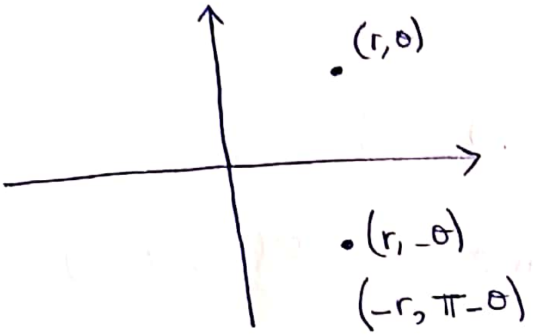
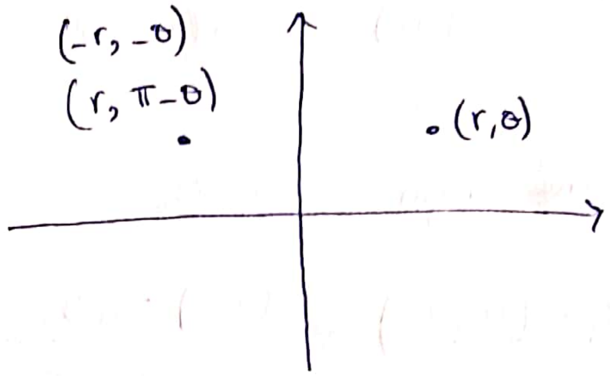
Section 11.4 – Graphing in Polar Coordinates

Polar curve $\boxed{r = f(\theta)}$

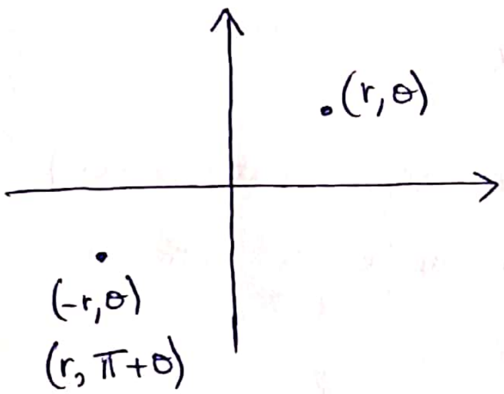
Symmetry:



symmetry about the x-axis



symmetry about the y-axis.



symmetry about the origin.

Slope of a Polar Curve:

$r = f(\theta)$

$y = r\sin\theta = f(\theta)\sin\theta$

$x = r\cos\theta = f(\theta)\cos\theta$

$$\boxed{\text{slope} = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}}$$ → parametric formula for the slope

$$\boxed{\text{Slope} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}}$$

<u>at the origin</u>: $\theta = \theta_0$ $\quad r = 0 \quad (f(\theta) = 0)$.

$$\text{Slope}\,(\theta = \theta_0) = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0 \qquad (f'(\theta_0) \neq 0)$$

$=$ slope of the tangent line at the origin.

<u>Equation of tangent line</u>: $y - y_A = \text{slope}\,(x - x_A)$

At origin $A(0,0)$ $\Rightarrow y = (\tan\theta_0)\, x$

The tangent line to the curve at the origin (at $\theta = \theta_0$) is the line through the origin moving an angle θ_0 with the positive x-axis. (the line $\theta = \theta_0$).

<u>Graphing Polar Curves</u>: $r = f(\theta)$

① <u>Interval</u>: Always start with $[-\pi, \pi]$

② Symmetry:

1) Replace θ by $-\theta$:

- In the equation $r=f(\theta)$, if we obtain r $(r=f(-\theta))$
$\Rightarrow$ symmetry about the x-axis.
- If we obtain $-r$ $(-r=f(-\theta))$ $\Rightarrow$ symmetry about the y-axis.

Original interval: $[-\pi,\pi] \longrightarrow$ Reduce to $[0,\pi]$

2) Replace θ by $(\pi-\theta)$:

- If we obtain r $(r=f(\pi-\theta))$ $\Rightarrow$ symmetry about the y-axis
- If we obtain $-r$ $(-r=f(\pi-\theta))$ $\Rightarrow$ symmetry about the x-axis.

Original interval: $[-\pi,\pi] \longrightarrow$ Reduce to $[\frac{-\pi}{2},\frac{\pi}{2}]$

3) Replace θ by $(\pi+\theta)$:

- If we get r $(r=f(\pi+\theta))$ $\Rightarrow$ symmetry about the origin
reduce $[-\pi,\pi]$ to $[0,\pi]$

4) If $-r=f(\theta)$ $\Rightarrow$ symmetry about the origin
($(-r,\theta)$ satisfies the equation $r=f(\theta)$). Reduce $[-\pi,\pi]$ to $[0,\pi]$

③ $\boxed{r=0} \Rightarrow f(\theta)=0$

Solve for the values θ_0 for which the curve cuts the origin.
The graph passes through the origin, tangent to the lines $\theta=\theta_0$.

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④ $\frac{dr}{d\theta}$

θ	
$dr/d\theta$	
r	$\nearrow \searrow \nearrow \ldots$

⑤

θ					
r					

(Take about 6 values)

⑥ Graph:

- First, sketch the lines $\theta = \theta_0$ tangent to the graph at the origin.
- Second, sketch the graph.

