Section 10.2 _ Infinite Series:

Def: let $\{a_n\}$ be a sequence.

The sum: $a_0 + a_1 + a_2 + \dots = \sum_{n=0}^{\infty} a_n$ is called an infinite series

$$\left.\begin{aligned} S_0 &= a_0 \\ S_1 &= a_0 + a_1 \\ S_2 &= a_0 + a_1 + a_2 \\ &\vdots \\ S_n &= a_0 + a_1 + a_2 + \dots + a_n \end{aligned}\right\}$$

$\{S_n\}$ sequence of partial sums of the series.

$*$ $S_n = a_0 + a_1 + \dots + a_n$

As $n \to \infty$, $S_n \longrightarrow \sum_{n=0}^{\infty} a_n$

$\{S_n\}$ converges to $L \iff \sum_{n=0}^{\infty} a_n = L$

$\lim_{n\to\infty} S_n = L \iff \sum_{n=0}^{\infty} a_n = L$

$\{S_n\}$ diverges $\iff \sum_{n=0}^{\infty} a_n$ diverges.

Example: Telescoping series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \qquad (A = 1 \text{ and } B = -1)$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots$$

$$S_1 = a_1 = 1 - \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$$

$$S_3 = a_1 + a_2 + a_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$S_n = a_1 + a_2 + \dots + a_n = 1 - \frac{1}{n+1}$$

$\lim_{n\to\infty} S_n = 1 - 0 = 1$. Therefore $\sum_{n=1}^{\infty} a_n = 1$ (series converges to 1)

Geometric Series:

$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots + \frac{1}{3^n} + \dots$

ratio $= \frac{1}{3}$

* Geometric series are series of the form:

$a + ar + ar^2 + \dots + ar^n + \dots = \sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1}$

(1st term: a, ratio: r)

* r can be positive: $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$ $\left(r = \frac{1}{3}\right)$

or negative: $\frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \dots$ $\left(r = -\frac{1}{2}\right)$

* For $r = 1$: $a + a + a + a + \dots$

$$\left.\begin{aligned} S_1 &= a \\ S_2 &= a + a = 2a \\ S_3 &= 3a \\ &\vdots \\ S_n &= na \end{aligned}\right\}$$

$\lim_{n\to\infty} S_n = \infty \Rightarrow \{S_n\}$ diverges.

$\Rightarrow$ the series diverges.

* For $r = -1$: $a - a + a - a + a - \dots$

$$\left.\begin{aligned} S_1 &= a \\ S_2 &= 0 \\ S_3 &= a \\ S_4 &= 0 \end{aligned}\right\}$$

$\lim_{n\to\infty} S_n$ does not exist $\Rightarrow \{S_n\}$ diverges

$\Rightarrow$ the series diverges.

* Geometric series converges to $\frac{\text{1st term}}{1-r}$ if $\boxed{|r|<1}$ $(-1<r<1)$

sum $= \frac{\text{1st term}}{1-r}$ (for $-1<r<1$)

* Geometric series diverges if $|r| \geq 1$ $(r \leq -1$ or $r \geq 1)$

Examples:

1) $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots$

Geometric series with $r = \frac{1}{5}$.

Since $-1 < r < 1$, then the series converges.

$$\text{sum} = \frac{1^{st}\ \text{term}}{1-r} = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{\frac{1}{5}}{\frac{4}{5}} = \frac{1}{4}$$

2) $\sum_{n=0}^{\infty} \left(\frac{\pi}{2}\right)^n = 1 + \frac{\pi}{2} + \left(\frac{\pi}{2}\right)^2 + \cdots$

Geometric series with $r = \frac{\pi}{2} > 1$, then the series diverges.

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \longrightarrow 0$

(The sequence $\{a_n\}$ converges to 0).

The n^{th} term test for divergence:

* If the sequence $\{a_n\}$ diverges or converges to L where $L \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

* In other words, if $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Note: If $\lim_{n\to\infty} a_n = 0$, then no conclusion about $\sum_{n=1}^{\infty} a_n$

ex: $\sum_{n=1}^{\infty} \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0 \implies$ No conclusion about $\sum_{n=1}^{\infty} \frac{1}{n}$

(We will show that it diverges in 10.3)

Examples:

1) $\sum_{n=1}^{\infty} \frac{n}{n+2}$: $\lim_{n\to\infty} \frac{n}{n+2} = 1 \neq 0 \Rightarrow \sum_{n=1}^{\infty} \frac{n}{n+2}$ diverges by n^{th} term test.

