# Section 8.8 - Improper Integrals:

An integral $\int_a^b f(x)\,dx$ is said to be an "improper integral" in any of the following cases

1) Type I: a or b (or both) is infinite (infinite limits of integration)

ex: $\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x} = \lim_{b\to\infty} -\left[e^{-x}\right]_1^b$

$= \lim_{b\to\infty} -\left[e^{-b} - e^{-1}\right] = 0 + e^{-1} = \frac{1}{e}$

2) Type II: f(x) has an infinite discontinuity (a vertical asymptote) at a or b or at one (or more) points in the interval of integration.

ex: $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \left[2\sqrt{x}\right]_a^1$

$= \lim_{a\to 0^+} \left[2 - 2\sqrt{a}\right] = 2 - 0 = 2$

## Improper Integrals: Type I:

1) If $f(x)$ is continuous on $[a,\infty)$, then: $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$

2) If $f(x)$ is continuous on $(-\infty, b]$, then: $\int_{-\infty}^b f(x)\,dx = \lim_{a\to -\infty} \int_a^b f(x)\,dx$

3) If $f(x)$ is continuous on $(-\infty, +\infty)$, then:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{+\infty} f(x)\,dx$$

$$= \lim_{a\to -\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx$$

Remark: If the limit is finite and equal to $l$, then we say that the improper integral converges to $l$, otherwise, we say that it diverges.

Examples:

1) $$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{0} \frac{dx}{x^2+1} + \int_{0}^{\infty} \frac{dx}{x^2+1}$$

$$= \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{x^2+1} + \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{x^2+1}$$

$$= \lim_{a \to -\infty} \left[\tan^{-1} x\right]_a^0 + \lim_{b \to \infty} \left[\tan^{-1} x\right]_0^b$$

$$= \lim_{a \to -\infty} \left[\tan^{-1} 0 - \tan^{-1} a\right] + \lim_{b \to \infty} \left[\tan^{-1} b - \tan^{-1} 0\right]$$

$$= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 = \pi$$

$\tan^{-1} 0 = 0$

$\lim_{a \to -\infty} (\tan^{-1} a) = -\frac{\pi}{2}$

$\lim_{a \to \infty} (\tan^{-1} a) = \frac{\pi}{2}$

2) $$\int_{1}^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{\ln x}{x^2}\, dx$$

$$= \lim_{b \to \infty} \left[\left(\frac{-1}{x} \ln x\right)_1^b + \int_1^b \frac{1}{x^2}\, dx\right]$$

$$= \lim_{b \to \infty} \left[\frac{-1}{x} \ln x - \frac{1}{x}\right]_1^b$$

$$= \lim_{b \to \infty} \left[\left(\frac{-1}{b} \ln b - \frac{1}{b}\right) - \left(\frac{-1}{1} \ln 1 - \frac{1}{1}\right)\right]$$

$$= 0 - 0 - (0 - 1) = 1 \Rightarrow \text{it converges to } 1.$$

By parts:

$u = \ln x \qquad dv = \frac{1}{x^2}\, dx$

$du = \frac{1}{x}\, dx \qquad v = \frac{-1}{x}$

Recall that:

$$\lim_{b\to\infty}\left(\frac{\ln b}{b}\right)=0$$

$$\lim_{b\to 0^+}(\ln b)=-\infty$$

$$\lim_{b\to\infty}(\ln b)=\infty$$

$$\lim_{b\to\infty}\left(\frac{\ln b}{b^n}\right)=0$$

$$\lim_{b\to\infty}(e^b)=\infty$$

$$\lim_{b\to-\infty}(e^b)=0$$

$$\lim_{b\to\infty}\left(\frac{e^b}{b^n}\right)=\infty \quad (n\geq 1)$$

$$\lim_{b\to 0^+}(b\ln b)=0$$

3) $$\int_1^\infty \frac{1}{x^p}\,dx=\lim_{b\to\infty}\int_1^b x^{-p}\,dx=\lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b \quad (p\neq 1)$$

$$=\lim_{b\to\infty}\left[\frac{b^{-p+1}}{-p+1}-\frac{1}{-p+1}\right]$$

* if $-p+1>0$, $\boxed{p<1}$ : as $b\to\infty$, $b^{-p+1}\longrightarrow\infty$, then $\int_1^\infty \frac{1}{x^p}dx$ diverges.