Examples:

1) $r = 1 + \cos\theta$

Step 1:

Interval $[-\pi, \pi]$

Step 2: Symmetry

a) Replace θ by $-\theta$:

$1 + \cos(-\theta) = 1 + \cos\theta = r \implies$ symmetry about the x-axis

Original interval $[-\pi, \pi] \longrightarrow$ reduce to $[0, \pi]$

b) Replace θ by $\pi - \theta$:

$1 + \cos(\pi - \theta) = 1 - \cos\theta \neq r$ and $\neq -r \implies$ no conclusion.

c) Replace θ by $\pi+\theta$:

$1+\cos(\pi+\theta) = 1-\cos\theta \neq r \implies$ no conclusion.

d) $(-r,\theta)$: $-r = -1-\cos\theta \neq r \implies$ no symmetry w.r.t origin

<u>Step 3:</u> $r=0 \implies 1+\cos\theta = 0$

$\cos\theta = -1 \implies \boxed{\theta=\pi}$

$\implies$ graph cuts the origin, tangent to the line $\theta=\pi$ (negative x-axis).

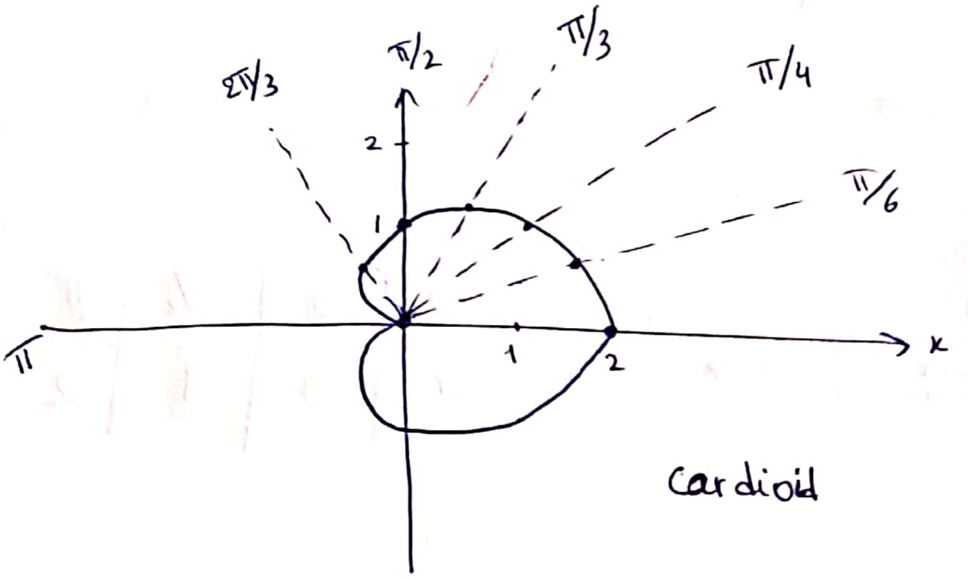
<u>Step 4:</u> $\dfrac{dr}{d\theta} = -\sin\theta$ $\qquad -\sin\theta = 0$ for $\boxed{\theta = 0, \pi}$

θ	0		π
$dr/d\theta$	0	$-$	0
r		$\searrow$	

<u>Step 5:</u> $r = 1+\cos\theta$

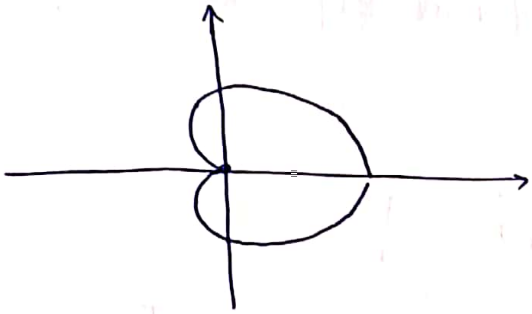
θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	2	1.8	1.7	1.5	1	½	0

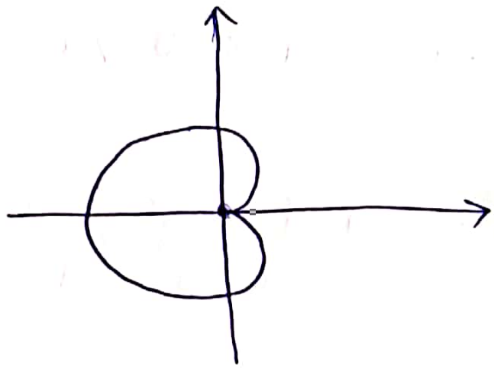
<u>Step 6:</u>



Cardioids:

* $r = a(\pm 1 \pm \cos\theta)$ $\quad a > 0$

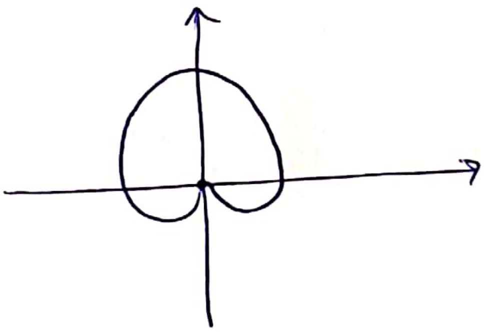


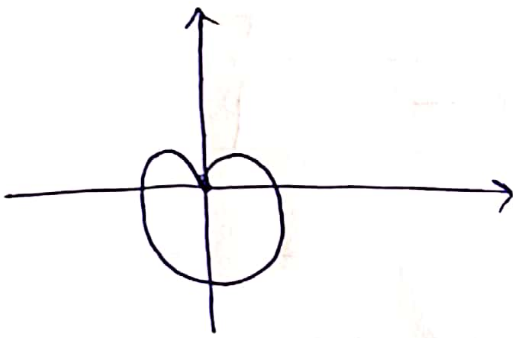


Check for the values $\theta = 0$, $\theta = \frac{\pi}{2}$, $\theta = \pi$

Symmetry with respect to x-axis.

