Chapter 4: Force System Resultants

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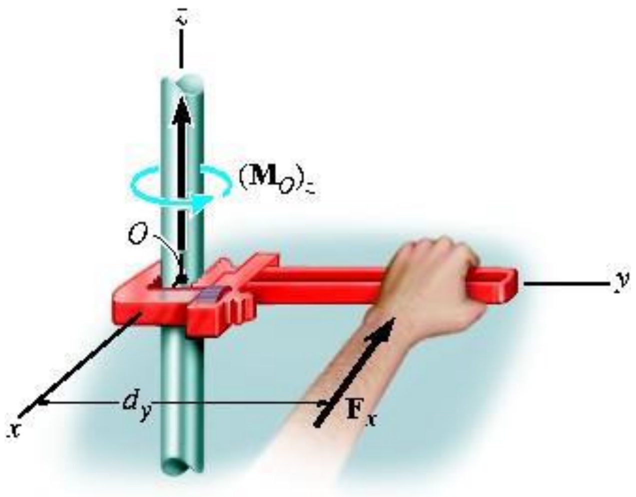
Textbook: R.C. Hibbeler, *Engineering Mechanics: Statics*, 14th Edition, Pearson Prentice Hall, ISBN 978-981-06-8134-0, 2010

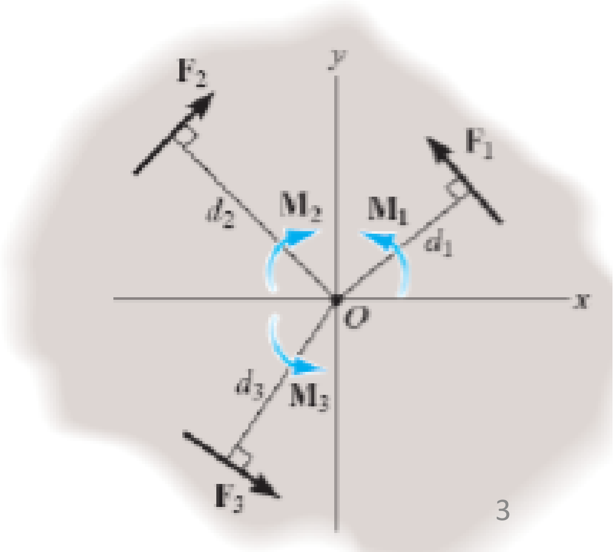
Plan

- Moment of a force-Scalar formulation
- Cross product
- Moment of a force-Vector formulation
- Principle of moments
- Moment of a force about a specified axis
- Moment of a couple
- Simplification of a force and couple system
- Reduction of a simple distributed loading

Moment of a force-Scalar formulation

- **Moment of a force**: tendency of a force to rotate a rigid body about any defined axis.
- **Torque or twist moment**: tendency of rotation caused by F_x or simple moment $(M_O)_z$
- **Magnitude** M_O=F.d (N.m) where d: perpendicular distance from O to its line of action
- **Direction** using “right hand rule”: positive CCW and negative CW
- **Resultant moment** $M_{Ro}=\Sigma Fd$





Cross Product

- Cross product of vectors **A** and **B** yields vector **C**, written as:

$$\mathbf{C}=\mathbf{A} \times \mathbf{B}=(AB \sin\theta)\mathbf{u}_C$$

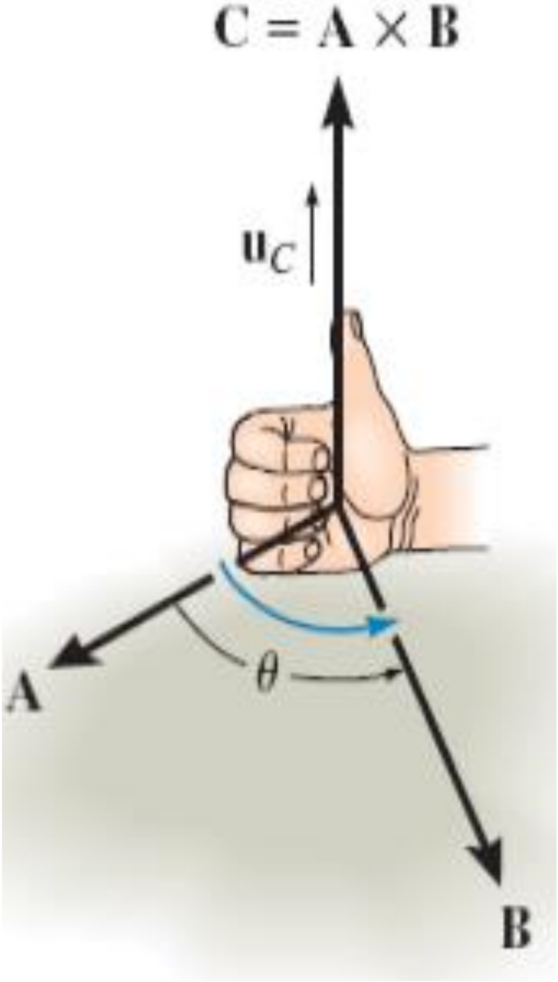
- **Law of operations**:

