Section 10.9 - Convergence of Taylor Series

$$f(x) = \underbrace{P_n(x)}_{\text{Taylor polynomial of order } n} + \underbrace{R_n(x)}_{\text{Remainder Error}}$$

Taylor's Formula:

If f has derivatives of all orders in an open interval I containing a, then for every positive integer n and every x in I:

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{P_n(x)} + R_n(x)$$

where $\boxed{R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}$ for some c between a and x.

* If when $n \to \infty$, $R_n \to 0$ for all x in I, we say that the Taylor series generated by $f(x)$ at a converges to $f(x)$ on I, and we can write:

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k} = \text{Taylor series} \quad (x \in I)$$

Example: Show that the Taylor series of $\sin x$ at 0 converges to $\sin x$ for all x.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n+1}(x).$$

where $R_{2n+1}(x) = \frac{f^{(2n+2)}(c)}{(2n+2)!} x^{2n+2}$

$$|R_{2n+1}(x)| = \frac{|f^{(2n+2)}(c)|}{(2n+2)!} |x|^{2n+2}$$

Since $f(x) = \sin x$, then $f^{(2n+2)}(x) = \pm \sin x$ or $\pm \cos x$

$f^{(2n+2)}(c) = \pm \sin c$ or $\pm \cos c$

$$|f^{(2n+2)}(c)| \leq 1$$

$$|R_{2n+1}(x)| = \frac{|f^{(2n+2)}(c)|}{(2n+2)!} |x|^{2n+2} \leq \frac{1 . |x|^{2n+2}}{(2n+2)!}$$

$$0 \leq |R_{2n+1}(x)| \leq \frac{|x|^{2n+2}}{(2n+2)!}$$

as $n \to \infty$, $\frac{|x|^{2n+2}}{(2n+2)!} \longrightarrow 0$ for all x

Then by Sandwich theorem, $|R_{2n+1}(x)| \longrightarrow 0$ for all x in $(-\infty, \infty)$

Then $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x in $(-\infty, \infty)$

* Similarly, we can show that the Maclaurin series for $\underline{\cos x}$ converges to $\cos x$ for all x in $(-\infty, \infty)$.

* Also, Maclaurin series for $\underline{e^x}$ converges to e^x for all x in $(-\infty, \infty)$

$$\left.\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots
\end{aligned}\right\} -\infty < x < \infty$$

Combining Taylor Series:

Inside the intersection of their interval of convergence, Taylor Series can be added, substracted, multiplied by constants or powers of x.

* $\sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \quad (-\infty < x < \infty)$

* $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \quad (-\infty < x < \infty)$

* $\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left(e^x - e^{-x}\right)$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots + \frac{(-1)^n x^n}{n!} + \dots \quad (-\infty < x < \infty)$$

$$\sinh x = \frac{1}{2}\left(0 + 2x + 0 + 2\frac{x^3}{3!} + 0 + 2\frac{x^5}{5!} + \dots + \dots\right)$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

or using the definition of Taylor series (Section 10.8 n°= 20)

Exercises:

Find the Maclaurin series of:

13) xe^x

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \quad (-\infty < x < \infty)$$

$$xe^x = x + x^2 + \frac{x^3}{2!} + \dots + \frac{x^{n+1}}{n!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

15) $\dfrac{x^2}{2} - 1 + \cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\frac{x^2}{2} - 1 + \cos x = \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$= \sum_{n=2}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

19) $\cos^2 x$

$$\cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6} \cdots + \frac{(-1)^n (2x)^{2n}}{(2n)!} + \cdots$$

$$\frac{1}{2}\cos 2x = \frac{1}{2} - \frac{2x^2}{2!} + \frac{2^3 x^4}{4!} - \frac{2^5 x^6}{6!} + \cdots + \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!} + \cdots$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} - \frac{2x^2}{2!} + \frac{2^3 x^4}{4!} + \cdots + \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!} + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}$$

41) For approximately what values of x can you replace $\sin x$ by $\left(x - \frac{x^3}{6}\right)$ with an error of magnitude no greater than 5×10^{-4} ?

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$$

Alternating series for all x and satisfies conditions of the alternating series estimation theorem. Then:

$|\text{Error}| <$ 1st unused term

$$|\text{Error}| < \frac{|x|^5}{5!}$$

So $\frac{|x|^5}{5!} < 5\times10^{-4}$ so, $|x|^5 < 5!\,5\times10^{-4}$

$$|x| < \sqrt[5]{5!\,5\times10^{-4}}$$

42) If $\cos x$ is replaced by $1-\frac{x^2}{2}$ and $|x|<0.5$,

* what estimate can be made of the error?

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

Alternating series for all x in $]-0.5, 0.5[$ which satisfies the conditions for A.S.T

$|\text{Error}| <$ 1st unused term

$|\text{Error}| < \frac{x^4}{4!}$ and $|x|<0.5$

then $|\text{Error}| < \frac{0.5^4}{4!}$

* Does $\left(1-\frac{x^2}{2}\right)$ tend to be larger or smaller than $\cos x$?

$\cos x = 1 - \frac{x^2}{2} + \text{Error}$

$\text{Error} = \cos x - \left(1-\frac{x^2}{2}\right)$

Error has the sign of 1st unused term $\left(\frac{x^4}{4!}\right)$ which is +

then $\cos x - \left(1-\frac{x^2}{2}\right) > 0$

then $\cos x > \left(1-\frac{x^2}{2}\right)$

so, $\left(1-\frac{x^2}{2}\right) < \cos x$

43) How close is the approximation $\sin x = x$ when $|x| < 10^{-3}$?

For which of these values of x is $x < \sin x$?

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

Alternating series for all $x \in]-10^{-3}, 10^{-3}[$

Satisfies conditions of AST.

$|\text{Error}| <$ 1st unused term.

$|\text{Error}| < \frac{x^3}{3!}$

$|\text{Error}| < \frac{(10^{-3})^3}{3!}$

$|\text{Error}| < \frac{10^{-9}}{3!}$

$\sin x = x + \text{Error}$

$\text{Error} = \sin x - x$

sign of error is that of $\frac{x^3}{3!}$ (-)

So $\sin x - x < 0$ when error is -

and $\sin x - x > 0$ when Error is +

$\sin x > x$ when $-\frac{x^3}{3!} > 0$

$\sin x > x$ when $x < 0$

$-10^{-3} < x < 0$

46) When $x<0$, the series for e^x is an alternating series. What is the error that results from replacing e^x by $\left(1+x+\frac{x^2}{2}\right)$ when $-0.1<x<0$?

$e^x = 1+x+\frac{x^2}{2} + \frac{x^3}{3!} + \cdots$ (Alternating series satisfying AST conditions for $-0.1<x<0$)

$|\text{Error}| <$ 1st unused term

$|\text{Error}| < \frac{x^3}{3!}$ where $|x|<0.1$

$|\text{Error}| < \frac{0.1^3}{3!}$