Chapter - 10 - Sequences and Series

Section 10.1 - Sequences:

ex: 2, 4, 6, 8, 10, 12, ... (List of numbers in a given order)

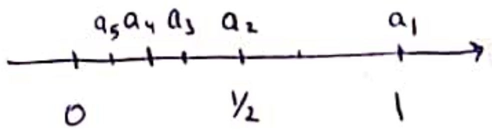
$a_n = 2n$ for $n = 1, 2, 3, 4, \dots$

Definition: An infinite Sequence of numbers is a function whose domain is the set of integers greater than or equal to some integer n_0 (usually $n \geq 0$, $n \geq 1$)

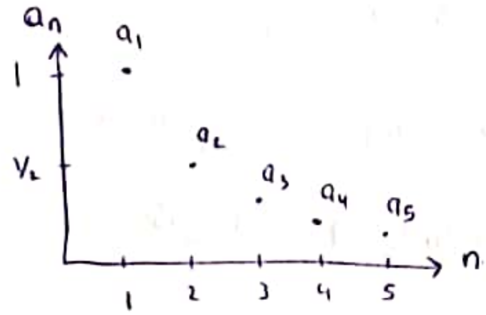
Graphically, we can represent sequences in 2 ways:

ex: $a_n = \frac{1}{n}$, $n \geq 1$

$a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, ...

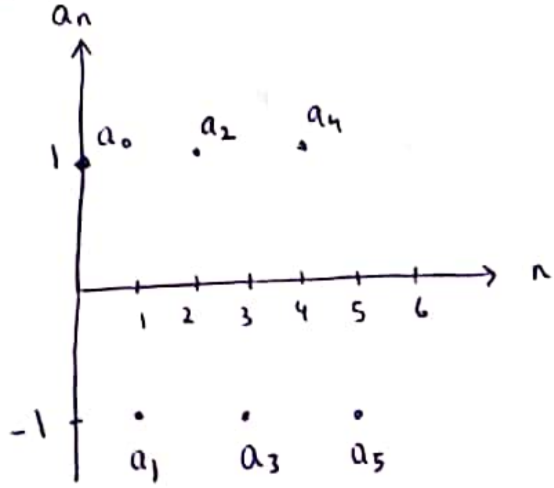


converges to 0.



converges to 0

ex: $a_n = (-1)^n$ $n = 0, 1, 2, 3, \dots$



alternating sequences.

Definition of Convergence:

A sequence $\{a_n\}$ is said to converge to a finite number L if for every positive number ε $(\varepsilon > 0)$, there corresponds an integer N such that:

for all $n > N$, $\quad |a_n - L| < \varepsilon$

$$-\varepsilon < a_n - L < \varepsilon$$

If no such number L exists, we say $\{a_n\}$ diverges.

Notation: If $\{a_n\}$ converges to L, we write $a_n \longrightarrow L$

or $\lim\limits_{n\to\infty} a_n = L$

Limit Laws for Sequences:

Let $\{a_n\}$, $\{b_n\}$ be sequences such that $a_n \longrightarrow A$ and $b_n \longrightarrow B$

1) $\lim\limits_{n\to\infty} (a_n \pm b_n) = A \pm B$

2) $\lim\limits_{n\to\infty} (a_n \cdot b_n) = A \cdot B$

3) $\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B} \quad (B \neq 0)$

4) $\lim\limits_{n\to\infty} (k a_n) = K.A$

example: $\lim\limits_{n\to\infty} \left(\frac{1}{n} + \frac{1}{\sqrt{2n}}\right) = 0 + 0 = 0$

$$\lim_{n\to\infty} \left(\frac{4 - 7n^6}{n^6 + 3}\right) = \lim_{n\to\infty} \left(\frac{\frac{4}{n^6} - 7}{1 + \frac{3}{n^6}}\right) = \frac{-7}{1} = -7$$

The Sandwich theorem:

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of real numbers such that:

$\boxed{a_n \le b_n \le c_n}$ for all $n \geq N$.