1. Commutative law is not valid

   $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$; $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$
2. Multiplication by a scalar

   $a(\mathbf{A} \times \mathbf{B})=(a\mathbf{A}) \times \mathbf{B}= \mathbf{A} \times (a\mathbf{B})=(\mathbf{A} \times \mathbf{B})a$
3. Distributive law

   $A \times (B+D)=(A \times B)+ (A \times D)$

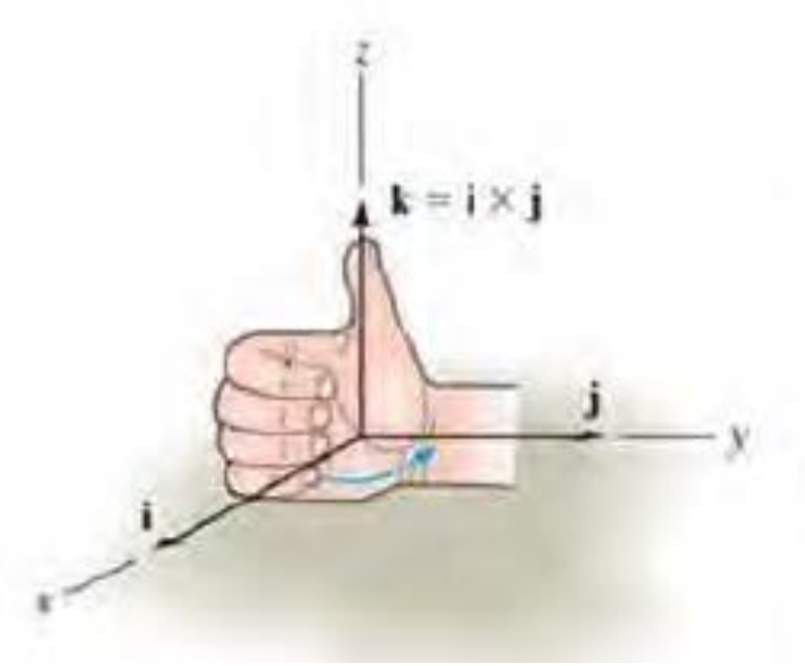


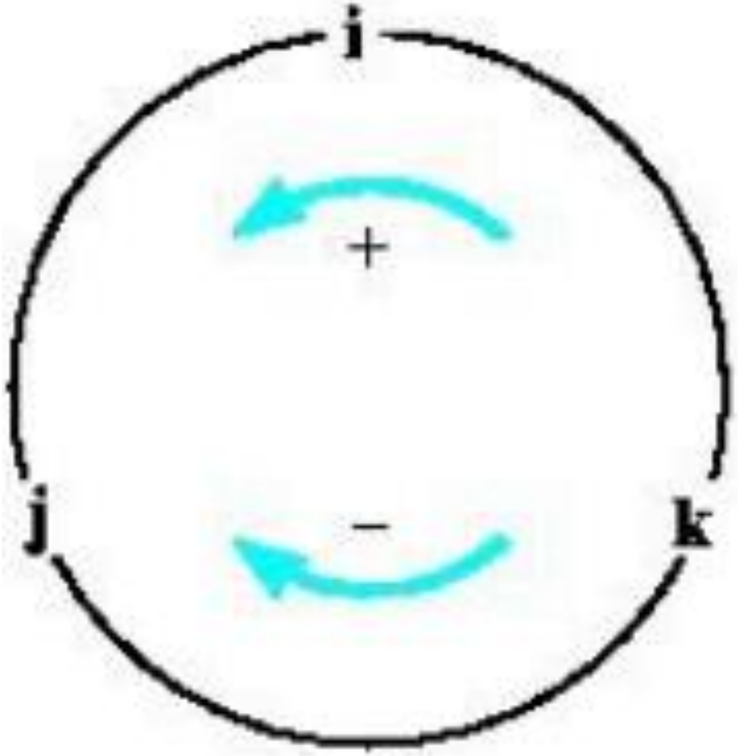
Proper order of the cross product must be maintained

Cross Product

Cartesian Vector Formulation

- Determinant form **A** x **B**= $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

