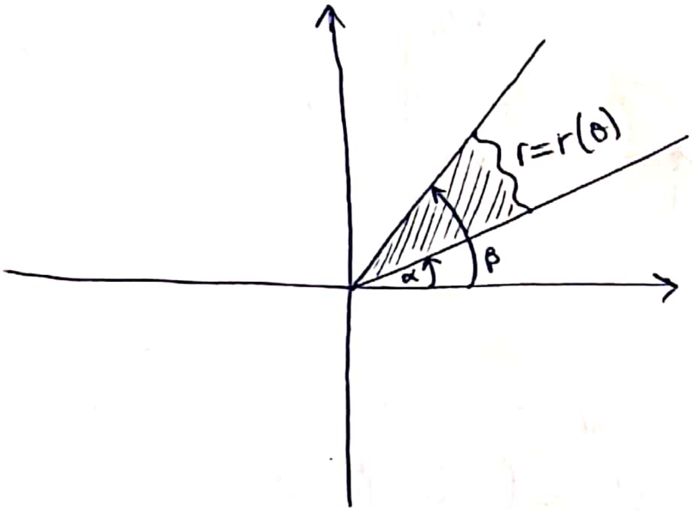
Section 11.5 – Areas in Polar Coordinates



The area of the region between the origin and the curve $r=f(\theta)$ for $\alpha \leq \theta \leq \beta$ is:

$$\boxed{\text{Area} = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} \left(r(\theta)\right)^2 d\theta}$$

Example: Find the area inside 1 leaf of the rose

$r = \cos(2\theta)$.

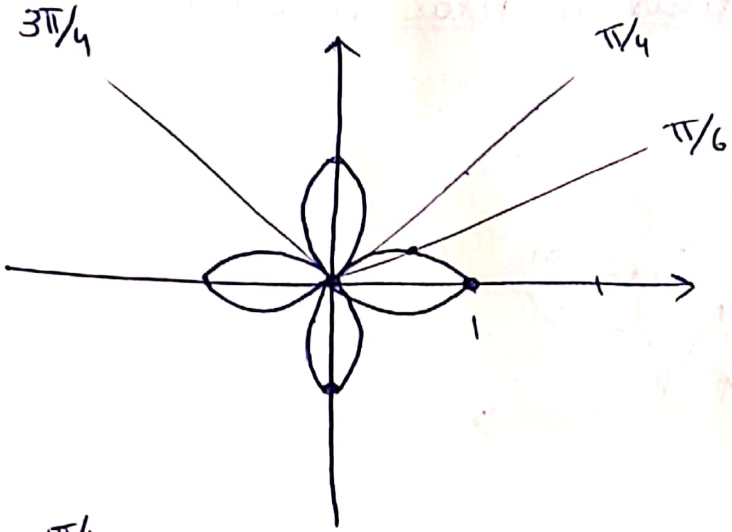
* Check for sym: x-axis, y-axis, origin. (reduce interval to $[0, \frac{\pi}{2}]$)

+ $r = 0 \iff \cos(2\theta) = 0$ $\quad 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$

curve is tangent to the line $\theta = \pi/4$ at the origin.

θ	0	$\frac{\pi}{6}$	$\pi/4$	$\pi/2$
r	1	0.8	0	-1



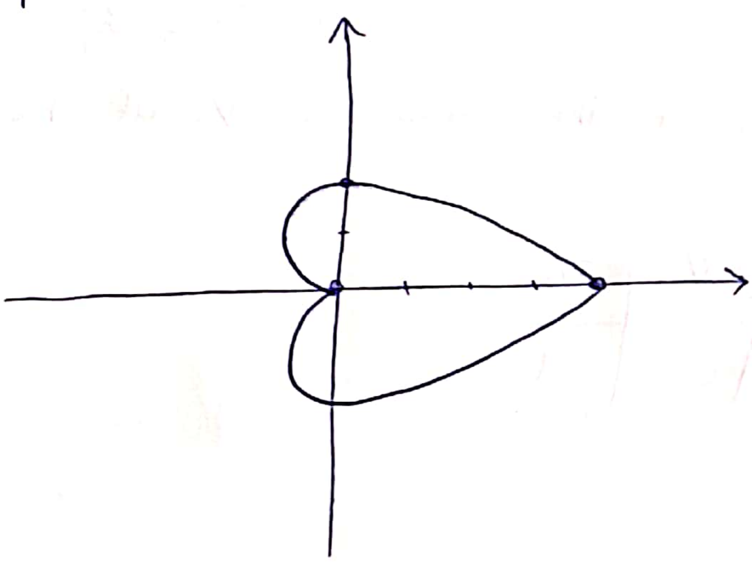
$$\text{Area} = 2\int_0^{\pi/4} \frac{1}{2} r^2 \, d\theta = 2\int_0^{\pi/4} \frac{1}{2} (\cos(2\theta))^2 \, d\theta$$