* if $-p+1<0$, $\boxed{p>1}$ : as $b\to\infty$, $b^{-p+1}\longrightarrow 0$, then :

$$\int_1^\infty \frac{dx}{x^p} \text{ converges to } \frac{1}{p-1}$$

* if $\boxed{p=1}$: $\int_1^\infty \frac{1}{x}dx=\lim_{b\to\infty}[\ln x]_1^b=\lim_{b\to\infty}[\ln b-\ln 1]=\infty$

$\Rightarrow$ it diverges.

p-integrals: $\displaystyle\int_1^\infty \frac{dx}{x^p}$ — converges if $p>1$; diverges if $p\leq 1$.

Ex: $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$ p-integral with $p=\frac{1}{2}<1 \Rightarrow$ it diverges.

$\displaystyle\int_1^\infty \frac{dx}{x^3}$ p-integral with $p=3>1 \Rightarrow$ it converges.

Improper integrals: type II:

1) If $f(x)$ is continuous on $(a,b]$, then:

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$

2) If $f(x)$ is continuous on $[a,b)$, then:

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$$

3) If $f(x)$ is continuous on $[a,c)\cup(c,b]$, then:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{d\to c^-}\int_a^d f(x)\,dx + \lim_{e\to c^+}\int_e^b f(x)\,dx$$

Example: $\displaystyle I = \int_0^3 \frac{dx}{(x-1)^{2/3}}$ (infinite discontinuity at $x=1$)

$$= \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 (x-1)^{-2/3}\,dx + \int_1^3 (x-1)^{-2/3}\,dx$$

$$= \lim_{d \to 1^-} \int_0^d (x-1)^{-2/3}\, dx + \lim_{e \to 1^+} \int_e^3 (x-1)^{-2/3}\, dx$$

$$= \lim_{d \to 1^-} \left[ 3(x-1)^{1/3} \right]_0^d + \lim_{e \to 1^+} \left[ 3(x-1)^{1/3} \right]_e^3$$

$$= \lim_{d \to 1^-} \left[ 3(d-1)^{1/3} - 3(-1) \right] + \lim_{e \to 1^+} \left[ 3(2)^{1/3} - 3(e-1)^{1/3} \right]$$

$$= 0 + 3 + 3\sqrt[3]{2} + 0 = 3 + 3\sqrt[3]{2}$$

<u>Testing for Convergence</u>:
- → direct comparison test (DCT)
- → limit comparison test (LCT)

<u>Direct Comparison test:</u> (DCT)

Let $f(x)$ and $g(x)$ be continuous on $[a, \infty)$

Suppose $0 \leq f(x) \leq g(x)$ for $x \geq a$,

1) If $\int_a^\infty g(x)\, dx$ converges, then $\int_a^\infty f(x)\, dx$ converges

2) If $\int_a^\infty f(x)\, dx$ diverges, then $\int_a^\infty g(x)\, dx$ diverges

<u>Note</u> : If $\int_a^\infty g(x)\, dx$ diverges, no conclusion about $\int_a^\infty f(x)\, dx$

If $\int_a^\infty f(x)\, dx$ converges, no conclusion about $\int_a^\infty g(x)\, dx$

Examples:

1) $\displaystyle\int_1^{\infty} \frac{dx}{x^3+1}$

$0 < \underbrace{\frac{1}{x^3+1}}_{f(x)} < \underbrace{\frac{1}{x^3}}_{g(x)}$ for $x \geq 1$

