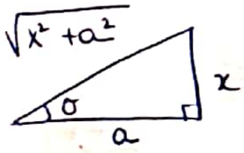
Section 8.4 – Trigonometric Substitutions

We will use trigonometric substitutions to find integrals involving $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ $(a>0)$, where all the previous techniques do not work.

We will use 3 types of substitutions:

① For $\sqrt{a^2+x^2}$: let $x = a\tan\theta \implies \theta = \tan^{-1}\left(\frac{x}{a}\right)$

$dx = a\sec^2\theta \, d\theta$ $\qquad \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

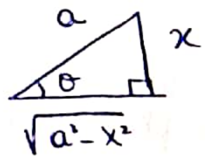


$$\sqrt{a^2+x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)}$$

$$= a\sqrt{\sec^2\theta} = a\left|\sec\theta\right| = a\sec\theta$$

② For $\sqrt{a^2-x^2}$: let $x = a\sin\theta \implies \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$dx = a\cos\theta \, d\theta$ $\qquad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



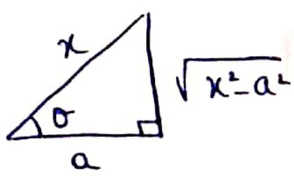
$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)}$$

$$= a\sqrt{\cos^2\theta} = a\left|\cos\theta\right| = a\cos\theta.$$

③ For $\sqrt{x^2-a^2}$: let $x = a\sec\theta \implies \theta = \sec^{-1}\left(\frac{x}{a}\right)$

$dx = a\sec\theta\tan\theta \, d\theta$ $\qquad \begin{cases} \theta \in \left[0, \frac{\pi}{2}\right[& \text{if } \frac{x}{a} \geq 1 \\ \theta \in \left]\frac{\pi}{2}, \pi\right[& \text{if } \frac{x}{a} \leq -1 \end{cases}$

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)}$$



$$= a\sqrt{\tan^2\theta} = a\left|\tan\theta\right|$$

$$= \begin{cases} a\tan\theta & \text{if } \frac{x}{a} \geq 1 \\ -a\tan\theta & \text{if } \frac{x}{a} \leq -1. \end{cases}$$

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Examples:

1) $$I = \int \frac{dx}{\sqrt{4+x^2}}$$

$$\boxed{\begin{array}{ll} \text{let } x = 2\tan\theta & \Rightarrow \tan\theta = \frac{x}{2} \\ dx = 2\sec^2\theta\, d\theta & \theta = \tan^{-1}\left(\frac{x}{2}\right) \\ & \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\end{array}}$$

$$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta} = 2\sqrt{1+\tan^2\theta} = 2\sqrt{\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$$

$$\Rightarrow I = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c$$

2) $$J = \int \frac{x^2}{\sqrt{9-x^2}}\, dx$$

$$\boxed{\begin{array}{ll} \text{let } x = 3\sin\theta & \Rightarrow \sin\theta = \frac{x}{3} \\ dx = 3\cos\theta\, d\theta & \theta = \sin^{-1}\left(\frac{x}{3}\right) \\ & \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{array}}$$

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{\cos^2\theta}$$
$$= 3|\cos\theta| = 3\cos\theta$$

$$J = \int \frac{9\sin^2\theta \;\, 3\cos\theta\, d\theta}{3\cos\theta} = 9\int \sin^2\theta\, d\theta = 9\int \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{9}{2}\int (1-\cos 2\theta)\, d\theta = \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) = \frac{9}{2}\left(\theta - \sin\theta\cos\theta\right) + c$$

$$= \frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + c = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + c.$$

3) $$K = \int \frac{dx}{\sqrt{25x^2-4}}, \quad x > \frac{2}{5}$$

$$\boxed{\begin{array}{lll} \text{let } x = \frac{2}{5}\sec\theta & \Rightarrow \sec\theta = \frac{5x}{2} & \Rightarrow \theta = \sec^{-1}\left(\frac{5x}{2}\right) \\ dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta & & \theta \in \left[0, \frac{\pi}{2}\right[\text{ since } \frac{5x}{2} > 1 \end{array}}$$

$$\sqrt{25x^2-4} = \sqrt{4\sec^2\theta - 4} = 2\sqrt{\sec^2\theta - 1} = 2\sqrt{\tan^2\theta}$$

$$= 2|\tan\theta| = 2\tan\theta \qquad \left(\text{since } \theta \in \left[0, \frac{\pi}{2}\right[\right).$$