$$= \int_0^{\pi/4} (\cos(2\theta))^2 \, d\theta = \int_0^{\pi/4} \frac{1+\cos 4\theta}{2} \, d\theta = \dots\dots = \frac{\pi}{8} \, u^2$$

ex: Find the area of the region enclosed by the cardioid $r = 2(1+\cos\theta)$

$r = 2(1+\cos\theta)$: symmetry wrt x-axis. (interval is $[0,\pi]$)

θ	0	$\pi/2$	π
r	4	2	0



$$\text{Area} = 2\int_0^{\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi} 4(1+\cos\theta)^2 \, d\theta = 4\int_0^{\pi} 1+2\cos\theta+\cos^2\theta$$

$$= 4\int_0^{\pi}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta = 4\int_0^{\pi}\left(1+2\cos\theta+\frac{1}{2}+\frac{\cos 2\theta}{2}\right)d\theta$$

$$= 4\int_0^{\pi}\left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right)d\theta$$

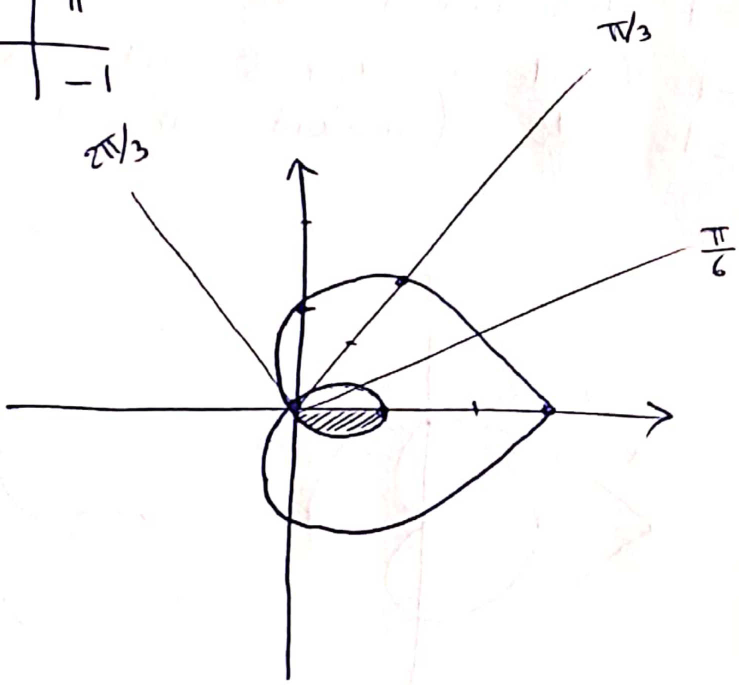
$$= 4\left[\frac{3}{2}\theta+2\sin\theta+\frac{\sin 2\theta}{4}\right]_0^{\pi} = 6\pi \ u^2.$$

ex: Set up an integral for the area inside the smaller loop for the limaçon $r = 1+2\cos\theta$

(symmetry wrt x-axis) : reduce interval to $[0,\pi]$

$r=0 \iff 1+2\cos\theta=0 \iff \cos\theta = -\frac{1}{2} \iff \theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \dots$