* $r = a(\pm 1 \pm \sin\theta)$ $\quad a > 0$

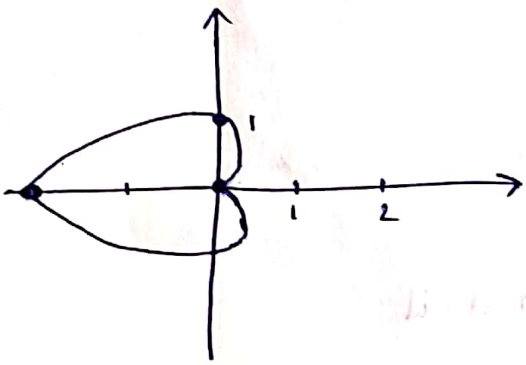




Check for the values $\theta = \frac{-\pi}{2}$, $\theta = 0$, $\theta = \frac{\pi}{2}$

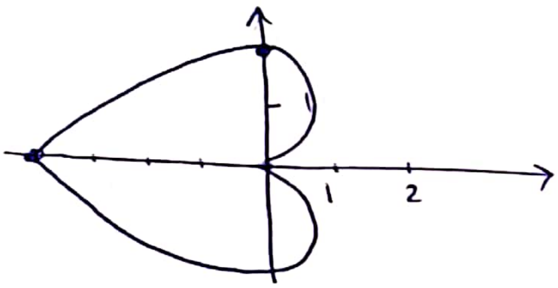
Symmetry with respect to y-axis.

Ex: • $r = 1 - \cos\theta$



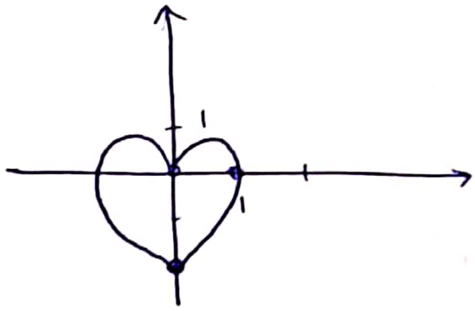
θ	0	$\pi/2$	π
r	0	1	2

- $r = 2 - 2\cos\theta$



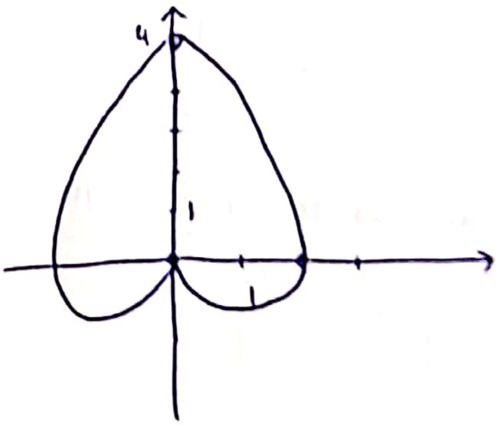
θ	0	$\pi/2$	π
r	0	2	4

- $r = 1 - \sin\theta$



θ	$-\frac{\pi}{2}$	0	$\pi/2$
r	2	1	0

- $r = 2 + 2\sin\theta$



θ	$-\frac{\pi}{2}$	0	$\pi/2$
r	0	2	4

<u>Limaçons with inner loop</u>:

$r = a + b\cos\theta$ <u>or</u> $r = a + b\sin\theta$

where $|a| < |b|$

Example: $r = \frac{1}{2} + \cos\theta$

1) Interval $[-\pi, \pi]$

2) Symmetry:

a) Replace θ by $-\theta$:

$\frac{1}{2} + \cos(-\theta) = \frac{1}{2} + \cos\theta = r$ $\Rightarrow$ sym wrt x-axis

original interval $[-\pi, \pi] \longrightarrow [0, \pi]$

b) Replace θ by $\pi - \theta$:

$\frac{1}{2} + \cos(\pi - \theta) = \frac{1}{2} - \cos\theta \neq r \neq -r$

$\Rightarrow$ no sym wrt y-axis.

c) Replace θ by $\pi + \theta$:

$\frac{1}{2} + \cos(\pi + \theta) = \frac{1}{2} - \cos\theta \neq r$

d) $(-r, \theta)$ does not satisfy the equation $\Rightarrow$ no origin sym.