2) $\sum_{n=0}^{\infty} (n^2+1)$: $\lim_{n\to\infty} (n^2+1) = \infty \neq 0 \Rightarrow \sum_{n=0}^{\infty} (n^2+1)$ diverges by n^{th} term test.

3) $\sum_{n=0}^{\infty} \left((-1)^n+1\right)$: $\lim_{n\to\infty} [(-1)^n+1]$ doesn't exist $\Rightarrow \sum_{n=0}^{\infty} [(-1)^n+1]$ diverges by n^{th} term test.

Theorem: If $\sum a_n$ converges to A $\left(\sum a_n = A\right)$ and $\sum b_n$ converges to B, $\left(\sum b_n = B\right)$, then:

1) $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$

2) $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$

3) $\sum K a_n = K \sum a_n = K A$.

Examples:

1) $$\sum_{n=1}^{\infty} \frac{2^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{2^{n-1}}{6^{n-1}} - \frac{1}{6^{n-1}}\right) = \sum_{n=1}^{\infty} \left[\left(\frac{2}{6}\right)^{n-1} - \left(\frac{1}{6}\right)^{n-1}\right]$$

$$= \sum_{n=1}^{\infty} \left[\left(\frac{1}{3}\right)^{n-1} - \left(\frac{1}{6}\right)^{n-1}\right]$$

* $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$: geometric series with $|r| = \frac{1}{3} < 1 \Rightarrow$ the series converges

* $\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1}$: geometric series with $|r| = \frac{1}{6} < 1 \Rightarrow$ the series converges.

Then: $$\sum_{n=1}^{\infty} \left[\left(\frac{1}{3}\right)^{n-1} - \left(\frac{1}{6}\right)^{n-1}\right] = \frac{1}{1-\frac{1}{3}} - \frac{1}{1-\frac{1}{6}} = \frac{3}{2} - \frac{6}{5} = \frac{3}{10}$$

2) $\sum_{n=1}^{\infty} \frac{7}{3^n} = \sum_{n=1}^{\infty} 7\left(\frac{1}{3}\right)^n = 7 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

Geometric series with $|r| = \frac{1}{3} < 1 \Rightarrow$ the series converges

$$7\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = 7 \times \frac{1/3}{1-\frac{1}{3}} = \frac{7}{2}$$

Corollaries:

1) If $\sum a_n$ diverges, then $K \sum a_n = \sum K a_n$ also diverges for $K \neq 0$.

2) If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ diverges and $\sum (a_n - b_n)$ diverges.

Note: If $\sum a_n$ diverges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ may converge or diverge.

Exercises:

Find the sum of the following series or show that it diverges.

12) $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right) = 5 \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n}_{\substack{\text{geometric series} \\ \text{with } |r| = \frac{1}{2} < 1 \\ \Rightarrow \text{converges}}} - \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n}_{\substack{\text{geometric series} \\ \text{with } |r| = \frac{1}{3} < 1 \\ \Rightarrow \text{converges}}}$

Then, $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$ converges

and $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right) = 5 \times \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{3}} = 10 - \frac{3}{2} = \frac{17}{2}$

45) $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$ $\left(\lim_{n \to \infty} a_n = 0 \implies \text{no conclusion} \right)$

$$\frac{4}{(4n-3)(4n+1)} = \frac{A}{4n-3} + \frac{B}{4n+1} \quad (A = 1 \text{ and } B = -1)$$

$$\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{4n-3} - \frac{1}{4n+1} \right) = \left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \dots$$

$$S_1 = a_1 = 1 - \frac{1}{5}$$

$$S_2 = a_1 + a_2 = \left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right)$$

$$S_3 = a_1 + a_2 + a_3 = \left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right)$$

$$\vdots$$

$$S_n = a_1 + \dots + a_n = 1 - \frac{1}{4n+1}$$

$\lim_{n \to \infty} S_n = 1 - 0 = 1 \implies \{S_n\}$ converges to 1

$\implies \sum_{n=1}^{\infty} a_n$ converges to 1.

