Section 10.6 – Alternating Series, Absolute and Conditional Convergence

Alternating Series:

A series in which terms are alternating positive and negative.

Examples: * $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (Alternating harmonic)

(Recall that $\sum_{n=1}^{\infty} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\dots$ is harmonic)

* $\sum_{n=1}^{\infty} (-1)^{n+1} n = 1-2+3-4+5 \dots$

* $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$.

The Alternating Series Test: (Leibniz's theorem)

the series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$

converges if all of the 3 conditions are satisfied:

1) u_n are positive $(u_n > 0)$
2) $u_n \geq u_{n+1}$ (non-increasing)
3) $\lim_{n\to\infty} u_n = 0$

Remark: * If these conditions are not true $\longrightarrow$ no conclusion.

* If the 3rd condition is not true $\longrightarrow$ series diverges by n^{th} term test.

($\lim_{n\to\infty} u_n \neq 0$, then $\lim_{n\to\infty} (-1)^{n+1} u_n$ does not exist and $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ diverges by n^{th} term test.)

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Examples: 1) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, $u_n = \frac{1}{n}$

- $u_n = \frac{1}{n} > 0$
- $\frac{1}{n+1} < \frac{1}{n}$ then u_n is decreasing
- $\lim_{n \to \infty} u_n = 0$

$\Bigg\}$ then, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by AST.

2) $\sum_{n=1}^{\infty} (-1)^{n+1} n$

$\lim_{n \to \infty} u_n = \lim_{n \to \infty} n = \infty$, Then $\lim_{n \to \infty} (-1)^{n+1} n$ does not exist

Then $\sum_{n=1}^{\infty} (-1)^{n+1} n$ diverges by the n^{th} term test.

The Alternating Series Estimation theorem:

If $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ satisfies conditions 1, 2, and 3 of the AST and so converges to a sum L, then the n^{th} partial sum:

$S_n = u_1 + \dots + (-1)^{n+1} u_n$ approximates the sum L with an error E_n such that $|E_n| < u_{n+1}$

$$|E_n| = |L - S_n| < u_{n+1} \longrightarrow 1^{st} \text{ unused term.}$$

Also, the error E_n has the same sign as the one preceeding the 1^{st} unused term.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

upper bound for the magnitude of the error involved in using the sum of the first 7 terms to approximate the sum of the series. What is the sign of the error?

Alternating series with conditions 1, 2, and 3 satisfied.

By the Alternating Series theorem : $|E_7| <$ 1st unused term.

$$|E_7| < \frac{1}{8}$$

E_7 has the same sign as u_8 which is negative.

Absolute and Conditional Convergence

Definition: A series $\sum a_n$ converges absolutely (or is absolutely convergent) if $\sum_{n}^{\infty} |a_n|$ converges.

Ex: 1) $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n$

$\sum |a_n| = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges (geometric series with $|r| = \frac{1}{2} < 1$)

Then $\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n$ converges absolutely.

2) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series with $p = 3 > 1$)

Then $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ converges absolutely.

Theorem: If $\sum |a_n|$ converges, then $\sum a_n$ converges
(absolute convergence $\longrightarrow$ convergence).

Ex: $\sum \frac{(-1)^{n+1}}{n^3}$ converges absolutely $\Longrightarrow$ it converges.

$$\sum \left| \frac{(-1)^{n+1}}{n^3} \right| \text{ converges} \Longrightarrow \sum \frac{(-1)^{n+1}}{n^3} \text{ converges.}$$

Note: Converse of theorem is not true.

convergence of a series $\xrightarrow{X}$ absolute convergence

Many series $\sum a_n$ are convergent without being absolutely convergent.

ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by AST (shown previously)

but $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$).

Definition: A series $\sum a_n$ <u>converges conditionally</u> if $\sum |a_n|$ <u>diverges</u>

and $\sum a_n$ converges.

ex: The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ (Alternating p-series)

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\leq 1. \end{cases}$$

* If $p>1$: $\sum \frac{(-1)^{n+1}}{n^p}$ converges absolutely.

* If $p \leq 1$: $\sum a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$, $u_n = \frac{1}{n^p}$

① $u_n > 0$

② $\frac{1}{(n+1)^p} < \frac{1}{n^p} \Rightarrow u_n$ decreasing

③ $\lim_{n \to \infty} u_n = 0$

$\Longrightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ converges by AST

and $\sum |a_n|$ diverges, then $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

<u>Summary: Alternating p-series:</u>

$\sum \frac{(-1)^{n+1}}{n^p}$ $(p>0)$
- → converges absolutely if $p>1$
- → converges conditionally if $p \leq 1$.

<u>ex:</u> $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely $(p=2>1)$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges conditionally $(p=\frac{1}{2}<1)$

<u>Summary to check if $\sum a_n$ conv. absolutely? conv. conditionally? div?</u>

① <u>n^{th} term test:</u> $\lim_{n \to \infty} a_n \neq 0 \Longrightarrow \sum a_n$ diverges

If $\lim_{n \to \infty} a_n = 0$, move to step ②.

② Geometric series? p- series? Do their rules.

If not, move to step ③.

③ $\sum |a_n|$. Apply DCT, LCT, ratio test, root test, integral test (if they can be applied).

- If $\sum |a_n|$ converges, then $\sum a_n$ converges also and <u>stop</u>.
- If $\sum |a_n|$ diverges, then move to step ④.