3) $r = 0 \Leftrightarrow \frac{1}{2} + \cos\theta = 0 \Leftrightarrow \cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\Rightarrow$ The graph cuts the origin, tangent with the lines $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

4) $\frac{dr}{d\theta} = -\sin\theta$

θ	0		π
$dr/d\theta$		$-$	
r	$3/2$	$\longrightarrow$	$-1/2$

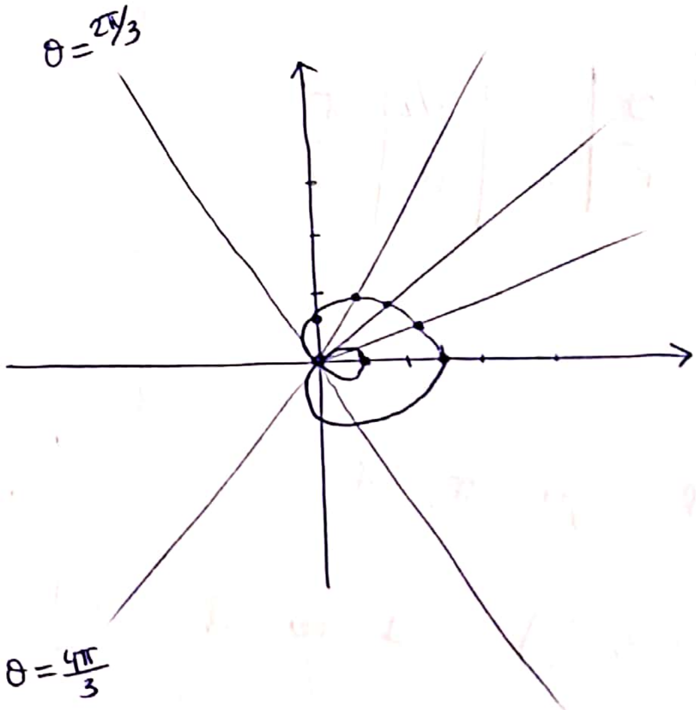
$\sin\theta = 0$

$\theta = 0, \pi$

5)

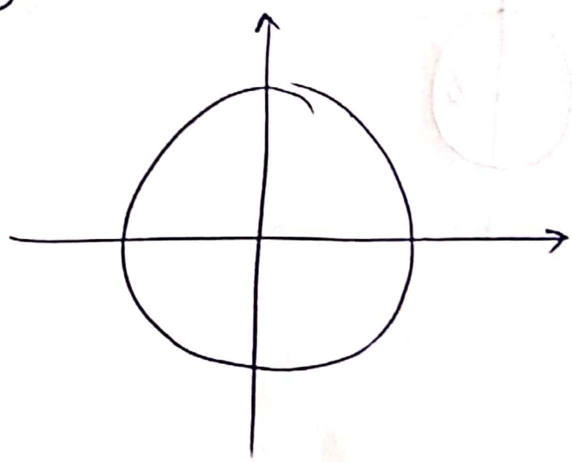
θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	1.5	1.3	1.2	1	$1/2$	0	$-1/2$

6)



Circles

① $r = a$

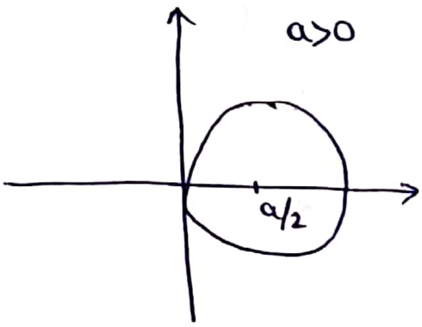


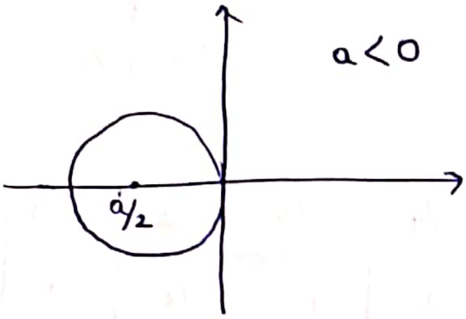
Centered at origin

Radius $= |a|$

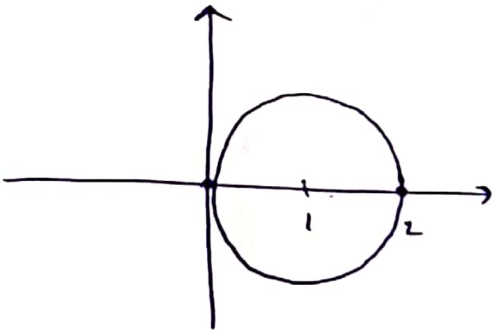
Trace completely with an interval of θ of length 2π.

② $r = a\cos\theta \longrightarrow$ center $\left(\frac{a}{2}, 0\right)$ and radius $= \frac{|a|}{2}$





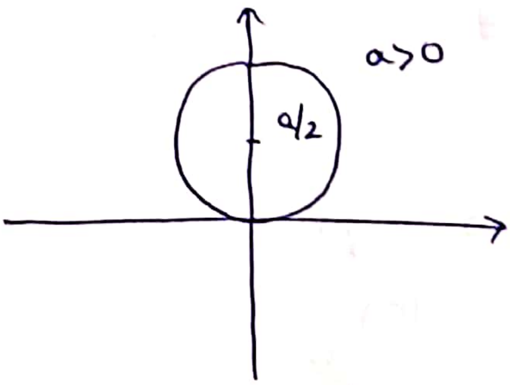
ex: $r = 2\cos\theta$

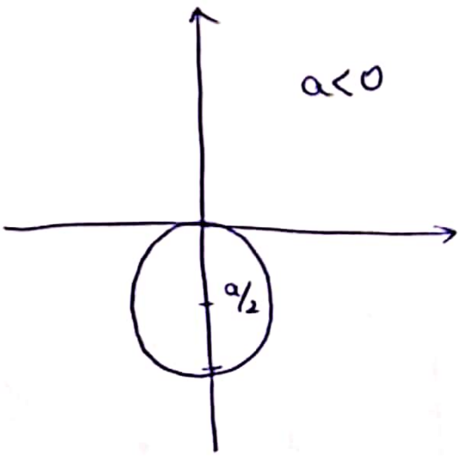


θ	0	$\pi/2$	π
r	2	0	-2

Traced with an interval of length π of θ.