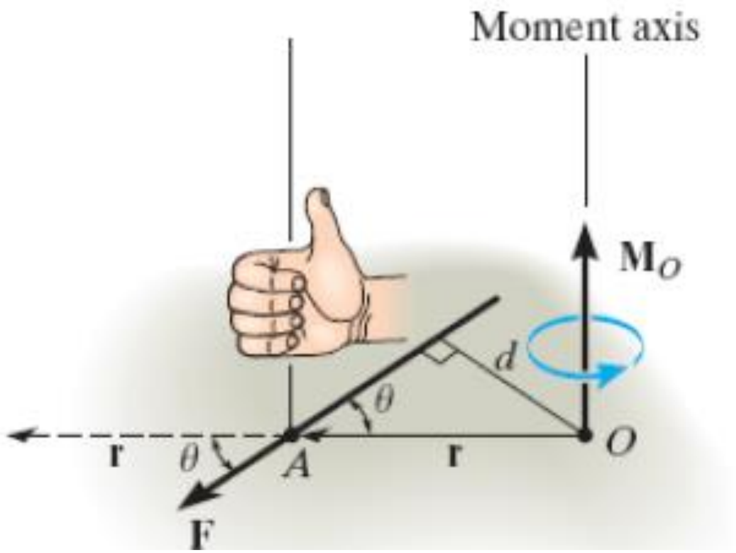




Moment of a force-Vector formulation

- Moment of a force using a cross product

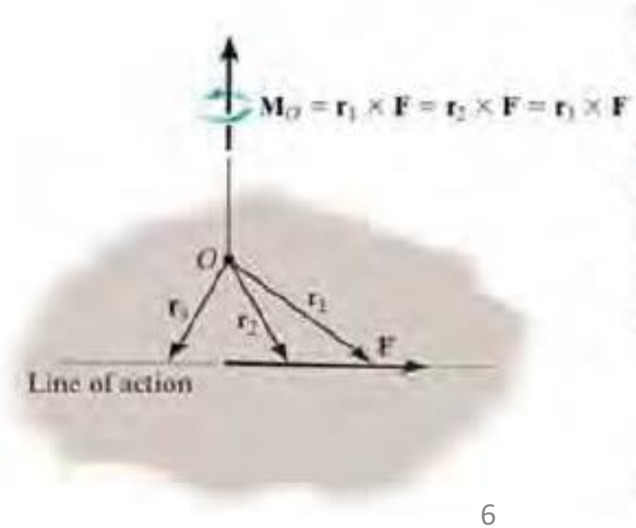
$$\boldsymbol{M_o} = \boldsymbol{r}\; x\; \boldsymbol{F}$$



- Magnitude of cross product M_O=rF sinθ
- With d=r sinθ ⇒ M_O=rF sinθ=F (r sinθ)=Fd

- Principle of transmissibility

$$\boldsymbol{M_o} = \boldsymbol{r}_1\; x\; \boldsymbol{F} = \boldsymbol{r}_2\; x\; \boldsymbol{F} = \boldsymbol{r}_3\; x\; \boldsymbol{F}$$

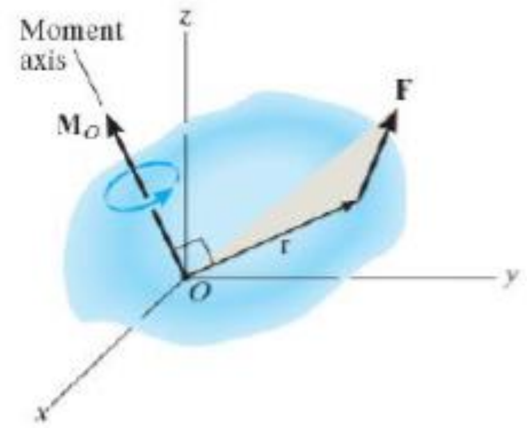


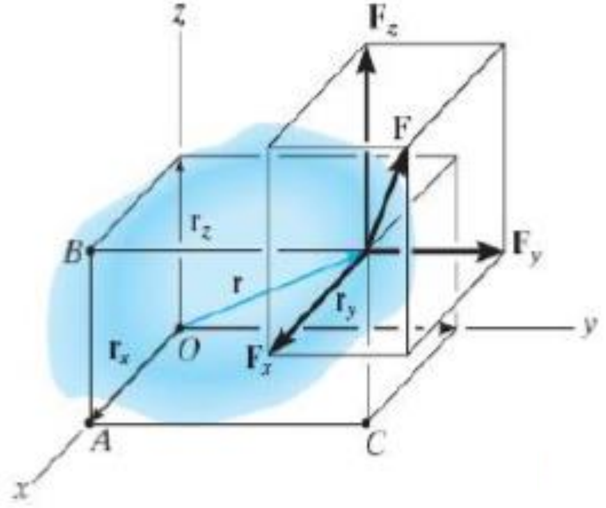
Moment of a force-Vector formulation

Cartesian Vector Formulation

- For force F and position vector r expressed in Cartesian form,

$$\mathbf{M_O} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$





If the determinant is expanded:

$\mathbf{M_O}=(r_yF_z-r_zF_y)\mathbf{i}-(r_xF_z-r_zF_x)\mathbf{j}+(r_xF_y-r_yF_x)\mathbf{k}$

With r_x, r_y, r_z: the x, y , z components of the position vector drawn from point O to any point of the line of action of the force.