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	3	2	1	0	-1

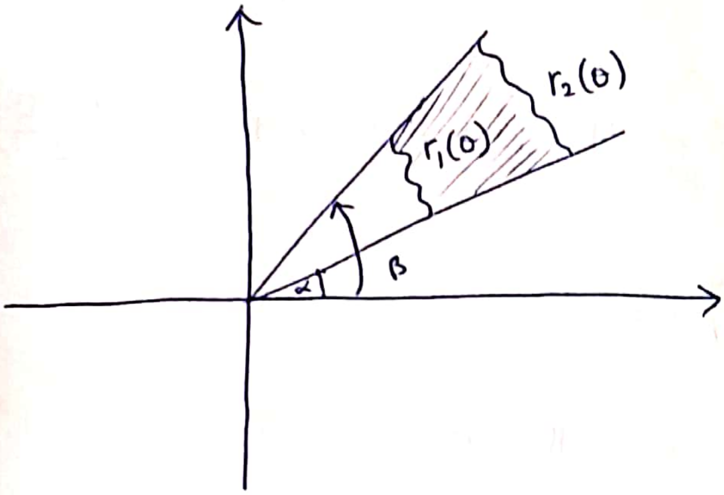


$$\underline{\text{Area}}: 2\int_{2\pi/3}^{\pi} \frac{1}{2} r^2 \, d\theta$$

$$= 2\int_{2\pi/3}^{\pi} \frac{1}{2}(1+2\cos\theta)^2 \, d\theta$$

$$= \int_{2\pi/3}^{\pi} (1+2\cos\theta)^2 \, d\theta$$

Area Between Polar Curves:



$$\int_\alpha^\beta \frac{1}{2} r_2^2 \, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 \, d\theta$$

$$= \int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

Area of region : $0 \leq r_1(\theta) \leq r(\theta) \leq r_2(\theta)$

for $\alpha \leq \theta \leq \beta$. is given by:

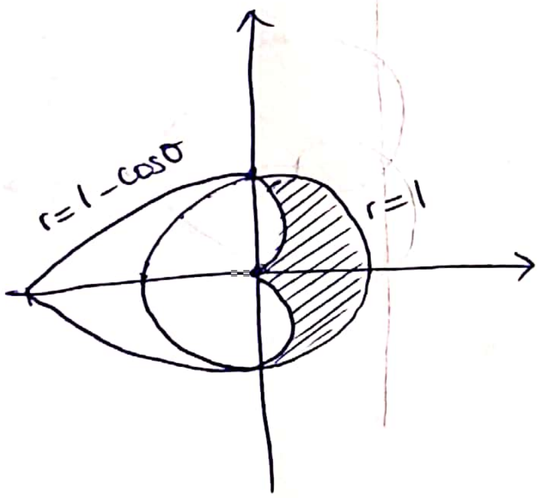
$$\boxed{\text{Area} = \int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta}$$

Example: Set up integral(s) for the area inside the curve $r=1$ and outside the curve $r = 1 - \cos\theta$

$r=1$ (circle of center O and radius 1)

$r = 1 - \cos\theta$ (cardioid sym wrt x-axis).

θ	0	$\pi/2$	π
r	0	1	2



$r = r$

$1 - \cos\theta = 1$

$\cos\theta = 0$

$$\boxed{\theta = \frac{\pi}{2}}$$

$$\text{Area} = 2\int_0^{\pi/2} \frac{1}{2}\left[1^2 - (1-\cos\theta)^2\right] d\theta$$

$$= \int_0^{\pi/2} \left[1 - (1-\cos\theta)^2\right] d\theta$$

* If asked about the area shared:

$$\text{Area} = 2\left[\int_0^{\pi/2} \frac{1}{2}(1-\cos\theta)^2 d\theta + \int_{\pi/2}^{\pi} \frac{1}{2}(1^2)\, d\theta\right]$$

$$= \int_0^{\pi/2} (1-\cos\theta)^2 d\theta + \int_{\pi/2}^{\pi} d\theta.$$

Exercises:

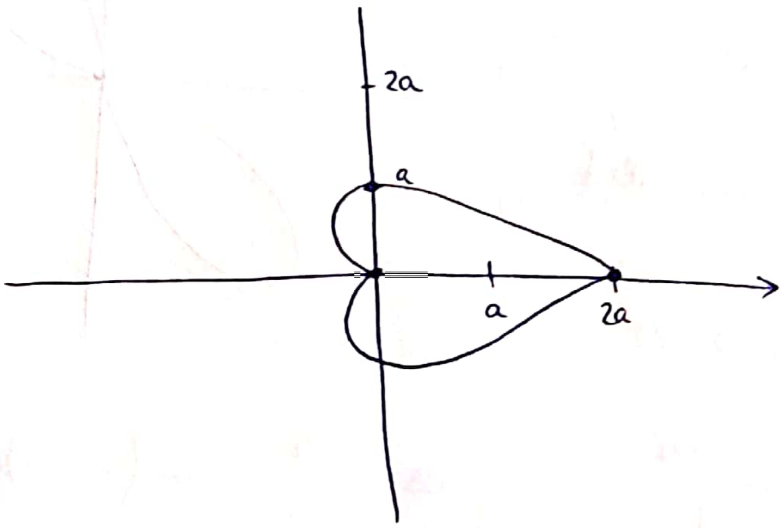
Find the areas of the regions:

4) Inside the cardioid $r = a(1+\cos\theta)$ $a>0$

* sym wrt x-axis : interval is $[0,\pi]$

* $r=0 \iff 1+\cos\theta = 0 \iff \cos\theta = -1 \iff \theta = \pi$

θ	0	$\pi/2$	π
r	$2a$	a	0



$$\text{Area} = 2\int_0^{\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi} a^2 (1+\cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^{\pi} \left(1 + 2\cos\theta + \frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta = a^2 \left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= a^2 \left[\frac{3}{2}\pi\right] = \frac{3a^2\pi}{2} \; u^2$$

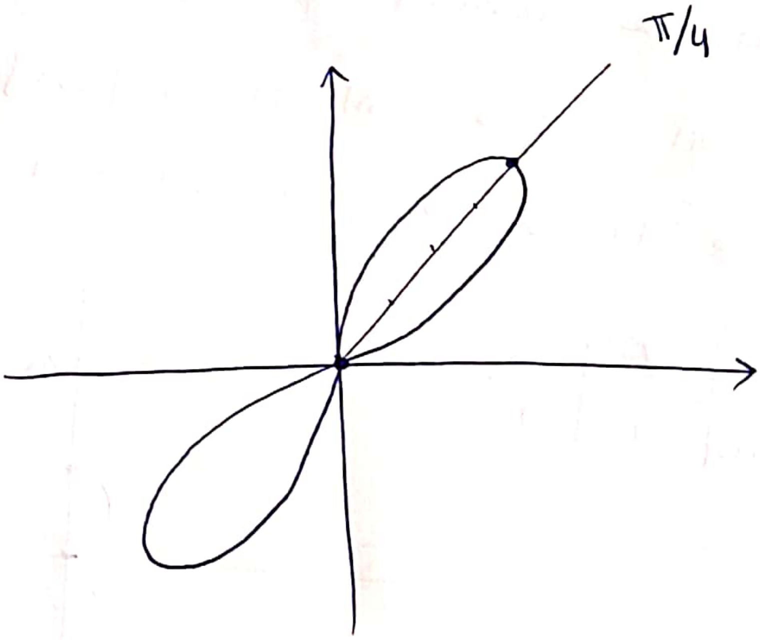
5) Inside one leaf of the rose $r = \cos(2\theta)$ (done in lecture)

7) Inside one loop of the lemniscate $r^2 = 4\sin 2\theta$

* $0 \leq 2\theta \leq \pi$
  $0 \leq \theta \leq \frac{\pi}{2}$

* symmetry about the origin
* $r = 0 \iff \sin 2\theta = 0 \iff 2\theta = 0$ or $2\theta = \pi$
  $\theta = 0$ $\quad \theta = \pi/2$

