Section 10.10 – Applications of Taylor Series

① Evaluating Non-Elementary Integrals:

② Evaluating Indeterminate forms of limits.

① Evaluating Nonelementary Integrals:

Taylor series can be used to represent nonelementary integrals using series:

1) Express $\int \sin(x^2)\,dx$ as a power series.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \dots + (-1)^n \frac{x^{4n+2}}{(2n+1)!} + \dots$$

$$\int \sin(x^2)\,dx = C + \frac{x^3}{3} - \frac{x^7}{7\times 3!} + \frac{x^{11}}{11\times 5!} + \dots + (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!} + \dots$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!}$$

2) Estimate $\int_0^1 \sin(x^2)\,dx$ with an error $< 10^{-3}$.

$$\int_0^1 \sin(x^2)\,dx = \left[\frac{x^3}{3} - \frac{x^7}{7\times 3!} + \frac{x^{11}}{11\times 5!} + \dots + (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!} + \dots \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{7\times 3!} + \frac{1}{11\times 5!} + \dots + (-1)^n \frac{1}{(4n+3)(2n+1)!} + \dots$$

Alternating series satisfying the conditions of the AST

by the theorem: $|\text{Error}| <$ 1st unused term

$\frac{1}{7\times 3!} = \frac{1}{42} > 10^{-3}$ so $\frac{1}{7\times 3!}$ is not the 1st unused term.

$\frac{1}{11\times 5!} = 0.75\times 10^{-3} < 10^{-3}$ so $\frac{1}{11\times 5!}$ is the 1st unused term

So if we approximate $\int_0^1 \sin(x^2)dx$ with $\frac{1}{3} - \frac{1}{42}$, then

$|\text{Error}| < \frac{1}{11\times 5!} < 10^{-3}$, Then $\int_0^1 \sin(x^2)dx \cong \frac{1}{3} - \frac{1}{42}$

3) Find the lowest degree polynomial that approximates $F(x) = \int_0^x \sin(t^2)\,dt$ on $[0,1]$ with $|\text{error}| < 10^{-3}$ and $|x|<1$

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots + (-1)^n \frac{t^{2n+1}}{(2n+1)!} + \dots$$

$$\sin t^2 = t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} + \dots + (-1)^n \frac{t^{4n+2}}{(2n+1)!} + \dots$$

$$\int \sin(t^2)\,dt = \frac{t^3}{3} - \frac{t^7}{7\times 3!} + \frac{t^{11}}{11\times 5!} + \dots + (-1)^n \frac{t^{4n+3}}{(4n+3)(2n+1)!} + \dots$$

$$\int_0^x \sin(t)^2\,dt = \frac{x^3}{3} - \frac{x^7}{7\times 3!} + \frac{x^{11}}{11\times 5!} + \dots + (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!} + \dots$$

- $\frac{|x|^7}{7\times 3!} \leq \frac{1}{42}$ but $\frac{1}{42} > 10^{-3}$ so not first unused term.
- $\frac{|x|^{11}}{11\times 5!} \leq \frac{1}{11\times 5!} < 10^{-3}$ so 1st unused term.

Alternating series satisfying the conditions of AST

So, by the theorem $|Error| <$ 1st unused term.

If $F(x) \simeq \frac{x^3}{3} - \frac{x^7}{7 \times 3!}$

$$|Error| < \frac{|x|^{11}}{11 \times 5!} < 10^{-3}$$

So lowest degree polynomial is $\frac{x^3}{3} - \frac{x^7}{7 \times 3!} = \frac{x^3}{3} - \frac{x^7}{42}$

Table: Important Maclaurin Series:

- $e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + \dots = \sum_{n=0}^{\infty} x^n \quad (-1 < x < 1)$

  geometric series : 1st term $= 1$ and ratio $= x$
- $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots = \sum_{n=0}^{\infty} (-1)^n x^n \quad (|x| < 1)$

  geometric series : 1st term $= 1$ and ratio $= -x$
- $\ln(x+1) = c + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^n \frac{x^{n+1}}{n+1} + \dots$

$$\boxed{\ln(x+1) = \int \frac{1}{1+x}\, dx}$$

For $x = 0$, $\ln 1 = c + 0 \Rightarrow c = 0$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^n \frac{x^{n+1}}{n+1} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

• $\tan^{-1} x = ?$

Replace x by x^2 in the series $\frac{1}{1+x}$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + \dots$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots$$

For $x = 0$, $\tan^{-1} 0 = c + 0 \implies c = 0$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

(2) Evaluating Indeterminate forms of limits:

1) $\lim\limits_{x \to 0} \frac{\sin x - \tan x}{x^3}$ using Maclaurin series.