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$$\Rightarrow K = \int \frac{\frac{2}{5}\sec\theta \tan\theta \, d\theta}{2\tan\theta} = \frac{1}{5}\int \sec\theta \, d\theta$$

$$= \frac{1}{5}\ln|\sec\theta + \tan\theta| + c = \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + c.$$

Exercises:

8) $\int \sqrt{1-9t^2}\, dt = 3\int \sqrt{\frac{1}{9} - t^2}\, dt$

$$\boxed{\begin{array}{ll} \text{let } t = \frac{1}{3}\sin\theta & \Rightarrow \sin\theta = 3t \\ & \theta = \sin^{-1}(3t) \\ dt = \frac{1}{3}\cos\theta \, d\theta & \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \end{array}}$$

$$\sqrt{1-9t^2} = \sqrt{1-\sin^2\theta}$$

$$= \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$$

$$\Rightarrow \int \sqrt{1-9t^2}\, dt = \int \cos\theta \, \frac{1}{3}\cos\theta \, d\theta = \frac{1}{3}\int \cos^2\theta \, d\theta$$

$$= \frac{1}{3}\int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{6}\left(\theta + \frac{1}{2}\sin 2\theta\right) + c = \frac{1}{6}\left(\theta + \sin\theta\cos\theta\right) + c$$

$$= \frac{1}{6}\left(\sin^{-1}(3t) + 3t\sqrt{1-9t^2}\right) + c$$

13) $I = \int \frac{dx}{x^2\sqrt{x^2-1}}$, $x > 1$

$$\boxed{\begin{array}{ll} \text{let } x = \sec\theta & \Rightarrow \theta = \sec^{-1} x \\ dx = \sec\theta\tan\theta \, d\theta & \theta \in \left[0, \frac{\pi}{2}\right[\\ & \text{since } x > 1 \end{array}}$$

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta \quad \left(\text{since } \theta \in \left[0, \tfrac{\pi}{2}\right[\right)$$

$$I = \int \frac{\sec\theta\tan\theta \, d\theta}{\sec^2\theta \tan\theta} = \int \frac{1}{\sec\theta} d\theta = \int \cos\theta \, d\theta = \sin\theta + c$$

$$= \frac{\sqrt{x^2-1}}{x} + c.$$

$\sec\theta = x \Rightarrow \cos\theta = \frac{1}{x}$

$\tan\theta = \sqrt{x^2-1}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \sin\theta = \tan\theta . \cos\theta = \sqrt{x^2-1} . \frac{1}{x}.$

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17) $$\int \frac{x^3\,dx}{\sqrt{x^2+4}}$$

$$\boxed{\begin{array}{ll} x = 2\tan\theta & \Longrightarrow \quad \tan\theta = \frac{x}{2} \\ dx = 2\sec^2\theta\, d\theta & \qquad \theta = \tan^{-1}\left(\frac{x}{2}\right) \\ & \qquad \theta \in \left]\frac{-\pi}{2}, \frac{\pi}{2}\right[\end{array}}$$

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta+4} = 2\sqrt{\tan^2\theta+1} = 2\sqrt{\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$$

$$\int \frac{x^3\,dx}{\sqrt{x^2+4}} = \int \frac{8\tan^3\theta \; 2\sec^2\theta\, d\theta}{2\sec\theta} = 8\int \tan^3\theta \sec\theta\, d\theta$$

$$= 8\int \tan^2\theta \;\tan\theta\sec\theta\, d\theta = 8\int (\sec^2\theta - 1)\tan\theta\sec\theta\, d\theta$$

$$= 8\int (u^2-1)\,du = 8\left(\frac{u^3}{3} - u\right) + c \qquad \boxed{\begin{array}{l} u = \sec\theta \\ du = \tan\theta\sec\theta\, d\theta \end{array}}$$

$$= 8\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + c = \frac{8}{3}\sec^3\theta - 8\sec\theta + c$$

$$= \frac{8}{3}\,\frac{(x^2+4)\sqrt{x^2+4}}{8} - 8\,\frac{\sqrt{x^2+4}}{2} + c = \frac{(x^2+4)\sqrt{x^2+4}}{3} - 4\sqrt{x^2+4} + c$$

