Section 10.7 – Power Series

Definition: A power series about a (centered at a) is a series of the form: $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots\ldots$

(center is a ; coefficients are $c_0, c_1, \ldots$)

A power series about 0 (centered at 0) is a series of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots \quad c_n x^n.$$

Examples:

1) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} = 1 + (x-1) + \frac{(x-1)^2}{2!} + \ldots\ldots$

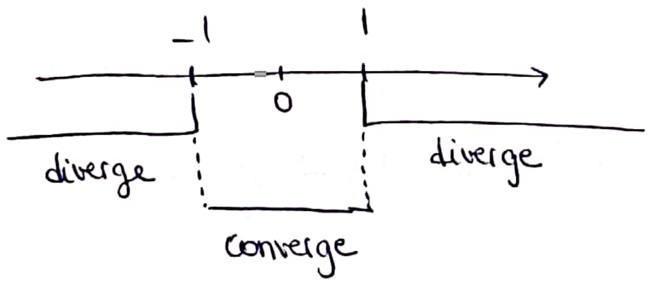
center $= 1$ and $c_n = \frac{1}{n!}$

2) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots$

center $= 0$ and $c_n = 1$

Geometric series with ratio $= x$

It converges for $-1 < x < 1$ and diverges for $x \leq -1$ and $x \geq 1$.



Theorem: Corollary of the convergence theorem of power series, there are 3 possibilities for the power series

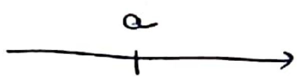
$$\sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{(where } a \text{ is the center)}$$

① There is a positive finite number R $(R>0)$ such that $\sum c_n (x-a)^n$ converges absolutely for :

$$|x-a|<R \implies \boxed{a-R<x<a+R}$$ → interval of absolute convergence.

* the series diverges for $x<a-R$ or $x>a+R$
* At the end points : at $x=a-R$ and $x=a+R$ the behavior of the series is unknown.

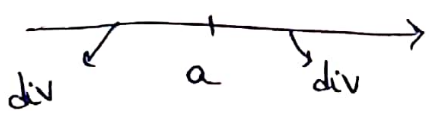
② The series converges absolutely for all real values of x.



$\boxed{R=\infty}$

Interval of absolute Convergence is $]-\infty,+\infty[$

③ The series converges only at $x=a$ and diverges elsewhere



$\boxed{R=0}$

Finding the Radius and Interval of Convergence:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

① Apply the ratio or root test on $\boxed{\sum_{n=0}^{\infty} \left| c_n (x-a)^n \right|}$ to find the interval of absolute convergence and the radius of convergence R.

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② If R is finite and $\neq 0$, then test for convergence at endpoints $(x = a-R,\ x = a+R)$ and deduce the interval of convergence.

ex: 1) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

Find the interval of convergence, the radius of convergence of the power series.

Determine for what values of x the series converges absolutely, converges conditionally, or diverges.

$$\sum \left| (-1)^{n-1} \frac{x^n}{n} \right| = \sum_{n=1}^{\infty} \frac{|x|^n}{n}$$

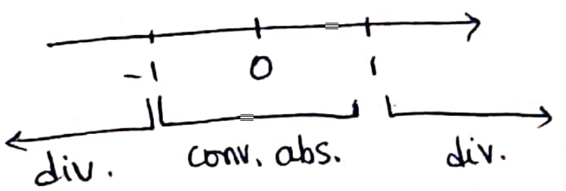
Root test: $\lim_{n\to\infty} \sqrt[n]{\frac{|x|^n}{n}} = \lim_{n\to\infty} \frac{|x|}{n^{1/n}} = \frac{|x|}{1}$

The series converges absolutely if $|x|<1$ $\boxed{-1 < x < 1}$

interval of absolute convergence

center $=0$

$\boxed{R=1}$



$x=-1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} = \sum \frac{-1}{n}$

$= -\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series $p=1$)

$\underline{x=1:}$ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (Alternating harmonic series)

alternating p-series with $p=1$ $\Longrightarrow$ converges conditionally

* interval of convergence: $]-1, 1]$
* For $-1<x<1$, converges absolutely.
* For $x \leq -1$ or $x>1$, diverges.
* for $x=1$, converges conditionally.

2) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n!}$

$\sum_{n=1}^{\infty} \frac{|x-3|^n}{n!}$

$\lim_{n\to\infty} \frac{|x-3|^{n+1}}{(n+1)!} \times \frac{n!}{|x-3|^n} = \lim_{n\to\infty} \frac{|x-3|}{n+1} = 0<1$ for all x

Then, it converges absolutely for all x (center $=3$ and $R=\infty$)

<u>Term-by-term Differentiation Theorem:</u>

If the power series $\sum c_n (x-a)^n$ converges for $a-R<x<a+R$ then it defines a function $\boxed{f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n}$ for $a-R<x<a+R$

The function $f(x)$ has derivatives of all orders inside the interval of convergence.