$\displaystyle\int_1^{\infty} \frac{1}{x^3}\,dx$ converges (p-integral with $p=3>1$)

Then, $\displaystyle\int_1^{\infty} \frac{1}{x^3+1}\,dx$ also converges by the DCT

2) $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x^2-0.1}}$

$0 < \sqrt{x^2-0.1} < \sqrt{x^2}$

$0 < \sqrt{x^2-0.1} < x$

(for $x \geq 1$)

$0 < \underbrace{\frac{1}{x}}_{f(x)} < \underbrace{\frac{1}{\sqrt{x^2-0.1}}}_{g(x)}$

$\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ diverges (p-integral with $p=1$)

Then, $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x^2-0.1}}\,dx$ also diverges by the DCT.

## The Limit Comparison Test (LCT)

let $f(x)$ and $g(x)$ be positive continuous functions on $[a, \infty)$

If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$ where $\boxed{0 < L < \infty}$

Then: $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge or both diverge.

Note: If $L = 0$ or $L = \infty$, then no conclusion.

Example: $\int_2^\infty \frac{dx}{e^x - 5}$

$\ln x < x < e^x$ for $x > 0$

* If we try to apply DCT :

$0 < e^x - 5 < e^x \qquad (x \geq 2)$

$0 < \underbrace{\frac{1}{e^x}}_{f(x)} < \underbrace{\frac{1}{e^x - 5}}_{g(x)}$

$\int_2^\infty \frac{1}{e^x}\,dx = \int_2^\infty e^{-x}\,dx = \lim_{b \to \infty} \left[-e^{-x}\right]_2^b = \lim_{b \to \infty} -\left[e^{-b} - e^{-2}\right] = e^{-2} \Rightarrow$ converges

but no conclusion with DCT.

* Apply LCT:

$\lim_{x \to \infty} \frac{\frac{1}{e^x}}{\frac{1}{e^x - 5}} = \lim_{x \to \infty} \frac{e^x - 5}{e^x} = \lim_{x \to \infty} \frac{e^x\left(1 - \frac{5}{e^x}\right)}{e^x} = 1 - 0 = 1$

Both converge or both diverge

$\int_2^\infty \frac{1}{e^x}\,dx$ converges, then $\int_2^\infty \frac{1}{e^x - 5}\,dx$ converges by LCT

Exercises:

Evaluate the following integrals:

1) $$\int_0^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b$$

$$= \lim_{b\to\infty}\left[\tan^{-1}b - \tan^{-1}0\right] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

2) $$\int_1^\infty \frac{dx}{x^{1.001}} = \lim_{b\to\infty}\int_1^b x^{-1.001}\,dx = \lim_{b\to\infty}\left[\frac{x^{-0.001}}{-0.001}\right]_1^b$$

$$= \lim_{b\to\infty}\left[\frac{b^{-0.001}}{-0.001} + \frac{1}{0.001}\right] = 0 + \frac{1}{0.001} = 1000$$

6) $$\int_{-8}^1 \frac{dx}{x^{1/3}} = \int_{-8}^0 x^{-1/3}dx + \int_0^1 x^{-1/3}dx = \lim_{b\to0^-}\int_{-8}^b x^{-1/3}dx + \lim_{a\to0^+}\int_a^1 x^{-1/3}dx$$

$$= \lim_{b\to0^-}\left[\frac{3}{2}x^{2/3}\right]_{-8}^b + \lim_{a\to0^+}\left[\frac{3}{2}x^{2/3}\right]_a^1$$

$$= \lim_{b\to0^-}\left[\frac{3}{2}\sqrt[3]{b^2} - \frac{3}{2}\sqrt[3]{8^2}\right] + \lim_{a\to0^+}\left[\frac{3}{2}1 - \frac{3}{2}\sqrt[3]{a^2}\right]$$

$$= 0 - \frac{3}{2}\times 4 + \frac{3}{2} - 0 = -6 + \frac{3}{2} = \frac{-9}{2}$$

