Section 10.4 – Comparison Tests

The Direct Comparison Test:

let $\sum a_n$ be a series with non-negative terms $(a_n \geq 0)$.

a) If there is a convergent series $\sum c_n$ such that $\boxed{0 \leq a_n \leq c_n}$ for all $n \geq N$, then $\sum a_n$ converges.

b) If there is a divergent series $\sum d_n$ such that $\boxed{a_n \geq d_n \geq 0}$ for all $n \geq N$, then $\sum a_n$ diverges.

Examples:

1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + 7n}$ $\qquad$ $0 < \dfrac{1}{3^n + 7n} < \dfrac{1}{3^n}$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (geometric series with $|r| = \frac{1}{3} < 1$).

Then, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + 7n}$ also converges by the DCT.

2) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$ $\qquad$ $\dfrac{5}{5n-1} > \dfrac{5}{5n} > 0$ for $n \geq 1$.

$\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$).

Then, $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$ diverges by the D.C.T.

Limit Comparison Test:

Suppose that $\boxed{a_n > 0}$ and $\boxed{b_n \geq 0}$ for $n \geq N$.

1) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$ where $\boxed{0 < L < \infty}$

then, $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges

3) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges

Examples:

1) $\sum\limits_{n=2}^{\infty} \frac{3}{n^3-2}$ (Apply LCT with $\sum\limits_{n=2}^{\infty} \frac{1}{n^3}$)

$$\lim_{n\to\infty} \frac{\frac{3}{n^3-2}}{1/n^3} = \lim_{n\to\infty} \frac{3n^3}{n^3-2} = \lim_{n\to\infty} \frac{3n^3}{n^3} = 3 \quad (\text{and } 0<3<\infty)$$

We know that $\sum\limits_{n=2}^{\infty} \frac{1}{n^3}$ converges (p-series with $p=3>1$)

Then, $\sum\limits_{n=2}^{\infty} \frac{3}{n^3-2}$ also converges by L.C.T.

2) $\sum\limits_{n=1}^{\infty} \frac{2n+1}{n^2+3n+2}$ (L.C.T with $\sum\limits_{n=1}^{\infty} \frac{2}{n}$)

$$\lim_{n\to\infty} \frac{\frac{2n+1}{n^2+3n+2}}{\frac{2}{n}} = \lim_{n\to\infty} \frac{2n+1}{n^2+3n+2} \times \frac{n}{2} = \lim_{n\to\infty} \frac{2n^2}{2n^2} = 1$$

(and $0<1<\infty$)

We know that $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges (p- series with $p=1$).

Then $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+3n+2}$ diverges also by L.C.T.

3) $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$ (Apply LCT with $\sum_{n=1}^{\infty} \frac{1}{n}$).

$$\lim_{n\to\infty} \frac{\frac{1}{1+\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{1+\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \frac{1}{0} = \infty$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (p-series with $p=1$).

Then $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$ also diverges by L.C.T.

Exercises:

Which of the series converge and which diverge?

18) $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ (Apply LCT with $\sum_{n=1}^{\infty} \frac{1}{n}$).

$$\lim_{n\to\infty} \frac{\frac{3}{n+\sqrt{n}}}{1/n} = \lim_{n\to\infty} \frac{3n}{n+\sqrt{n}} = 3 \quad (\text{and } 0<3<\infty)$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p- series with $p=1$).

Then $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ also diverges by the L.C.T.

20) $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$

$-1 \leq \cos n \leq 1$

$0 \leq 1+\cos n \leq 2$

$0 \leq \frac{1+\cos n}{n^2} \leq \frac{2}{n^2}$

$\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges (p-series with $p=2>1$).

Then, $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$ converges by D.C.T

24) $\sum_{n=3}^{\infty} \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$ (Apply LCT with $\sum_{n=3}^{\infty} \frac{5}{n^2}$)

$$\lim_{n\to\infty} \frac{\frac{5n^3-3n}{n^2(n-2)(n^2+5)}}{\frac{5}{n^2}} = \lim_{n\to\infty} \frac{5n^3-3n}{n^2(n-2)(n^2+5)} \times \frac{n^2}{5} = \lim_{n\to\infty} \frac{5n^5}{5n^5} = 1$$

(and $0<1<\infty$)

$\sum_{n=3}^{\infty} \frac{5}{n^2}$ converges (p-series with $p=2>1$)

Then $\sum_{n=3}^{\infty} \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$ also converges by LCT.

27) $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

$\ln n < n$ (for $n \geq 3$)

$\ln(\ln n) < \ln n < n$

$0 < \frac{1}{n} < \frac{1}{\ln(\ln n)}$

$\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$)

Then, $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ diverges as well by the D.C.T.

31) $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$ (done in lecture)

33) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ (Apply LCT with $\sum_{n=2}^{\infty} \frac{1}{n^2}$)

$$\lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2\left(1-\frac{1}{n^2}\right)}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}\sqrt{1-\frac{1}{n^2}}} = \frac{1}{\sqrt{1-0}} = 1. \text{ (and } 0<1<\infty)$$

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges (p-series with $p=2>1$).

Then, $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges as well by L.C.T.

58) If $\sum_{n=1}^{\infty} a_n$ is a convergent series of non-negative numbers, can anything be said about $\sum_{n=1}^{\infty} \left(\frac{a_n}{n}\right)$? Explain.

$0 \le \frac{a_n}{n} \le a_n$ (for $n \ge 1$).

$\sum_{n=1}^{\infty} a_n$ convergent, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ convergent by D.C.T.

or

$$\lim_{n\to\infty} \frac{\frac{a_n}{n}}{a_n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

and $\sum_{n=1}^{\infty} a_n$ convergent, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ convergent by LCT.

60) Prove that if Σa_n is convergent series of nonnegative terms, then Σa_n^2 converges.

$a_n \geq 0$ and $a_n^2 \geq 0$ Then, we can use LCT.

$$\lim_{n\to\infty} \frac{a_n^2}{a_n} = \lim_{n\to\infty} a_n$$

Σa_n converges $\Longrightarrow \lim\limits_{n\to\infty} a_n = 0$ (Theorem in section 10.2)

Then $\lim\limits_{n\to\infty} \dfrac{a_n^2}{a_n} = 0$

and Σa_n converges, then Σa_n^2 converges by LCT.