③ $r = a\sin\theta \longrightarrow$ center $\left(0, \frac{a}{2}\right)$ and radius $= \frac{|a|}{2}$





Lemniscates :

$r^2 = b\sin(2\theta)$ or $r^2 = b\cos(2\theta)$

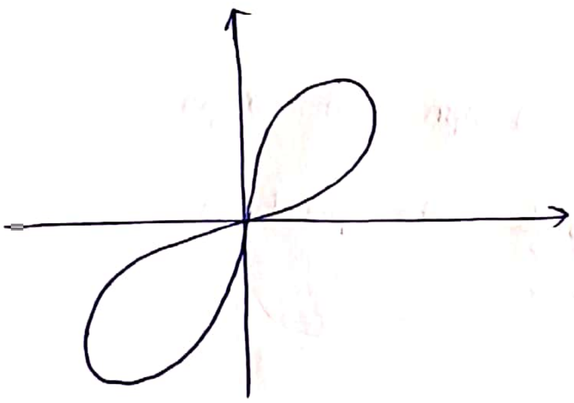
$r^2 = b\sin(2\theta)$ ↓ satisfy origin symmetry

$r^2 = b\cos(2\theta)$ ↓ 3 types of symmetry

* $\boxed{r^2 = b\sin(2\theta)}$

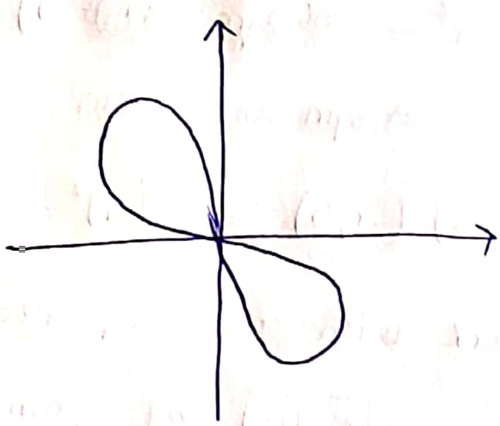
$b > 0$

$0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta \le \frac{3\pi}{2}$



$b < 0$

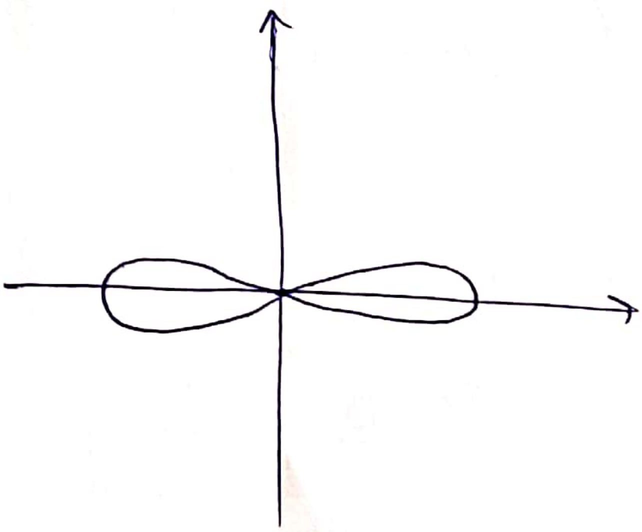
$\frac{\pi}{2} \le \theta \le \pi$ or $\frac{3\pi}{2} \le \theta \le 2\pi$



* $\boxed{r^2 = b\cos(2\theta)}$

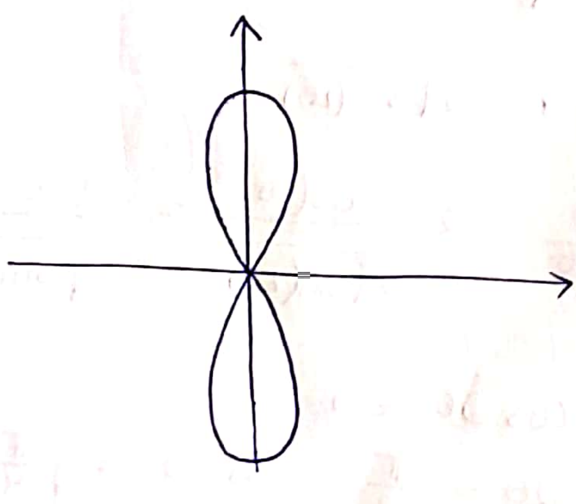
$b > 0$

$\frac{-\pi}{4} \le \theta \le \frac{\pi}{4}$ or $\frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}$



$b < 0$

$\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$ or $\frac{5\pi}{4} \le \theta \le \frac{7\pi}{4}$



Ex: $r^2 = 4\sin(2\theta)$

1) Interval : $4\sin(2\theta) \geq 0 \Rightarrow 0 \leq 2\theta \leq \pi$
$0 \leq \theta \leq \frac{\pi}{2} \Rightarrow [0, \frac{\pi}{2}]$

2) Symmetry:

a) θ by $-\theta$: inapplicable

b) θ by $\pi - \theta$: inapplicable

c) θ by $\pi + \theta$:

$r^2 = 4\sin(2(\pi+\theta)) = 4\sin(2\pi + 2\theta) = 4\sin(2\theta) = r$

$\Rightarrow$ sym wrt origin

d) $(-r, \theta)$: $(-r)^2 = 4\sin(2\theta) = r^2 \Rightarrow$ sym wrt origin

Work with $r = +2\sqrt{\sin(2\theta)}$ and complete the part of $r = -2\sqrt{\sin(2\theta)}$ by symmetry.