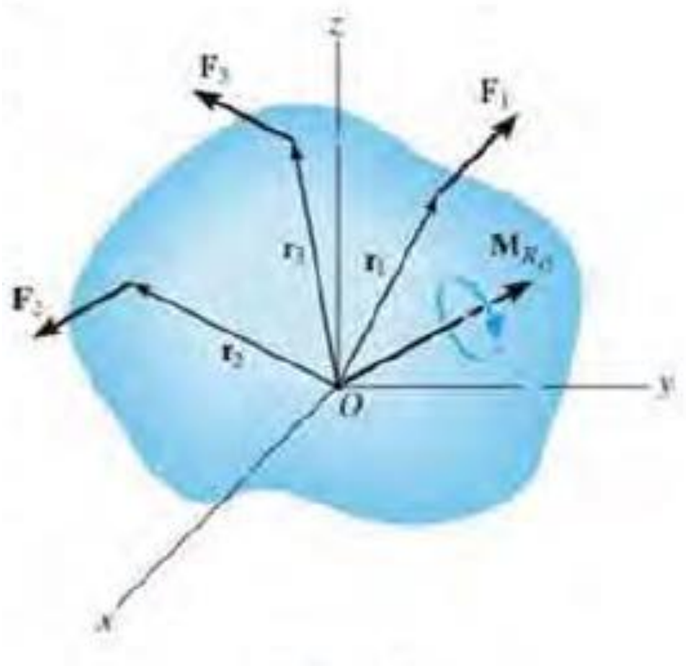
F_x, F_y, F_z: the x, y, z components of the force vector

Moment of a force-Vector formulation

Resultant moment of a system of forces

- The resultant moment of a system of forces about point O can be determined by vector addition of the moment of each force:

$$\mathbf{M}_O = \Sigma\,(\,\mathbf{r} \times \mathbf{F}\,)$$

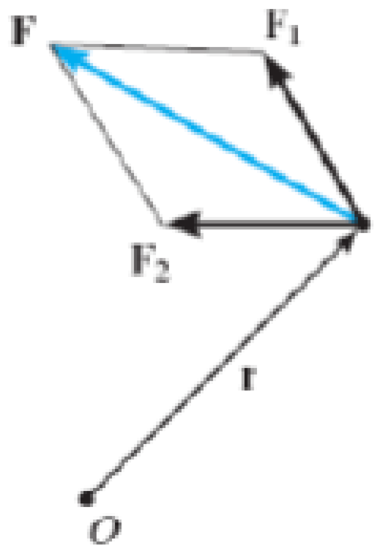


Principle of moments

Varignon's Theorem

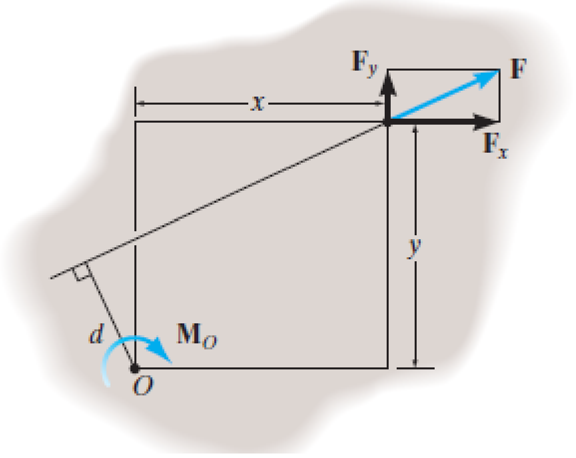
- The moment of a force about a point is equal to the sum of the moments of the components of the force about the point.
- Since $\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2$,

$$\mathbf{Mo}=\mathbf{r} \times \mathbf{F}=\mathbf{r} \times (\mathbf{F}_1+\mathbf{F}_2)=\mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$



