2 x \tan^2 x \, dx$

$= \int (\tan^2 x + 1) \tan^2 x \sec^2 x \, dx$

$$\boxed{\begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \end{array}}$$

$= \int (u^2 + 1) u^2 \, du = \int (u^4 + u^2) \, du$

$= \frac{u^5}{5} + \frac{u^3}{3} + c = \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + c$

39) $\int_{-\pi/3}^{0} 2 \sec^3 x \, dx = 2 \int_{-\pi/3}^{0} \sec^3 x \, dx = 2 \int_{-\pi/3}^{0} \sec x \sec^2 x \, dx$

$I = 2 \left[\sec x \tan x \right]_{-\pi/3}^{0} - 2 \int_{-\pi/3}^{0} \sec x \tan^2 x \, dx$

$$\boxed{\begin{array}{ll} u = \sec x & dv = \sec^2 x \, dx \\ du = \sec x \tan x \, dx & v = \tan x \end{array}}$$

$= 2 \left[\sec x \tan x \right]_{-\pi/3}^{0} - 2 \int_{-\pi/3}^{0} \sec x (\sec^2 x - 1) \, dx$

$= 2 \left[\sec x \tan x \right]_{-\pi/3}^{0} - 2 \int_{-\pi/3}^{0} \sec^3 x \, dx + 2 \int_{-\pi/3}^{0} \sec x \, dx$

$I = 2 \left[\sec x \tan x \right]_{-\pi/3}^{0} - I + 2 \left[\ln | \sec x + \tan x | \right]_{-\pi/3}^{0}$

$$2I = 2\left[\sec 0 \tan 0 - \sec(-\pi/3)\tan\left(\frac{-\pi}{3}\right)\right] + 2\left[\ln|\sec 0 + \tan 0| - \ln\left|\sec\left(\frac{-\pi}{3}\right) + \tan\left(\frac{-\pi}{3}\right)\right|\right]$$

$$2I = 2[+2\sqrt{3}] + 2[\ln 1 - \ln(2-\sqrt{3})]$$

$$2I = 4\sqrt{3} - 2\ln(2-\sqrt{3})$$

$$I = 2\sqrt{3} - \ln(2-\sqrt{3})$$

51) $\int \sin 3x \cos 2x \, dx = \frac{1}{2}\int \left(\sin(3-2)x + \sin(3+2)x\right) dx$

$= \frac{1}{2}\int (\sin x + \sin 5x)\,dx = \frac{1}{2}\left(-\cos x - \frac{1}{5}\cos 5x\right) + C$

55) $\int \cos 3x \cos 4x \, dx = \frac{1}{2}\int \left(\cos(3-4)x + \cos(3+4)x\right) dx$

$= \frac{1}{2}\int \left(\cos(-x) + \cos 7x\right) dx = \frac{1}{2}\int (\cos x + \cos 7x)\, dx$

$= \frac{1}{2}\left(\sin x + \frac{1}{7}\sin 7x\right) + C.$