If $\lim\limits_{n \to \infty} a_n = L$ and $\lim\limits_{n \to \infty} c_n = L$, then: $\lim\limits_{n \to \infty} b_n = L$

example: $a_n = \dfrac{(-1)^n}{n}$, $n \geq 1$

$$\frac{-1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}$$

Since $\frac{-1}{n} \longrightarrow 0$ and $\frac{1}{n} \longrightarrow 0$, then $\frac{(-1)^n}{n} \longrightarrow 0$ by Sandwich theorem

The Continuous Function theorem for Sequences:

Let $\{a_n\}$ be a sequence such that $a_n \longrightarrow L$. Let $f(x)$ be a function defined at each a_n and continuous at L. Then:

$f(a_n) \longrightarrow f(L)$

Example: Show that $2^{\frac{1}{n}} \longrightarrow 1$

$\frac{1}{n} \longrightarrow 0$

$f(x) = 2^x \Rightarrow 2^{\frac{1}{n}} \longrightarrow 2^0 = 1.$

Theorem: Let $f(x)$ be a function defined for all $x \geq n_0$.

Let $\{a_n\}$ be a sequence such that $f(n) = a_n$ for $n \geq n_0$

Then: $\lim\limits_{x \to \infty} f(x) = L \Longrightarrow \lim\limits_{n \to \infty} a_n = L.$

Examples:

1) $a_n = \frac{\ln n}{n}$

$\lim\limits_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty} \xRightarrow{\text{L'Hopital's}} \lim\limits_{x\to\infty} \frac{1/x}{1} = \frac{1}{\infty} = 0$

$\Rightarrow \lim\limits_{n\to\infty} \frac{\ln n}{n} = 0 \quad (a_n \longrightarrow 0)$

2) $b_n = \frac{2^n}{5n}$

$\lim\limits_{n\to\infty} \frac{2^n}{5n} = \frac{\infty}{\infty} \Rightarrow \lim\limits_{n\to\infty} \frac{2^n \ln 2}{5} = \infty \quad$ (b_n diverges)

Indeterminate forms for limits:

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty$.

$\boxed{1^\infty, \infty^0, 0^0} \longrightarrow$ use $\ln$

3) $a_n = \left(\frac{n+1}{n-1}\right)^n$

$\lim\limits_{n\to\infty} a_n = 1^\infty \Rightarrow$ use $\ln$.

$\ln a_n = \ln\left[\left(\frac{n+1}{n-1}\right)^n\right]$

$= n \ln\left(\frac{n+1}{n-1}\right)$

$\lim\limits_{n\to\infty} \ln a_n = \infty \times 0 \quad \underline{\text{ind}} \Rightarrow \lim\limits_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} = \frac{0}{0}$ (Apply l'hopitals)

$= \lim\limits_{n\to\infty} \frac{\left(\frac{n+1}{n-1}\right)' \Big/ \frac{n+1}{n-1}}{\frac{-1}{n^2}} = \lim\limits_{n\to\infty} \frac{\frac{n-1-(n+1)}{(n-1)^2} \times \frac{n-1}{n+1}}{\frac{-1}{n^2}}$

$= \lim\limits_{n\to\infty} \frac{\frac{-2}{(n-1)(n+1)}}{\frac{-1}{n^2}} = \lim\limits_{n\to\infty} \frac{+2n^2}{(n-1)(n+1)} = \lim\limits_{n\to\infty} \frac{2n^2}{n^2} = 2$

If $\ln a_n \longrightarrow 2$, then $a_n \longrightarrow e^2$

Frequently Faced Limits:

1) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2) $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = 1$

Proof: $\ln\left(n^{\frac{1}{n}}\right) = \frac{1}{n}\ln n \longrightarrow 0$

Then $n^{\frac{1}{n}} \longrightarrow e^0 = 1$

3) $\lim_{n\to\infty} x^{\frac{1}{n}} = 1 \quad (x>0)$

4) $\lim_{n\to\infty} x^n = 0 \quad (-1<x<1)$

5) $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

Proof: $\ln\left(1+\frac{x}{n}\right)^n = n\ln\left(1+\frac{x}{n}\right) = \frac{\ln\left(1+\frac{x}{n}\right)}{1/n} \cdots\cdots$

$\ln\left(1+\frac{x}{n}\right)^n \longrightarrow x$, then $\left(1+\frac{x}{n}\right)^n \longrightarrow e^x$

6) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (for any x)

