Section 10.5 - The Ratio and Root Tests

The Ratio Test:

Let $\sum a_n$ be a series with $a_n > 0$ (positive terms)

Assume : $\boxed{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \mathcal{L}}$

1) If $\mathcal{L} < 1$, the series $\sum a_n$ converges.
2) If $\mathcal{L} > 1$, the series $\sum a_n$ diverges.
3) If $\mathcal{L} = 1$, the test has no conclusion.

Example: $\sum_{n=1}^{\infty} \frac{1}{n!}$

$n! = 1 \times 2 \times 3 \times \dots \times (n-1) \times n.$

$(n+1)! = \underbrace{1 \times 2 \times 3 \times \dots (n-1)\, n}_{n!}\,(n+1) = n!\,(n+1)$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{n!\,(n+1)} = 0 .$$

$\mathcal{L} = 0 < 1$. Then the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges by ratio test.

The Root Test:

Let $\sum a_n$ be a series with $a_n \geq 0$. $\boxed{\text{Assume } \lim_{n\to\infty} \sqrt[n]{a_n} = \mathcal{L}}$

1) If $\mathcal{L} < 1$, the series $\sum a_n$ converges.
2) If $\mathcal{L} > 1$, the series $\sum a_n$ diverges.
3) If $\mathcal{L} = 1$, the test has no conclusion.

Examples:

1) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{n^{2/n}}{2} = \lim_{n\to\infty} \frac{(n^{1/n})^2}{2} = \frac{1}{2} = \mathcal{L} < 1$$

Then $\sum \frac{n^2}{2^n}$ converges by the root test.

2) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)! \big/ (n+1)!\,(n+1)!}{(2n)! \big/ n!\,n!} = \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)!\,(n+1)!} \times \frac{n!\,n!}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim_{n\to\infty} \frac{4n^2}{n^2} = 4 = \mathcal{L} > 1$$

Then $\sum \frac{(2n)!}{n!\,n!}$ diverges by the Ratio test.

Exercises:

Determine if the series converges or diverges.

20) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \times \frac{10^n}{n!} = \lim_{n\to\infty} \frac{(n+1)}{10} = \infty = \mathcal{L} > 1$$

Then, the series $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ diverges by the ratio test.

21) $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^{10}}{10^n}} = \lim_{n\to\infty} \frac{(n^{1/n})^{10}}{10} = \frac{1}{10} = \mathcal{L} < 1$$

Then, $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ converges by the root test.

22) $\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n-2}{n}\right)^n} = \lim_{n\to\infty} \frac{n-2}{n} = 1 \implies \text{no conclusion.}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^n = e^{-2} \neq 0$$

Then, the series diverges by n^{th} term test.

30) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n} = \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 = \mathcal{L} < 1$$

Then, series converges by root test.

45) $\sum_{n=3}^{\infty} \frac{2^n}{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}} = \lim_{n\to\infty} \frac{2}{(n^{1/n})^2} = \frac{2}{1} = 2 = \mathcal{L} > 1$$

Then $\sum_{n=3}^{\infty} \frac{2^n}{n^2}$ diverges by the root test.

47) $\sum_{n=1}^{\infty} a_n$ where $a_1 = 2$, $a_{n+1} = \frac{1+\sin n}{n} a_n$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1+\sin n}{n} a_n}{a_n} = \lim_{n\to\infty} \frac{1+\sin n}{n} = 0 = \mathcal{L} < 1$$

(Sandwich theorem)

Then $\sum_{n=1}^{\infty} a_n$ converges by the ratio test.