11) $$\int_2^\infty \frac{2}{v^2-v}dv = \lim_{b\to\infty}\int_2^b \frac{2}{v^2-v}dv = \lim_{b\to\infty}\int_2^b \frac{2}{v(v-1)}dv$$

partial fractions

$$\lim_{b\to\infty}\int_2^b\left(\frac{-2}{v} + \frac{2}{v-1}\right)dv = \lim_{b\to\infty}\left[-2\ln|v| + 2\ln|v-1|\right]_2^b$$

$$= \lim_{b\to\infty} \left[ \ln|v-1|^2 - \ln|v|^2 \right]_2^b = \lim_{b\to\infty} \left[ \ln\left(\frac{v-1}{v}\right)^2 \right]_2^b$$

$$= \lim_{b\to\infty} \left[ \ln\left(\frac{b-1}{b}\right)^2 - \ln\left(\frac{1}{4}\right) \right] = 0 - \ln\left(\frac{1}{4}\right) = \ln 4.$$

13) $$\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = \int_{-\infty}^{0} \frac{2x\,dx}{(x^2+1)^2} + \int_{0}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$$

$$= \lim_{a\to-\infty} \int_a^0 2x\,(x^2+1)^{-2}\,dx + \lim_{b\to\infty} \int_0^b 2x\,(x^2+1)^{-2}\,dx$$

$$= \lim_{a\to-\infty} \left[\frac{-1}{x^2+1}\right]_a^0 + \lim_{b\to\infty} \left[\frac{-1}{x^2+1}\right]_0^b$$

$$= \lim_{a\to-\infty} \left[\frac{-1}{1} + \frac{1}{a^2+1}\right] + \lim_{b\to\infty} \left[\frac{-1}{b^2+1} + \frac{1}{1}\right]$$

$$= -1 + 0 - 0 + 1 = 0.$$

20) $$\int_0^{\infty} \frac{16\tan^{-1}x}{1+x^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{16\tan^{-1}x}{1+x^2}\,dx$$

$$\boxed{\begin{aligned} u &= \tan^{-1}x \\ du &= \frac{1}{1+x^2}\,dx \\ \int 16u\,du &= 8u^2 \\ &= 8\left(\tan^{-1}x\right)^2 \end{aligned}}$$

$$= \lim_{b\to\infty} \left[ 8\left(\tan^{-1}x\right)^2 \right]_0^b$$

$$= \lim_{b\to\infty} \left[ 8\left(\tan^{-1}b\right)^2 - 8\tan^{-1}0 \right]$$

$$= 8\left(\frac{\pi}{2}\right)^2 - 0 = 2\pi^2$$

23) $\int_{-\infty}^{0} e^{-|x|}\,dx = \int_{-\infty}^{0} e^{x}\,dx = \lim_{a\to-\infty} \int_{a}^{0} e^{x}\,dx$

$= \lim_{a\to-\infty} \left[e^{x}\right]_{a}^{0} = \lim_{a\to-\infty} \left[e^{0} - e^{a}\right] = 1 - 0 = 1$

27) $\int_{0}^{2} \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to2^-} \int_{0}^{b} \frac{ds}{\sqrt{4-s^2}} = \lim_{b\to2^-} \frac{1}{2}\int_{0}^{b} \frac{ds}{\sqrt{1-\left(\frac{s}{2}\right)^2}}$

$= \lim_{b\to2^-} \left[\sin^{-1}\left(\frac{s}{2}\right)\right]_{0}^{b}$

$= \lim_{b\to2^-} \left[\sin^{-1}\left(\frac{b}{2}\right) - \sin^{-1} 0\right]$

$= \sin^{-1} 1 - 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

$u = \frac{s}{2}$

$du = \frac{1}{2}\,ds$

$\frac{1}{2}\int \frac{ds}{\sqrt{1-\left(\frac{s}{2}\right)^2}} = \int \frac{du}{\sqrt{1-u^2}}$

$= \sin^{-1} u = \sin^{-1}\left(\frac{s}{2}\right)$

In the following, use integration, DCT, or LCT to test the integrals for convergence.

