# Section 10.3 - Integral test

**Theorem:** let $\{a_n\}$ be a sequence with positive terms. Assume that $a_n = f(n)$ where $f(x)$ is continuous, positive, decreasing function of $x$ for all $x \geq N$.

then, the series $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

**Note:** If they both converge, they may not converge to the same value.

**Examples:**

1) Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ ; $f(x) = \frac{1}{x}$

* For $x \geq 1$, $f(x)$ is continuous, positive and decreasing since for $x_1 < x_2 \Rightarrow \frac{1}{x_2} < \frac{1}{x_1}$

* $\int_1^{\infty} \frac{1}{x}\,dx$ diverges ($p$- integral with $p=1$)

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the integral test.

2) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $(p>0)$ : $p$ - series ($p$ is a positive real nb)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p \leq 1. \end{cases}$$

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$f(x) = \frac{1}{x^p}$ $(p>0)$ is positive, continuous, decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^p}\,dx \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p \leq 1. \end{cases}$$

$*$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ : diverges $\left(\text{p-series with } p = \frac{1}{2} < 1\right)$

$*$ $\sum_{n=1}^{\infty} \frac{2}{n^3} = 2 \sum_{n=1}^{\infty} \frac{1}{n^3}$ : converges $\left(\text{p-series with } p = 3 > 1\right)$

<u>Exercises:</u>

Which of the following series converge and which diverge?

21) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ $\left( \lim_{n\to\infty} \frac{\ln n}{n} \overset{H.R}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0 \Rightarrow \text{no conclusion} \right)$

$f(x) = \frac{\ln x}{x}$ $(f(x) \text{ is positive and continuous for } x \geq 2)$

$f'(x) = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1-\ln x}{x^2} < 0$ for $\boxed{\begin{array}{c} 1-\ln x<0 \\ \ln x>1 \\ x>e \end{array}}$

$\Rightarrow f(x)$ decreasing for $x>e$

$$\int_2^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \left[\frac{(\ln x)^2}{2}\right]_2^b = \lim_{b\to\infty} \left[\frac{(\ln b)^2}{2} - \frac{(\ln 2)^2}{2}\right] = \infty$$

$\int u\,du = \frac{u^2}{2}$

$\Rightarrow \int_2^{\infty} \frac{\ln x}{x}\,dx$ diverges

Then, $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges by the integral test.

34) $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$

$f(x) = \frac{1}{x(1+\ln^2 x)}$ is positive and continuous for $x \geq 1$

For $x_1 < x_2$ , $\ln^2 x_1 < \ln^2 x_2$

$(1+\ln^2 x_1) < (\ln^2 x_2 + 1)$

$x_1(1+\ln^2 x_1) < x_2(1+\ln^2 x_2)$

$\frac{1}{x_2(1+\ln^2 x_2)} < \frac{1}{x_1(1+\ln^2 x_1)} \Rightarrow f(x)$ is decreasing.

$\int_1^{\infty} \frac{dx}{x(1+\ln^2 x)} = \lim_{b\to\infty} \left[\tan^{-1}(\ln x)\right]_1^b = \lim_{b\to\infty} \left[\tan^{-1}(\ln b) - \tan^{-1}(\ln 1)\right]$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$

$u = \ln u \Rightarrow \int \frac{du}{1+u^2}$

$du = \frac{1}{x}dx \quad = \tan^{-1} u$

So $\int_1^{\infty} \frac{dx}{x(1+\ln^2 x)}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$ converges by the integral test.

43) $\sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1+n^2}$

$f(x) = \frac{\tan^{-1} x}{1+x^2}$ (positive and continuous for $x \geq 1$)

$f'(x) = \frac{\frac{1}{1+x^2}(1+x^2) - 2x\tan^{-1} x}{(1+x)^2} = \frac{1 - 2x\tan^{-1} x}{(1+x^2)^2} < 0$

$\Rightarrow f(x)$ decreasing.

$$\int_1^\infty \frac{\tan^{-1}x}{1+x^2}\,dx = \lim_{b\to\infty}\left[\frac{(\tan^{-1}x)^2}{2}\right]_1^b = \lim_{b\to\infty}\left[\frac{(\tan^{-1}b)^2}{2} - \frac{(\tan^{-1}1)^2}{2}\right]$$

$\int u\,du$ where $u = \tan^{-1}x$, $du = \frac{1}{1+x^2}dx$

$$= \frac{(\pi/2)^2}{2} - \frac{(\pi/4)^2}{2} = \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3\pi^2}{32}$$

So $\int_1^\infty \frac{\tan^{-1}x}{1+x^2}\,dx$ converges

$\Longrightarrow \sum_{n=1}^{\infty} \frac{\tan^{-1}n}{1+n^2}$ converges by integral test $\Longrightarrow \sum_{n=1}^{\infty}\frac{8\tan^{-1}n}{1+n^2}$ converges

61) a) Show that the improper integral $\int_2^\infty \frac{dx}{x(\ln x)^p}$ ($p$ positive constant) converges if and only if $p>1$. (Logarithmic $p$-integral).

$\boxed{p\neq 1}$: $\int_2^\infty \frac{dx/x}{(\ln x)^p} = \lim_{b\to\infty}\left[\frac{(\ln x)^{-p+1}}{-p+1}\right]_2^b$

$\int \frac{du}{u^p} = \frac{u^{-p+1}}{-p+1}$

$$= \lim_{b\to\infty}\left[\frac{(\ln b)^{-p+1}}{-p+1} - \frac{(\ln 2)^{-p+1}}{-p+1}\right]$$

* If $p>1$: $-p+1<0 \Rightarrow \lim_{b\to\infty}(\ln b)^{-p+1} = 0 \Rightarrow$ integral converges
* If $p<1$: $-p+1>0 \Rightarrow \lim_{b\to\infty}(\ln b)^{-p+1} = \infty \Rightarrow$ integral diverges.

$\boxed{p=1}$: $\int_2^\infty \frac{dx/x}{\ln x} = \lim_{b\to\infty}\left[\ln(\ln x)\right]_2^b = \infty$

$\int \frac{du}{u} = \ln u$ where $u = \ln x$, $du = \frac{dx}{x}$

$\Rightarrow$ integral diverges

b) What can you say about the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ ? (Logarithmic p-series).

$$f(x) = \frac{1}{x(\ln x)^p} \quad \text{(positive and continuous for } x \geq 2)$$

$$x_1 < x_2 \Rightarrow \frac{1}{x_1(\ln x_1)} > \frac{1}{x_2(\ln x_2)} \Rightarrow f(x) \text{ decreasing.}$$

$\Rightarrow$ By the integral test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ → converges for $p>1$
→ diverges for $p \leq 1$.

62) Use the result in exercise 61 to determine which of the following series converge and which diverge?

a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ : diverges (Logarithmic p-series with $p=1$)

b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.01}}$ : converges (Logarithmic p-series with $p=1.01>1$)

c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ : Converges (Logarithmic p-series with $p=3>1$)

d) $\sum_{n=2}^{\infty} \frac{1}{n \ln(n^3)} = \sum_{n=2}^{\infty} \frac{1}{3n \ln n}$ : diverges (Logarithmic p-series $p=1$).