- For two-dimensional problems,

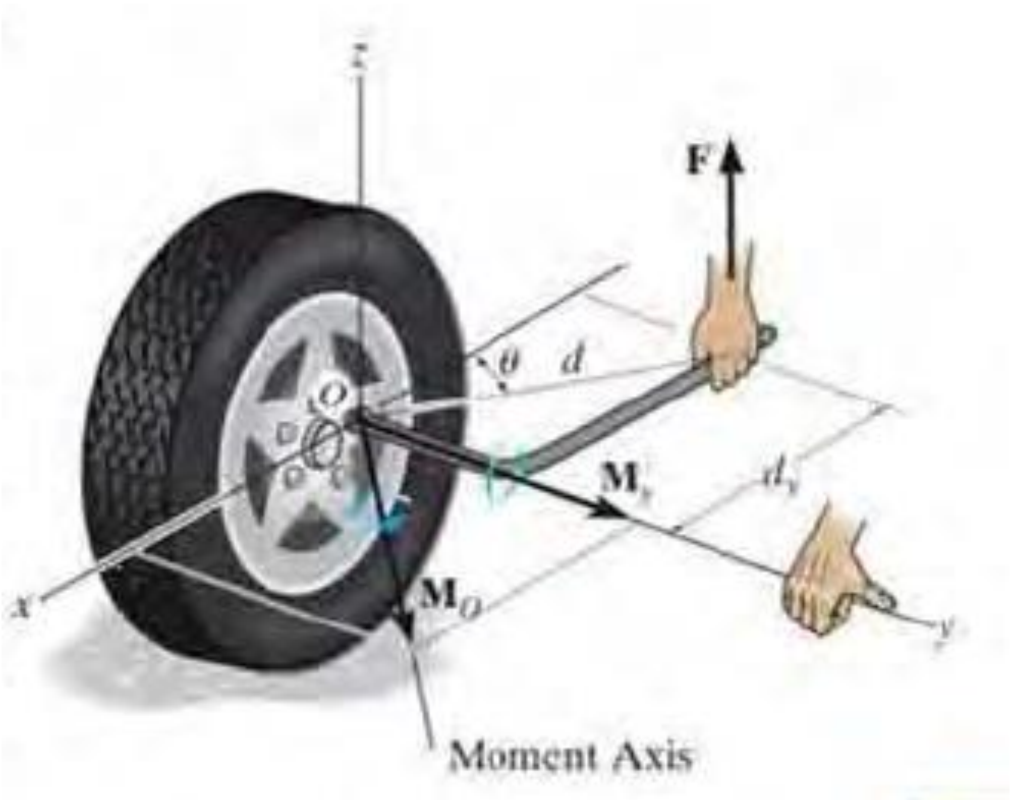
$$M_O=F_x y - F_y x$$



Moment of a force about a specified axis

Scalar Analysis

- The moment of **F** about y axis $M_y = F\ d_y = F(d \cos\theta)$
- For any axis a, $\mathbf{M_a = F\ d_a}$

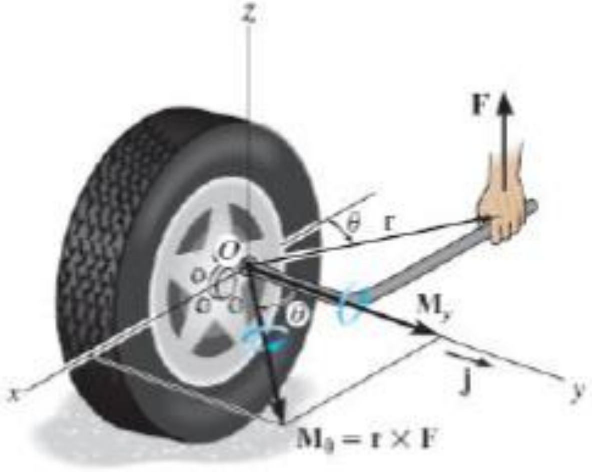


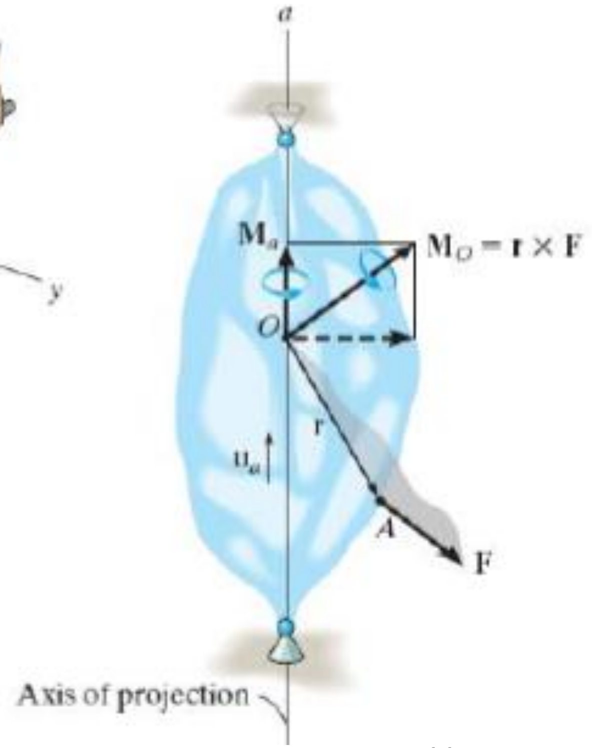
Moment of a force about a specified axis

Vector Analysis

- For magnitude of M_a, $M_a = M_O \cos\theta = \mathbf{M}_O . \mathbf{u}_a$ with $\mathbf{u}_a$: unit vector
- In determinant form:

$$|\boldsymbol{M}_a| = \boldsymbol{u}_a . (\boldsymbol{r}\ x\ \boldsymbol{F}) = \begin{vmatrix} u_{ax} & u_{ay} & u_{az} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$