3) $r = 0$

$2\sqrt{\sin(2\theta)} = 0 \Rightarrow \sin(2\theta) = 0$ where $0 \leq 2\theta \leq \pi$

$2\theta = 0, \pi$

$\boxed{\theta = 0}$ $\boxed{\theta = \frac{\pi}{2}}$

Graph cuts the origin tangent to the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$

4) $r = 2\sqrt{\sin(2\theta)}$

$$\frac{dr}{d\theta} = 2\,\frac{2\cos(2\theta)}{2\sqrt{\sin(2\theta)}} = \frac{2\cos(2\theta)}{\sqrt{\sin(2\theta)}}$$

$\cos 2\theta = 0$

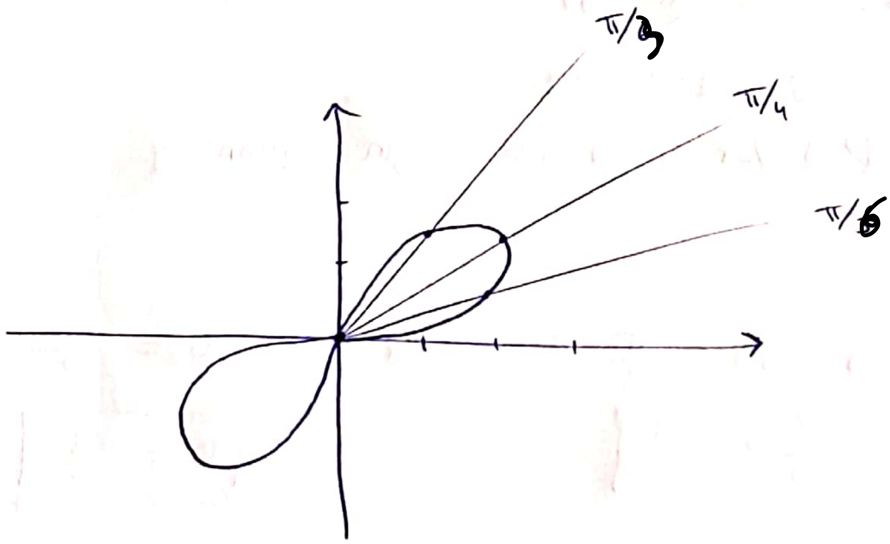
$2\theta = \frac{\pi}{2} \Rightarrow \theta = \boxed{\frac{\pi}{4}}$

θ	0		$\frac{\pi}{4}$		$\pi/2$
2θ	0		$\pi/2$		π
$\frac{dr}{d\theta}$		$+$	0	$-$	
r	0	$\nearrow$	2	$\searrow$	0

5)

θ	0	$\pi/2$	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/12$
r	0	0	1.86	2	1.86	1.4

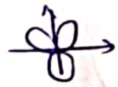
6)

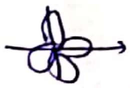


Roses:

$r = \cos(n\theta)$ where n is an integer ≥ 2

or $r = \sin(n\theta)$

n odd $\longrightarrow$ n leaves ($n = 3$ )

n even $\longrightarrow$ $2n$ leaves ($n = 2$ )

Ex: $r = \cos(3\theta)$

1) Interval $[-\pi, \pi]$

2) Symmetry:

a) θ by $-\theta$: $\cos(-3\theta) = \cos(3\theta) = r$

$\Rightarrow$ sym wrt x-axis $\Rightarrow$ reduce interval to $[0, \pi]$

b) θ by $\pi - \theta$: $\cos(3(\pi - \theta)) = \cos(3\pi - 3\theta) = \cos(\pi - 3\theta)$

$= -\cos(3\theta) = -r \Rightarrow$ sym wrt x-axis

$\Rightarrow$ reduce interval to $[0, \frac{\pi}{2}]$

c) θ by $\pi + \theta$: $\cos(3(\pi + \theta)) = \cos(3\pi + 3\theta) = -\cos(3\theta) \neq r$

d) $-r = -\cos(3\theta) \neq r \Rightarrow$ no origin symmetry.