- $f'(x),\ f''(x),\ f'''(x), \dots, f^{(n)}(x), \dots$ exist over the interval of convergence and $f'(x), f''(x), \dots$ are power series which can be obtained by differentiating $\sum_{n=0}^{\infty} c_n(x-a)^n$ term by term.

- The series representing $f'(x), f''(x), \dots$ converge at every interior point of the interval of convergence of $\sum c_n(x-a)^n$ $(a-R<x<a+R)$.

$$* \quad \sum_{n=0}^{\infty} c_n(x-a)^n = f(x) \qquad a-R<x<a+R$$

$$* \quad \sum_{n=1}^{\infty} n c_n(x-a)^{n-1} = f'(x) \qquad a-R \overset{?}{<} x \overset{?}{<} a+R$$

$$* \quad \sum_{n=2}^{\infty} n(n-1) c_n(x-a)^{n-2} = f''(x) \qquad a-R \overset{?}{<} x \overset{?}{<} a+R$$

<u>example:</u> $f(x) = \dfrac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n \quad (-1<x<1)$

- Find a power series representation for $\dfrac{1}{(1-x)^2}$

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \dots \quad (-1 \overset{?}{<} x \overset{?}{<} 1)$$

<u>$x=-1$:</u> $\sum_{n=1}^{\infty} (-1)^{n-1} n$ diverges by the n^{th} term test.

<u>$x=1$:</u> $\sum_{n=1}^{\infty} n$ diverges by the n^{th} term test.

- Find a power series representation for $\dfrac{1}{(1-x)^3}$

$$f''(x) = \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = 2+6x+12x^2+\ldots \quad (-1 \overset{?}{<} x \overset{?}{<} 1)$$

$$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2} = 1+3x+6x^2+\ldots \quad (-1 \overset{?}{<} x \overset{?}{<} 1)$$

Term-by-Term Integration theorem:

Assume $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$. for $a-R<x<a+R$

then $\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ converges for $a-R<x<a+R$

and $\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} + c$ for $a-R \overset{?}{<} x \overset{?}{<} a+R$

Examples:

1) $\frac{1}{1+x} = 1-x+x^2-x^3+\ldots = \sum_{n=0}^{\infty} (-1)^n x^n \quad (-1<x<1)$

Geometric series : $r=-x$
1st term $=1$

Find a power series for $\ln(1+x)$; $(1+x)>0$

$$\int \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + c$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + c$$

For $x=0$: $\ln 1 = 0 + c \Rightarrow c=0$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

2) Identify the function : $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad (-1<x\leq 1).$$

$$f'(x) = 1 - x^2 + x^4 - x^6 + \dots \qquad = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad -1 \overset{?}{<} x \overset{?}{<} 1$$

Geometric series with $r = -x^2$

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2} \qquad (-1<x<1)$$

$$f(x) = \int \frac{1}{1+x^2}\, dx = \tan^{-1} x + c$$

$$f(x) = \tan^{-1} x + c = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

For $x=0$, $\tan^{-1} 0 + c = 0 \implies \boxed{c=0}$

Then $f(x) = \tan^{-1} x$

<u>Exercises:</u>

Find the series' radius and interval of convergence.

For what values of x does it converge absolutely, cond? div?

8) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$ center $= -2$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{(x+2)^n}{n}$$

root test: $\displaystyle\lim_{n\to\infty} \sqrt[n]{\frac{(x+2)^n}{n}} = \lim_{n\to\infty} \frac{|x+2|}{n^{1/n}} = |x+2|$

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The series conv. absolutely for $|x+2|<1$

$-1 < x+2 < 1$

$\boxed{-3 < x < -1}$ Interval of absolute convergence

$\boxed{R=1}$

At $x=-3$: $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$

diverges (p-series with $p=1$).

At $x=-1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally

Alternating p-series with $p=1$.

* conv. absolutely : $-3<x<-1$
* conv. cond : $x=-1$
* diverges : $x \leq -3$ or $x>-1$

$\Big\}$ interval of conv. $]-3,-1]$

18) $\sum_{n=0}^{\infty} \frac{nx^n}{4^n(n^2+1)}$

$\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} \frac{n|x|^n}{4^n(n^2+1)}$ (center $=0$)

$\lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{4^{n+1}((n+1)^2+1)} \times \frac{4^n(n^2+1)}{n|x|^n} = \lim_{n\to\infty} \frac{|x|(n+1)(n^2+1)}{4n((n+1)^2+1)}$

$= \lim_{n\to\infty} \frac{n^3}{4n^3}|x| = \frac{|x|}{4} \Rightarrow$ conv. absolutely for $\frac{|x|}{4}<1$

conv. absolutely for $-1 < \frac{x}{4} < +1$ so $\boxed{-4<x<4}$

$\boxed{R=4}$ Interval of absolute conv.

For $x=4$: $\sum_{n=0}^{\infty} \frac{n}{n^2+1}$ LCT with $\sum_{n=0}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{n/n^2+1}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 = L \quad (0 < L < \infty)$$

and $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$).

Then $\sum_{n=0}^{\infty} \frac{n}{n^2+1}$ diverges by L.C.T.