Examples:

1) $\lim_{n\to\infty} n^{3/n} = \lim_{n\to\infty} \sqrt[n]{n^3} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^3 = 1^3 = 1$

2) $\lim_{n\to\infty} \left(\frac{1}{5}\right)^n = 0$

3) $\lim_{n\to\infty} \left(\frac{n-3}{n}\right)^n = \lim_{n\to\infty} \left(1-\frac{3}{n}\right)^n = e^{-3}$

4) $\lim_{n\to\infty} \sqrt[n]{5n} = \lim_{n\to\infty} \sqrt[n]{5} \cdot \sqrt[n]{n} = 1 \times 1 = 1$

5) $\lim_{n \to \infty} \frac{100^n}{n!} = 0$

Recursive Definitions:

We define a sequence recursively by using:

1) the value of the initial term(s).
2) the recursion formula.

example: $\begin{cases} a_1 = 1 \\ a_n = n a_{n-1} & n \geq 2 \end{cases}$

$a_1 = 1, \quad a_2 = 2, \quad a_3 = 6, \quad a_4 = 24, \quad \ldots \quad, \boxed{a_n = n!}$

Bounded Monotonic Sequences:

Definition:
- A sequence $\{a_n\}$ such that $a_n \leq a_{n+1}$ is said to be nondecreasing for all n.
- A sequence $\{a_n\}$ such that $a_n \geq a_{n+1}$ is said to be nonincreasing for all n.

} monotonic sequences

Definition:
- A sequence $\{a_n\}$ is bounded from above if there exists a number M such that $\boxed{a_n \leq M}$ for all n.

  M: upper bound for $\{a_n\}$
- A sequence $\{a_n\}$ is bounded from below if there exists a number m such that $\boxed{a_n \geq m}$ for all n.

  m: lower bound for $\{a_n\}$.
- $\{a_n\}$ is bounded if it is bounded from above and from below.

<u>Remark:</u> not every bounded sequence is convergent.

for ex, $a_n = (-1)^n$ is bounded but not convergent.

<u>Exercises:</u>

Find a formula for the n-th term of the sequence.

22) $2, 6, 10, 14, 18, \ldots$

$a_n = 2 + 4n \qquad (n = 0, 1, 2, \ldots)$

25) $1, 0, 1, 0, 1, \ldots$

$a_n = \dfrac{(-1)^n + 1}{2} \qquad (n = 0, 1, 2, \ldots)$

Which of the sequences $\{a_n\}$ converge, and which diverge? Find the limit of each convergent sequence.

32) $a_n = \dfrac{n+(-1)^n}{n}$

$-\dfrac{1}{n} \leq \dfrac{(-1)^n}{n} \leq \dfrac{1}{n}$

$-\dfrac{1}{n} \to 0$ and $\dfrac{1}{n} \to 0$

then $\dfrac{(-1)^n}{n} \to 0$

$a_n = 1 + \dfrac{(-1)^n}{n}$

$\lim\limits_{n\to\infty} a_n = 1 + 0 = 1$

convergent to 1, by sandwich theorem

40) $a_n = (-1)^n\left(1-\dfrac{1}{n}\right)$

If n even: $\lim\limits_{n\to\infty} a_n = 1\left(1-\dfrac{1}{\infty}\right) = 1$

If n odd: $\lim\limits_{n\to\infty} a_n = -1\left(1-\dfrac{1}{\infty}\right) = -1$

$\Big\}$ $\lim\limits_{n\to\infty} a_n$ doesn't exist $\Rightarrow \{a_n\}$ diverges.

41) $a_n = \left(\dfrac{n+1}{2n}\right)\left(1-\dfrac{1}{n}\right)$

$\lim\limits_{n\to\infty} \left(\dfrac{n+1}{2n}\right)\left(1-\dfrac{1}{n}\right) = \dfrac{1}{2} \times 1 = \dfrac{1}{2}$ $\Rightarrow$ convergent to $\dfrac{1}{2}$

50) $a_n = \dfrac{\sin^2 n}{2^n}$

$-1 \leq \sin n \leq 1$

$0 \leq \sin^2 n \leq 1$

$0 \leq \dfrac{\sin^2 n}{2^n} \leq \dfrac{1}{2^n}$

$\lim\limits_{n\to\infty} 0 = 0$ and $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$

Then $\lim\limits_{n\to\infty} a_n = 0$ $\Rightarrow$ convergent to 0, by Sandwich theorem.