θ	0	$\pi/4$	$\pi/2$
r	0	2	0



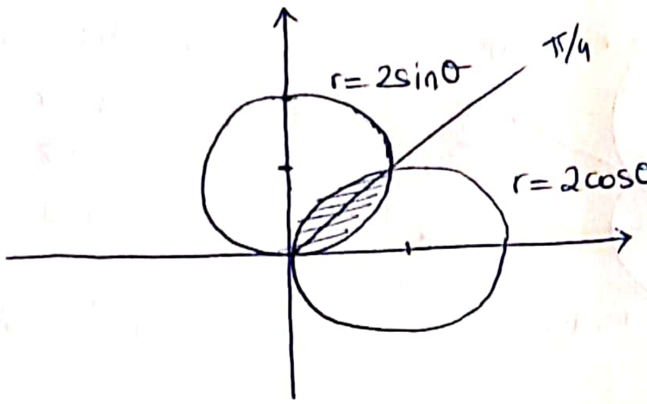
$$\text{Area} = \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2} \cdot 4\sin 2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$= 2\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2} = -\left[\cos 2\theta\right]_0^{\pi/2} = -(-1-1) = 2 \; u^2$$

9) Shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$



$r = 2\cos\theta$ (center $(1,0)$ radius 1)

θ	0	$\pi/2$	π
r	2	0	-2

$r = 2\sin\theta$ (center $(0,1)$ radius 1)

θ	0	$\pi/2$	π
r	0	2	0

* intersection: $r = r$

$2\cos\theta = 2\sin\theta \Rightarrow \boxed{\theta = \frac{\pi}{4}}$

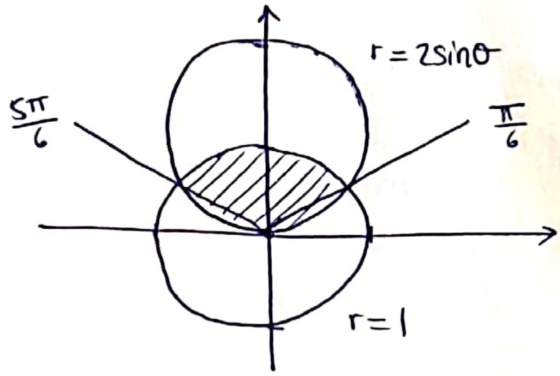
$$\text{Area} = \int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2\,d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$$

$$= 2\int_0^{\pi/4} \sin^2\theta\,d\theta + 2\int_{\pi/4}^{\pi/2} \cos^2\theta\,d\theta = 2\int_0^{\pi/4} \frac{1-\cos 2\theta}{2}\,d\theta + 2\int_{\pi/4}^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4} + \left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2} = \left[\frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \left(\frac{\pi}{2} - 1\right)u^2$$

10) Shared by the circles $r=1$ and $r=2\sin\theta$.



* intersection:

$r=r$

$2\sin\theta=1$

$\sin\theta=\frac{1}{2}$

$\boxed{\theta=\frac{\pi}{6}}$

$$\text{Area} = 2\left[\int_0^{\pi/6}\frac{1}{2}(2\sin\theta)^2\,d\theta + \int_{\frac{\pi}{6}}^{\pi/2}\frac{1}{2}(1^2)\,d\theta\right]$$

$$= \int_0^{\pi/6} 4\sin^2\theta\,d\theta + \int_{\frac{\pi}{6}}^{\pi/2} d\theta = 4\int_0^{\pi/6}\frac{1-\cos 2\theta}{2}\,d\theta + \int_{\pi/6}^{\pi/2} d\theta$$

$$= \left[2\theta - \sin 2\theta\right]_0^{\pi/6} + \left[\theta\right]_{\pi/6}^{\pi/2} = \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\pi}{2} - \frac{\pi}{6} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)u^2$$

13) Inside the lemniscate $r^2 = 6\cos(2\theta)$ and outside the circle $r=\sqrt{3}$

* $r^2 = 6\cos(2\theta)$ lemniscate : sym wrt x-axis, y-axis, origin.