④ If the series is alternating, apply AST.

$$\left.\begin{array}{l} 1)\ u_n>0 \\ 2)\ u_n \searrow \\ 3)\ \lim\limits_{n\to\infty} u_n=0 \end{array}\right\} \sum a_n \text{ conv.}$$

In step 4, the series will converge conditionally.

<u>Exercises:</u>

Determine if the alternating series converge or diverge.

2) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{3/2}}$

alternating p- series with $p=\frac{3}{2}>1$

$\Rightarrow$ It converges absolutely, then $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{3/2}}$ converges

12) $\sum\limits_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$, $u_n = \ln\left(1+\frac{1}{n}\right)$

① $1+\frac{1}{n}>1$ for $n\geq 1 \Rightarrow \ln\left(1+\frac{1}{n}\right)>0 \Rightarrow u_n>0$

② $1+\frac{1}{n+1} < 1+\frac{1}{n} \Rightarrow \ln\left(1+\frac{1}{n+1}\right) < \ln\left(1+\frac{1}{n}\right) \Rightarrow u_n \searrow$

③ $\lim\limits_{n\to\infty} u_n = \ln 1 = 0 \Rightarrow$ By AST, $\sum\limits_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$ converges

Which series converge absolutely, which converge, and which conditionally diverge?

16) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{0.1^n}{n}$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{0.1^n}{n}$

$\lim_{n\to\infty} \sqrt[n]{\frac{0.1^n}{n}} = \lim_{n\to\infty} \frac{0.1}{n^{1/n}} = \frac{0.1}{1} = 0.1 = L < 1$

Then $\sum_{n=1}^{\infty} \frac{0.1^n}{n}$ converges by the root test

Then, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{0.1^n}{n}$ converges absolutely.

19) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^3+1}$

$0 < n^3 < n^3+1$

$0 < \frac{n}{n^3+1} < \frac{n}{n^3}$

$\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series $p=2$)

Then $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ converges by D.C.T

Then, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$ converges absolutely.

25) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$

$$0 < n < 1+n$$
$$0 < \frac{n}{n^2} < \frac{1+n}{n^2}$$
$$0 < \frac{1}{n} < \frac{1+n}{n^2}$$

* $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1+n}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$)

Then $\sum_{n=1}^{\infty} \frac{1+n}{n^2}$ diverges by D.C.T

Then $\sum_{n=1}^{\infty} |a_n|$ diverges.

* Now, for $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{n^2}$, $u_n = \frac{n+1}{n^2}$

① $\frac{n+1}{n^2} > 0 \implies u_n > 0$

② $u_{n+1} - u_n = \frac{n+2}{(n+1)^2} - \frac{n+1}{n^2} = \frac{n^3 + 2n^2 - (n+1)(n^2+2n+1)}{n^2(n+1)^2}$

$= \frac{\cancel{n^3} + \cancel{2n^2} - \cancel{n^3} - \cancel{2n^2} - n - n^2 - 2n - 1}{n^2(n+1)^2} = \frac{-n^2 - 3n - 1}{n^2(n+1)^2} < 0 \implies u_n \downarrow$

③ $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Then by AST, the series converges

$\implies \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$ converges conditionally.

31) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$, $u_n = \frac{n}{n+1}$

$\lim_{n\to\infty} u_n = 1 \neq 0 \implies \lim_{n\to\infty} (-1)^n \frac{n}{n+1}$ does not exist

Then $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$ diverges by the n^{th} term test.

33) $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!} = \sum_{n=1}^{\infty} (-1)^n \frac{100^n}{n!}$

$\sum_{n=1}^{\infty} \left|\frac{(-100)^n}{n!}\right| = \sum_{n=1}^{\infty} \frac{100^n}{n!}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{100^{n+1}}{(n+1)!} \times \frac{n!}{100^n} = \lim_{n\to\infty} \frac{100}{n+1} = 0 = \mathcal{L} < 1$

Then, $\sum_{n=1}^{\infty} \frac{100^n}{n!}$ converges by the ratio test.

Then, $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ converges absolutely.

35) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$

Alternating p-series with $p = \frac{3}{2} > 1$

Then $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}}$ converges absolutely.

41) $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$

If we try $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0 \implies$ no conclusion.

$\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{1 \quad (\sqrt{n+1} + \sqrt{n})} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} - \sqrt{n}}$

Apply L.C.T with $\sum \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n+1}{n}}+1} = \frac{1}{2}$$

$0 < \frac{1}{2} < \infty$

We know that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series with $p = \frac{1}{2} < 1$)

Then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges by L.C.T.

Then $\sum_{n=1}^{\infty} |a_n|$ diverges

* Try AST for $u_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$

① $u_n > 0$ for $n > 1$

② $\frac{1}{\sqrt{n+1}+\sqrt{n+2}} < \frac{1}{\sqrt{n+1}+\sqrt{n}}$, then $u_n \searrow$

③ $\lim_{n\to\infty} u_n = 0$

By AST, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}+\sqrt{n}}$ converges

Then $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1}-\sqrt{n})$ converges conditionally.

51) Estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{0.01^n}{n} \quad \text{(Alternating series)}$$

By the Alt. series Estimation theorem:

$|\text{Error}| < 1^{st}$ unused term

$|\text{Error}| < u_5$

$|\text{Error}| < \frac{0.01^5}{5}$

Sign of error $=$ sign of u_5 which is positive. $\left((-1)^{5+1} = 1\right)$