62) $a_n = (n+4)^{\frac{1}{n+4}}$

$\ln a_n = \ln (n+4)^{\frac{1}{n+4}} = \frac{1}{n+4} \ln(n+4)$

$\lim_{n\to\infty} \ln a_n = \frac{\infty}{\infty} \overset{H.R}{=} \lim_{n\to\infty} \frac{\frac{1}{n+4}}{1} = 0$

then $a_n \longrightarrow e^0 = 1 \implies$ convergent to 1.

64) $a_n = \ln n - \ln(n+1)$

$a_n = \ln\left(\frac{n}{n+1}\right)$

$\lim_{n\to\infty} a_n = \ln 1 = 0 \implies$ convergent to 0.

67) $a_n = \frac{n!}{n^n}$ (Hint: Compare with $\frac{1}{n}$).

$\frac{1}{n} \frac{\times n^{n-1}}{\times n^{n-1}} = \frac{n^{n-1}}{n^n}$

$n! < n^{n-1}$

$\frac{n!}{n^n} < \frac{n^{n-1}}{n^n}$

$a_n < \frac{1}{n}$ and $a_n > 0$

So $0 < a_n < \frac{1}{n}$

$\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$. Then $\lim_{n\to\infty} a_n = 0$

$\implies$ convergent to 0 by the Sandwich theorem.

82) $a_n = \left(1 - \frac{1}{n^2}\right)^n$

$a_n = \left(1 - \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)^n$

$\lim_{n\to\infty} a_n = e^{-1} \cdot e^{1} = e^{0} = 1 \implies$ convergent to 1.

or use ln.

94) $a_n = \sqrt[n]{n^2+n} = (n^2+n)^{1/n}$

$\lim_{n\to\infty} a_n = \infty^{0}$

$\ln a_n = \ln (n^2+n)^{1/n} = \frac{1}{n} \ln (n^2+n)$

$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln(n^2+n)}{n} = \frac{\infty}{\infty} \overset{H.R}{=} \lim_{n\to\infty} \frac{\frac{2n+1}{n^2+n}}{1}$

$= \lim_{n\to\infty} \frac{2n}{n^2} = \frac{2}{\infty} = 0$

$\ln a_n \longrightarrow 0$, then $a_n \longrightarrow e^{0} = 1 \implies$ convergent to 1.

97) $a_n = n - \sqrt{n^2 - n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\left(n - \sqrt{n^2-n}\right) \times \left(n + \sqrt{n^2-n}\right)}{1 \times \left(n + \sqrt{n^2-n}\right)}$

$= \lim_{n\to\infty} \frac{n^2 - (n^2-n)}{n + \sqrt{n^2-n}} = \lim_{n\to\infty} \frac{n}{n + \sqrt{n^2\left(1 - \frac{1}{n}\right)}} = \lim_{n\to\infty} \frac{n}{n + n\sqrt{1 - \frac{1}{n}}}$

$= \lim_{n\to\infty} \frac{n}{n\left(1 + \sqrt{1 - \frac{1}{n}}\right)} = \frac{1}{1 + \sqrt{1}} = \frac{1}{2} \implies$ convergent to $\frac{1}{2}$

129) $a_n = \left[(-1)^n + 1 \right] \left(\frac{n+1}{n} \right)$

If n even: $\lim\limits_{n\to\infty} a_n = 2 \times 1 = 2$

If n odd: $\lim\limits_{n\to\infty} a_n = 0 \times 1 = 0$

$\Big\}$ $\lim\limits_{n\to\infty} a_n$ doesn't exist

$\Rightarrow a_n$ diverges.

133) $a_n = \dfrac{4^{n+1} + 3^n}{4^n} = \dfrac{4^{n+1}}{4^n} + \dfrac{3^n}{4^n} = 4 + \left(\dfrac{3}{4}\right)^n$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(4 + \left(\frac{3}{4}\right)^n \right) = 4 + 0 = 4$

$\Rightarrow$ convergent to 4.

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