interval : $-\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$

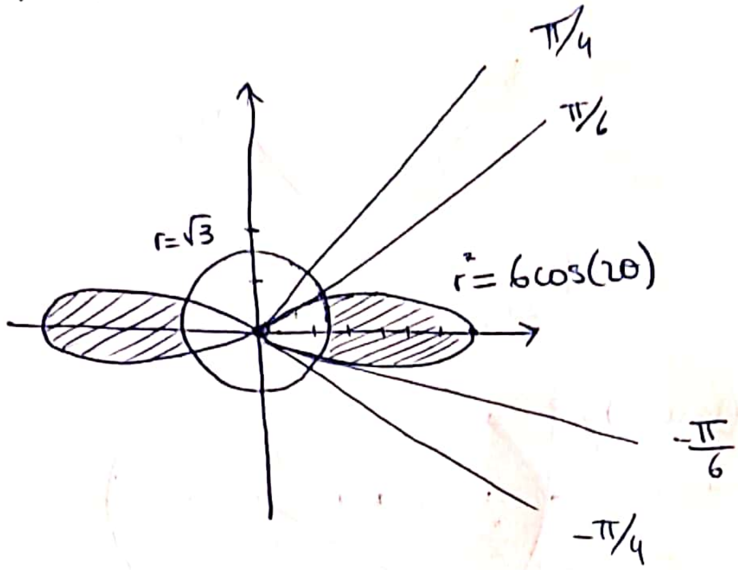
$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

$r=0 \iff \cos(2\theta)=0$

$2\theta = -\frac{\pi}{2}$ or $2\theta = \frac{\pi}{2}$

$\boxed{\theta = -\frac{\pi}{4}}$ $\boxed{\theta = \frac{\pi}{4}}$

θ	$-\pi/4$	0	$\pi/4$
r	0	$\sqrt{6}$	0



Intersection:

$$r^2 = r^2$$
$$3 = 6\cos(2\theta)$$
$$\cos(2\theta) = \frac{1}{2}$$
$$2\theta = \frac{\pm\pi}{3} \Rightarrow \boxed{\theta = \frac{\pm\pi}{6}}$$

$$\text{Area} = 2\int_{-\pi/6}^{\pi/6} \frac{1}{2}\left(6\cos 2\theta - 3\right) d\theta = \int_{-\pi/6}^{\pi/6} (6\cos 2\theta - 3)\, d\theta$$

$$= \left[3\sin 2\theta - 3\theta\right]_{-\pi/6}^{\pi/6} = 3\frac{\sqrt{3}}{2} - \frac{\pi}{2} + 3\frac{\sqrt{3}}{2} - \frac{\pi}{2} = (3\sqrt{3} - \pi)\, u^2$$

14) Inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1+\cos\theta)$ $a > 0$.

* $r = 3a\cos\theta$ (circle of center $\left(\frac{3a}{2}, 0\right)$ and radius $\frac{3a}{2}$)