3) $r = 0$

$\cos(3\theta) = 0 \Rightarrow 3\theta = \frac{\pi}{2}$ or $3\theta = \frac{3\pi}{2}$

$0 \leq \theta \leq \frac{\pi}{2}$ $\quad \boxed{\theta = \frac{\pi}{6}} \quad \boxed{\theta = \frac{\pi}{2}}$

$0 \leq 3\theta \leq \frac{3\pi}{2}$

4) $\frac{dr}{d\theta} = -3\sin(3\theta) = 0$

$\sin(3\theta) = 0 \Rightarrow 3\theta = 0$ $\quad 3\theta = \pi$

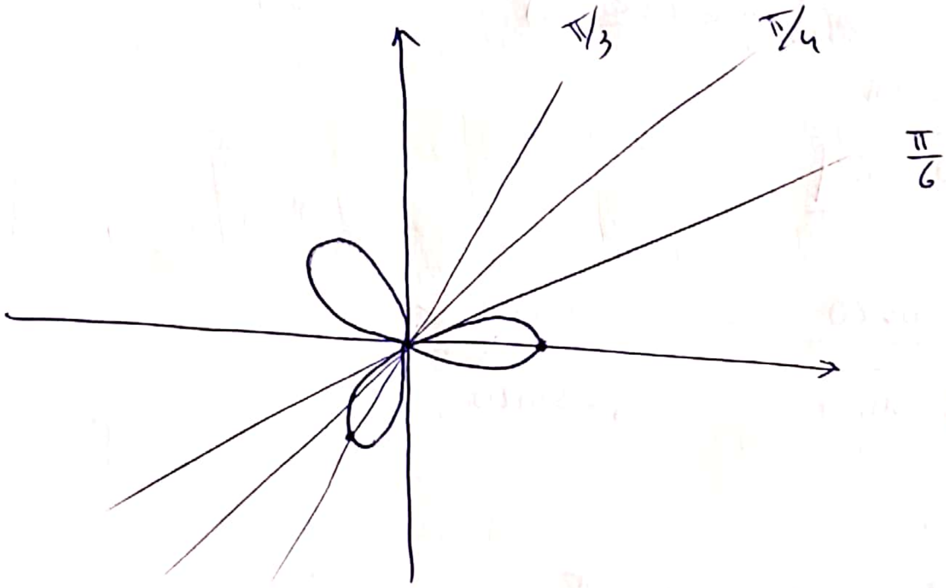
$\boxed{\theta = 0} \quad \boxed{\theta = \frac{\pi}{3}}$

θ	0		$\pi/3$		$\pi/2$
3θ	0		π		$3\pi/2$
$\frac{dr}{d\theta}$		$-$	0	$+$	
r	1	$\searrow$	-1	$\nearrow$	0

5)

θ	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
r	1	0.7	0	-0.7	-1	0

6)



Exercises:

1) $r = 1 + \cos\theta$ (done in lecture)

15) $r^2 = -\sin 2\theta$ (Lemniscate)

1) Interval: $-\sin 2\theta > 0$

$\sin 2\theta < 0$

$\pi \leq 2\theta \leq 2\pi$

$\frac{\pi}{2} \leq \theta \leq \pi \implies [\pi/2, \pi]$

2) Symmetry:

a) θ by $-\theta$: inapplicable

b) θ by $\pi - \theta$: inapplicable

c) θ by $\pi + \theta$: $r^2 = -\sin(2(\pi+\theta)) = -\sin 2\theta = r^2$

$\implies$ sym wrt origin.

d) $(-r)^2 = r^2 \Rightarrow$ sym wrt origin.

Work with $r = \sqrt{-\sin 2\theta}$ and find $r = -\sqrt{-\sin 2\theta}$ by symmetry.

3) $r = 0$

$\sqrt{-\sin 2\theta} = 0$

$\sin 2\theta = 0$ $\qquad (\pi \leq 2\theta \leq 2\pi)$

$2\theta = \pi$ $\qquad$ $2\theta = 2\pi$

$\boxed{\theta = \frac{\pi}{2}}$ $\qquad$ $\boxed{\theta = \pi}$

4) $\frac{dr}{d\theta} = \frac{-2\cos 2\theta}{2\sqrt{-\sin 2\theta}} = \frac{-\cos 2\theta}{\sqrt{-\sin 2\theta}}$

$-\cos 2\theta = 0$

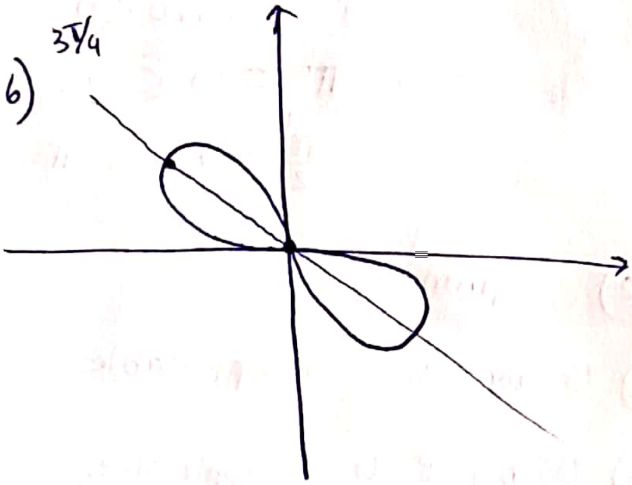
$2\theta = \frac{3\pi}{2} \Rightarrow \boxed{\theta = \frac{3\pi}{4}}$

θ	$\pi/2$		$3\pi/4$		π
2θ	π		$3\pi/2$		2π
$dr/d\theta$		+	0	−	
r	0	$\nearrow$	1	$\searrow$	0

5)

θ	$\pi/2$	$3\pi/4$	π
r	0	1	0

6)



18) $r = -1 + \sin\theta$ (Cardioid)

1) Interval $[-\pi, \pi]$

2) Symmetry:

a) θ by $-\theta$: $-1 + \sin(-\theta) = -1 - \sin\theta \neq r$ and $\neq -r$

b) θ by $\pi - \theta$: $-1 + \sin(\pi - \theta) = -1 + \sin\theta = r$

$\Rightarrow$ symmetry about y- axis

$\Rightarrow$ reduce interval to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

c) θ by $(\pi + \theta)$: $-1 + \sin(\pi + \theta) = -1 - \sin\theta \neq r$

d) $-r = -(-1 + \sin\theta) = 1 - \sin\theta \neq r$

$\Rightarrow$ no sym wrt origin.