For $x=-4$: $\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2+1}$ (n^{th} term test, no conclusion)

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n n}{n^2+1} \right| = \sum_{n=0}^{\infty} \frac{n}{n^2+1}$$ diverges (shown above).

$\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2+1}$ (Try AST with $u_n = \frac{n}{n^2+1}$)

① $u_n > 0$ for $n > 0$

② $$u_{n+1} - u_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{(n+1)(n^2+1) - n(n^2+2n+2)}{((n+1)^2+1)(n^2+1)}$$

$$= \frac{n^3 + n + n^2 + 1 - n^3 - 2n^2 - 2n}{((n+1)^2+1)(n^2+1)} = \frac{-n^2 - n + 1}{((n+1)^2+1)(n^2+1)} < 0 \text{ for } n > 0$$

$\Rightarrow u_n$ is decreasing.

③ $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$$

$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2+1}$ conv. by AST

$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2+1}$ conv. conditionally.

* conv. abs: $-4<x<4$
* conv. cond: $x=-4$.
* diverges : $x<-4$ or $x\geq 4$

$\}$ interval of conv: $[-4,4[$

23) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n x^n$

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n |x|^n$$

$$\lim_{n\to\infty}\sqrt[n]{\left(1+\frac{1}{n}\right)^n|x|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)|x| = |x|$$

converges absolutely when $|x|<1$, $\boxed{-1<x<1}$ interval of absolute convergence.

$\boxed{R=1}$

<u>For $x=1$:</u> $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1 \neq 0 \implies$ diverges by n^{th} term test.

<u>For $x=-1$:</u> $\sum_{n=1}^{\infty}(-1)^n\left(1+\frac{1}{n}\right)^n$

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1 \neq 0$ Then $\lim_{n\to\infty}(-1)^n\left(1+\frac{1}{n}\right)^n$ doesn't exist

$\implies$ diverges by n^{th} term test.

* conv. abs : $-1<x<1$
* div : $x\leq -1$ or $x\geq 1$

$\}$ interval of convergence $]-1,1[$

24) $\sum_{n=1}^{\infty} (\ln n) x^n$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} (\ln n) |x|^n$$

$$\lim_{n\to\infty} \frac{\ln(n+1)\, |x|^{n+1}}{\ln n \; |x|^n} = \lim_{n\to\infty} |x| \frac{\ln(n+1)}{\ln n} \overset{H.R}{=} \lim_{n\to\infty} |x| \frac{\frac{1}{n+1}}{\frac{1}{n}} = |x|$$

conv. absolutely for $|x|<1$, $\boxed{-1<x<1} \rightarrow$ interval of absolute convergence

$$\boxed{R=1}$$

<u>For $x=1$:</u> $\sum_{n=1}^{\infty} \ln n$

$\lim_{n\to\infty} \ln n = \infty \neq 0 \Rightarrow$ diverges by n^{th} term test.

<u>For $x=-1$:</u> $\sum_{n=1}^{\infty} (-1)^n \ln n$

$\lim_{n\to\infty} \ln n = \infty \Rightarrow \lim_{n\to\infty} (-1)^n \ln n$ does not exist

$\Rightarrow$ diverges by n^{th} term test.

$$\left.\begin{array}{ll} * \text{ conv. abs}: & -1<x<1 \\ * \text{ div}: & x\leq -1 \text{ or } x\geq 1 \end{array}\right\} \text{interval of convergence }]-1,1[$$

26) $\sum_{n=0}^{\infty} n!\,(x-4)^n$

$$\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} n!\,|x-4|^n$$

$$\lim_{n\to\infty} \frac{(n+1)!\,|x-4|^{n+1}}{n!\,|x-4|^n} = \lim_{n\to\infty} (n+1)\,|x-4| = \infty > 1 \text{ for all } x$$

except at $x=4$.

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For all $x \neq 4$: Series diverges

At $x = 4$: series conv. absolutely (center)

$\boxed{R = 0}$

interval of convergence $= \{4\}$.

48) Find the series interval of convergence and within this interval, the sum of the series as a function of x.

$$\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n$$

Geometric series with $r = \frac{x^2-1}{2}$

It converges for $-1 < \frac{x^2-1}{2} < 1$

x		$-\sqrt{3}$		$\sqrt{3}$	
x^2-3	$+$	0	$-$	0	$+$

$$-2 < x^2 - 1 < 2$$

$$-1 < x^2 < 3$$

$$0 < x^2 < 3$$

$$x^2 < 3$$

$$-\sqrt{3} < x < \sqrt{3}$$

$\left.\begin{array}{l} x = -\sqrt{3} \\ x = \sqrt{3} \end{array}\right\}$ $r = 1$ $\Longrightarrow$ The series diverges

interval of convergence : $]-\sqrt{3}, \sqrt{3}[$

$$\text{sum} = \frac{1^{st}\text{ term}}{1 - r} = \frac{1}{1 - \left(\frac{x^2-1}{2}\right)} = \frac{2}{3 - x^2}$$