* $r = a(1+\cos\theta)$

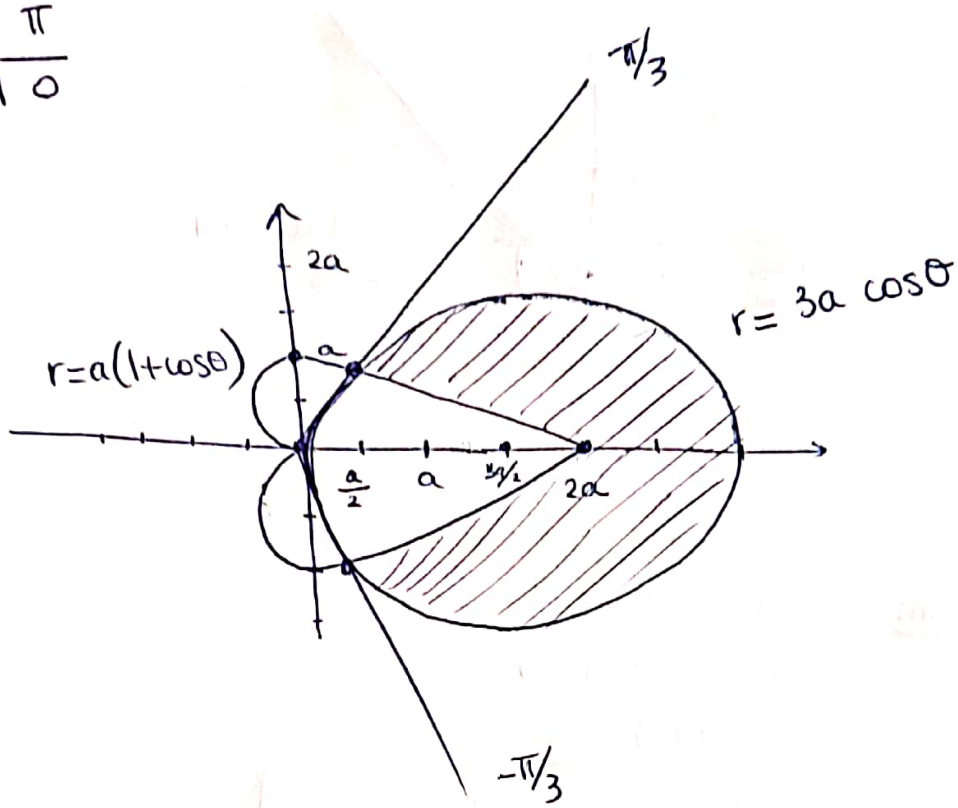
sym wrt x-axis

$\Rightarrow$ interval is $[0, \pi]$

$r=0 \iff 1+\cos\theta = 0$

$\cos\theta = -1 \implies \boxed{\theta = \pi}$

θ	0	$\pi/2$	π
r	$2a$	a	0



<u>Intersection:</u> $r=r$

$3a\cos\theta = a(1+\cos\theta)$

$3\cos\theta = 1+\cos\theta$

$2\cos\theta = 1 \implies \cos\theta = \frac{1}{2} \implies \boxed{\theta = \frac{\pi}{3}}$

$$\text{Area} = 2\int_0^{\pi/3} \frac{1}{2}\left(9a^2\cos^2\theta - a^2(1+\cos\theta)^2\right)d\theta$$

$$= \int_0^{\pi/3}\left(9a^2\cos^2\theta - a^2 - 2a^2\cos\theta - a^2\cos^2\theta\right)d\theta$$

$$= a^2\int_0^{\pi/3}(8\cos^2\theta - 1 - 2\cos\theta)\,d\theta = a^2\int_0^{\pi/3}(4+4\cos 2\theta - 1 - 2\cos\theta)\,d\theta$$

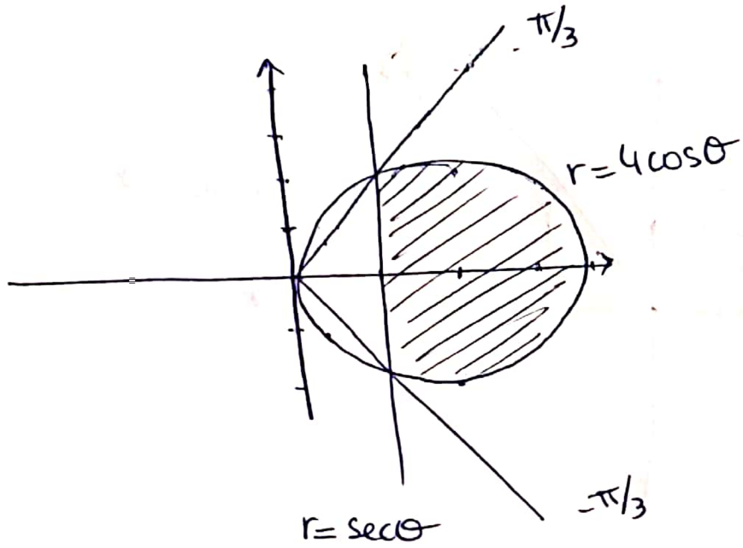
$$= a^2\left[3\theta + 2\sin 2\theta - 2\sin\theta\right]_0^{\pi/3} = a^2\left[\pi + \sqrt{3} - \sqrt{3}\right]$$

$$= (a^2\pi)\ u^2$$

17) Inside the circle $r = 4\cos\theta$ and to the right of the vertical line $r = \sec\theta$

* $r = 4\cos\theta$: circle of center $(2,0)$ and radius 2.