20) $$\int \frac{\sqrt{9-w^2}}{w^2}\,dw$$

$$\boxed{\begin{array}{ll} w = 3\sin\theta & \Longrightarrow \quad \sin\theta = \frac{w}{3} \\ dw = 3\cos\theta\, d\theta & \qquad \theta = \sin^{-1}\left(\frac{w}{3}\right) \\ & \qquad \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \end{array}}$$

$$\sqrt{9-w^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{\cos^2\theta}$$

$$= 3|\cos\theta| = 3\cos\theta$$

$$\Longrightarrow \int \frac{\sqrt{9-w^2}}{w^2}\,dw = \int \frac{3\cos\theta \; 3\cos\theta\, d\theta}{9\sin^2\theta} = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\, d\theta$$

$$= \int (\csc^2 - 1)\,d\theta = -\cot\theta - \theta + c = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + c$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{9-w^2}/3}{w/3} = \frac{\sqrt{9-w^2}}{w}$$

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26) $\displaystyle\int \frac{x^2\,dx}{(x^2-1)^{5/2}}$, $x>1$

$$\boxed{\begin{array}{ll} x=\sec\theta & \Rightarrow \quad \theta=\sec^{-1}x \\ dx=\sec\theta\tan\theta\,d\theta & \quad \theta\in[0,\tfrac{\pi}{2}[\\ & \quad \text{since } x>1 \end{array}}$$

$\displaystyle\int \frac{x^2\,dx}{(x^2-1)^2\sqrt{x^2-1}}$, $x>1$

$$\sqrt{x^2-1}=\sqrt{\sec^2\theta-1}=\sqrt{\tan^2\theta}=|\tan\theta|=\tan\theta \quad \left(\text{since } \theta\in[0,\tfrac{\pi}{2}[\right)$$

$$\Rightarrow \int \frac{x^2\,dx}{(x^2-1)^2\sqrt{x^2-1}}=\int \frac{\sec^2\theta\,\sec\theta\tan\theta\,d\theta}{(\sec^2\theta-1)^2\,\tan\theta}=\int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta$$

$$=\int \frac{1/\cos^3\theta}{\sin^4\theta/\cos^4\theta}\,d\theta=\int \frac{1}{\cos^3\theta}\cdot\frac{\cos^4\theta}{\sin^4\theta}\,d\theta=\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$$

$$=\int \frac{du}{u^4}=\int u^{-4}\,du=\frac{u^{-3}}{-3}+C=\frac{-1}{3\sin^3\theta}+C \qquad \boxed{\begin{array}{l} u=\sin\theta \\ du=\cos\theta\,d\theta \end{array}}$$

$$=\frac{-1}{3\left(\frac{\sqrt{x^2-1}}{x}\right)^3}+C=\frac{-x^3}{3\left(\sqrt{x^2-1}\right)^3}+C$$

$\sec\theta=x \Rightarrow \cos\theta=\frac{1}{x}$

$\tan\theta=\sqrt{x^2-1}$

$\Rightarrow \sin\theta=\tan\theta\cos\theta=\frac{\sqrt{x^2-1}}{x}$

27) $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^6}$

$$\boxed{\begin{array}{ll} x=\sin\theta & \Rightarrow \quad \theta=\sin^{-1}x \\ dx=\cos\theta\,d\theta & \quad \theta\in\,]-\tfrac{\pi}{2},\tfrac{\pi}{2}[\end{array}}$$

$$=\int \frac{(1-x^2)\sqrt{1-x^2}}{x^6}\,dx$$

$$\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\sqrt{\cos^2\theta}=|\cos\theta|=\cos\theta$$