51) $\int_{1}^{\infty} \frac{dx}{x^3+1}$ (done in lecture)

54) $\int_{0}^{\infty} \frac{d\theta}{1+e^{\theta}}$

let $f(\theta) = \frac{1}{e^{\theta}}$ and $g(\theta) = \frac{1}{1+e^{\theta}}$ (positive and continuous for $\theta \geq 0$)

$\lim_{\theta\to\infty} \frac{f(\theta)}{g(\theta)} = \lim_{\theta\to\infty} \frac{\frac{1}{e^{\theta}}}{\frac{1}{1+e^{\theta}}} = \lim_{\theta\to\infty} \frac{1+e^{\theta}}{e^{\theta}} = 1 = L$

$(0 < L < \infty)$

Then $\int_0^\infty \frac{d\theta}{e^\theta}$ and $\int_0^\infty \frac{d\theta}{1+e^\theta}$ both converge or both diverge.

$$\int_0^\infty \frac{d\theta}{e^\theta} = \lim_{b\to\infty} \int_0^b e^{-\theta}\, d\theta = \lim_{b\to\infty} \left[-e^{-\theta}\right]_0^b = \lim_{b\to\infty} -\left[e^{-b} - e^0\right]$$

$$= 0 + 1 = 1 \Rightarrow \int_0^\infty \frac{d\theta}{e^\theta} \text{ converges}$$

Then $\int_0^\infty \frac{d\theta}{1+e^\theta}$ converges by LCT.

57) $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\, dx$

$\left( f(x) = \frac{\sqrt{x+1}}{x^2},\ g(x) = \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}} \right.$

$\left. f \text{ and } g \text{ are positive and continuous for } x \geq 1 \right)$

$$\lim_{x\to\infty} \left( \frac{\sqrt{x+1}/x^2}{1/x^{3/2}} \right)$$

$$= \lim_{x\to\infty} \frac{\sqrt{x+1}}{x^{1/2}} = \lim_{x\to\infty} \sqrt{\frac{x+1}{x}} = 1 = L \qquad (0 < L < \infty)$$

$\int_1^\infty \frac{1}{x^{3/2}}\, dx$ converges ( p-integral with $p = \frac{3}{2} > 1$)

Then $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\, dx$ also converges by LCT.

60) $\int_\pi^\infty \frac{1+\sin x}{x^2}$

**Remark**: Try avoiding LCT with sin and cos

$$-1 \leq \sin x \leq 1$$

$$0 \leq \sin x + 1 \leq 2$$

$$0 \leq \underbrace{\frac{\sin x + 1}{x^2}}_{f(x)} \leq \underbrace{\frac{2}{x^2}}_{g(x)} \qquad (x \geq \pi)$$

$f(x)$ and $g(x)$ are continuous for $x \geq \pi$.

$$\int_{\pi}^{\infty} \frac{2}{x^2}\,dx = 2 \lim_{b\to\infty} \int_{\pi}^{b} x^{-2}\,dx = 2 \lim_{b\to\infty} \left[\frac{-1}{x}\right]_{\pi}^{b}$$

$$= 2 \lim_{b\to\infty} \left[\frac{-1}{b} + \frac{1}{\pi}\right] = 2\left(0 + \frac{1}{\pi}\right) = \frac{2}{\pi} \Rightarrow \text{converges} \begin{pmatrix}\text{p-integral} \\ p = 2 > 1\end{pmatrix}$$

Then, $\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\,dx$ converges by DCT.