* $r = \sec\theta \iff r\cos\theta = 1 \iff x = 1$: vertical line.



Intersection: $r = r$

$4\cos\theta = \sec\theta \iff 4\cos^2\theta = 1 \iff \cos^2\theta = \frac{1}{4} \iff \cos\theta = \frac{1}{2}$

$\iff \boxed{\theta = \frac{\pm\pi}{3}}$

$$\text{Area} = 2\int_0^{\pi/3} \frac{1}{2}\left(16\cos^2\theta - \sec^2\theta\right) d\theta = \int_0^{\pi/3} \left(8 + 8\cos 2\theta - \sec^2\theta\right) d\theta$$

$$= \left[8\theta + 4\sin 2\theta - \tan\theta\right]_0^{\pi/3} = \frac{8\pi}{3} + 4\frac{\sqrt{3}}{2} - \sqrt{3}$$

$$= \left(\frac{8\pi}{3} + \sqrt{3}\right) u^2$$

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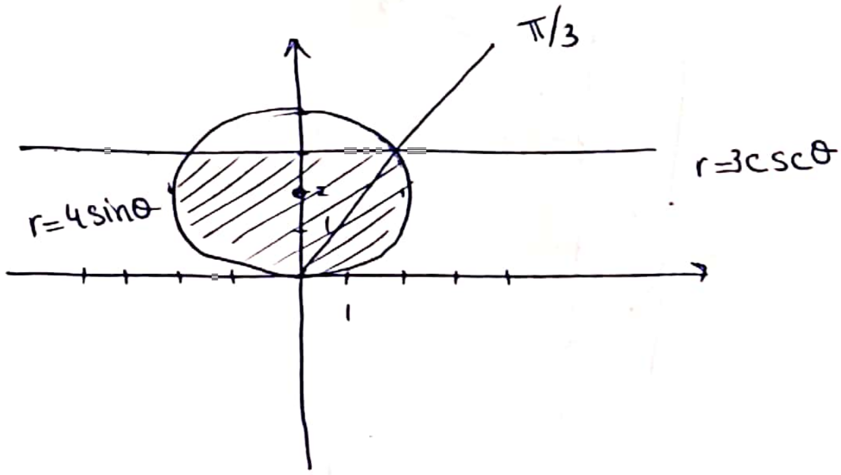
18) Inside the circle $r = 4\sin\theta$ and below the horizontal line $r = 3\csc\theta$.

* $r = 4\sin\theta$: circle of center $(0,2)$ and radius 2.
* $r = 3\csc\theta \iff r\sin\theta = 3 \iff y = 3$: horizontal line



Intersection: $r = r$

$4\sin\theta = 3\csc\theta \iff \sin^2\theta = \frac{3}{4} \iff \sin\theta = \frac{\sqrt{3}}{2} \iff \boxed{\theta = \frac{\pi}{3}}$

$$\text{Area} = 2\left[\int_0^{\pi/3} \frac{1}{2}\cdot 16\sin^2\theta \, d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}\, 9\csc^2\theta \, d\theta\right]$$

$$= \int_0^{\pi/3} 16\sin^2\theta \, d\theta + \int_{\pi/3}^{\pi/2} 9\csc^2\theta \, d\theta = \int_0^{\pi/3} (8 - 8\cos 2\theta)d\theta + \int_{\pi/3}^{\pi/2} 9\csc^2\theta \, d\theta$$

$$= \Big[8\theta - 4\sin 2\theta\Big]_0^{\pi/3} - 9\Big[\cot\theta\Big]_{\pi/3}^{\pi/2}$$

$$= \frac{8\pi}{3} - \frac{4\sqrt{3}}{2} - 9\left(0 - \frac{\sqrt{3}}{3}\right) = \frac{8\pi}{3} - 2\sqrt{3} + 3\sqrt{3} = \left(\frac{8\pi}{3} + \sqrt{3}\right)u^2$$