52) $\sum_{n=1}^{\infty} \left(\tan^{-1} n - \tan^{-1}(n+1) \right) = \left(\tan^{-1} 1 - \tan^{-1} 2 \right) + \left(\tan^{-1} 2 - \tan^{-1} 3 \right) + \dots$

$$S_1 = a_1 = \tan^{-1} 1 - \tan^{-1} 2$$

$$S_2 = a_1 + a_2 = \left(\tan^{-1} 1 - \tan^{-1} 2 \right) + \left(\tan^{-1} 2 - \tan^{-1} 3 \right)$$

$$S_n = \tan^{-1} 1 - \tan^{-1}(n+1)$$

$$\lim_{n \to \infty} S_n = \tan^{-1} 1 - \tan^{-1} \infty = \frac{\pi}{4} - \frac{\pi}{2} = \frac{-\pi}{4}$$

$\{S_n\}$ converges to $\frac{-\pi}{4} \implies \sum_{n=1}^{\infty} \left(\tan^{-1} n - \tan^{-1}(n+1) \right)$ converges to $\frac{-\pi}{4}$.

54) $\sum_{n=0}^{\infty} (\sqrt{2})^n = 1 + \sqrt{2} + \sqrt{2}^2 + \dots$

Geometric series with $|r| = \sqrt{2} > 1 \Rightarrow$ the series diverges.

58) $\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{-1}{5}\right)^n$

Geometric series with $|r| = \frac{1}{5} < 1 \Rightarrow$ the series converges

$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n} = \frac{1}{1+\frac{1}{5}} = \frac{1}{\frac{6}{5}} = \frac{5}{6}.$$

60) $\sum_{n=1}^{\infty} \ln\left(\frac{1}{3^n}\right)$

$\lim_{n\to\infty} \ln\left(\frac{1}{3^n}\right) = \ln 0^+ = -\infty \Rightarrow \lim_{n\to\infty} a_n \neq 0$

$\Rightarrow$ the series diverges by n^{th} term test.

64) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$

$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = e^{-1} \neq 0 \Rightarrow$ the series diverges by the n^{th} term test.

66) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ (In section 10.1, n° 67, $\frac{n!}{n^n} \to 0$)

$\lim_{n\to\infty} \frac{n^n}{n!} = \frac{1}{0} = \infty \Rightarrow \lim_{n\to\infty} \frac{n^n}{n!} \neq 0 \Rightarrow$ the series diverges by the n^{th} term test.

69) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ $\left(\lim_{n\to\infty} a_n = \ln 1 = 0 \Rightarrow \text{no conclusion}\right)$

$$= \sum_{n=1}^{\infty} [\ln n - \ln(n+1)] = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots$$

$S_1 = \ln 1 - \ln 2$

$S_2 = (\ln 1 - \ln 2) + (\ln 2 - \ln 3)$

$S_3 = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4)$

$S_n = a_1 + a_2 + \cdots + a_n = \ln 1 - \ln(n+1) = -\ln(n+1)$

$\lim_{n\to\infty} S_n = -\infty \Rightarrow \{S_n\}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ diverges

71) $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$

Geometric series with $|r| = \frac{e}{\pi} < 1 \Rightarrow$ the series converges.

$$\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n = \frac{1}{1-\frac{e}{\pi}} = \frac{\pi}{\pi - e}$$

92) Find convergent geometric series $A = \sum a_n$ and $B = \sum b_n$ that illustrate the fact that $\sum a_n b_n$ may converge without being equal to AB.

* $\sum a_n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ $\left(\text{Geometric series with } |r| = \frac{1}{2} < 1 \Rightarrow \text{converges to } \frac{1}{1-\frac{1}{2}} = 2\right)$

* $\sum b_n = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$ $\left(\text{Geometric series with } |r| = \frac{1}{3} < 1 \Rightarrow \text{converges to } \frac{1}{1-\frac{1}{3}} = \frac{3}{2}\right)$

* $\sum a_n b_n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n$

Geometric series with $|r| = \frac{1}{6} < 1$

$\Rightarrow$ converges to $\frac{1}{1-\frac{1}{6}} = \frac{6}{5}$

* $A \cdot B = \sum a_n \cdot \sum b_n = 2 \times \frac{3}{2} = 3 \neq \frac{6}{5}$.

So $\sum a_n b_n \neq A \cdot B$.

Q4) If $\sum a_n$ converges where $a_n > 0$ for all n, can anything be said about $\sum \frac{1}{a^n}$? Give reasons for your answer.

$\sum a_n$ converges $\longrightarrow$ $\{a_n\}$ converges to 0 (theorem)

$\Rightarrow \lim_{n \to \infty} a_n = 0$

Therefore $\lim_{n \to \infty} \frac{1}{a^n} = \frac{1}{0} = \infty \Rightarrow \lim_{n \to \infty} \frac{1}{a^n} \neq 0$

Then, $\sum \frac{1}{a^n}$ diverges by the n^{th} term test.