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$$\Rightarrow \int \frac{(1-x^2)\sqrt{1-x^2}}{x^6}\,dx = \int \frac{(1-\sin^2\theta)\cos\theta\cos\theta\,d\theta}{\sin^6\theta}$$

$$= \int \frac{\cos^4\theta\,d\theta}{\sin^6\theta} = \int \frac{\cos^4\theta}{\sin^4\theta}\cdot\frac{1}{\sin^2\theta}\,d\theta = \int \cot^4\theta \cdot \csc^2\theta\,d\theta$$

$$= \int -u^4\,du = \frac{-u^5}{5} + c \qquad \boxed{\begin{array}{l} u = \cot\theta \\ du = -\csc^2\theta\,d\theta \end{array}}$$

$$= \frac{-\cot^5\theta}{5} + c = \frac{-1}{5}\,\frac{\cos^5\theta}{\sin^5\theta} + c = \frac{-1}{5}\,\frac{(\sqrt{1-x^2})^5}{x^5} + c$$

35) $$I = \int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$$

$$\boxed{\begin{array}{ll} \text{let } x = e^t & t = 0 \Rightarrow x = 1 \\ dx = e^t\,dt & t = \ln 4 \Rightarrow x = e^{\ln 4} = 4 \end{array}}$$

$$= \int_1^4 \frac{dx}{\sqrt{x^2+9}}$$

$$\boxed{\begin{array}{lll} \text{let } x = 3\tan\theta & \Rightarrow & \tan\theta = \frac{x}{3} \\ dx = 3\sec^2\theta\,d\theta & & \theta = \tan^{-1}\left(\frac{x}{3}\right) \\ & & \theta \in \left]\frac{-\pi}{2}, \frac{\pi}{2}\right[\end{array}}$$

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sqrt{\sec^2\theta} = 3|\sec\theta| = 3\sec\theta$$

$$x = 1 \Rightarrow \theta = \tan^{-1}\left(\frac{1}{3}\right)$$

$$x = 4 \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\Rightarrow I = \int_{\tan^{-1}(\frac{1}{3})}^{\tan^{-1}(\frac{4}{3})} \frac{3\sec^2\theta\,d\theta}{3\sec\theta} = \int_{\tan^{-1}(\frac{1}{3})}^{\tan^{-1}(\frac{4}{3})} \sec\theta\,d\theta = \Big[\ln|\sec\theta + \tan\theta|\Big]_{\tan^{-1}(\frac{1}{3})}^{\tan^{-1}(\frac{4}{3})}$$

$$= \left[\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right|\right]_1^4 = \ln\left|\frac{\sqrt{16+9}}{3} + \frac{4}{3}\right| - \ln\left|\frac{\sqrt{1+9}}{3} + \frac{1}{3}\right|$$

$$= \ln(3) - \ln\left(\frac{\sqrt{10}+1}{3}\right)$$

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37) $$\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t} + 4t\sqrt{t}}$$

$$\boxed{\begin{array}{ll} \text{let } x = \sqrt{t} & t = \frac{1}{12} \Rightarrow x = \frac{1}{2\sqrt{3}} \\ dx = \frac{1}{2\sqrt{t}}\,dt & t = \frac{1}{4} \Rightarrow x = \frac{1}{2} \end{array}}$$

$$= \int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}\,(1+4t)} = \int_{\frac{1}{2\sqrt{3}}}^{1/2} \frac{4\,dx}{1+4x^2} = 2\int_{1/2\sqrt{3}}^{1/2} \frac{2\,dx}{1+(2x)^2}$$

$$\boxed{\begin{array}{l} u = 2x \\ du = 2\,dx \\ x = \frac{1}{2\sqrt{3}} \Rightarrow u = \frac{1}{\sqrt{3}} \\ x = \frac{1}{2} \Rightarrow u = 1 \end{array}}$$

$$= 2\int_{1/\sqrt{3}}^{1} \frac{du}{1+u^2} = 2\left(\tan^{-1} u\right]_{1/\sqrt{3}}^{1}$$

$$= 2\left(\tan^{-1} 1 - \tan^{-1}\frac{1}{\sqrt{3}}\right) = 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{6}.$$

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