Given the Maclaurin series of $\tan x$:

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots$$

$$\lim_{x \to 0} \frac{\sin x - \tan x}{x^3} = \frac{0}{0} \text{ ind.}$$

$$= \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) - \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots\right)}{x^3}$$

$$= \lim_{x \to 0} \frac{\left(\frac{-1}{3!} - \frac{1}{3}\right)x^3 + \left(\frac{1}{5!} - \frac{2}{15}\right)x^5 + \left(\frac{-1}{7!} - \frac{17}{315}\right)x^7 + \dots}{x^3}$$

$$= \lim_{x \to 0} \left(\frac{-1}{3!} - \frac{1}{3}\right) + \left(\frac{1}{5!} - \frac{2}{15}\right)x^2 + \left(\frac{-1}{7!} - \frac{17}{315}\right)x^4 + \dots$$

$$= \frac{-1}{3!} - \frac{1}{3} = \frac{-1}{6} - \frac{1}{3} = \frac{-3}{6} = \frac{-1}{2}$$

2) $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1} = \dfrac{0}{0}$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln x = \ln(1+(x-1)) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$$

$$\lim_{x\to 1} \frac{\ln x}{x-1}$$

$$= \lim_{x\to 1} \frac{(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots}{x-1}$$

$$= \lim_{x\to 1} 1 - \frac{(x-1)}{2} + \frac{(x-1)^2}{3} + \cdots = 1-0+0+\cdots = 1+0 = 1$$

3) $\lim\limits_{x\to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right) = \infty - \infty$ ind.

$$\lim_{x\to 0} \left(\frac{x-\sin x}{x\sin x}\right) = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}{x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}$$

$$= \lim_{x\to 0} \frac{x^3/3! - x^5/5! + \frac{x^7}{7!} + \cdots}{x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots}$$

$$= \lim_{x\to 0} \frac{x^2\left(\frac{x}{3!} - \frac{x^3}{5!} + \frac{x^5}{7!} + \cdots\right)}{x^2\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)}$$

$$= \lim_{x\to 0} \frac{\frac{x}{3!} - \frac{x^3}{5!} + \frac{x^5}{7!} + \cdots}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots} = \frac{0}{1} = 0$$

Exercises:

Find a lowest degree polynomial that will approximate $F(x)$ throughout the given interval with an error of magnitude less than 10^{-3}.

25) $F(x) = \int_0^x \sin t^2 \, dt$ (done in lecture).

28) $F(x) = \int_0^x \frac{\ln(1+t)}{t} \, dt$

a) $[0, 0.5]$

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots + (-1)^{n-1}\frac{t^n}{n} + \dots$$

$$\frac{\ln(1+t)}{t} = 1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \dots + (-1)^{n-1}\frac{t^{n-1}}{n} + \dots$$

$$F(x) = \int_0^x \frac{\ln(1+t)}{t} \, dt = \left[t - \frac{t^2}{2 \times 2} + \frac{t^3}{3 \times 3} - \frac{t^4}{4 \times 4} + \dots + (-1)^{n-1}\frac{t^n}{n \times n} + \dots \right]_0^x$$

$$= x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \dots + (-1)^{n-1}\frac{x^n}{n^2} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n^2}$$

For $x \in [0; 0.5]$, alternating series satisfying conditions of AST:

$\Rightarrow |\text{Error}| < 1^{st}$ unused term.

$$\frac{|x|^4}{16} < \frac{0.5^4}{16} = 3.9 \times 10^{-3} \quad (> 10^{-3})$$

$$\frac{|x|^5}{25} < \frac{0.5^5}{25} = 1.25 \times 10^{-3} \quad (> 10^{-3})$$

$$\frac{|x|^6}{36} < \frac{0.5^6}{36} = 0.43 \times 10^{-3} \quad (< 10^{-3}) \Rightarrow 1^{st} \text{ unused term.}$$

If $F(x) \simeq x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \frac{x^5}{25}$ (lowest degree polyn).

then $|\text{Error}| < \frac{|x|^6}{36} < \frac{0.5^6}{36} < 10^{-3}$.

b) $[0,1]$.

