Section 10.8 - Taylor and Maclaurin Series

Definition: Let $f(x)$ be a function with derivatives of all orders on some interval containing "a" as an interior point.

* The Taylor series generated by $f(x)$ at "a" is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$+ \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

* The Maclaurin series of f is the Taylor series generated by $f(x)$ at 0 $(a=0)$.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

Examples:

1) Use the definition of Taylor series to find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a=2$

$f(x) = \frac{1}{x}$; $f(2) = \frac{1}{2}$

$f'(x) = \frac{-1}{x^2}$; $f'(2) = \frac{-1}{4}$

$f''(x) = \frac{2}{x^3}$; $f''(2) = \frac{2}{8} = \frac{1}{4}$

$f'''(x) = \frac{-6}{x^4}$; $f'''(2) = \frac{-6}{2^4} = \frac{-3}{8}$

$\vdots$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \dots\dots$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{4}\cdot\frac{1}{2!}(x-2)^2 - \frac{3}{8}\cdot\frac{1}{3!}(x-2)^3 + \dots$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \dots\dots$$

Geometric series : $r = \frac{-1}{2}(x-2)$

General term : $\sum_{n=0}^{\infty} \frac{1}{2}\cdot\left[\frac{-1}{2}(x-2)\right]^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n$

2) Find the Maclaurin series of $f(x) = \sin x$.

$f(x) = \sin x$; $f(0) = 0$

$f'(x) = \cos x$; $f'(0) = 1$

$f''(x) = -\sin x$; $f''(0) = 0$

$f'''(x) = -\cos x$; $f'''(0) = -1$

$f^{(4)}(x) = \sin x$; $f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x$; $f^{(5)}(0) = 1$

$\vdots$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots\dots$$

$$= 0 + 1x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7 + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

3) Maclaurin series for $f(x) = \cos x$

$$\vdots$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

4) Maclaurin series for $f(x) = e^x$

$f(x) = e^x$; $f(0) = 1$

$f'(x) = e^x$; $f'(0) = 1$

$f''(x) = e^x$; $f''(0) = 1$

$$\vdots$$

$$f^{(n)}(0) = 1$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Definition:

Taylor polynomial of order n generated by $f(x)$ at a is:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$P_1(x) = f(a) + f'(a)(x-a)$

$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}$

equation of tangent line to curve of $f(x)$ at a

$P_1(x) =$ standard linear approximation of $f(x)$

example: $f(x) = e^x$, $f(0) = 1$

$f'(x) = e^x$; $f'(0) = 1$

Taylor polynomials of order 1 and 2 generated by $f(x)$ at 0:

$$P_1(x) = f(0) + f'(0)x$$

$$= 1 + 1x$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$= 1 + x + \frac{x^2}{2!}$$

Exercises:

15) Find the Maclaurin series of the function $\sin 3x$.

$f(x) = \sin 3x$; $f(0) = 0$

$f'(x) = 3\cos 3x$; $f'(0) = 3$

$f''(x) = -9\sin 3x$; $f''(0) = 0$

$f'''(x) = -27\cos 3x$; $f'''(0) = -27$

$f^{(4)}(x) = 81\sin 3x$; $f^{(4)}(0) = 0$

$f^{(5)}(x) = 243\cos 3x$; $f^{(5)}(0) = 243$

$\vdots$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$= 0 + 3x + 0 - \frac{27}{3!}x^3 + 0 + \frac{243}{5!}x^5 + \dots$$

$$= 3x - \frac{27}{3!}x^3 + \frac{243}{5!}x^5 - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(3x)^{2n+1}(-1)^n}{(2n+1)!}$$

20) Find the Maclaurin series of $\sinh x = \frac{e^x - e^{-x}}{2}$

$f(x) = \frac{e^x - e^{-x}}{2}$, $f(0) = \frac{1-1}{2} = 0$

$f'(x) = \frac{e^x + e^{-x}}{2}$; $f'(0) = \frac{1+1}{2} = 1$

$f''(x) = \frac{e^x - e^{-x}}{2}$; $f''(0) = 0$

$f'''(x) = \frac{e^x + e^{-x}}{2}$; $f'''(0) = 1$

$\vdots$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$= 0 + 1x + 0 + \frac{1}{3!}x^3 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

29) Find the Taylor series generated by f at $x = a$.

$f(x) = \frac{1}{x^2}$, $a = 1$.

$f(x) = \frac{1}{x^2}$; $f(1) = \frac{1}{1} = 1$

$f'(x) = \frac{-2}{x^3}$; $f'(1) = -2$

$f''(x) = \frac{6}{x^4}$; $f''(1) = 6$

$f'''(3) = \frac{-24}{x^5}$; $f'''(3) = -24$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)\,(x-1)^k}{k!} = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \dots$$

$$= 1 - 2(x-1) + \frac{6}{2!}(x-1)^2 - \frac{24}{3!}(x-1)^3 + \dots$$

$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 + \dots$$

$$= \sum_{n=0}^{\infty} (n+1)(x-1)^n(-1)^n = \sum_{n=0}^{\infty} (n+1)(1-x)^n$$