3) $r = 0$

$-1 + \sin\theta = 0$

$\sin\theta = 1 \Rightarrow \boxed{\theta = \frac{\pi}{2}}$

4) $\frac{dr}{d\theta} = \cos\theta = 0$

$\boxed{\theta = -\frac{\pi}{2}}$ $\boxed{\theta = \frac{\pi}{2}}$

θ	$-\pi/2$		$\pi/2$
$dr/d\theta$		$+$	
r	-2	$\nearrow$	0

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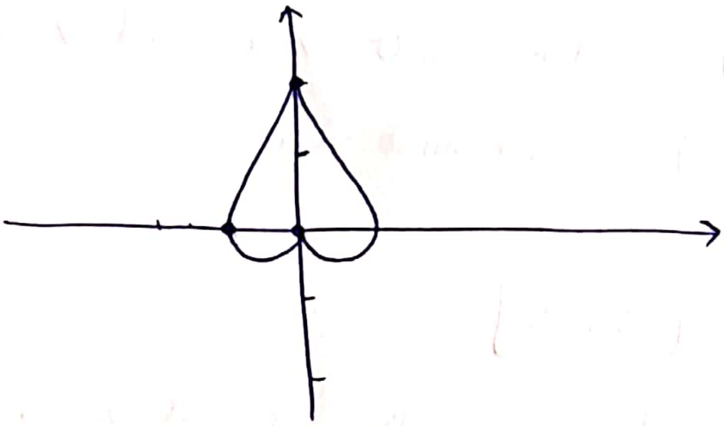
5)

θ	$-\pi/2$	0	$\pi/2$
r	-2	-1	0

6)



19) $r = \sin(2\theta)$ (Roses)

$n = 2$ (even) $\Longrightarrow$ $2 \times 2 = 4$ leaves

1) Interval $[-\pi, \pi]$

2) symmetry:

a) θ by $-\theta$: $\sin(-2\theta) = -\sin 2\theta = -r$

$\Longrightarrow$ sym wrt y-axis $\Longrightarrow$ reduce interval to $[0, \pi]$

b) θ by $\pi - \theta$: $\sin(2(\pi - \theta)) = \sin(2\pi - 2\theta) = -\sin 2\theta = -r$

$\Longrightarrow$ sym about x-axis $\Longrightarrow$ reduce interval to $[0, \pi/2]$

c) θ by $\pi + \theta$: $\sin(2(\pi + \theta)) = \sin(2\pi + 2\theta) = \sin 2\theta = r$

sym wrt origin

3) $r=0$ $\quad\quad 0 \leq \theta \leq \frac{\pi}{2}$

$\quad\quad 0 \leq 2\theta \leq \pi$

$\sin(2\theta)=0$

$2\theta=0 \quad\quad 2\theta=\pi$

$\boxed{\theta=0} \quad\quad \boxed{\theta=\pi/2}$

4) $\frac{dr}{d\theta} = 2\cos(2\theta)$

$\cos(2\theta)=0$

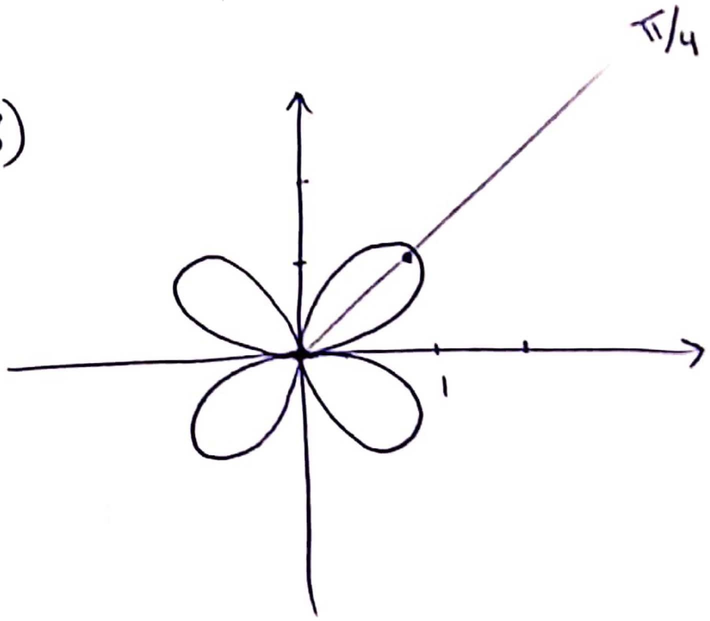
$2\theta=\frac{\pi}{2} \Longrightarrow \boxed{\theta=\frac{\pi}{4}}$

θ	0		$\pi/4$		$\pi/2$
2θ	0		$\pi/2$		π
$dr/d\theta$		+	0	$-$	
r	0	$\nearrow$	1	$\searrow$	0

5)

θ	0	$\pi/4$	$\pi/2$
r	0	1	0

6)



25) a) $r=\frac{1}{2}+\cos\theta$ (Limaçons)

(done in lecture)