Moment of a couple

- Couple: two parallel forces of the same magnitude but opposite direction separated by perpendicular distance d

- Scalar formulation:

Magnitude of couple moment M=Fd

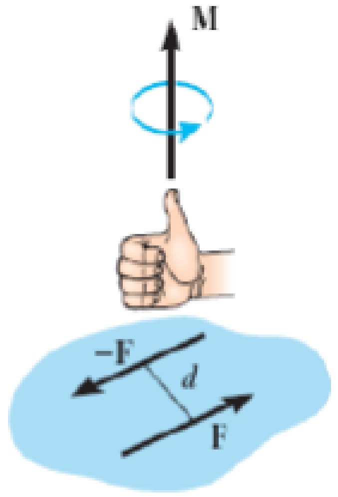
M acts perpendicular to plan containing the forces

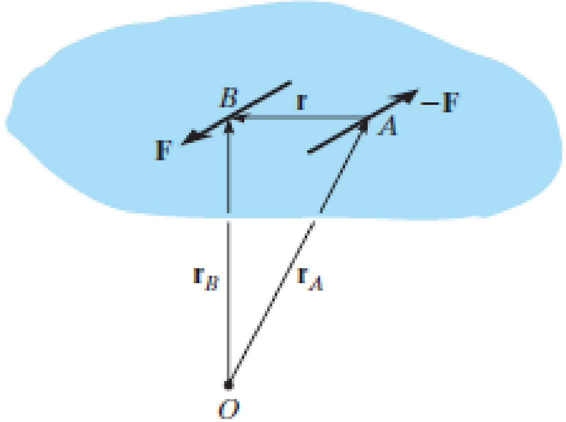
- Vector formulation

For couple moment, $\mathbf{M}=\mathbf{r}_B \times \mathbf{F} + \mathbf{r}_A \times (-\mathbf{F}) = (\mathbf{r}_B - \mathbf{r}_A) \times \mathbf{F}$

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

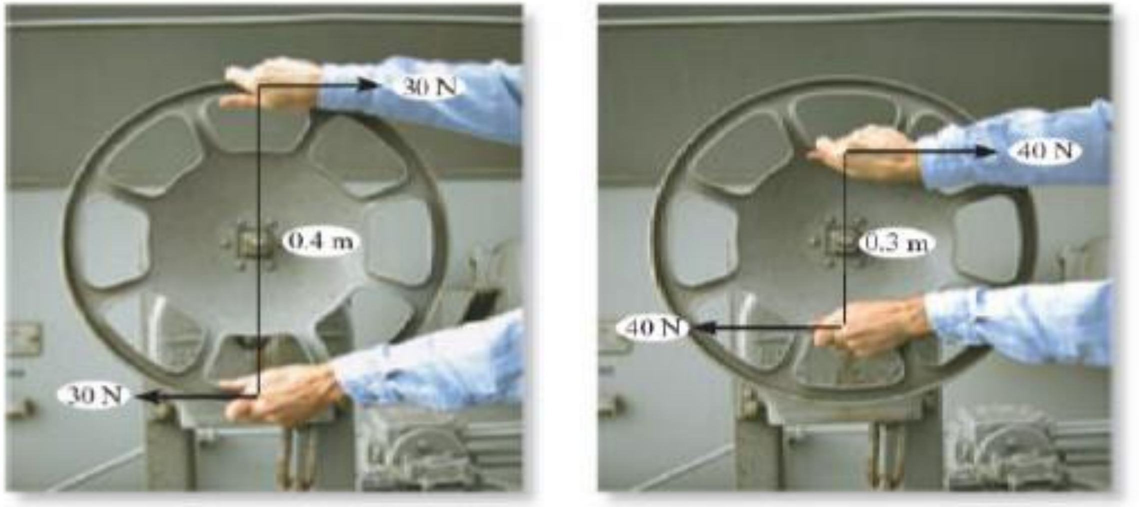
The couple moment is a free vector, it can act at any point since M depends only upon the position vector **r** directed between the forces.