$|\text{Error}| <$ 1st unused term.

$$\frac{|x|^n}{n^2} < 10^{-3}$$

$$\frac{1}{n^2} < 10^{-3} \Rightarrow n^2 > \frac{1}{10^{-3}}$$

$$n^2 > 1000$$

$$n > \sqrt{1000}$$

$$n \geq 31.6$$

The first unused term corresponds to $n = 32$.

$\Rightarrow$ the lowest degree polynomial is:

$$x - \frac{x^2}{4} + \frac{x^3}{9} - \cdots\cdots + \frac{x^{31}}{31^2}$$

35) Use series to evaluate the limits

$$\lim_{x\to\infty} x^2\left(e^{-1/x^2} - 1\right) = \infty \times 0 \quad \text{ind.}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^{-1/x^2} = 1 - \frac{1}{x^2} + \frac{1}{2!\,x^4} - \frac{1}{3!\,x^6} + \frac{1}{4!\,x^8} + \cdots + (-1)^n \frac{1}{n!\,x^{2n}} + \cdots$$

$$e^{-1/x^2} - 1 = \frac{-1}{x^2} + \frac{1}{2!\,x^4} - \frac{1}{3!\,x^6} + \frac{1}{4!\,x^8} + \cdots$$

$$x^2\left(e^{-1/x^2}-1\right) = -1 + \frac{1}{2!\,x^2} - \frac{1}{3!\,x^4} + \frac{1}{4!\,x^6} - \cdots$$

$$\lim_{x\to\infty} x^2\left(e^{-1/x^2}-1\right) = -1 + 0 - 0 + \cdots = -1+0 = -1$$

45) Use the table of Maclaurin series to find the sum of each series.

$$\frac{\pi}{3} - \frac{\pi^3}{3^3 . 3!} + \frac{\pi^5}{3^5\, 5!} - \frac{\pi^7}{3^7\, 7!} + \cdots$$

$$= \frac{\pi}{3} - \frac{(\pi/3)^3}{3!} + \frac{(\pi/3)^5}{5!} - \frac{(\pi/3)^7}{7!} + \cdots$$

$$= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

50) $$x^2 - 2x^3 + \frac{2^2 x^4}{2!} - \frac{2^3 x^5}{3!} + \frac{2^4 x^6}{4!} - \cdots$$

$$= x^2\left(1 - 2x + \frac{2^2 x^2}{2!} - \frac{2^3 x^3}{3!} + \frac{2^4 x^4}{4!} - \cdots\right)$$

$$= x^2\left(1 - 2x + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} - \cdots\right)$$

$$= x^2 e^{-2x}$$

<u>Recall:</u> $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-2x} = 1 - 2x + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \cdots$$

54) How many terms of the Taylor series for $\ln(1+x)$ should you add to be sure of calculating $\ln(1.1)$ with an error of magnitude less than 10^{-8}?

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1.1) = \ln(1+0.1) = 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \frac{0.1^4}{4} + \cdots + (-1)^{n-1} \frac{0.1^n}{n} + \cdots$$

Alternating series satisfying conditions of AST

$|\text{Error}| <$ 1st unused term.

For $n=7$: $\quad \frac{0.1^7}{7} > 10^{-8}$

For $n=8$: $\quad \frac{0.1^8}{8} = \frac{10^{-8}}{8} < 10^{-8}$ (1st unused term)

Then $\ln(1.1) = 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \cdots + \frac{0.1^7}{7}$

with $|\text{Error}| < \frac{0.1^8}{8} < 10^{-8}$ (Use 7 terms).

55) According to the alternating series Estimation theorem, how many terms of the Taylor series for $\tan^{-1} 1$ would you have to add to be sure of finding $\pi/4$ with an error of magnitude less than 10^{-3}?

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots$$

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \dots + \frac{(-1)^n}{2n+1} + \dots$$

Alternating series satisfying conditions of AST

$|\text{Error}| < 1^{st}$ unused term $< 10^{-3}$

$$\frac{1}{2n+1} < 10^{-3}$$

$$2n+1 > \frac{1}{10^{-3}}$$

$$2n+1 > 1000$$

$2n > 999 \Rightarrow n \geq 499.5 \Rightarrow 1^{st}$ unused term at $n = 500$.

$$\tan^{-1} = \frac{\pi}{4} \approx \underbrace{1}_{\boxed{n=0}} - \underbrace{\frac{1}{3}}_{\boxed{n=1}} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \dots + \underbrace{(-1)^{499} \frac{1}{2(499+1)}}_{\boxed{n=499}}$$

Then, we should add 500 terms