62) $\int_{2}^{\infty} \frac{1}{\ln x}\,dx$

$0 < \ln x < x \qquad (x \geq 2)$

$0 < \underbrace{\frac{1}{x}}_{f(x)} < \underbrace{\frac{1}{\ln x}}_{g(x)}$ ( $f(x)$ and $g(x)$ are continuous for $x \geq 2$)

$\int_{2}^{\infty} \frac{1}{x}\,dx$ diverges (p-integral with $p=1$)

$\Rightarrow \int_{2}^{\infty} \frac{1}{\ln x}\,dx$ diverges by DCT

63) $\int_{1}^{\infty} \frac{e^x}{x}\,dx$ $\qquad 0 < 1 < e^x$ for $(x \geq 1)$

$0 < \underbrace{\frac{1}{x}}_{f(x)} < \underbrace{\frac{e^x}{x}}_{g(x)}$

$f(x)$ and $g(x)$ are continuous for $x \geq 1$.

$\int_1^\infty \frac{1}{x}\,dx$ diverges ( p-integral with $p=1$).

Then $\int_1^\infty \frac{e^x}{x}\,dx$ diverges by the DCT.

66) $\int_1^\infty \frac{1}{e^x - 2^x}\,dx$ $\left( f(x) = \frac{1}{e^x} \;,\; g(x) = \frac{1}{e^x - 2^x} \text{ positive and continuous for } x \geq 1 \right)$

$$\lim_{x \to \infty} \frac{\frac{1}{e^x}}{\frac{1}{e^x - 2^x}} = \lim_{x \to \infty} \frac{e^x - 2^x}{e^x} = \lim_{x \to \infty} \frac{1 - \left(\frac{2}{e}\right)^x}{1} = \frac{1-0}{1} = 1 = L$$

where $0 < L < \infty$

Now, $\int_1^\infty \frac{1}{e^x}\,dx = \lim_{b \to \infty} \int_1^b e^{-x}\,dx = \lim_{b \to \infty} -\left[e^{-x}\right]_1^b$

$= \lim_{b \to \infty} -\left[e^{-b} - e^{-1}\right] = +\frac{1}{e} \Rightarrow$ converges

Then, $\int_1^\infty \frac{1}{e^x - 2^x}\,dx$ converges by LCT.

70) $\int_{-\infty}^\infty f(x)\,dx$ may not equal $\lim_{b \to \infty} \int_{-b}^b f(x)\,dx$.

Show that $\int_0^\infty \frac{2x\,dx}{x^2+1}$ diverges.

Hence, that $\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$ diverges

Then, show that $\lim_{b \to \infty} \int_{-b}^b \frac{2x\,dx}{x^2+1} = 0$

* $\displaystyle\int_0^{\infty} \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty}\int_0^b \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty}\left[\ln(x^2+1)\right]_0^b$

$\displaystyle= \lim_{b\to\infty}\left(\ln(b^2+1) - \ln 1\right) = \infty \implies$ diverges

* $\displaystyle\int_{-\infty}^{\infty} \frac{2x\,dx}{x^2+1} = \int_{-\infty}^{0} \frac{2x\,dx}{x^2+1} + \underbrace{\int_0^{\infty} \frac{2x\,dx}{x^2+1}}_{\text{diverges}}$

$\displaystyle\implies \int_{-\infty}^{\infty} \frac{2x\,dx}{x^2+1}$ diverges

* $\displaystyle\lim_{b\to\infty}\int_{-b}^{b} \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty}\left[\ln(x^2+1)\right]_{-b}^{b}$

$\displaystyle= \lim_{b\to\infty}\left[\ln(b^2+1) - \ln(b^2+1)\right] = \lim_{b\to\infty} 0 = 0$

So, $\displaystyle\int_{-\infty}^{\infty} \frac{2x\,dx}{x^2+1} \neq \lim_{b\to\infty}\int_{-b}^{b} \frac{2x\,dx}{x^2+1}$