Moment of a couple

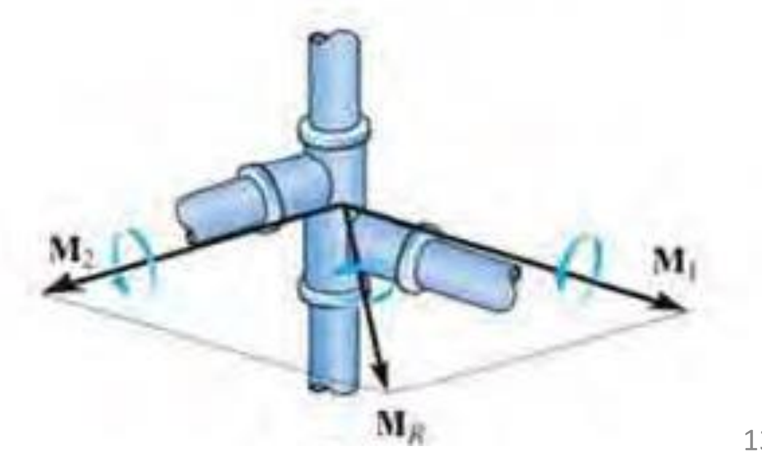
- Equivalent Couples



Two couples are equivalent if they produce the same moment
Forces of equal couples lie on the same plane or plane parallel to one another

- Resultant couple moment:

$$\mathbf{M}_R = \Sigma\, \mathbf{r} \times \mathbf{F}$$



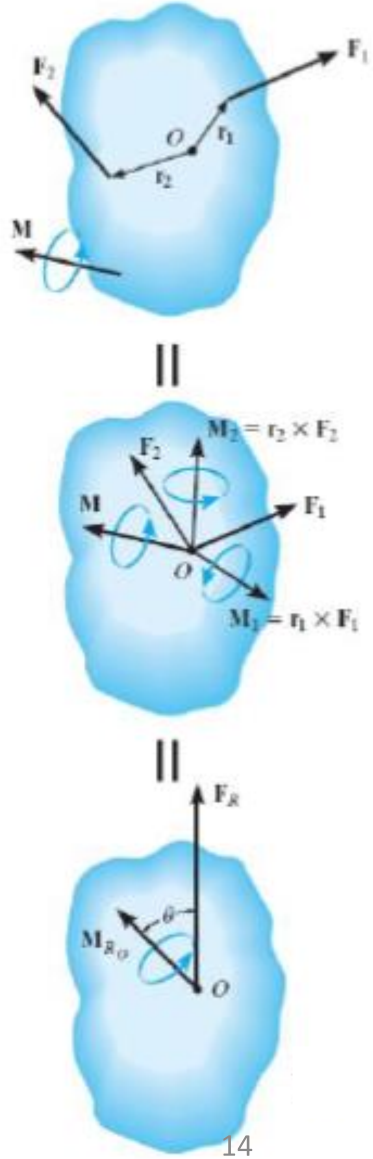
Simplification of a force and couple system

- Equivalent resultant force acting at point O and a resultant couple moment is expressed as

$$\mathbf{F_R} = \Sigma\, \mathbf{F}$$
$$(\mathbf{M}_R)_O = \Sigma\, \mathbf{M_O} + \Sigma\, \mathbf{M}$$

- If the force system lies in the x-y plane, the couple moments are perpendicular to this plane :

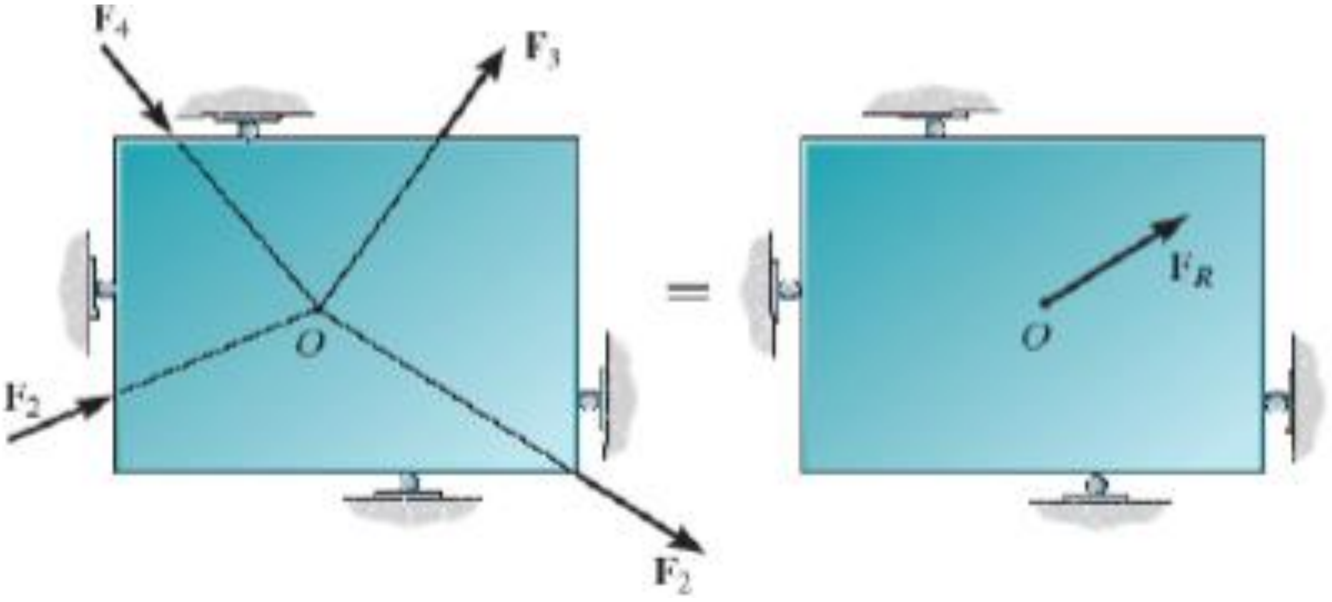
$$(\mathbf{F_R})_x = \Sigma\, \mathbf{F_x}$$
$$(\mathbf{F_R})_y = \Sigma\, \mathbf{F_y}$$
$$(\mathbf{M}_R)_O = \Sigma\, \mathbf{M_O} + \Sigma\, \mathbf{M}$$



Simplification of a force and couple system

- Concurrent Force system

A *concurrent force system* is where lines of action of all the forces intersect at a common point O



Simplification of a force and couple system

- Coplanar Force system

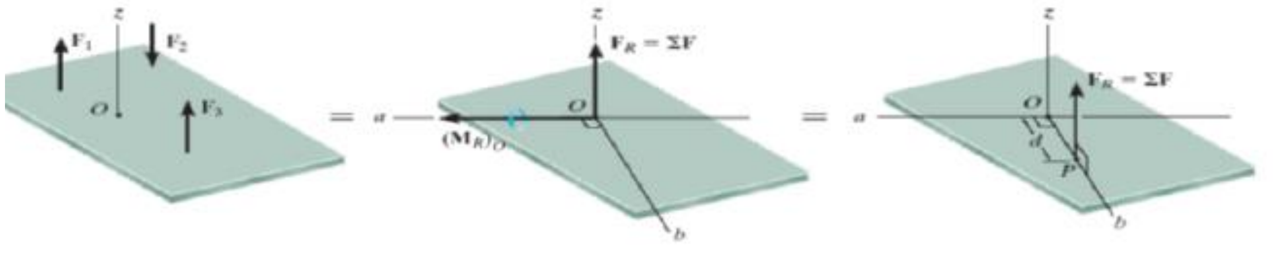
Lines of action of all the forces lie in the same plane

Resultant force of this system also lies in this plane

- Parallel Force system

$F_R = \Sigma F$

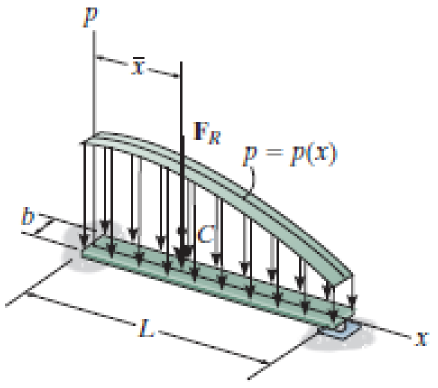
$(M_R)_O = F_R d = \Sigma M_O$ or $d = \Sigma M_O / F_R$



Reduction of a simple distributed loading

- Large surface area of a body may be subjected to distributed loadings, often defined as pressure measured in Pascal (Pa):
  1 Pa = 1N/m^2

- **Loading Along a single axis**: for a constant width b, w(x)=p(x) b (N/m). This coplanar parallel force system can be replaced with an equivalent resultant force F_R acting at a specific location on the beam.



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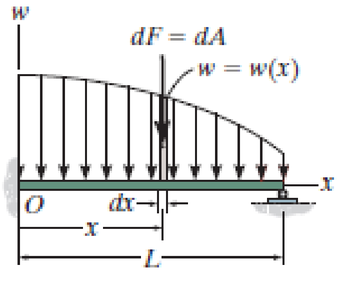

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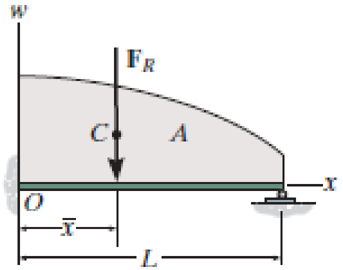

(c)

Reduction of a simple distributed loading

- **Magnitude of resultant force**: $dF=w(x)\,dx=dA$, with $F_R=\Sigma F \Rightarrow$

$$\boldsymbol{F_R = \int_L w(x)dx = \int_A dA = A}$$

- **Location of the resultant force:**

$$M_{RO}=\Sigma M_O \Rightarrow -\bar{x}F_R = -\int_L xw(x)dx$$

The location $\bar{x}$ of the line of action of F_R,

$$\boldsymbol{\overline{x} = \frac{\int_L xw(x)dx}{\int_L w(x)dx} = \frac{\int_A xdA}{\int_A dA}}$$

The resultant force has a line of action which passes through the centroid C (geometric center) of the area under the loading diagram.